PERSPECTIVES IN LOGIC

Edited by
J. Barwise
S. Feferman

MODEL-THEORETIC LOGICS

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Model-Theoretic Logics

Since their inception, the Perspectives in Logic and Lecture Notes in Logic series have published seminal works by leading logicians. Many of the original books in the series have been unavailable for years, but they are now in print once again.

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PERSPECTIVES IN LOGIC

The *Perspectives in Logic* series publishes substantial, high-quality books whose central theme lies in any area or aspect of logic. Books that present new material not now available in book form are particularly welcome. The series ranges from introductory texts suitable for beginning graduate courses to specialized monographs at the frontiers of research. Each book offers an illuminating perspective for its intended audience.

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Preface to the Series

Perspectives in Mathematical Logic

(Edited by the Ω-group for "Mathematische Logik" of the Heidelberger Akademie der Wissenschaften)

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematize the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought, and that, by developing certain* themes, *it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory, some highly specialized. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book in with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R. O. Gandy, A. Levy, G. H. Müller, G. E. Sacks, D. S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F. K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of the series. We established our centre* *in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over-all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.*

Oberwolfach, September 1975

Acknowledgements. *In starting our enterprise we essentially were relying on the personal confidence and understanding of Professor Martin Barner of the Mathematisches Forschungsinstitut Oberwolfach, Dr. Klaus Peters of Springer-Verlag and Dipl.-Ing. Penschuck of the Stiftung Volkswagenwerk. Through the Stiftung Volkswagenwerk we received a generous grant (1970–1973) as an initial help which made our existence as a working group possible.*

Since 1974 the Heidelberger Akademie der Wissenschaften (Mathematisch-Naturwissenschaftliche Klasse) has incorporated our enterprise into its general scientific program. The initiative for this step was taken by the late Professor F. K. Schmidt and the former President of the Academy, Professor W. Doerr.

Through all the years, the Academy has supported our research project, especially our meetings and the continuous work on the Logic Bibliography, in an outstandingly generous way. We could always rely on their readiness to provide help wherever it was needed.

Assistance in many various respects was provided by Drs. U. Felgner and K. Gloede (till 1975) and Drs. D. Schmidt and H. Zeitler (till 1979). Last but not least, our indefatigable secretary Elfriede Ihrig was and is essential in running our enterprise.

We thank all those concerned.

Heidelberg, September 1982

R. O. Gandy *H. Hermes*
A. Levy *G. H. Müller*
G. E. Sacks *D. S. Scott*

Preface

The subject matter of this book constitutes a merging of several directions of work in general model theory over the last 25 years. Three main lines can be distinguished: first, that initiated by Andrzej Mostowski on *cardinality quantifiers* in the late 1950s; second, the work of Alfred Tarski, his colleagues and students on *infinitary languages* into the mid-1960s; and, finally, that stemming from the results of Per Lindström on *generalized quantifiers* and *abstract characterizations of first-order logic* in the late 1960s. The subject of *abstract model theory* blossomed from that as a unified and illuminating framework in which to organize, compare, and seek out the properties of the many stronger logics which had then come to be recognized.

Interest in abstract model theory and extended logics was intense in the early 1970s, particularly as a result of the work of Jon Barwise on *infinitary admissible languages*. Where the previous developments had largely connected up model theory with set theory, this added ideas from extended recursion theory in an essential way, e.g., to yield successful infinitary generalizations of the compactness theorem. It also turned out that proof theory—including such consequences as the interpolation and definability theorems—could be successfully generalized to these languages. Thus, one was here witness to an exciting confluence of all the main branches of mathematical logic. These successes led to promising research programs for further interactive development, but the hopes they raised, especially with respect to the treatment of uncountable languages, were not realized. A number of us (including Barwise and myself) who had been involved at that stage of the subject turned to other interests in the latter part of the 1970s and gave little attention to its ongoing progress. As it happens, all through that period (at least) the set-theoretic and model-theoretic aspects of the subject were continuing to develop at a rapid rate. Looking again at the field in 1980 we found a body of work that was quite staggering.

The re-examination of the area at that time had come in response to repeated urging by the editors of the Ω-group (particularly by Gert Müller) for Barwise and/or me to write a volume for the series on the general subject of model-theoretic logics. This might have been conceivable in the early 1970s, but owing to the intervening growth in the field, it was clearly beyond us eight years later. On the one hand, the field seemed to be in such a state of advanced and intense development that the idea of writing a relatively finished text did not seem appropriate—

even if undertaken by someone working directly in the subject. On the other hand, it became more and more apparent that unless some effort were made to provide an exposition of the field as currently understood, many potential researchers would simply be left behind.

An alternative idea then presented itself: namely, to knit together a number of individual contributions which would provide substantial coverage of the field and would constitute an introduction to the main ideas, examples, and results of the literature. Barwise and I made this proposal with some trepidation at the meeting of the Ω-group in Patras in August, 1980. We wanted it clearly understood that we were not suggesting writing a "handbook," each part of which would be reporting on a relatively finished or settled topic. Rather, we wanted to present a picture of a rapidly evolving subject, in which much that has been accomplished so far must be digested if one is to contribute to further progress. The aim of the project would be to give an entry into the field for anyone sufficiently equipped in general model theory and set theory, and thereby to bring them closer to the frontiers of research. This proposal, together with our preliminary table of contents and suggested list of individual authors of chapters, was agreed to enthusiastically by the editors of the Ω-group. As it happened, a number of those we had suggested as prospective authors were attending the Patras Logic Symposium. We quickly gathered from them enough expressions of willingness to participate so that the viability of the project could be assured.

Thanks to the support of the Ω-Group and the financial generosity of the Heidelberger Akademie der Wissenschaften, the organization of work on this book was able to proceed in a unique cooperative way. Almost all the authors and editors met together as a group on two occasions, first in Freiburg during the period 21–27 June 1981 and then at Stanford during the period 28 March–4 April 1982. At the first of these meetings, authors brought plans, outlines and, in many instances, first drafts of their chapters. These materials were explained, discussed, and circulated. In effect, this constituted a very high-level interchange on matters of substance, style, approach, and exposition. (It emerged that three additional chapters were needed to round out the coverage for which, fortunately, authors could be secured.) The participants found the Freiburg meeting extremely exciting and stimulating, and left with high confidence in the success of the project. Afterward, preparation continued at a quicker pace than originally expected. At the second meeting in Stanford, authors brought semi-final drafts of their chapters, and continued the process established in Freiburg. Then each chapter was circulated in the summer of 1982 to two other authors and/or editors for reading, detailed comments and suggestions. On the basis of these comments, chapters were brought to final form and submitted to the editors by early 1983. Soon after, a small working editorial group (meeting at Stanford) organized the manuscript in final form, touching up and smoothing out the chapters, preparing explanatory introductions on the various parts and completing the work on a unified bibliography. In doing so, no effort was made to impose uniformity of style or thought. The aim was to bring out the individual contributions in the best and most understandable and useful form possible.

The book is divided into six parts (A–F), each consisting of two to four chapters. Part A provides an introduction to the subject as a whole as well as to the basic theory and examples. In particular, Chapter I, by Barwise, presents a general discussion of the background and aims of our subject. Each part is preceded by a detailed introduction summarizing its contents. From that material the reader will learn which chapters can be read for general purposes and which for more special research interests, together with the background required in each case. It will be seen that many of the chapters can be read independently and that moving back and forth between them can be rewarding. Parts B–F of the book take up, in turn: finitary languages with additional quantifiers, infinitary languages, second-order logic, logics of topology and analysis, and advanced topics in abstract model theory.

An explanation is needed for the form of the bibliography for this book. As the chapters were being written, it soon became apparent that the individual bibliographies would be a valuable source of references and history taken in combination. Scott volunteered to oversee the collection of the materials to make a single, unified listing. What has been incorporated are all the contributions from the authors (with many additions and corrections which they sent in), an early bibliography started by Barwise, and many selections from the Omega Logic Bibliography at Heidelberg. At the start no one had any idea that the listing would have 1,261 items—or how vexing it would be to run down certain items. Fortunately, after coming to Carnegie-Mellon University in 1981, Scott was able to arrange that the bibliography be put on the computer, which was also used to make camera-ready copy; otherwise, without computer aids, a task of this size would have been virtually impossible. In the event, this project turned out to be more labor intensive than had been anticipated; it could not have been carried out without the full collaboration of Charles McCarty and John Horty, who spent many hours over many months checking sources, working in the library, typing into terminals, and proofreading many versions. The compilers were also aided by many other people at CMU: W. L. Scherlis and Roberto Minio gave us constant help with the TEX type-setting system, and Todd Knoblock, Lars W. Ericson, and John Aronis at various times served as programmers on the project; without their expertise and help on many small problems, nothing could have been done. Marko Petkovsek also very ably assisted with the final proofreading. The editors would thus like to take the opportunity to convey their warm thanks to all these people for their efforts, to the Computer-Science Department of Carnegie-Mellon University for the support of McCarty and Horty and the programmers and particularly for the generous use of their facilities, and to the authors, who helped assemble and check details of the bibliography. Alas, as it stands the bibliography is not complete historically, but, even so, the editors and compliers hope it will materially aid future students and researchers in learning about this work and continuing the investigations.

In addition to a unified bibliography, the idea of having a unified open problem section had also been given serious consideration. However, it was finally decided that such problems are best appreciated in the specific contexts in which they arise, and that no general rule need be followed as to their location in the individual chapters.

We also wish to express our thanks to the students Ian Mason and Sergio Fajardo, who read and made useful comments on various chapters; to Priscilla Feigen and Isolde Field, for their great assistance in many ways during and after the meetings at Stanford; likewise to Elfriede Ihrig for assistance at Freiburg and Heidelberg; to the secretarial staffs of the many institutions represented by the editors and authors of this book who helped in its preparation; to the University of Freiburg and Stanford University for providing us with facilities for our meetings; to the publisher, Springer-Verlag, and particularly the assistance of its editors; and finally (once more) to the Heidelberger Akademie der Wissenschaften, without whose support nothing like the present volume would have been possible.

Stanford, 1 *March* 1983 *Solomon Feferman*

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Part A

Introduction, Basic Theory and Examples

This part of the book provides a basic setting for the chapters that follow, by isolating examples and concepts that have emerged as central and by presenting some of the more basic methods and results. Chapter I discusses how the subject of model-theoretic logics got started, both the parts that have to do with extended logics, and the part having to do with abstract model theory. The chapter presupposes familiarity with only the most basic parts of first-order model theory, its syntax and semantics.

In Chapter II the basic concept of a logic is presented, with many examples, as well as the concepts of elementary and projective class and compactness, Löwenheim–Skolem and definability properties. The notion of one logic being stronger than another is introduced and studied. Examples discussed include higher-order logics, logics with cardinality and cofinality quantifiers, infinitary logics and other logics with generalized quantifiers and logical operations.

Given any particular logic $\mathscr{L}$ one central problem is that of understanding when two structures are $\mathscr{L}$-equivalent, that is, satisfy the same $\mathscr{L}$-sentences. Among the basic results of Chapter II is a characterization of $\mathscr{L}$-equivalence in terms of partial isomorphisms, for a wide range of $\mathscr{L}$. Here we have a good example of a method borrowed from first-order logic which really comes into its own only in the more general setting. Another important method presented in Chapter II is the use of projective classes (PC) for establishing countable compactness and recursive axiomatizability for a host of logics.

Chapter III begins with an exposition of Lindstrom's theorem, which shows that first-order logic is the strongest logic (of ordinary structures) which satisfies the compactness and Löwenheim–Skolem properties. First-order logic is also shown to be maximal with respect to other combinations of familiar properties. The methods used are those of partial isomorphisms and projective classes.

Lindstrom's theorem has become a paradigm for characterizing other logics. Among those discussed in Chapter III are certain infinitary logics and logics with added quantifiers. Chapter III ends with an abstract characterization theorem which covers Lindstrom's theorem as well as logics for other types of structures, like topological structures. This connects with work in Chapter XV.

These chapters are meant to be accessible to anyone with a knowledge of basic model theory for first-order logic. They provide the reader with the basic notions and viewpoint needed to appreciate what follows.

Chapter I
Model-Theoretic Logics: Background and Aims

by J. Barwise

Two aspects to the study of model-theoretic logics are represented in this volume. First, there is the isolation and study of specific model-theoretic languages, or *logics* as they are called here, for the study of various mathematical properties. Second, there is the investigation into relations between these logics. These two parts of the subject are called *extended model theory* and *abstract model theory*, respectively, and are the two subjects of the two main sections of this chapter.

In writing this chapter I hope to give a perspective from which to view the study of model-theoretic logics. First (in Section 1.2) I will contrast the view of logic implicit in this endeavor with what I call the *first-order thesis*, a view of logic and mathematics which claims that logic *is* first-order logic. Then (in the rest of Sections 1 and 2) I will discuss some of the motivation, ideas, aims, and preconceptions of early workers in the subject. The first is needed to appreciate the most basic definitions. The second is needed to judge the progress made against those early hopes and preconceptions. Where it is natural, I will point ahead to later chapters, but more specific introductions to the chapters will be found at the beginning of each part of the book.

1. Logics Embodying Mathematical Concepts

In extended model theory one asks, "What is the logic of specific mathematical concepts?" More explicitly, given a particular mathematical property (like being a finite, infinite, countable, uncountable, or open set, or being a well-ordering or a continuous function, or having probability greater than some real number r), what is the logic implicit in the mathematician's use of the property? What sorts of mathematical structures isolate the property most naturally? What sorts of languages best mirror the mathematician's talk about the property? What forms of reasoning about it are legitimate? Which other properties are implicit in it or are presupposed by it?

1.1. Logic, Structures and Logics

A word of explanation is in order about the way we are using the words "logic", "structure" and "logics" here. For the person in the street, logic is the study of valid forms of reasoning, from the most mundane uses in our day-to-day lives to the most sophisticated uses in science and mathematics. If you and I are discussing some topic, like fixing the roof, a law of genetics, or the solution to some partial differential equation, and I say "The logic of that escapes me", what I mean is that I do not see how the conclusion you have come to follows from our shared assumptions and concepts, including the conception of the task at hand. How does it follow from the properties of roofs, or the laws of genetics that we both accept, or the concepts involved in differential equations? When I talk of *logic* as I have above, I am referring to this common sense, person-in-the-street notion.

On the common sense view of logic, all the concepts we use to cope with and organize our world have their own logic. As logicians, we are perfectly entitled to delve into their logic. However, as mathematical logicians, or metamathematicians, our interest is more specialized. What we seek to understand is the logic of precise mathematical concepts. Extended model theory makes a frontal attack on this problem by, where appropriate, building "logics" to get answers to some of the questions listed above.

We assume that the reader of this volume is familiar with first-order logic, its syntax, semantics and basic model theory, because first-order logic is the inspiration for extended model theory. The basic idea of model theory, first-order and beyond, is that one can profit by paying attention to the relationship between some *mathematical structures* and some collection of expressions of a language used to describe properties of such structures. The basic notion is that of *satisfaction*: $\mathfrak{M} \models \phi$ if the expression ϕ is *true of*, or satisfied by, the structure $\mathfrak{M}$. First-order logic considers mathematical structures of a particularly algebraic sort, domains of individuals with arbitrary sets and functions to serve as interpretations for various predicate and function symbols. It allows expressions that build in the concepts *and*, *or*, *not*, *every* and *some*, and concepts that can be expressed in terms of them, but nothing else.

First-order model theory is the study of the semantics of this language, and it has become a very sophisticated branch of mathematics, full of its own concepts and theorems, some of extraordinary beauty and complexity. These theorems give insight into and enrichment for those parts of mathematics that happen to fit the shoe of first-order logic. This includes a fairly extensive part of modern algebra. The book Chang and Keisler [1973] provides an excellent introduction to the model theory of first-order logic. In extended model theory, we take the basic idea and expand it in various ways, by allowing richer mathematical structures or richer expressive power in the language, or both.

As used in this book, then, *a logic* consists of a collection of mathematical structures, a collection of formal expressions, and a relation of satisfaction between the two. We are primarily interested in logics where the class of structures are those where some important mathematical property is built in, and where the language gives us a convenient way of formalizing the mathematician's talk about the property. We might say, then, that a logic is something we construct to study the logic of some part of mathematics.

1.2. The First-Order Thesis

If first-order logic is the inspiration for much of extended model theory, it is also its nemesis. The common sense, mathematician-in-the-street view of logic implicit in this subject is at variance with what we teach our students in basic logic courses. There we attempt to draw a line between "logical concepts", as embodied in the so-called "logical constants", and all the rest of the concepts of mathematics. In extended model theory we do not so much question the placement of this line, as question whether there is such a line, or whether all mathematical concepts have their own logic, something that can be investigated by the tools of mathematics.

To give ourselves a foil, let us call the view that attempts to define logic as the logic implicit in the "logical constants" the *first-order thesis.* (Among the numerous past and present adherents to this thesis there is a slight disagreement as to whether identity should be counted as a "logical constant".) Another way to state this view is to claim that logic is first-order logic, so that anything that cannot be defined in first-order logic is outside the domain of logic.

The reasons for the widespread, often uncritical, acceptance of the first-order thesis are numerous. Partly it grew out of interest in and hopes for Hilbert's program. Partly is was spawned by the great success in the formalization of parts of mathematics in first-order theories like Zermelo–Fraenkel set theory. And partly, it grew out of a pervasive nominalism in the philosophy of science in the mid-twentieth century, led by Quine, among others. As late as 1953, well after the Gödel incompleteness theorems, Quine wrote in his book *From a Logical Point of View*:

> The bulk of logical reasoning takes place on a level which does not presuppose abstract entities. Such reasoning proceeds mostly by quantification theory, the laws of which can be represented through schemata involving no quantification over class variables. Much of what is commonly formulated in terms of classes, relations, and even number, can easily be reformulated schematically within quantification theory plus perhaps identity theory. Quine [1953, p. 116].

As logicians we do our subject a disservice by convincing others that logic is first-order logic and then convincing them that almost none of the concepts of modern mathematics can really be captured in first-order logic. Paging through any modern mathematics book, one comes across concept after concept that cannot be expressed in first-order logic. Concepts from set-theory (like *infinite set, countable set*), from analysis (like *set of measure* 0 or *having the Baire property*), from topology (like *open set* and *continuous function*), and from probability theory (like *random variable* and *having probability greater than some real number r*), are central notions in mathematics which, on the mathematician-in-the-street view, have their own logic. Yet none of them fit within the domain of first-order logic. In some cases the basic presuppositions of first-order logic about the kinds of mathematical structures one is studying are inappropriate (as the examples from topology or analysis show). In other cases, the structures dealt with are of the sort studied in first-order logic, but the concepts themselves cannot be defined in terms of the "logical constants." For example, by the Löwenheim–Skolem theorem, any countable set of first-order sentences which is true in some structure is true in some countable structure. This shows that the complementary concepts of *countable* and *uncountable* cannot be defined in first-order logic. The compactness theorem, stated below, shows that the concepts of finite and infinite cannot be captured in first-order logic.

Extended model theory adds a new dimension and new tools to the study of the logic of mathematics. The first-order thesis, by contrast, confuses the subject matter of logic with one of its tools. First-order logic is just an artificial language constructed to help investigate logic, much as the telescope is a tool constructed to help study heavenly bodies. From the perspective of the mathematician in the street, the first-order thesis is like the claim that astronomy is the study of the telescope. Extended model theory attempts to take the experience gained in first-order model theory and apply it in ever broader contexts, by allowing richer structures and richer ways of building expressions. It attempts to build languages similar to the first-order predicate calculus to study concepts that are banned from logic by the first-order thesis.

It is not always straightforward to come up with the best language to capture a given concept. For example, the "best" one for studying the concepts of finite and infinite is not at all the one that first came to mind, as we shall see. Similarly, finding the "best" logic of topological structures was a process of successive approximations. In both cases the class of structures is clear: ordinary structures in the first case, topological structures in the second; but the choice of just the right language is difficult. In other cases, even finding just the right collection of structures has been problematic. Finding natural logics takes trial, error and experience. Part of the accumulated experience is discussed in the section on abstract model theory, below.

1.3. The Completeness Problem

Similarly, there is nothing straightforward about knowing the best questions to ask about a given logic. They will depend, in general, on the concepts it captures. But one question always suggests itself just by virtue of being a study of logic, the *completeness problem*: is there any kind of completeness theorem that goes with the logic, analogous to the completeness theorem for first-order logic? That is, given a logic $\mathscr{L}$, is there an effective list of axioms that are valid in all structures of the logic and a list of valid rules of inference that, together with the axioms, generate all valid theorems of the logic, i.e., the set of sentences that hold in all its structures?

Using the language of recursion theory, the completeness problem can be phrased quite abstractly (or crudely, depending on one's point of view). For if it has a positive solution, then the set of valid sentences is recursively enumerable. And, conversely, if the set of valid sentences is recursively enumerable, then in principle we can find such a completeness theorem. However, this does not give one a simple set of axioms and rules of inference which generate the valid sentences. Thus, up to aesthetic considerations, the first question about a logic $\mathscr{L}$ that we usually ask is: Is the set of valid sentences recursively ennumerable? This is sometimes called "abstract completeness."

The completeness problem ties up with the first-order thesis and an even older view of logic, where it was seen as the study of axioms and rules of inference. Of the logics studied here, some have a completeness theorem, some don't. If one thinks of logic as limited to the study of axioms and rules of inference, then logics without an abstract completeness theorem will not seem part of logic. But if you think of logic as the mathematician in the street, then the logic in a given concept is what it is, and if there is no set of rules which generate all the valid sentences, well, that is just a fact about the complexity of the concept that has to be lived with. It is this latter point of view that is implicit in the study of model-theoretic logics.

1.4. Compactness

A major theme in the early days of extended model theory was the search for compact logics, logics which satisfied the following (1) or (2), or some appropriate analogue of them where the concept of finite is replaced by a different notion of small.

(1) (Strong Compactness Property.) If T is any set of sentences of the logic, and if every finite subset of T has a model (i.e., is true in some structure of the logic) then T has a model.
(2) (Countable Compactness Property.) Same as (1), but only for countable sets T.

There are two reasons for interest in these results. One is closely related to the completeness problem. Usually a completeness theorem establishes that if ϕ is a logical consequence of some set (or perhaps countable set) T of assumptions, then it is derivable from some finite subset of T. In particular, if T is inconsistent and so has no models whatsoever, then some contradictory sentence is a consequence of T, in which case some finite subset of T will be inconsistent. That is, usually (1) or (2) fall out of a completeness theorem, if there is one.

Secondly, in first-order model theory, the compactness theorem is a ubiquitous tool, applied at almost every turn. It was natural that it should have been deemed a crucial property for a logic to have, if one wanted to exploit experience gained in first-order model theory.

For some logics, like the infinitary logics discussed below, it was realized that finite was the wrong property, because proofs themselves could have infinitely many hypotheses, so various analogues of compactness were sought where *finite* was replaced by some other notion of small set. First attempts were in terms of cardinality. Later, and more successful attempts brought in notions of small from generalized recursion theory.

1.5. Mostowski's Proposal and Generalized Quantifiers

One of the first explicit proposals for studying extensions of first-order logic by the methods of model theory came in Mostowski [1957]. His idea was that since various concepts like finitely many and countably many are not definable in first-order logic but are important in modern mathematics, we should add quantifiers embodying such concepts directly. He suggested having a new syntactic rule:

$$\text{if } \phi(x) \text{ is a formula, so is } Qx\phi(x),$$

where x is not free in the new formula. This formation rule is added in such a way that it can be iterated along with "and", "or", "not", "everything" and "something". The meaning of Q depends on a new semantic rule. In fact, given any cardinal number $\aleph_\alpha$ one has a logic $\mathscr{L}(Q_\alpha)$ defined by giving the semantics:

$$\mathscr{M} \models_\alpha Qx\phi(x) \quad \text{iff} \quad \begin{array}{l}\text{there are at least } \aleph_\alpha \text{ elements} \\ b \text{ such that } \mathscr{M} \models_\alpha \phi(b).\end{array}$$

In words, $Qx\phi(x)$ is true just in case there are at least $\aleph_\alpha$ elements b such that $\phi(b)$ is true. The logics $\mathscr{L}(Q_\alpha)$ all have the very same syntax but have different semantics assigning different meanings to the quantifier symbol Q.

The logic $\mathscr{L}(Q_0)$ builds in the finite/infinite distinction missing from first-order logic. It is a notion at the heart of much mathematics, especially in modern algebra. Using it one can define notions like torsion group, finitely generated group, finite-dimensional vector space, and one can define the natural numbers.

The logic $\mathscr{L}(Q_1)$ on the other hand, builds in the countable/uncountable distinction missing from first-order logic, but it does not include $\mathscr{L}(Q_0)$. Using it one can define notions like countably generated groups, uncountable structures, and the like.

One of the first surprises in extended model theory was the extent to which $\mathscr{L}(Q_1)$ is better behaved than the logic $\mathscr{L}(Q_0)$. For example, while there is no completeness theorem for $\mathscr{L}(Q_0)$ there is one for $\mathscr{L}(Q_1)$. Vaught [1964] proved a "two-cardinal theorem" of first-order model theory which had as a corollary an abstract completeness theorem for $\mathscr{L}(Q_1)$. The problem of finding a concrete completeness theorem for $\mathscr{L}(Q_1)$ was left open until a very elegant complete set of axioms and rules was found by Keisler [1970]. Similarly, Fuhrken [1965] used the proof of Vaught's two-cardinal theorem to show that $\mathscr{L}(Q_1)$ is countably compact; $\mathscr{L}(Q_0)$ is not. This is also an immediate consequence of Keisler's completeness theorem. To prove his result, Keisler had to develop much more refined techniques of building uncountable models than had been available before, techniques which have been incorporated into the heart of the subject. They are discussed in Kaufmann's chapter in Part B.

A great deal of effort has gone into studying the logics $\mathscr{L}(Q_\alpha)$ in general, and especially $\mathscr{L}(Q_1)$, as well as closely related logics. But cardinality is only one rather crude distinction between sets. Mostowski's idea of imposing various properties on definable sets has had a liberating effect on logic and has been extended in many different directions. Quantifiers based on measure theory, on probability and on other measures of size have been studied, for example. Lindstrom [1966a] proposed a very general definition of a quantifier, so that one could use practically any class K of structures to define a new quantifier Q_K that captures membership in that class. The notion of a Lindstrom quantifier is defined in Chapter II. Adding quantifiers to first-order logic is a central theme of extended model theory, and provides the focus of Part B of this book.

Most work in extended model theory assumes that one wants to study logics that are stronger than first-order logic, stronger in the sense of containing first-order logic. However, in investigating the logic of probability spaces, Keisler realized that to get the right logic, one wants to have all definable sets measurable, and that these measurability considerations dictate that the logic is strictly incomparable with first-order logic, since one cannot in general assume closure under the ordinary quantifiers "everything" and "something". Instead one has quantifiers of the form

$$(Px \geq r)\phi(x)$$

meaning that the probability of ϕ is at least r. But this logic has a rather weak expressive power unless one takes advantage of countable additivity by allowing infinitary propositional operations, as had already been studied in the more classical setting. (See the next subsection.) Besides the interesting applications, such logics give us a new kind of testing ground for our basic ideas about what a logic is and what, if anything, is so special about first-order logic.

1.6. *Infinitary Logics*

The logic $\mathscr{L}(Q_0)$ embodying the finite/infinite distinction turned out to have less than satisfactory properties. A number of logics more or less equivalent to $\mathscr{L}(Q_0)$ (e.g. weak second-order logic, that allows quantification over finite sets, and ω-logic, that allows quantification directly over the natural numbers) were worked on until they were gradually replaced by the study of logics with infinitely long formulas.

Actually, the investigation of such languages is older than that dealing with generalized quantifiers (see Zermelo [1931], Novikoff [1939, 1943], Bochvar [1940]), but had fallen on hard times until the late 1950's and early 1960's, when work of Tarski, Henkin, Karp, Scott, Lopez-Escobar, Hanf, and Keisler revitalized the subject. Part C of this book is devoted to infinitary languages and their applications.

Early work on infinitary logics dealt with certain languages $\mathscr{L}_{\kappa,\lambda}$ which were generated by allowing conjunctions and disjunctions of size less than κ and homogeneous strings of quantifiers of length less than λ. The early work looked for analogues of the compactness, completeness and Löwenheim–Skolem theorems. Initial results were discouraging, in that compactness was found to exist only under the rarest of circumstances. Indeed, work of Hanf [1964] showed that it required strong new set-theoretical assumptions to prove that there were any logics $\mathscr{L}_{\kappa,\lambda}$ that were compact in the hoped-for sense, of being κ-compact, where "finite" is replaced by "size less than κ" in the statement of compactness.

Completeness results were a little easier to come by. Building on work of Scott and Tarski [1958], Karp [1964] gave a completeness theorem for the logic $\mathscr{L}_{\omega_1\omega}$. Notice, though, that since the syntactic expressions are infinite, the recursion-theoretic formulation in terms of recursively ennumerable sets had to be abandoned—or better—generalized. What one wanted was a recursion theory over infinitary objects to capture the sense in which one notion of proof might be seen as appropriately effective, another not. Such generalized recursion theories were being developed at about this time (by Takeuti, Levy and Machover, Kripke, Kreisel and Sacks, and Platek) for independent reasons, but then led to a fruitful interaction with the work on infinitary logics.

One of the reasons for favoring an infinitary language over $\mathscr{L}(Q_0)$ had to do with the failure of the Craig interpolation theorem and its consequence, the Beth definability theorem. (The latter says that any notion that is implicitly definable in first-order logic is also explicitly definable in first-order logic.) Mostowski [1968] showed that there is a principled reason for the failure of these results in logics like $\mathscr{L}(Q_0)$, weak second-order logic, and ω-logic. What he showed was that any logic where the syntax is finite but where the notion of finite is definable has sets that are implicitly definable but not explicitly definable. Hence the obvious analogues of the Beth and Craig results fail. More to the point, though, his results show that such logics fail to capture all that is implicit in the logic of finiteness.

The moral is that if you want a logic where the notions of finite and infinite are expressible, and if you want it to be closed under implicit definability, then the syntax is going to have to be infinitary—in some sense. This is not the original motivation for the study of infinitely long formulas, but it is a sound one. The logic $\mathscr{L}_{\omega_1\omega}$ studied by Karp, Scott, Lopez-Escobar is a different way of building the notion of finite into a logic, one that *does* satisfy the obvious analogues of the Beth and Craig theorems, as shown in Lopez-Escobar [1965b]. It allows arbitrary countable conjunctions and disjunctions of formulas to be formulas. The logic $\mathscr{L}(Q_0)$ is a "sublogic", since "there exist infinitely many" can be defined by the following countable conjunction:

$$\bigwedge_{n\geq 0} \forall x_1 \ldots x_n \exists y(\phi(y) \wedge y \neq x_1 \wedge \cdots \wedge y \neq x_n).$$

Lopez-Escobar gave a completeness proof for a Gentzen-style system for $\mathscr{L}_{\omega_1\omega}$, from which he was able to derive an interpolation theorem, and an analogue of the Beth definability theorem.

One of the notions that has emerged as central to logic is that of an inductive definition, i.e., one of the form: the smallest relation R satisfying some closure condition. The notion comes up in the very definition of the syntax and semantics of specific logics, in recursion theory, and in various other branches of mathematics. It is only natural that logicians would look for logics where such implicit forms of definability were made explicit. Infinitely long formulas emerged again in this connection. Moschovakis [1972] showed that any inductive definition could be made explicit by using a formula with an infinite string of alternating quantifiers:

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \ldots \bigwedge_n \phi_n(x_1, y_1, \ldots, x_n, y_n).$$

This generalized a theorem of Svenonius [1965] about PC-classes on countable models. Various suggestions for logics admitting such infinite alternating strings have been forthcoming. The most useful now appears to be the Vaught formulas built into the logic $\mathscr{L}$ studied in Kolaitis's chapter. Such infinite strings also have connections with work in game theory, higher recursion theory and descriptive set theory.

1.7. Second-Order Logic

Actually, there was another extension of first-order logic that was around for a long time before Mostowski's suggestion. Everyday mathematical experience shows us that the concepts of arbitrary set and function are important and powerful. Notions like finite, infinite, countable, uncountable, well-ordering, the natural and real numbers, are all definable in terms of these notions. Second-order logic is the extension of first-order logic where these concepts are built in by allowing quantifiers not just over individuals in the domain $\mathfrak{M}$, but also over subsets of that domain and over relations and functions on the domain.

Judged by the standards of first-order logic, the model theory of second-order logic was deemed unmanageable. None of the basic theorems of first-order logic extended to second order logic. There were no completeness, compactness, interpolation or Löwenheim–Skolem theorems. For many years the model-theory of second-order logic was thus largely ignored. In fact, in the early days of extended model theory, many of us saw ourselves as chipping away manageable fragments of second-order logic. However, the way we judged what it was to be a manageable theory was by comparing it to first-order model theory. In retrospect, this seems unimaginative, since there has turned out to be quite a rich model theory for second-order logic, once the right questions started being asked.

Second-order logic permits quantification over arbitrary functions on the domain of discourse as well as quantification over the elements in the domain of discourse. Since sets and relations can be represented by their characteristic functions, second-order logic embodies quantification over arbitrary sets and relations, as well. There is an obvious equivalence between functions and relations, but allowing quantification only over sets turns out to be weaker than full second-order logic. It is called "monadic" second-order logic, and it is much more expressive than first-order logic while being manageable enough to provide many interesting decidability and undecidability results. Some of these are discussed in Gurevich's chapter. For example, he discusses a classification of ordered abelian groups by means of properties definable in the monadic second-order logic of such groups. He also presents a proof of the famous result due to Rabin on the decidability of the monadic theory of the infinite binary tree.

Shelah [1973c] investigated what other types of restricted second-order quantifiers there were, besides the restriction to monadic quantification, but where the restrictions considered had to be first-order definable. He proved a striking and difficult result: there are only four first-order definable second-order quantifiers. Baldwin's chapter takes advantage of more recent work in model theory to give a simplified presentation of the result. The structural results implicit in the proof of the four definable second-order quantifiers theorem emphasize the importance of studying three theories in monadic logic: (i) the monadic theory of order, (ii) the monadic theory of the tree $\lambda^{\leq\omega}$, and (iii) the monadic theory of the tree $\lambda^{<\omega}$.

Both Baldwin's and Gurevich's chapters emphasize the importance for monadic logic of a basic result for first-order logic which does extend to this situation: the Feferman–Vaught theorem.

1.8. Applications to Mathematics

There are many kinds of applications of logic to mathematics. The most striking (at least the ones that strike most people) are those where some specific theorem or method from logic gives an outright solution to some open question in mathematics. Eklof surveys a number of applications of this sort, of infinitary logics within algebra. Keisler's chapter contains some applications of this sort to probability theory.

A second kind of application of logic is in the realm of independence results where it is shown that certain problems cannot be settled on the basis of the first-order axioms of set-theory. These results are really about the limitations of first-order logic, and so are outside the scope of this book, except to the extent that they have an impact on extended model theory itself. (See Section 2.6 below.)

Most important in the long run, it seems, is where logic contributes to mathematics by leading to the formation of concepts that allow the right questions to be asked and answered. A simple example of this sort stems from "back and forth arguments" and leads to the concept of partially isomorphic structures, which plays such an important role in extended model theory. For example, there is a classical theorem of Erdos, Gillman and Henriksen; two real-closed fields of order type η_1 and cardinality $\aleph_1$ are isomorphic. However, this way of stating the theorem makes it vacuous unless the continuum hypothesis is true, since without this hypothesis there are no fields which satisfy both hypotheses. But if one looks at the proof, there is obviously something going on that is quite independent of the size of the continuum, something that needs a new concept to express. This concept has emerged in the study of logic, first in the work of Ehrenfeucht and Fraissé in first-order logic, and then coming into its own with the study of infinitary logic. And so in his chapter Dickmann shows that the theorem can be reformulated using partial isomorphisms as: Any two real-closed fields of order-type η_1, of any cardinality whatsoever, are strongly partially isomorphic. There are similar results in the theory of abelian torsion groups which place Ulm's theorem in its natural setting.

Notice the shift of perspective here. While we started with the idea of taking concepts that were already explicit in mathematics and studying their logic, we now see the possibility of exploring concepts that are only implicit in existing mathematics, making them explicit, and using them to go back and re-examine and enrich mathematics itself. Isolating the notions of inductive definability implicit in so much of mathematics is another example mentioned above. The results mentioned from Keisler's and Gurevich's chapters are also of this nature, bringing in new concepts with which the right questions can be asked and answered. Similarly, much of Shelah's work in extended model theory can be seen in this light, taking some important construction from mathematics or logic and building the construction into a new logic. Extended model theory provides a framework within which to understand existing mathematics and push it forward with new concepts and tools.

2. *Abstract Model Theory*

Once there are lots of similar structures around one begins to study the relationships that exist between them. And so it is with extended model theory. Once there are lots of logics around, one begins to study their interrelationships. This part of the subject is known as *abstract model theory.*

2.1. *Lindstrom's Theorem*

One of the first equations that must be settled is, just what makes a logic natural? What are the guiding principles which help one find interesting and useful logics?

Here the experience built up with many examples suggests three principles:

(1) build into the semantics natural and important notions from some particular domain of mathematical activity;
(2) keep the semantics constrained so that it embodies just those notions one intends to study, and notions implicit in them; and
(3) find a syntax in which the basic notions of the logic find natural expression.

It was obvious from the start that there is a trade-off in the construction of logics. You can't build in some concept that goes beyond first-order logic without paying the piper. For example, if some particular theorem about first-order model theory shows that adding a new quantifier is a genuine strengthening of first-order logic, then the obvious analogue of that theorem will fail for the new logic. For example, the countable compactness of first-order logic has as an easy corollary that the quantifier "there exists at most finitely many" is not definable therein. It follows from the proof that $\mathscr{L}(Q_0)$ and $\mathscr{L}_{\omega_1\omega}$ are not countably compact. Similarly, the Löwenheim–Skolem theorem (if a countable set of sentences has a model, it has one that is at most countable) has as a corollary that "there exist uncountably many" is not definable in first-order logic. Hence the analogous statement will fail for the logic $\mathscr{L}(Q_1)$.

There is an important theorem lurking here, one discovered by Lindstrom [1969]; it is a result that opened up a new aspect to the study of logic. What Lindstrom showed is that what we have just observed in these two cases is in fact quite general. Any attempt to build a logic that is more expressive than first-order logic will fail to satisfy the obvious analogue of either the countable compactness theorem or the Löwenheim–Skolem theorem. Or, to state it more positively, first-order logic can be characterized as the strongest logic satisfying the following two properties:

(1) (Countable Compactness Property.) If a countable set of sentences has no model then some finite subset has no model; and
(2) (Löwenheim Property.) If a sentence has an infinite model, it has a countable model.

$\mathscr{L}(Q_1)$ is countably compact; $\mathscr{L}_{\omega_1\omega}$ satisfies the Löwenheim property. This striking result has led to much important research after lying largely unnoticed for several years. It was the rediscovery of the result and its widespread circulation in Friedman [1970a] that in many ways woke logicians to the potential in abstract model theory. A proof of Lindstrom's theorem is contained in Chapter III.

Characterizing a given logic $\mathscr{L}$ as the strongest logic with some property presupposes an understanding of just what a logic is. What kinds of syntactic and semantic closure conditions does one build into the notion of a logic? Obviously the more one builds in, the fewer logics there are and so the weaker a characterization theorem becomes. On the other hand, for the other aims of extended model theory, one wants a notion that captures the important examples and systematizes the common assumptions.

Lindstrom and Friedman managed to side-step this problem. To get around the difficulties of saying just what a logic is, they dealt entirely with classes of structures and closure conditions on these classes, thinking of the classes definable in some logic. That is, they avoided the problem of formulating a notion of a logic in terms of syntax, semantics, and satisfaction, and dealt purely with their semantic side. From the point of view of logic, this is at best a stop-gap measure, to be replaced by an analysis of just what makes up a logic. But the task of coming up with a general definition of just what constitutes a logic has been a large one, one that may still be not entirely settled. The one given in this book has emerged as fairly stable over time, and most useful for a variety of investigations.

2.2. *Characterization Theorems*

The compactness and Löwenheim–Skolem theorems are two of the most striking results in first-order model theory—and probably the most frequently used tools of the first-order model-theorist. This made Lindstrom's characterization theorem of first-order logic somewhat disheartening, initially at least, since it says that the model-theorist interested in extensions of first-order logic is going to have to give up at least one of his most cherished tools. Luckily, however, there had already been enough success in the model theory of $\mathscr{L}_1$, $\mathscr{L}(Q_1)$, $\mathscr{L}_{\omega_1\omega}$, and some other logics to whet the appetites of those interested in extensions of first-order logic and to convince them that there was room to maneuver around the failures of these results. And there was enough intrinsic interest in these logics that workers attempted to find Lindstrom-style characterization theorems for them.

There have been some successes finding such characterizations, but they have been few and far between. What there are can be found in the chapters by Flum and Väänänen. But there are still no satisfactory characterizations of $\mathscr{L}_{\omega_1\omega}$ or $\mathscr{L}(Q_1)$. Indeed, search for such results has led to the study of even stronger logics that are based on the same sorts of mathematical concepts, but there is no satisfactory characterization of these stronger logics either.

2.3. *Uses of Abstract Model Theory*

Abstract model theory has turned out to have more to say about the *relations between* various properties of logics than about the characterization of logics by their properties. In general, abstraction can serve many different masters. It can be used to systematize a body of examples, notions and results, and in this organization, help us to understand more explicitly what we already know. This usually leads to the emergence of new concepts for unifying properties of the material, concepts which are overlooked in specific cases. And new problems and theorems that can be formulated in terms of the new concepts that emerge.

Studying only the model theory of first-order logic would be analogous to the study of real analysis never knowing of any but the polynomial functions: core concepts like continuity, differentiability, analyticity, and their relations would remain at best vaguely perceived. It is only in the study of more general functions that one sees the importance of these notions, and their different roles, even for the simple case.

One of the aims of abstract model theory is develop an analogous classification of logics by means of their most important properties. This entails understanding the relationships between these properties. Properties of logics that are co-extensive in the first-order case often have quite different extensions in the general setting. For example, in first-order logic, the interpolation theorem and the Robinson consistency theorem appear to be equivalent results. However, in general, the latter is much more powerful than the former. $\mathscr{L}_{\omega_1\omega}$, for example, has the interpolation property but not the Robinson consistency property. So too, the difference between strong compactness and countable compactness is not too noticeable in first-order logic, because of the Löwenheim–Skolem theorem. In general, however, countable compactness is much weaker.

Like properties of logics, so too methods of proof that seem more or less equivalent in the context of first-order model theory often split and come into their own in abstract model theory. For example, the Ehrenfeucht–Fraissé partial isomorphism method has come to the fore in two ways. First, it generalizes in different ways to a host of model-theoretic logics. Second, it is used as a means of classifying logics, into those that have and those that do not have the "Karp property". In the next subsections, we discuss three particularly important links that come up repeatedly in extended and abstract model theory, the Δ-closure of a logic, and the least ordinal pinned down by a bounded logic, and the Hanf number of a logic. In each we have a property of first-order logic that is largely overlooked until put in the context of the more general theory.

2.4. The Interpolation Theorem and the Δ-Closure

The interpolation theorem illustrates a number of the issues discussed above. The Craig interpolation theorem (stated below) shows that first-order logic is closed under a very general form of implicit definability, so that the concepts embodied in first-order logic are all given explicitly. Closure under implicit definability is obviously a highly desirable result from the perspective of defining logics that embody a given mathematical notion. Craig's result was discovered about the same time as the Robinson consistency theorem, and they were widely perceived to be more or less the same result, one that implied the Beth definability theorem.

As mentioned above, the Robinson consistency property turns out to be a much stronger property of logics than the Craig interpolation property in the context of extended model theory. In fact, as long as the number of symbols in any single sentence is finite, or at all reasonable in size, one can say that a logic has the Robinson consistency property just in case it satisfies *both* the compactness property and the Craig interpolation property (see Chapter XVIII).

Neither $\mathscr{L}(Q_0)$ nor $\mathscr{L}(Q_1)$ satisfy the Craig interpolation theorem. But whereas Mostowski found a principled reason for the failure of interpolation for $\mathscr{L}(Q_0)$, there is no such explanation known for $\mathscr{L}(Q_1)$. (Keep in mind that $\mathscr{L}(Q_1)$ is not in any sense an extension of $\mathscr{L}(Q_0)$. The logic $\mathscr{L}(Q_1)$ satisfies the countable compactness property so "finite" is not definable in this logic.) Rather, the counter-examples that were found to the Craig and Beth theorems for $\mathscr{L}(Q_1)$ and related logics have repeatedly suggested additional concepts that were in the constellation of notions around countability but that were not definable in $\mathscr{L}(Q_1)$. That is, the counter-examples all suggested that we just did not yet have the right logic, rather than that there was an essential obstacle. This is presumably part of the reason there is no convincing characterization theorem for any of these logics.

The problem of finding a countably-compact logic extending $\mathscr{L}(Q_1)$ with the interpolation property has become known as Feferman's problem. It has led to the study of many interesting and useful extensions of $\mathscr{L}(Q_1)$—extensions that remedy various deficiencies in $\mathscr{L}(Q_1)$ by building in other notions that seem still in the spirit of the countable/uncountable distinction. Some of these extensions are discussed in Kaufmann's chapter. Nevertheless, there is still no conclusive solution to Feferman's problem either positively, or negatively by a result that shows, under some reasonable assumption, that an essential obstacle exists.

Feferman's motivation in stating the problem goes back to the issue of completeness. For first-order logic, there are both model-theoretic and proof-theoretic proofs of the interpolation theorem, the latter deriving the theorem from the completeness of Gentzen's cut-free set of axioms and rules. (Gentzen's rule of "cut" is the analogue of modus ponens for his system. He showed that this rule is redundant in his system.) For $\mathscr{L}_{\omega_1\omega}$, it was this latter proof that Lopez-Escobar managed to generalize. It was harder to find a purely model-theoretic proof. The basic idea of the proof-theoretic proof is that if you are able to prove $\psi(R, T)$ from $\phi(R, S)$, where R, S and T are relation symbols, and if the proof does not use "cut", then there should be a proof that only uses the common symbol R in an essential way, in that you should be able to isolate a sentence $\theta(R)$ so that both $\phi(R, S) \to \theta(R)$ and $\theta(R) \to \psi(R, T)$ are provable.

One can use the interpolation property as a yardstick for measuring whether there is a good proof theory. In the case of $\mathscr{L}(Q_1)$, knowing that interpolation fails shows that one is not going to have a good Gentzen style proof theory for $\mathscr{L}(Q_1)$. What Feferman was after was a richer logic that had a better completeness theorem in this sense, and he was using the interpolation property as a model-theoretic test for such a better theorem.

The proof theory of strong logics has not kept pace with their model theory, partially due to the interests of the people working in the field, partially due to the fact that proof theory is not seen as being particularly central to the subject since many of the logics do not have an r.e. set of valid sentences. And from a model-theoretic point of view, it has turned out that interpolation is not a particularly important or natural property for a logic to have. Interpolation is a much stronger property than is needed for a logic to be closed under implicit definability. The notion that has turned out to be more important in this respect is that of a Δ-*closed logic.*

A class K of structures is called PC (or Σ_1^1) in a logic $\mathscr{L}$ if there is a class K' of structures that is definable in $\mathscr{L}$ so that $\mathfrak{M} \in K$ if and only if some expansion

$\mathfrak{M}'$ of $\mathfrak{M}$ is in K'. The interpolation theorem can be restated as: If K_0 and K_1 are disjoint PC classes then there is a definable class K containing one and disjoint from the other. An obvious consequence is that *if a class K is both PC and co-PC (that is, its complement is PC) in $\mathscr{L}$ then K is definable in $\mathscr{L}$.* A logic with this property is called Δ-*closed.* Any logic satisfying the interpolation property is automatically Δ-closed, but not conversely. And whereas there is no known way to start with a logic $\mathscr{L}$ where interpolation fails and find a smallest extension where it holds, there is a way to define a smallest logic $\Delta(\mathscr{L})$ containing $\mathscr{L}$ and Δ-closed, called the Δ-*closure* of $\mathscr{L}$. This operation on logics preserves many of the nice properties of the original logic.

The Δ-closure is completely overlooked in first-order logic because we have so much more. And Δ-closure, rather than the stronger interpolation property, is really what shows us that we have a well-rounded logic.

A frequent use of the Δ-closure is to show that two logics $\mathscr{L}$ and $\mathscr{L}'$ are really the same up to implicit definability by showing that $\Delta(\mathscr{L}) = \Delta(\mathscr{L}')$. Several such results appear in Chapters VI and XVII. For example, the various logics $\mathscr{L}(Q_0)$, weak second-order logic (where one quantifies over finite sets) and ω-logic are the same up to implicit definability. Makowsky [1975a] and I (Barwise [1974a]) independently noticed that Mostowski's result, that no logic with finitary syntax that can define finite and infinite has the interpolation property, could be turned into a characterization of the common Δ-closure of these logics as a certain infinitary logic, the "hyperarithmetic" fragment of $\mathscr{L}_{\omega_1\omega}$ (see Chapter XVII for a proof of this result).

2.5. *Pinning Down Ordinals*

Another property of first-order logic that goes all but unnoticed in that setting, but assumes a central place in the general theory, is the undefinability of well-orderings. The distinction between logics where well-ordering is undefinable and those where it is definable turns out to be an important one.

A logic $\mathscr{L}$ is said to be *bounded* by an ordinal α if α is greater than all ordinals that can be "characterized" in the logic. Second-order-like logics are those where the notion of well-ordering is definable and so are unbounded.

First-order logic is bounded by ω, the first infinite ordinal, as the (countable) compactness theorem shows. Indeed, any extension of first-order logic that is countably compact will be bounded by ω. For example, $\mathscr{L}(Q_1)$ is bounded by ω. $\mathscr{L}(Q_0)$, by contrast, is bounded not by ω but by a certain countable ordinal ω_1^c, the least non-recursive ordinal. $\mathscr{L}_{\omega_1\omega}$ is bounded by ω_1, the least uncountable ordinal. On the other hand, second-order logics $\mathscr{L}_{\omega_1\omega_1}$, and logic with the game quantifier are not bounded.

For some applications, the failure of the compactness theorem can be circumvented in applications by knowing that the logic is bounded. For example, first-order logic can be characterized in terms of the Löwenheim–Skolem theorem and the assumption that the logic is bounded by ω. Similarly, for many "Hanf number" calculations (see the next subsection) one needs to know a bound for the logic.

In first-order logic, the fact that the logic is bounded by ω is such a simple consequence of compactness, that we do not even notice that the property is important. In more general logics, this notion assumes its rightful place in the web of properties of logics.

2.6. *Hanf Numbers*

In elementary textbooks on logic one often finds the Löwenheim–Skolem theorem for first-order logic stated as: If a theory has an infinite model, then it has models of all infinite cardinalities. The proof, however, when given, always breaks into two parts. There is a "downward" half, that allows one to get smaller models from bigger, and an "upward" half that allows one to get bigger from smaller. The downward version uses some form of submodel argument, the upward a compactness argument. Not surprisingly, these two arguments generalize quite differently, to different logics.

Many logics have some form of downward Löwenheim–Skolem theorem, with a proof analogous to the usual one, with the difference being just how small the submodel can be. But almost no logics have a simple analogue of the upward version. In $\mathscr{L}(Q_0)$, for example, one can define theories with model of quite large infinite cardinalities, but without arbitrarily large models. Hanf observed, however, that as long as the expressions of a logic $\mathscr{L}$ form a set, as opposed to a proper class, that one can show quite easily, though very non-constructively, that there must be *some* cardinal κ such that if a sentence ϕ of $\mathscr{L}$ has a model of size at least κ, then it has arbitrarily large models. The least such cardinal has come to be known as the *Hanf number* $h(\mathscr{L})$ of $\mathscr{L}$.

A fair amount of work has gone into calculating the Hanf number of various logics. The reader can find a number of such calculations for infinitary logics in Chapter IX. For bounded logics, the Hanf number is often related to the least ordinal that cannot be pinned down in the following manner. Define

$$\beth_0 = \aleph_0,$$

$$\beth_{\alpha+1} = 2^{\beth_\alpha},$$

and, for limit ordinals λ,

$$\beth_\lambda = \sup_{\alpha<\lambda} \beth_\alpha.$$

Then for many logics $\mathscr{L}$, like $\mathscr{L}(Q_0)$, $\mathscr{L}_{\omega_1\omega}$, $\mathscr{L}_{\kappa,\omega}$, one has $h(\mathscr{L}) = \beth_\lambda$, where λ is the least ordinal that cannot be pinned down by the logic. For logics that are not bounded, there is very little that can be said about the size of the Hanf number.

Shelah has suggested a structural explanation for the relation between the ease of computing the Hanf number and the boundedness of the logic. The situation is clearer if we consider the Hanf number $h(T, L)$ of a countable theory T in a logic L, the least κ such that for any L-sentence ϕ if $T \cup \{\phi\}$ has a model of power κ then $T \cup \{\phi\}$ has arbitrarily large models. (Setting T as the "empty" theory we specialize to $h(L)$.) Similarly, we can define T to be bounded or unbounded in the logic L.

The important structural distinction can be expressed by considering the class of models of T. Each model of T can be decomposed as a "product" of countable models if and only if T is bounded if and only if the Hanf number of T can be easily computed. The proof of this result for logics with definable second-order quantifiers, a characterization of theories according to this classification, and an account of the ensuing computation of Hanf numbers occurs in Chapter XII. Shelah has identified a similar dichotomy between superstable theories with and without the dimensional order property. The resulting structure theory also analyzes a model of power λ in terms of countable models and subtrees of $\lambda^{\leq \omega}$.

2.7. *Strong Logics and First-Order Set Theory*

There is an older approach to the study of the relationship between logic and concepts that lie outside of first-order logic, one subscribed to by those who accept the first-order thesis. One gives a first-order approximation to one's meta-theory T, something like Zermelo–Fraenkel set theory (ZF) in which all the notions in question can be defined *relative to* the notion of set, or perhaps a weaker or stronger metatheory. To the extent that one can view some branch of mathematics as consequences of this theory, one has an account of that part of mathematics.

This has become something like the orthodox position of remaining mathematical formalists, those who see mathematics as the working out of consequences of some formal first-order theory by means of the axioms and rules of first-order logic. In particular, one can step back and look at extended model theory itself from this perspective. We can define many of the logics discussed here relative to the notion of set in ZF set theory. Hence, we can examine the relationship between the properties of logics and their definitions in set theory. This is an approach which I initiated in Barwise [1972a], motivated by an acceptance of the first-order thesis. While it now seems to me that my motivations were misguided, the approach has led to some very interesting work on the relationship between strong logics and set theory, work that is discussed in Chapter XVII.

From the early days of infinitary logic there has been a close interplay between strong logics and set-theoretic principles that go beyond ZF set-theory in various ways, especially so called "large cardinal" assumptions. These are assumptions that are not justified by clear-cut intuitions about sets, at least not by intuitions shared by the silent mathematical majority. Weakly and strongly compact cardinals κ are defined in terms of the associated infinitary logic $\mathscr{L}_{\kappa,\kappa}$ satisfying an analogue of the countable or full compactness property, for example. The assumption that there are such cardinals goes beyond the intuitions about sets built into ZF. Measurable cardinals come up in the discussion of the Robinson consistency property. It, too, is a strong assumption that goes beyond ZF. An even stronger assumption, Vopenka's principle, is equivalent to the statement that every finitely generated logic has a strong compactness cardinal, that is, has a cardinal κ so that any inconsistent theory T of the logic has a subset of size less than κ which is inconsistent. These and related results are discussed in Part F.

It is not clear what to make of results like these. Luckily, most of them have to do with very abstract logics, or with abstract logic itself, not with the concrete logics that arise from natural mathematical concepts.

2.8. *Other Types of Structures*

Lindstrom's theorem poses a dilemma: Give up either compactness or Löwenheim–Skolem. However, there is an escape from the horns of the dilemma mentioned earlier. Implicit in the discussion in this section has been the assumption that we were discussing logics that have the same basic sort of syntax and semantics as first-order logic. There is always the possibility of violating one or both of these assumptions by studying logics that have different sorts of structures, or have syntactic rules that are stronger in some ways than first-order logic but weaker in others.

Part E of the book is devoted to the study of some of the logics that have been developed for different kinds of mathematical structures. The most extensively studied class of structures is the class of topological models, models where there is an underlying topology. In this setting there has been a great deal of effort that has gone into discovering the analogue of first-order logic.

Harvey Friedman initiated the study of logic on the real numbers incorporating the notions of measure and category, a topic pursued in Chapter XVI. Keisler, on the other hand, initiated the investigation into the logic of probability spaces. These logics are interesting not just for what they say about the logic of the reals and the logic of probability, but also because they force us to examine additional assumptions that are usually implicit in extended model theory, assumptions that do not hold in these settings.

2.9. *Unnatural Logics*

We should give a word of warning about some of the logics one will meet in this book. Recall that the aim of extended model theory is to discover natural logics that embody important mathematical notions. This leads to abstract model-theory and the study of the relationships between properties of logics. There are a number of logics that have arisen simply as counterexamples to show that some one property of logics does not imply some other, not with the real goals of extended model-theory in mind at all. And, too, some of the logics that seemed superficially natural turned out not to be. $\mathscr{L}(Q_0)$ is one such. Time will tell which logics are truly significant. There is no more point in getting bogged down in the study of purely artificial and unnatural logics than there is in the study of hemi-demi-semi-groups with chain conditions.

3. Conclusion

The reader of this volume will find many topics that have not been discussed above, for the book, like the subject, is a large one. Even so, there are topics in the field of extended model theory and abstract logics that could not be included in this volume, for one reason or another. Beyond that, there are many topics that fit under the general heading described by the title of this book, "model-theoretic logics," but which are not usually considered part of extended model theory since they do not fit so well under the general framework that has been developed in abstract model theory. Consequently, we have not attempted to include this work here.

The most glaring omission of this sort is work on the semantics and logic of computer languages. This is a rich domain of research that would need a volume of at least equal size to treat adequately. In the long run, it seems that a unified view of logic and semantics will require us to come up with a framework that encompasses both fields, but we are far from such a conception at present.

The semantics of computer languages, and the differences that emerge in that work from more traditional model theory, points to a shortcoming in the latter, namely its failure to come to grips with activity, as opposed to objects and static relations between them. This same shortcoming causes problems with traditional attempts to apply model theory to human languages, another topic not treated here.

Traditional model theory focuses on truth (and satisfaction) of sentences, and so leaves out the use of language to affect change. This is a shortcoming that has been emphasized by Austin and other writers on natural language in the tradition of "speech act" theory. This power of language to effect change (e.g., in so-called "side effects") is one of the things that makes the semantics of computer languages strikingly different.

Another area where work on computer and human languages makes the traditional work in logic appear too static is in the treatment of inference. Inference, whether by man or machine, is an activity, a process of extracting information, whereas the tradition attempts to reduce inference to objects (proofs, strings of symbols). In another paper I have discussed the need to place the study of logic within a setting where traditional inference is seen as just one form of information preserving activity. I think such an approach has much to contribute to the understanding of mathematical activity, and hence to mathematical logic, but the development of these ideas will have to take place elsewhere. Even the traditional approach to inference in logic has not made great inroads in extended model theory. There are few genuine completeness theorems and even fewer extensions of proof theory.

Mathematicians often lose patience with logic simply because so many notions from mathematics lie outside the scope of first-order logic, and they have been told that that *is* logic. The study of model-theoretic logics should change that, by getting at the logic of the concepts mathematicians actually use, by finding applications, and by the isolation of still new concepts that enrich mathematics and logic. I do not know just how much of the work presented in this volume will find a permanent place in mathematics, because it is, after all, a young and vigorous subject. But whatever the fate of the particulars, one thing is certain. There is no going back to the view that logic is first-order logic.

Chapter II

Extended Logics: The General Framework

by H.-D. Ebbinghaus[1]

The contents of this chapter are intended to serve as preparation for the more specific or more advanced topics of the chapters that follow. We will pay equal attention to general notions and concrete systems. The first part of the material is concerned with basic notions and examples. In Section 1 we define general logical systems. Section 2 contains a description of numerous concrete examples together with an elaboration of their essential properties—as far as this can be given without greater effort. Section 3 is concerned with elementary and projective classes as a tool to compare the expressive power of logical systems. Applications include the systematic use of PC-reducibility for compactness proofs. In Section 4 numerous preceding examples are systematized by the notion of the Lindström quantifier, and an analogue of the Ehrenfeucht–Fraissé characterization of elementary equivalence for logics with monotone quantifiers is proved. The second part of the chapter is concerned with a more systematic representation of central model-theoretical notions, divided into three groups around compactness (Section 5), Löwenheim–Skolem phenomena (Section 6) and interpolation (Section 7).

We assume that the reader is acquainted with basic notions and facts of first-order model theory. In general we will consider only one-sorted structures; however, since in some cases many-sortedness leads to a methodological enrichment even for one-sorted model theory (see, for instance, Examples 7.1.2), we give the definitions for the many-sorted case (provided the many-sorted formulation is not too tedious and is of practical value). If not stated otherwise, examples, results and proofs refer to the one-sorted version. In most cases it is not hard to give the many-sorted extensions. For example, this can be done by reduction to the one-sorted version using additional predicates ("Unification of Domains", see Feferman [1968a, p. 13]). However, there are exceptions and the warning following Definition 2.1.1 should be consulted.

[1] I would like to thank the co-authors for their advice, suggestions and corrections. Special thanks go to Gert H. Müller for all his support and stimulating encouragement and to Jonathan Stavi whose thoughts on the subjects have influenced my views and are apparent particularly in Sections 3.2 and 5.3.

1. General Logics

What is a logic? The answer to this question is a pragmatic one: we collect some basic features common to well-known logical systems and use them as defining properties of a logic. In order to cover all important systems, we would have to be rather general. On the other hand we wish to provide convenient definitions to work with. In order to escape this dilemma we do not fix a single definition, but leave it to the working logician to choose a suitable notion according to the needs of specific situations. Having thus created the general framework, we then list some further properties of logics that serve as a means for describing numerous important examples of stronger logics in Section 2.

1.1. The Framework

For the purposes of exposition, we shall restrict ourselves to notions of logics based on conventional algebraic structures. For natural generalizations to other structures such as topological ones, see Chapters III and XV. We begin by listing our notational conventions and by recalling standard concepts from model theory.

Many-sorted *vocabularies* τ, $\sigma, \ldots$ are non-empty sets that consist of *sort symbols* $s, \ldots$, finitary *relation symbols* $P, R, \ldots$, finitary *function symbols* $f, g, \ldots$ and *constants* $c, d, \ldots$. Each constant and each function symbol of a vocabulary τ is equipped with a sort symbol of τ as are the argument places of relation and function symbols of τ.

Let R be a binary relation symbol whose argument places are equipped with sort symbols s_2, s_1, respectively, f be a unary function symbol equipped with s_2, whose argument place is equipped with s_1, and c be a constant equipped with s_1. Then

$$(*) \qquad \tau = \{s_1, s_2, s_3, R, f, c\}$$

is a vocabulary. The τ-*terms* are built up and equipped with a sort symbol in the obvious way. For instance, $f(c)$ is a τ-term. It is assigned the sort symbol s_2, the symbol with which f is equipped. $f(f(c))$ is not a τ-term because $f(c)$ is not equipped with s_1. In first-order logic the atomic τ-sentences are of shape Rt_0t_1 where t_0, t_1 are τ-terms equipped with s_2, s_1, respectively, or of shape $t_0 = t_1$ either for arbitrary τ-terms t_0, t_1 or—a variant that we shall adopt—only for τ-terms t_0, t_1 which are equipped with the same sort symbol.

We use self-explanatory denotations of vocabularies such as

$$\tau = \{s, \ldots, R, \ldots, f, \ldots, c, \ldots\}.$$

In the one-sorted case we drop the sort symbol, writing for instance

$$\tau = \{R, \ldots, f, \ldots, c, \ldots\}.$$

A many-sorted structure $\mathfrak{A}$ of vocabulary τ (called a "τ-*structure*") possesses non-empty domains $A_s, \ldots$, corresponding to the sort symbols $s, \ldots$ of τ, and interprets the other symbols in τ as usual. The elements of A_s are called the *elements of sort* s of $\mathfrak{A}$.

For instance, with τ as in $(*)$ above, a τ-structure $\mathfrak{A}$ consists of domains A_{s_1}, A_{s_2}, A_{s_3}, of a subset $R^{\mathfrak{A}}$ of $A_{s_2} \times A_{s_1}$, a function $f^{\mathfrak{A}}\colon A_{s_1} \to A_{s_2}$ and an element $c^{\mathfrak{A}} \in A_{s_1}$.

We denote structures in obvious ways such as

$$\mathfrak{A} = (A_s, \ldots, R^{\mathfrak{A}}, \ldots, f^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

in the many-sorted case, and

$$\mathfrak{A} = (A, R^{\mathfrak{A}}, \ldots, f^{\mathfrak{A}}, \ldots, c^{\mathfrak{A}}, \ldots)$$

in the one-sorted case. The class of τ-structures will be denoted by $\mathrm{Str}[\tau]$, and for any structure $\mathfrak{A}$ we let $\tau_{\mathfrak{A}}$ be the vocabulary of $\mathfrak{A}$.

If $\sigma \subseteq \tau$ and $\mathfrak{A} \in \mathrm{Str}[\tau]$, then we define $\mathfrak{A} \upharpoonright \sigma$, the σ-*reduct* of $\mathfrak{A}$, to be the σ-structure that arises from $\mathfrak{A}$ by "forgetting" A_s for $s \notin \sigma$ and $R^{\mathfrak{A}}, \ldots$ for $R, \ldots \notin \sigma$. If τ is as in $(*)$ above, then for instance

$$(A_{s_1}, A_{s_2}, A_{s_3}, R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}) \upharpoonright \{s_1, s_2, R\} = (A_{s_1}, A_{s_2}, R^{\mathfrak{A}}).$$

Let τ be one-sorted, $\mathfrak{A} \in \mathrm{Str}[\tau]$, and $C \subseteq A$. C is τ-*closed in* $\mathfrak{A}$ if $C \neq \emptyset$, if moreover $c^{\mathfrak{A}} \in C$ for $c \in \tau$, and C is closed under $f^{\mathfrak{A}}$ for $f \in \tau$. If C is not empty, $[C]^{\mathfrak{A}}$ denotes the substructure of $\mathfrak{A}$ generated by C, sometimes also written $\mathfrak{A}|C$ if C is τ-closed in $\mathfrak{A}$. If $P \in \tau$ is unary, $\sigma \subseteq \tau$, and $P^{\mathfrak{A}}$ σ-closed in $\mathfrak{A} \upharpoonright \sigma$, we can form the structure $(\mathfrak{A} \upharpoonright \sigma)|P^{\mathfrak{A}}$. This gives what is called a *relativized reduct* of $\mathfrak{A}$.

A map $\rho\colon \tau \to \sigma$ is called a *renaming* (*from* τ *onto* σ) if it is a bijection from τ onto σ that maps sort symbols onto sort symbols, relation symbols onto relation symbols of the same arity, function symbols onto function symbols of the same arity, and constants onto constants such that the sort symbols the latter ones are equipped with correspond via ρ. For instance, if $R \in \tau$ is as in $(*)$ above, then the argument places of $\rho(R)$ are equipped with $\rho(s_1)$, $\rho(s_2)$, respectively. Given a renaming $\rho\colon \tau \to \sigma$ and a τ-structure $\mathfrak{A}$, we can "rename" $\mathfrak{A}$ by ρ, thus obtaining the σ-structure $\mathfrak{B} = \mathfrak{A}^{\rho}$ with $B_{\rho(s)} = A_s$ for $s \in \tau$ and $\rho(\S)^{\mathfrak{B}} = \S^{\mathfrak{A}}$ for the other symbols § from τ.

With this preparation, we can now come to the central notion of this chapter.

1.1.1 Definition. A *logic* is a pair $(\mathscr{L}, \models_{\mathscr{L}})$, where $\mathscr{L}$ is a mapping defined on vocabularies τ such that $\mathscr{L}[\tau]$ is a class (the class of $\mathscr{L}$-*sentences* of vocabulary τ)

and $\models_{\mathscr{L}}$ (the *$\mathscr{L}$-satisfaction relation*) is a relation between structures and $\mathscr{L}$-sentences. Moreover, the following properties (i)–(v) hold:

(i) *If* $\tau \subseteq \sigma$, *then* $\mathscr{L}[\tau] \subseteq \mathscr{L}[\sigma]$;
(ii) *If* $\mathfrak{A} \models_{\mathscr{L}} \varphi$, *then* $\varphi \in \mathscr{L}[\tau_{\mathfrak{A}}]$;
(iii) *Isomorphism Property.* If $\mathfrak{A} \models_{\mathscr{L}} \varphi$ and $\mathfrak{B} \cong \mathfrak{A}$, then $\mathfrak{B} \models_{\mathscr{L}} \varphi$.
(iv) *Reduct Property.* If $\varphi \in \mathscr{L}[\tau]$ and $\tau \subseteq \tau_{\mathfrak{A}}$, then

$$\mathfrak{A} \models_{\mathscr{L}} \varphi \quad \text{iff} \quad \mathfrak{A} \restriction \tau \models_{\mathscr{L}} \varphi.$$

(v) *Renaming Property.* Let $\rho: \tau \to \sigma$ be a renaming. Then for each $\varphi \in \mathscr{L}[\tau]$ there is a sentence, say φ^{ρ}, from $\mathscr{L}[\sigma]$ such that for all τ-structures $\mathfrak{A}$,

$$\mathfrak{A} \models_{\mathscr{L}} \varphi \quad \text{iff} \quad \mathfrak{A}^{\rho} \models_{\mathscr{L}} \varphi^{\rho}.$$

Remark. The renaming property expresses the following simple fact: Given an $\mathscr{L}$-sentence φ of vocabulary $\tau = \{s, \ldots, R, \ldots\}$, then the symbols (and the sorts) in φ can be renamed in any reasonable way ρ, and the resulting $\{\rho(s), \ldots, \rho(R), \ldots\}$-sentence φ^{ρ} has, for any τ-structure $\mathfrak{A}$, the same meaning in the "renamed" $\{\rho(s), \ldots, \rho(R), \ldots\}$-structure $\mathfrak{B} = (B_{\rho(s)}, \ldots, \rho(R)^{\mathfrak{B}}, \ldots)$ as φ has in $\mathfrak{A}$.

The reader will have noticed here that we did not incorporate conditions concerning rules of inference or other "logical" properties in our definition. Hence it would seem more appropriate to use a term such as *model-theoretic language* (see Feferman [1974b]) instead of the term *logic*. However, the latter is shorter and has become customary. (See also Chapter I for a discussion concerning the choice of this terminology.)

In order to avoid overburdening the notation, we often denote logics simply by $\mathscr{L}, \mathscr{L}^*, \ldots$ and write "$\models$" instead of "$\models_{\mathscr{L}}$". Basic model-theoretic notions are introduced as usual. For instance, if $\varphi \in \mathscr{L}[\tau]$, we write $\mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi)$ (or simply $\mathrm{Mod}(\varphi)$, if τ and $\mathscr{L}$ are given) for $\{\mathfrak{A} \in \mathrm{Str}[\tau] \mid \mathfrak{A} \models_{\mathscr{L}} \varphi\}$; and for $\Phi \cup \{\varphi\} \subseteq \mathscr{L}[\tau]$ the $\mathscr{L}$-consequence relation is defined by

$$\Phi \models_{\mathscr{L}} \varphi \quad \text{iff for all } \mathfrak{A} \in \mathrm{Str}[\tau], \quad \mathfrak{A} \models_{\mathscr{L}} \Phi \text{ implies } \mathfrak{A} \models_{\mathscr{L}} \varphi.$$

Two τ-structures $\mathfrak{A}, \mathfrak{B}$ are *$\mathscr{L}$-equivalent*, $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, iff for all $\varphi \in \mathscr{L}[\tau]$, $\mathfrak{A} \models_{\mathscr{L}} \varphi$ iff $\mathfrak{B} \models_{\mathscr{L}} \varphi$. We write $\mathrm{Th}_{\mathscr{L}}(\mathfrak{A})$ for $\{\varphi \in \mathscr{L}[\tau_{\mathfrak{A}}] \mid \mathfrak{A} \models_{\mathscr{L}} \varphi\}$; it is called the *$\mathscr{L}$-theory* of $\mathfrak{A}$.

1.1.2 A First Variant. For some purposes it is especially convenient to have *variables* and *formulas* available in a logic. This can be accomplished by a natural generalization of Definition 1.1.1: For each sort symbol s we specify a class of variables $x^s, \ldots$ for objects of sort s and replace $\mathscr{L}$ by two functions $\mathrm{Sent}_{\mathscr{L}}$ and $\mathrm{Form}_{\mathscr{L}}$, where, for all τ, we let $\mathrm{Form}_{\mathscr{L}}[\tau]$ be the class of *$\mathscr{L}$-formulas* of vocabulary τ and $\mathrm{Sent}_{\mathscr{L}}[\tau]$ be the class of *$\mathscr{L}$-sentences* of vocabulary τ. Exact definitions, even including that for the free occurrence of variables, can be obtained along the lines of Definition 1.1.1 in a canonical way. For the general theory we will usually assume that logics are given without variables, although in most concrete examples we will follow the definition just sketched. Since free variables behave like constants, there are only minor differences between the two variants.

The traditional *first-order logic*, $\mathscr{L}_{\omega\omega}$, can be regarded as a logic in the forgoing sense. Moreover, it is also a logic in the sense of Definition 1.1.1, if, for any τ, we define $\mathscr{L}_{\omega\omega}[\tau]$ to be the set of first-order sentences of vocabulary τ. Similarly, *second-order logic*, $\mathscr{L}^2$, *weak second-order logic*, $\mathscr{L}^{w2}$, *infinitary logics* such as $\mathscr{L}_{\omega_1\omega}$ or $\mathscr{L}_{\infty\omega}$ and logics with *cardinality quantifiers* such as $\mathscr{L}_{\omega\omega}(Q_\alpha)$ (where Q_α is interpreted as *there are* $\aleph_\alpha$ *many*) are logics in both sense, with or without free variables.

1.1.3 A Second Variant. The so-called ω-*logic* arises from first-order logic by fixing a vocabulary $\{s, <\}$ and allowing only structures $\mathfrak{A}$ such that $\{s, <\} \subseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A} \upharpoonright \{s, <\}$ is isomorphic to the standard structure $(\omega, <)$ of the ordering of the natural numbers. Of course ω-logic does not fit into the present framework because the renaming property fails. In order to cover it by a notion of logic, we are led to a generalization of Definition 1.1.1: In addition we demand that a logic $\mathscr{L}$ have a further component, namely a map $\mathrm{Str}_{\mathscr{L}}$ defined on vocabularies where, for all τ, we let $\mathrm{Str}_{\mathscr{L}}[\tau]$ be a class of τ-structures, the τ-*structures admitted for* $\mathscr{L}$. Then we modify the basic properties of Definition 1.1.1 in the obvious way (see Section 2.6).

1.2. Basic Closure Properties

Practically all investigations of logics need stronger assumptions than those of the last section. The following closure properties are met by most of the well-known systems and provide much technical facilitation.

1.2.1 Definition. The purpose of the *basic closure properties* is to guarantee that we have at least the expressive power of first-order logic. We have:

(i) *Atom Property.* For all τ and all atomic $\varphi \in \mathscr{L}_{\omega\omega}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau]$ such that

$$\mathrm{Mod}^{\tau}_{\mathscr{L}}(\psi) = \mathrm{Mod}^{\tau}_{\mathscr{L}_{\omega\omega}}(\varphi);$$

(ii) *Negation Property.* For all τ and all $\varphi \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau]$ such that

$$\mathrm{Mod}^{\tau}_{\mathscr{L}}(\psi) = \mathrm{Str}[\tau] \backslash \mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi);$$

(iii) *Conjunction Property.* For all τ and all $\varphi_0, \varphi_1 \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau]$ such that

$$\mathrm{Mod}^{\tau}_{\mathscr{L}}(\psi) = \mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi_0) \cap \mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi_1);$$

(iv) *Particularization Property.* If c is of sort s, $c \in \tau$, then for any $\varphi \in \mathscr{L}[\tau]$ there is a sentence $\psi \in \mathscr{L}[\tau \backslash \{c\}]$ such that for all $(\tau \backslash \{c\})$-structures $\mathfrak{A}$, $\mathfrak{A} \models \psi$ iff $(\mathfrak{A}, a) \models \varphi$ for some $a \in A_s$.

If $\mathscr{L}$ has the *boolean property*, that is, (ii) and (iii) together, then we use $\neg\varphi$, $\varphi_0 \wedge \varphi_1$ to denote the required sentence ψ. If $\mathscr{L}$ has the particularization property, we write $\exists c\varphi$ for a corresponding ψ.

For technical convenience we formulate the following properties only in the one-sorted case.

1.2.2 Definition. All the basic examples of logics above (but not ω-logic) allow relativizations in the sense of:

Relativization Property. If $c \notin \tau \cup \sigma$, $\chi \in \mathscr{L}[\sigma \cup \{c\}]$ and $\varphi \in \mathscr{L}[\tau]$, then there is a sentence $\psi \in \mathscr{L}[\tau \cup \sigma]$ such that for all $(\tau \cup \sigma)$-structures $\mathfrak{B}$, if the set $\chi^{\mathfrak{B}} = \{b \in B \mid (\mathfrak{B}, b) \models \chi\}$ is τ-closed in $\mathfrak{B}$, then

$$\mathfrak{B} \models \psi \quad \text{iff} \quad (\mathfrak{B} \restriction \tau) | \chi^{\mathfrak{B}} \models \varphi.$$

Intuitively, ψ is the (more exactly, it is *a*) relativization of φ to $\{c \mid \chi(c)\}$, often written as $\varphi^{\{c|\chi(c)\}}$ or simply φ^P, if $\chi = Pc$.

If constants are present, relativizations can cause difficulties. For instance, if a vocabulary τ contains constants, it is impossible to represent two τ-structures with distinct domains as relativized reducts of a third structure. For the usual logics this difficulty can be overcome, because one can eliminate constants via descriptions by unary relations. We formulate this in a general context, giving an even stronger version that is needed on various occasions: the substitution property. In the simplest case this property guarantees that for any σ, τ the following holds[2]: If $R \notin \tau$ is n-ary and $\psi(\mathbf{c}) \in \mathscr{L}[\sigma \cup \{c_0, \ldots, c_{n-1}\}]$, then for every $\varphi \in \mathscr{L}[\tau \cup \{R\}]$ there is $\varphi[R/\lambda\mathbf{c}\psi(\mathbf{c})] \in \mathscr{L}[\tau \cup \sigma]$ with the meaning

$$\exists R(\forall \mathbf{c}(R\mathbf{c} \leftrightarrow \psi(\mathbf{c})) \wedge \varphi).$$

Similarly for n-ary $f \notin \tau$ and constants $c \notin \tau$, where for instance $\varphi[f/\lambda\mathbf{c}c\psi(\mathbf{c}, c)]$ has the meaning

$$\exists f(\forall \mathbf{c}c(f(\mathbf{c}) = c \leftrightarrow \psi(\mathbf{c}, c)) \wedge \varphi)$$

and $\varphi[c/\lambda d\psi(d)]$ has the meaning

$$\exists c(\forall d(c = d \leftrightarrow \psi(d)) \wedge \varphi).$$

1.2.3 Definition. In generality $\mathscr{L}$ has the *substitution property* iff for any τ, τ' with $\tau \subseteq \tau'$, if $\varphi \in \mathscr{L}[\tau']$ and, for all $R, \ldots, f, \ldots, c, \ldots \in \tau' \backslash \tau$, there are given predicates

[2] We use $\mathbf{a}, \ldots, \mathbf{c}, \ldots, \mathbf{x}, \ldots$ to stand for finite sequences of elements, constants, variables, respectively, of appropriate length.

$\psi_R(\mathbf{c}_R), \ldots$ then there exists an $\mathscr{L}$-sentence that arises from φ by simultaneously replacing $R, \ldots$ by $\lambda \mathbf{c}_R \psi(\mathbf{c}_R), \ldots$, respectively.

It is easy to give a more precise formulation of this definition and to see that any logic $\mathscr{L}$ with the atom property and the substitution property *allows elimination of function symbols* in the following sense: If $\boldsymbol{\sigma}$ arises from $\boldsymbol{\tau}$ by replacing any $f \in \boldsymbol{\tau}$, where f is n_f-ary, and any $c \in \boldsymbol{\tau}$ by new relation symbols R_f and R_c of arity $(n_f + 1)$ and 1, respectively, then for each $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ there exists $\psi \in \mathscr{L}[\boldsymbol{\sigma}]$ such that the $\boldsymbol{\sigma}$-models of ψ arise from the $\boldsymbol{\tau}$-models of φ by replacing the functions and constants by their graphs. A similar consideration shows that the substitution property yields the renaming property, at least in the one-sorted case. The many-sorted version needs a diligent treatment of sort symbols.

Logics satisfying the properties given in Definitions 1.2.1 to 1.2.3 are well-suited for general investigations, and we call them *regular logics*. A regular logic contains for each first-order sentence φ a sentence ψ of the same vocabulary and with the same models. When working with such a logic, it is convenient (and will be done tacitly) to assume that φ itself can be taken as such a ψ.

Further basic properties of logics can be of value in specific situations. One can, for example, demand that for any $\mathscr{L}$-sentence φ there is a smallest $\boldsymbol{\tau} = \boldsymbol{\tau}_\varphi$ such that $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ ("*occurrence property*"). Concerning questions of effectiveness it is reasonable to assume that $\boldsymbol{\tau}_\varphi$ exists and is finite. In order to have precise definitions of such notions for the examples that follow, we complete this section with a translation of crucial properties known from first-order logic into our general framework. More detailed definitions will follow in Sections 5 through 7.

1.2.4 Definition. Let $\mathscr{L}$ be a logic. Then

(i) For an infinite cardinal κ, $\mathscr{L}$ is *κ-compact* iff for all $\boldsymbol{\tau}$ and all $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$ of power $\leq \kappa$, if each finite subset of Φ has a model, then Φ has a model.

(ii) $\mathscr{L}$ is *compact* iff $\mathscr{L}$ is κ-compact for all infinite κ.

(iii) $\mathscr{L}$ is *effective* iff for all $\boldsymbol{\tau} \subseteq \mathrm{HF}$ (the set of hereditarily finite sets),

$$\mathscr{L}[\boldsymbol{\tau}] = \bigcup_{\substack{\boldsymbol{\tau}_0 \subseteq \boldsymbol{\tau} \\ \boldsymbol{\tau}_0 \text{ finite}}} \mathscr{L}[\boldsymbol{\tau}_0],$$

and for all $\boldsymbol{\tau}_0 \in \mathrm{HF}$, $\mathscr{L}[\boldsymbol{\tau}_0]$ is a recursive subset of HF. (Of course, it is the usual encoding of first-order formulas by hereditarily finite sets that leads to this definition.)

(iv) $\mathscr{L}$ is *effectively regular* iff $\mathscr{L}$ is regular and effective and all regularity properties are effective. For instance, effectiveness of the negation property means that for each $\boldsymbol{\tau}_0 \in \mathrm{HF}$ there is a recursive function $\dot{\neg}: \mathscr{L}[\boldsymbol{\tau}_0] \to \mathscr{L}[\boldsymbol{\tau}_0]$ such that for any $\varphi \in \mathscr{L}[\boldsymbol{\tau}_0]$, $\dot{\neg}(\varphi)$ is a negation of φ.

(v) $\mathscr{L}$ is *recursively enumerable for validity* iff $\mathscr{L}$ is effective and for all $\boldsymbol{\tau}_0 \in \mathrm{HF}$ the set $\{\varphi \in \mathscr{L}[\boldsymbol{\tau}_0] \mid \models \varphi\}$ is recursively enumerable.

(vi) $\mathscr{L}$ is *recursively enumerable for consequence* iff $\mathscr{L}$ is effective and for all $\boldsymbol{\tau}_0 \in \mathrm{HF}$ and all recursively enumerable subsets Φ of $\mathscr{L}[\boldsymbol{\tau}_0]$ the set $\{\varphi \in \mathscr{L}[\boldsymbol{\tau}_0] \mid \Phi \models \varphi\}$ is recursively enumerable.

(vii) $\mathcal{L}$ has the *Löwenheim–Skolem property* (*down to* κ) iff each satisfiable $\mathcal{L}$-sentence has a model of power $\leq\aleph_0(\leq\kappa)$. (Here, the power of a τ-structure $\mathfrak{A}$ is defined as $|A|$ in the one-sorted case and as $\sum_{s\in\tau}|A_s|$ in the many-sorted case.)

(viii) $\mathcal{L}$ has the *Craig* or *interpolation property* iff for all τ_0, τ_1: if $\varphi_i \in \mathcal{L}[\tau_i]$ $(i = 0, 1)$ and $\varphi_0 \models \varphi_1$, then there is an *interpolant*, that is, a sentence $\psi \in \mathcal{L}[\tau_0 \cap \tau_1]$ such that $\varphi_0 \models \psi$ and $\psi \models \varphi_1$ (provided—in the many-sorted case—that $\tau_0 \cap \tau_1$ contains at least one sort symbol).

(ix) $\mathcal{L}$ has the *Beth property* (that is, $\mathcal{L}$ satisfies Beth's definability theorem) iff for all τ, all symbols § from τ different from sort symbols and all $\varphi \in \mathcal{L}[\tau]$, if § is implicitly defined by φ, then § is explicitly definable relative to φ.

The notions of implicit and explicit definability are given, say for unary R according to the following definition.

1.2.5 Definition. (i) R is *implicitly defined by* φ, if every $(\tau\setminus\{R\})$-structure has at most one expansion to a τ-structure satisfying φ.

(ii) R is *explicitly definable relative to* φ, if for a new constant c of the same sort s as the argument place of R, there is a sentence $\psi(c)$ in $\mathcal{L}[(\tau\setminus\{R\}) \cup \{c\}]$ such that for all τ-structures $\mathfrak{A}$ with $\mathfrak{A} \models \varphi$ one has

$$R^{\mathfrak{A}} = \{a \in A_s | (\mathfrak{A}, a) \models \psi(c)\}.$$

Intuitively this last means that

$$\varphi \models \forall c(Rc \leftrightarrow \psi(c)).$$

Inspection shows that the usual proof in $\mathcal{L}_{\omega\omega}$ of Beth's theorem via the interpolation theorem needs only the regularity properties of $\mathcal{L}_{\omega\omega}$ given by (i)–(v) in Definition 1.1.1 together with the basic closure properties given in Definition 1.2.1. Hence, any regular logic $\mathcal{L}$ with the interpolation property has the Beth property. This simple fact may be considered as the first theorem of abstract model theory that we have met. And, of course, there is also a first problem: Under what conditions can one conclude that the definability property yields the interpolation property? For an answer, the reader is referred to Section XVIII.4.

Historical Remarks. The impetus to treat general logical systems goes back to Mostowski [1957]. Definitions similar to the ones above were given first by Lindström [1969] and H. Friedman [1970a]. Barwise [1974a] develops a more systematic approach in a categorical framework. A fairly general definition covering, for instance, topological logics is given in Mundici [1984b]. A thorough discussion of properties of logics—from basic ones to more specific ones—can be found in Feferman [1975].

2. *Examples of Principal Logics*

The study of general logics should provide us with means to investigate concrete logics. On the other hand the study of concrete systems can indicate paths that should be followed in the abstract theory. Led by this insight, we now briefly describe numerous systems beyond first-order logic and sketch their most important features. According to our agreement we restrict ourselves to the one-sorted case. An exception is the higher-order case in Section 2.1. More details can be found in Chapter VI.

2.1. *Logics of Higher Order*

Among the possible higher-order logics, we will restrict ourselves to those of just the next level.

2.1.1 Definition. *Second-order logic*, $\mathscr{L}^2$, is built up as usual, allowing for each sort s quantification over n-ary relations on the domain of sort s.

Obviously, $\mathscr{L}^2$ is regular. Its expressive power, however, contrasts with the fact that practically all useful model-theoretic properties of first-order logic fail. Moreover, because of our weakness in governing the notion of subset, we quickly run into set-theoretical dependencies as well. For instance, via a suitable formulation of the continuum hypothesis (CH), one can obtain an $\mathscr{L}^2$-sentence that is valid iff CH holds. Nevertheless the situation is not quite hopeless since many of the logics developed up to now can be considered as parts of $\mathscr{L}^2$. Hence investigations of stronger logics can be seen as aimed at providing a model-theoretic treatment for more and more of $\mathscr{L}^2$. In particular, Chapters XII and XIII will demonstrate that it is even possible to venture into the "real realm" of second-order logic.

Warning. We are usually correct in taking it for granted that properties of a logic do not change if we pass from the many-sorted case to the one-sorted case or vice versa; however, the interpolation property does fail for $\mathscr{L}^2$ in the two-sorted case, even though it is trivially true in the one-sorted case. The proof uses a far-reaching method that goes back to Craig [1965]. A version of it is given in Section 7.3, and a systematic treatment can be found in Section XVII.1.2.

2.1.2 Definition. *Weak second-order logic*, $\mathscr{L}^{w2}$, in contrast to $\mathscr{L}^2$, has the relation variables ranging only over *finite* relations.

It would appear that $\mathscr{L}^{w2}$ deprives the notion of subset of its teeth. In $\mathscr{L}^{w2}$, however, one can easily express the notion of finiteness, because the finiteness of the domain of sort s is guaranteed by the sentence $\exists X^s\, \forall x^s\, X^s x^s$. In this way, one can characterize, for example, the standard model of arithmetic, torsion groups, etc. Hence, $\mathscr{L}^{w2}$ is neither $\aleph_0$-compact nor recursively enumerable for validity. On the other hand, it is easy to prove the Löwenheim–Skolem property. As arithmetical $\mathscr{L}^{w2}$-truth is implicitly definable, it can be seen by the method mentioned in the warning above (see Section 7.3) that the Beth property and hence the interpolation property fail.

2.2. *Examples of Logics with Cardinality Quantifiers*

If $\mathscr{L}_{\omega\omega}$ is enlarged by a unary quantifier Q that is monotone in the sense defined for Theorem 4.2.3, then, according to Theorem III.4.1, the resulting logic $\mathscr{L}_{\omega\omega}(Q)$ is regular just in case Q is some Q_α. (For any ordinal α, $Q_\alpha x\varphi(x)$ means that there are at least $\aleph_\alpha$ many x such that $\varphi(x)$.) We shall deal here with the logic $\mathscr{L}_{\omega\omega}(Q_1)$ and some of its relatives. For considerably more information and historical notes see Chapter IV, and for $\mathscr{L}_{\omega\omega}(Q_\alpha)$ with $\alpha > 1$ see Chapter V.

Example 1. The logic $\mathscr{L}_{\omega\omega}(Q_1)$ has some useful properties: It is $\aleph_0$-compact (Fuhrken [1964]) and recursively enumerable for consequence (Vaught [1964]). Keisler [1970] gives a completeness proof using an elegant system of rules that arises from a complete first-order calculus by addition of the following four axiom schemata:

(i) "*2 is countable*": $\neg Q_1 x(x = y \vee x = z)$;
(ii) "*Q_1 is monotone*": $\forall x(\varphi \to \psi) \to (Q_1 x\varphi \to Q_1 x\psi)$;
(iii) "*Countable unions of countable sets are countable*": $Q_1 x\, \exists y\varphi \to \exists x\, Q_1 y\varphi \vee Q_1 y\, \exists x\varphi$;
(iv) *Renaming of bound variables*: $Q_1 x\varphi(x, \mathbf{x}) \leftrightarrow Q_1 y\varphi(y, \mathbf{x})$ for any y not free in $\varphi(x, \mathbf{x})$.

For details see Section IV.3. Alternative proofs will be given in Sections 3.1 and 3.2. As we shall see there, the expressive power of $\mathscr{L}_{\omega\omega}(Q_1)$ beyond first-order logic comes down to the characterization of $\aleph_1$-*like orderings*, i.e. structures $\mathfrak{A} = (A, <^{\mathfrak{A}})$ that are models of the axioms for linear orderings plus the sentence

$$Q_1 x\, x = x \wedge \forall y \neg Q_1 x\, x < y.$$

For the strength of $\mathscr{L}_{\omega\omega}(Q_1)$ in mathematical contexts, see Chapter VII.

The set $\{\neg Q_1 x\, x = x\} \cup \{\neg c_\alpha = c_\beta | 0 \le \alpha < \beta < \aleph_1\}$ shows us that $\mathscr{L}_{\omega\omega}(Q_1)$ is not $\aleph_1$-compact. Of course, the Löwenheim–Skolem property fails, but the Löwenheim–Skolem property down to $\aleph_1$ (even for sets of sentences of power $\le\aleph_1$) can be proved similarly to the downward Löwenheim–Skolem–Tarski theorem in $\mathscr{L}_{\omega\omega}$. Also, $\mathscr{L}_{\omega\omega}(Q_1)$ satisfies an omitting types theorem (cf. Section IV.3.3). But the hope of having found a useful logic was weakened by several points. For instance, by the up-to-now unsuccessful search for satisfactory preservation theorems, and by the failure of the interpolation property (Keisler 1971) and the Beth property (H. Friedman [1973]).

Keisler's counterexample to interpolation in $\mathscr{L}_{\omega\omega}(Q_1)$ can be described as follows. Let $\varphi_0(E, R)$ express that E is an equivalence relation with only uncountable equivalence classes and that R is a countable set of representatives. Furthermore, let $\varphi_1(E, S)$ express a similar statement with S being an *un*countable set of representatives. Then the entailment

$$(*) \qquad \varphi_0(E, R) \models \neg\varphi_1(E, S)$$

holds, but there is no $\mathscr{L}_{\omega\omega}(Q_1)$-interpolant (cf. 4.2.7).

Example 2. What might be called the "*Ramseyfication*" of the quantifier Q_1 leads to the regular logics $\mathscr{L}_{\omega\omega}(Q_1^n)$, for $n \geq 1$, and $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ of Magidor–Malitz [1977a]. Q_1^n is an n-ary quantifier, the meaning of which is defined by the following satisfaction condition:

$$\mathfrak{A} \models Q_1^n \mathbf{x}\varphi(\mathbf{x}) \quad \text{iff} \quad \begin{array}{l}\text{there is an uncountable subset } M \text{ of } A \text{ where}\\ \mathfrak{A} \models \varphi[\mathbf{b}] \text{ for all } \mathbf{b} \in M^n.\end{array}$$

Sometimes one uses the variation with "for all $\mathbf{b} \in M^n$" replaced by "for all *distinct* $b_0, \ldots, b_{n-1} \in M$"; however, the quantifiers resulting in either version can easily be defined from each other.

Assuming $V = L$ (or even $\diamondsuit_{\aleph_1}$), Magidor and Malitz showed that $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ is $\aleph_0$-compact. A proof is given in Section IV.5.2. On the other hand, according to a result of Shelah, it is consistent to assume that $\mathscr{L}_{\omega\omega}(Q_1^2)$ is not $\aleph_0$-compact. The dependence on set-theoretical principles beyond usual set theory (such as $\diamondsuit_{\aleph_1}$) becomes intelligible if one takes into consideration that Suslin trees, for instance, are characterizable in $\mathscr{L}_{\omega\omega}(Q_1^2)$ (see Example IV.5.1.4).

In $\mathscr{L}_{\omega\omega}(Q_1^2)$, the entailment $(*)$ of Example 1 has the interpolant

$$\varphi(E) \wedge \neg Q_1^2 xy(x = y \vee \neg Exy)$$

where $\varphi(E)$ states that E is an equivalence relation with only uncountable equivalence classes. Nevertheless, for no $n \geq 1$ does $\mathscr{L}_{\omega\omega}(Q_1^n)$ have the Beth property (see Badger [1980]). For a counterexample to interpolation see 7.1.3(b). Because $\mathscr{L}_{\omega\omega}(Q_1^2)$ overcomes Keisler's counterexample, it is strictly stronger than $\mathscr{L}_{\omega\omega}(Q_1)$; moreover, as was shown by Garavaglia and Shelah, the expressive power of $\mathscr{L}_{\omega\omega}(Q_1^{n+1})$ is greater than that of $\mathscr{L}_{\omega\omega}(Q_1^n)$, for all $n \geq 1$. Details and further results of this kind can be found in Rapp [1983], [1984].

Example 3. "*Positive*" *logic*, $\mathscr{L}_{\omega\omega}(\text{pos})$, and "*negative*" *logic*, $\mathscr{L}_{\omega\omega}(\text{neg})$. As has been pointed out, mainly by Feferman, it would be interesting to have a regular $\aleph_0$-compact extension of $\mathscr{L}_{\omega\omega}(Q_1)$ that is recursively enumerable for consequence and has the interpolation property. Such a logic would combine the usefulness of $\aleph_0$-compactness *and* interpolation with the expressive power of $\mathscr{L}_{\omega\omega}(Q_1)$. The search has been unsuccessful so far. (Reasons can be found, for instance, in Proposition XVII.2.4.6.) However, the attempts to date have led to various systems possessing all desired properties up to interpolation.

In order to find a candidate besides $\mathscr{L}_{\omega\omega}(Q_1^n | n \geq 1)$ we observe that $Q_1 x\varphi(x)$ means the same as

$$(*) \qquad \exists \text{ uncountable } X\, \forall x(\neg Xx \vee \varphi(x))$$

or as

$$(**) \qquad \neg\exists \text{ countable } X\, \forall x(Xx \vee \neg\varphi(x)).$$

Thus we are led to logics that arise from $\mathscr{L}_{\omega\omega}$ by allowing quantifications over either uncountable or over countable subsets. In both cases, however, $\aleph_0$-compactness fails, since we can characterize in these logics $(\omega_1, <)$ and $(\omega, <)$, respectively. For instance, a linear ordering is isomorphic to $(\omega_1, <)$ iff it is ω_1-like and each uncountable subset has a least element.

Let us say that a set variable X occurs *negatively* (*positively*) in a formula φ, if there is an occurrence of X in φ that lies in the scope of an odd (even) number of negation signs provided $\neg$, $\wedge$, $\vee$ are the only propositional connectives in φ. Obviously, X occurs only negatively in the matrix of $(*)$ and only positively in the matrix of $(**)$. Hence, in our second, and more modest attempt, we define the logics $\mathscr{L}_{\omega\omega}(\text{neg})$ and $\mathscr{L}_{\omega\omega}(\text{pos})$ that arise from $\mathscr{L}_{\omega\omega}$ by allowing existential quantifications such as $\exists X\varphi$, with the variable X ranging over uncountable (countable) subsets, only in case X occurs at most negatively (positively) in φ.

$\mathscr{L}_{\omega\omega}(\text{neg})$ extends $\mathscr{L}_{\omega\omega}(Q^n | n \geq 1)$, but, according to a result of Stavi, $(\omega_1, <)$ is still characterizable (see Theorem IV.5.1.2). On the other hand, $\mathscr{L}_{\omega\omega}(\text{pos})$ turns out to be $\aleph_0$-compact and recursively enumerable for consequence. It is strictly stronger than $\mathscr{L}_{\omega\omega}(Q_1)$, because the entailment $(*)$ in Example 1 has the $\mathscr{L}_{\omega\omega}(\text{pos})$-interpolant

$$\varphi(E) \wedge \exists X\, \forall y\, \exists x(Xx \wedge Exy).$$

An easy induction shows the validity of:

(***) *If $\varphi(X, \ldots)$ is an $\mathscr{L}_{\omega\omega}(\text{pos})$-formula and $\mathfrak{A} \models \varphi[M, \ldots]$ holds for some countable $M \subseteq A$, then for any countable M' such that $M \subseteq M' \subseteq A$, we have $\mathfrak{A} \models \varphi[M', \ldots]$.*

Intuitively, this means that $\mathscr{L}_{\omega\omega}(\text{pos})$ allows existential quantifications over large countable sets. The next example provides a natural generalization of this feature.

Example 4. *Stationary logic* is denoted by $\mathscr{L}_{\omega\omega}(\text{aa})$. Here we restrict ourselves to a short description that will be sufficient to give a compactness proof for models of power $\aleph_1$ in Section 3.2. A comprehensive treatment is given in Section IV.4.

We first need some set-theoretical terminology. For any set A, a subset S of the set $P_{\omega_1}(A)$ of countable subsets of A is *unbounded* (in $P_{\omega_1}(A)$), if for any $s \in P_{\omega_1}(A)$ there is some $s' \in S$ such that $s \subseteq s'$. The set S is *closed* (in $P_{\omega_1}(A)$), if the union of any countable $\subseteq$-chain in S belongs to S. The set S is said to be *cub*, if it is both closed and unbounded. The *cub filter*, $D(A)$, over A (and it really is a filter!) consists of those subsets of $P_{\omega_1}(A)$ which contain a cub set. If $A = \omega_1$, then those subsets of ω_1 which are closed and unbounded in the usual sense of ordinal number theory form a basis of $D(A)$. Intuitively, $D(A)$ may be considered as the set of those subsets of $P_{\omega_1}(A)$ which consist of "almost all" elements of $P_{\omega_1}(A)$.

The logic $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ arises from $\mathscr{L}_{\omega\omega}$ by adding new variables $X, Y, \ldots$ for countable subsets. These lead to new atomic formulas Xt (for first-order terms t). Besides the usual first-order operations, quantifications over set variables are allowed *only* by means of a new unary quantifier (aa). The meaning of (aa) is specified by the satisfaction condition:

$$\mathfrak{A} \models (\mathrm{aa})X\varphi(X) \quad \text{iff} \quad \{s \in P_{\omega_1}(A) \mid \mathfrak{A} \models \varphi[s]\} \in D(A).$$

In other words the condition means that $\mathfrak{A} \models \varphi[s]$ holds for "almost all" countable subsets s of A.

The name "stationary" suggests several features: For instance, the dual quantifier $\neg(\mathrm{aa})\neg$ to (aa) means intuitively "for stationary many" (where a stationary set is one intersecting every cub set). As the results in Chapter IV will illustrate, stationary logic is a nice resting point in the ladder of extensions of $\mathscr{L}_{\omega\omega}(Q_1)$. According to (***) of Example 3 above, any $\mathscr{L}_{\omega\omega}(\mathrm{pos})$-formula $\exists X\varphi$ has the same meaning as $(\mathrm{aa})X\varphi$. Therefore $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ can be considered as an extension of $\mathscr{L}_{\omega\omega}(\mathrm{pos})$. It is even a strict extension (see Remark IV.4.1.2(v)).

2.3. Cardinality Quantifiers with Complex Scopes

There are some interesting quantifiers which are applied to pairs of formulas. The *Rescher quantifier*, Q^{R}, from Rescher [1962], is defined by the satisfaction condition:

$$\begin{aligned}&\mathfrak{A} \models Q^{\mathrm{R}}xy[\varphi(x), \psi(y)] \quad \text{iff}\\ &\qquad |\{a \in A \mid \mathfrak{A} \models \varphi[a]\}| < |\{b \in A \mid \mathfrak{A} \models \psi[b]\}|.\end{aligned}$$

The *equicardinality* or *Härtig quantifier*, I, from Härtig [1965], is defined similarly but with " = " instead of " < ". Q^{R} and I lead to the regular logics $\mathscr{L}_{\omega\omega}(Q^{\mathrm{R}})$ and $\mathscr{L}_{\omega\omega}(I)$.

Clearly, the quantifier I can be expressed by Q^{R}. On the other hand it can be seen that there is no $\mathscr{L}_{\omega\omega}(I)$-sentence of vocabulary $\{U\}$ that has the same models as $Q^{\mathrm{R}}xy[Ux, \neg Uy]$. (See also Hauschild [1981].)

Since $(\omega, <)$ can be characterized in $\mathscr{L}_{\omega\omega}(I)$ by adjoining to the usual axioms of linear orderings without last element the sentence

$$\forall xy(x = y \leftrightarrow Iuv[u < x, v < y]),$$

we see that neither $\mathscr{L}_{\omega\omega}(I)$ nor $\mathscr{L}_{\omega\omega}(Q^{R})$ is $\aleph_0$-compact. Even more, if φ is the $\mathscr{L}_{\omega\omega}(I)$-sentence in the vocabulary $\{<, U\}$ formed from the axioms of a linear ordering by adjoining the sentence

$$\forall xy(Ux \wedge Uy \wedge Iuv[u < x, v < y] \rightarrow x = y)$$

then the $\{<\}$-reducts of the models of φ relativized to the predicate U form the class of all linear orderings that are isomorphic to the natural ordering on a set of cardinals, and this is nothing more than the class of *all* well-orderings. In the terminology to come (see Definition 3.1.1) the class of all well-orderings in RPC in $\mathscr{L}_{\omega\omega}(I)$ and hence in $\mathscr{L}_{\omega\omega}(Q^{R})$.

2.4. Logics with Cofinality Quantifiers

Is there a regular logic strictly stronger than first-order logic that is fully compact? In Shelah [1975d] one finds a variety of examples. We mention the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$, where $Q^{\mathrm{cf}\,\omega}$ is a binary quantifier the meaning of which is given by

$$\mathfrak{A} \models Q^{\mathrm{cf}\,\omega}xy\, \varphi(x, y) \quad \text{iff} \quad \{(a, b) \in A \times A \mid \mathfrak{A} \models \varphi[a, b]\} \text{ is a linear ordering of its field with cofinality } \omega.$$

In Section 3.2 we sketch a proof that $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$ is fully compact and recursively enumerable for consequence. For the failure of the interpolation property see Counterexample 7.1.3(c), and for larger cofinalities, see Chapter V.

2.5. Logics with Quantifiers of Partially Ordered Prefixes

A usual first-order prefix is of "linear character" in the sense that each existential variable depends on *all* preceding universal ones. This becomes obvious by the introduction of *Skolem functions*. For instance, a formula such as

$$\forall u\, \exists v\, \forall wx\, \exists y\, \varphi(u, v, w, x, y)$$

is equivalent to

$$\exists fg\, \forall uwx\, \varphi(u, f(u), w, x, g(u, w, x)),$$

where f is a unary and g a ternary function variable. One of the simplest examples of a prefix that is not of this kind leads to the 4-ary *Henkin-quantifier* Q^{H} (Henkin [1961]). Its meaning is given by:

$$\mathfrak{A} \models Q^{H}x_0\, y_0\, x_1\, y_1\, \varphi(x_0, y_0, x_1, y_1)$$
$$\text{iff there are functions } f_0, f_1 : A \rightarrow A \text{ such that for all } a_0, a_1 \in A \text{ we have } \mathfrak{A} \models \varphi[a_0, f_0(a_0), a_1, f_1(a_1)].$$

Usually $Q^{\mathrm{H}}x_0y_0x_1y_1\,\varphi(x_0, y_0, x_1, y_1)$ is written more intuitively as

$$\begin{matrix}\forall x_0\,\exists y_0\\ \forall x_1\,\exists y_1\end{matrix}\varphi(x_0, y_0, x_1, y_1),$$

in order to display the functional dependence of the variables.

The *Henkin logic* $\mathscr{L}_{\omega\omega}(Q^{\mathrm{H}})$ is regular. But, if φ is the sentence

$$\exists z\begin{matrix}\forall x_0\,\exists y_0\\ \forall x_1\,\exists y_1\end{matrix}(z \neq y_0 \wedge (y_0 = x_1 \rightarrow y_1 = x_0))$$

and $A \neq \emptyset$, then we have $A \models \varphi$ iff there are $a \in A$ and $f_0, f_1\colon A \rightarrow A$ such that $a \notin \mathrm{rg}(f_0)$ and $f_1(f_0(b)) = b$ for all $b \in A$. This simply means that A is infinite. Hence, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{H}})$ is not $\aleph_0$-compact. Moreover, the adjunction to $\mathscr{L}_{\omega\omega}$ of quantifiers like Q^{H} that stem from partially ordered prefixes leads to the full expressive power of second-order logic. Details can be found in Section VI.1. For the mathematical relevance of these quantifiers see Barwise [1976].

2.6. Logics with Standard Part

An immediate way to obtain a logic in which, say, $(\omega, <)$ is characterizable, is to incorporate $(\omega, <)$ into the semantics of first-order logic as done in ω-logic. The following definition provides a generalization.

Let $\mathscr{L}$ be a logic, τ_0 a vocabulary, U a unary relation symbol not in τ_0, and $\mathfrak{K}$ a class of τ_0-structures closed under isomorphism. We define a logic $\mathscr{L}(\mathfrak{K})$ in the sense of the generalization under 1.1.3 as follows:

$$\mathscr{L}(\mathfrak{K})[\tau] = \begin{cases}\mathscr{L}[\tau], & \text{if } \tau_0 \cup \{U\} \subseteq \tau;\\ \emptyset, & \text{otherwise,}\end{cases}$$

and

$$\mathrm{Str}_{\mathscr{L}(\mathfrak{K})}[\tau] = \begin{cases}\{\mathfrak{A} \in \mathrm{Str}[\tau] \mid U^{\mathfrak{A}}\ \tau_0\text{-closed in } \mathfrak{A} \text{ and}\\ \quad (\mathfrak{A} \restriction \tau_0) \mid U^{\mathfrak{A}} \in \mathfrak{K}\}, & \text{if } \tau_0 \cup \{U\} \subseteq \tau;\\ \emptyset, & \text{otherwise.}\end{cases}$$

$$\mathfrak{A} \models_{\mathscr{L}(\mathfrak{K})} \varphi \quad \text{iff} \quad \mathfrak{A} \in \mathrm{Str}_{\mathscr{L}(\mathfrak{K})}[\tau_{\mathfrak{A}}],\ \varphi \in \mathscr{L}(\mathfrak{K})[\tau_{\mathfrak{A}}], \text{ and } \mathfrak{A} \models_{\mathscr{L}} \varphi.$$

In the many-sorted case one can proceed similarly (and even dispense with the analogues of U by introducing new sorts, see also Remark 3.1.2).

If $\mathfrak{K} = \{\mathfrak{B} \mid \mathfrak{B} \cong \mathfrak{A}\}$, we write $\mathscr{L}(\mathfrak{A})$ instead of $\mathscr{L}(\mathfrak{K})$.

An interesting example in addition to ω-logic, $\mathscr{L}_{\omega\omega}(\omega, <)$, is $\mathscr{L}_{\omega\omega}(\mathfrak{K})$, where $\mathfrak{K}$ is the class of $\aleph_1$-like orderings. In both cases one can dispense with U, as the task of U can be taken over by the field of $<$.

The following fact plays a key role in the compactness proof for $\mathscr{L}_{\omega\omega}(Q_1)$ as given in Section 3.1.

2.6.1 Theorem. *Let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. Then $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ is $\aleph_0$-compact.*

Proof. Let $\tau_{\mathfrak{A}} = \{<\}$ for $\mathfrak{A} \in \mathfrak{K}$ and τ a fixed *countable* vocabulary, $< \in \tau$. The τ-*regularity scheme* $\Sigma = \Sigma(\tau)$ consists of the $\mathscr{L}_{\omega\omega}[\tau]$-sentences of the form

$$\forall \mathbf{x}\, \forall x\, \exists y\, \forall u < x(\exists v \in \mathit{field}(<)\varphi(u, v, \mathbf{x}) \to \exists v < y\, \varphi(u, v, \mathbf{x})).$$

It is sufficient to prove that for all $\Phi \subseteq \mathscr{L}_{\omega\omega}[\tau]$,

(*) Φ has an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model iff Ψ has an $\mathscr{L}_{\omega\omega}$-model,

where $\Psi = \Phi \cup \Sigma \cup \{<$ *is a linear ordering of its field without last element*$\}$.

The implication from left to right is clear, because $\aleph_1$ is regular.

For the other direction assume that Ψ has a τ-model $\mathfrak{A}$, where $\mathfrak{A}$ can be chosen countable. We show that there exists a countable τ-structure $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $<^{\mathfrak{B}}$ is a proper end extension of $<^{\mathfrak{A}}$. Then we can repeat this process $\aleph_1$-times, taking unions at limit stages, and arrive at an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model of Φ.

Let $\Delta(\mathfrak{A})$ be the elementary diagram of $\mathfrak{A}$ formulated with new constants $\underline{a}$ for $a \in A$, c a new constant, and let $\Xi = \Delta(\mathfrak{A}) \cup \{\underline{a} < c \mid a \in \mathrm{field}(<^{\mathfrak{A}})\}$. We have to show that Ξ has a model which, for all $a_0 \in \mathrm{field}(<^{\mathfrak{A}})$, omits the type $\{x \neq \underline{a} \mid a <^{\mathfrak{A}} a_0\} \cup \{x < \underline{a_0}\}$. In order to prove this, let $a_0 \in \mathrm{field}(<^{\mathfrak{A}})$ be given and a formula $\chi(x, y)$ of vocabulary $\tau \cup \{\underline{a} \mid a \in A\}$ be such that

(1) $\Xi \cup \{\exists x\, \chi(x, c)\}$ has a model.

We have to show that

$$(2) \qquad \Xi \cup \left\{\exists x\left(\chi(x, c) \wedge \left(\bigvee_{a <^{\mathfrak{A}} a_0} x = \underline{a} \vee \underline{a_0} \leq x \vee x \notin \mathit{field}(<)\right)\right)\right\}$$

has a model.

Let us write $\exists$ arb. lg. $w\psi(w, \ldots)$ for $\forall u \in \mathit{field}(<)\, \exists w > u\psi(w, \ldots)$. By an easy compactness argument we see that (1) is equivalent to:

(1′) $(\mathfrak{A}, (a)_{a \in A}) \models \exists$ arb. lg. $w\, \exists x\, \chi(x, w)$,

and that it is sufficient to prove instead of (2):

(2′)

$(\mathfrak{A}, (a)_{a \in A}) \models \bigvee_{a <^{\mathfrak{A}} a_0} \exists$ arb. lg. $w\chi(\underline{a}, w)$, or

$(\mathfrak{A}, (a)_{a \in A}) \models \exists$ arb.lg. $w\, \exists x(\chi(x, w) \wedge (\underline{a_0} \leq x \vee x \notin \mathit{field}(<)))$.

For a proof of (2′) assume the first disjunct to be false. Then for each $a <^{\mathfrak{A}} a_0$ there is some $b \in \text{field}(<^{\mathfrak{A}})$ such that for all $d \in \text{field}(<^{\mathfrak{A}})$, if $\chi(\underline{a}, \underline{d})$ holds in $(\mathfrak{A}, (a)_{a \in A})$, then $d <^{\mathfrak{A}} b$. As $\mathfrak{A}$ satisfies the τ-regularity scheme, there is a uniform bound b_0 of this kind for all $a <^{\mathfrak{A}} a_0$. Hence, because of (1′), the second disjunct must be true. ☐

The proof yields more. From (*) we obtain for $\Phi \cup \{\varphi\} \subseteq \mathscr{L}_{\omega\omega}(\mathfrak{K})[\tau]$:

$$\Phi \models_{\mathscr{L}_{\omega\omega}(\mathfrak{K})} \varphi \quad \text{iff} \quad \Psi \models_{\mathscr{L}_{\omega\omega}} \varphi.$$

If Φ is recursively enumerable, then so is Ψ. Thus we have:

2.6.2 Corollary. *Let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. Then $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ is recursively enumerable for consequence.* ☐

2.7. Infinitary Logics

We shall not go into details here. Infinitary logics of type $\mathscr{L}_{\kappa\lambda}$ and admissible fragments will be treated in Chapters VIII and IX. Infinitary quantifiers such as the game quantifier G are described in Chapter X. For $\mathscr{L}_{\infty\omega}$ and arguments for its naturalness, see Section III.3 and, in particular, Section XVII.2.2. Occasionally we shall also consider logics such as $\mathscr{L}_{\kappa\lambda}(Q_1)$.

In $\mathscr{L}_{\omega\omega}$, the set $\{\neg, \wedge, \vee\}$ forms a complete system of propositional connectives. Of course, in $\mathscr{L}_{\omega_1\omega}$, where we use only $\neg$ and the generalizations of $\wedge$, $\vee$, we are far away from propositional completeness. Hence the question arises whether there are other reasonable (infinitary) propositional connectives for $\mathscr{L}_{\omega_1\omega}$. The answer is, in some sense, positive; details can be found in the references given in Section III.3.8.

3. Comparing Logics

In the preceding section we intuitively compared logics with respect to their expressive power. The aim of this section is to give precise definitions for the comparison of logics by use of elementary and projective classes and to present some concrete examples that will illustrate the methodological importance of the latter notions.

3.1. Elementary and Projective Classes

We begin with a basic definition.

3.1.1 Definition. Let $\mathscr{L}$ be a logic and $\mathfrak{K}$ a class of τ-structures.

We say that $\mathfrak{K}$ is an *elementary class* in $\mathscr{L}$ (or that $\mathfrak{K}$ is EC in $\mathscr{L}$, or that $\mathfrak{K} \in \text{EC}_{\mathscr{L}}$) iff there is $\varphi \in \mathscr{L}[\tau]$ such that $\mathfrak{K} = \text{Mod}_{\mathscr{L}}^{\tau}(\varphi)$.

We say that $\mathfrak{K}$ is a *projective class* in $\mathscr{L}$ (or that $\mathfrak{K}$ is PC in $\mathscr{L}$, or that $\mathfrak{K} \in \mathrm{PC}_{\mathscr{L}}$) iff there is $\tau' \supseteq \tau$, having, in the many sorted case, the same sort symbols as τ, and a class $\mathfrak{K}'$ of τ'-structures, $\mathfrak{K}' \in \mathrm{EC}_{\mathscr{L}}$, such that $\mathfrak{K} = \{\mathfrak{A} \upharpoonright \tau \mid \mathfrak{A} \in \mathfrak{K}'\}$, the class of τ-reducts of $\mathfrak{K}'$.

On the other hand $\mathfrak{K}$ is a *relativized projective class* in $\mathscr{L}$ (or $\mathfrak{K}$ is RPC in $\mathscr{L}$, or $\mathfrak{K} \in \mathrm{RPC}_{\mathscr{L}}$) iff (in the one-sorted case) there is $\tau' \supseteq \tau$, a unary relation symbol $U \in \tau' \backslash \tau$, and a class $\mathfrak{K}'$ of vocabulary τ', $\mathfrak{K}' \in \mathrm{EC}_{\mathscr{L}}$, such that

$$\mathfrak{K} = \{(\mathfrak{A} \upharpoonright \tau) \mid U^{\mathfrak{A}} \mid \mathfrak{A} \in \mathfrak{K}' \text{ and } U^{\mathfrak{A}} \text{ is } \tau\text{-closed in } \mathfrak{A}\};$$

or (in the many-sorted case) there is $\tau' \supseteq \tau$ and a class $\mathfrak{K}'$ of τ'-structures, $\mathfrak{K}' \in \mathrm{EC}_{\mathscr{L}}$, such that $\mathfrak{K} = \{\mathfrak{A} \upharpoonright \tau \mid \mathfrak{A} \in \mathfrak{K}'\}$.

Using an intuitive notation, we can say for instance that $\mathfrak{K}$ is RPC in $\mathscr{L}$ in the many-sorted version, if there is some $\tau' \supseteq \tau$ and $\psi \in \mathscr{L}[\tau']$ such that $\mathfrak{K} = \mathrm{Mod}^{\tau}_{\mathscr{L}}(\exists_{\tau' \backslash \tau} \psi)$.

3.1.2 Remarks. For all usual logics $\mathscr{L}$ and classes $\mathfrak{K}$ of one-sorted structures, we have $\mathfrak{K} \in \mathrm{RPC}_{\mathscr{L}}$ in the one-sorted version iff $\mathfrak{K} \in \mathrm{RPC}_{\mathscr{L}}$ in the many-sorted version. The same is true for all regular logics, if we restrict ourselves to finite vocabularies. (The direction from right to left can be shown by unification of domains, and that from left to right by the dual procedure.) Obviously, we have "PC $\subseteq$ RPC" for any logic $\mathscr{L}$ containing sentences such as $\forall x U x$; the inclusion is strict for $\mathscr{L}_{\omega\omega}$, but not for $\mathscr{L}_{\omega_1\omega}$. For details concerning these and other well-known logics, see Oikkonen [1979c].

In general it is not true that every class PC in $\mathscr{L}$ is EC in $\mathscr{L}$, even if $\mathscr{L} = \mathscr{L}_{\omega\omega}$. A counterexample for $\mathscr{L}_{\omega\omega}$ is given by the class of infinite sets. The question whether any class $\mathfrak{K}$ of τ-structures such that $\mathfrak{K}$ *and* $\bar{\mathfrak{K}} = \mathrm{Str}[\tau] \backslash \mathfrak{K}$ are (R)PC in $\mathscr{L}$, is EC in $\mathscr{L}$, will lead to an interesting interpolation property, the so-called Δ-*interpolation* (see Section 7.2). The following simple equivalence shows that interpolation is a generalization of Δ-interpolation.

3.1.3 Proposition. *For any logic $\mathscr{L}$ having the negation property, the following are equivalent*:

(i) *$\mathscr{L}$ has the interpolation property.*
(ii) *For all τ, any two disjoint classes $\mathfrak{K}_0$, $\mathfrak{K}_1$ of τ-structures that are* PC *in $\mathscr{L}$ (one-sorted case) or* RPC *in $\mathscr{L}$ (many-sorted case), can be separated by an elementary class; that is, there is a class $\mathfrak{K} \in \mathrm{EC}_{\mathscr{L}}$ such that $\mathfrak{K}_0 \subseteq \mathfrak{K}$ and $\mathfrak{K}_1 \subseteq \bar{\mathfrak{K}}$.* □

What does it mean to say that a logic $\mathscr{L}^*$ is as strong as $\mathscr{L}$? The model-theoretical point of view offers several ways that lead to a precise definition, starting for example from the following concepts:

(*) *For any $\mathscr{L}$-sentence φ there is an $\mathscr{L}^*$-sentence φ^* having the same meaning as φ.*

(**) *Structures that can be distinguished in $\mathscr{L}$, can also be distinguished in $\mathscr{L}^*$.*

From (*) we obtain the usual definition if we identify the meaning of a sentence with its class of models:

3.1.4 Definition. Let $\mathscr{L}, \mathscr{L}^*$ be logics. We say that $\mathscr{L}^*$ is *as strong as* $\mathscr{L}$, in symbols $\mathscr{L} \leq \mathscr{L}^*$, iff every class EC in $\mathscr{L}$ is EC in $\mathscr{L}^*$. Similarly, $\mathscr{L}$ and $\mathscr{L}^*$ are *equally strong* or *equivalent*, in symbols $\mathscr{L} \equiv \mathscr{L}^*$, iff both $\mathscr{L} \leq \mathscr{L}^*$ and $\mathscr{L}^* \leq \mathscr{L}$. Finally, we say that $\mathscr{L}^*$ is *stronger than* $\mathscr{L}$, in symbols $\mathscr{L} < \mathscr{L}^*$, iff $\mathscr{L} \leq \mathscr{L}^*$ and not $\mathscr{L} \equiv \mathscr{L}^*$.

Obviously, $\leq$ is a partial ordering on logics.

Concept (**) can be made precise by the notion of $\mathscr{L}$-equivalence of structures:

3.1.5 Definition. $\mathscr{L} \leq_{\equiv} \mathscr{L}^*$ iff for all τ and all $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$, if $\mathfrak{A} \equiv_{\mathscr{L}^*} \mathfrak{B}$, then $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$.

When we compare the two notions, we immediately see that $\mathscr{L} \leq \mathscr{L}^*$ implies $\mathscr{L} \leq_{\equiv} \mathscr{L}^*$. The other direction can be false; for instance $\mathscr{L}_{\infty G} \leq_{\equiv} \mathscr{L}_{\infty\omega}$, as $\mathscr{L}_{\infty G}$ has the Karp property, but $\mathscr{L}_{\infty\omega} < \mathscr{L}_{\infty G}$ (see the remark following Theorem 4.3.2 and Section X.3.1). Whereas we shall refer to $\leq_{\equiv}$ only occasionally, the relation $\leq$ and its generalizations (see Definition 3.1.6 below) will actually turn out to be of great methodological importance.

From the examples in Section 2 and the results there stated, we obtain that

$$\mathscr{L}_{\omega\omega} < \mathscr{L}^{w2} < \mathscr{L}^2;$$

$$\mathscr{L}_{\omega\omega} < \mathscr{L}_{\omega\omega}(Q_1) < \mathscr{L}_{\omega\omega}(Q_1^2) < \cdots;$$

$$\mathscr{L}_{\omega\omega}(Q_1) < \mathscr{L}_{\omega\omega}(\mathrm{pos}) < \mathscr{L}_{\omega\omega}(\mathrm{aa});$$

$$\mathscr{L}_{\omega\omega} < \mathscr{L}_{\omega\omega}(I) < \mathscr{L}_{\omega\omega}(Q^{\mathrm{R}}).$$

Moreover, one can easily prove that $\mathscr{L}^{w2} < \mathscr{L}_{\omega_1\omega}$, but $\mathscr{L}^2 \nleq \mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\omega_1\omega} \nleq \mathscr{L}^2$. For the class $\mathfrak{K}$ of $\aleph_1$-like orderings we have $\mathscr{L}_{\omega\omega}(\mathfrak{K}) \leq \mathscr{L}_{\omega\omega}(Q_1)$. However, the other direction is false as can be seen from the sentence $Q_1 x\, x = x$. In order to remedy this situation to some extent, we introduce some new relations between logics, taking (relativized) projective classes instead of elementary ones in Definition 3.1.4.

3.1.6 Definition. For logics $\mathscr{L}$ and $\mathscr{L}^*$, $\mathscr{L} \leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}^*$ iff every class that is (R)PC in $\mathscr{L}$, is (R)PC in $\mathscr{L}^*$. Analogously $<_{(\mathrm{R})\mathrm{PC}}$ and $\equiv_{(\mathrm{R})\mathrm{PC}}$ can be defined.

Now we can state:

3.1.7 Proposition. *Let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. Then $\mathscr{L}_{\omega\omega}(Q_1) \leq_{\mathrm{RPC}} \mathscr{L}_{\omega\omega}(\mathfrak{K})$, provided that for $\mathscr{L}_{\omega\omega}(Q_1)$ we do not allow the symbol $<$ that is used for the orderings in $\mathfrak{K}$.*

Proof. Let $\varphi \in \mathscr{L}_{\omega\omega}(Q_1)[\tau]$ be given such that φ contains a subformula $Q_1 x\, \psi(x, \mathbf{y})$ with Q_1 not in ψ. We take an appropriate new function symbol f and then, writing $f_{\mathbf{y}}(x)$ for $f(x, \mathbf{y})$, we replace $Q_1 x \psi(x, \mathbf{y})$ in φ by a formula $\chi = \chi(\mathbf{y})$ expressing

$\{f_{\mathbf{y}}(x) \mid \psi(x, \mathbf{y})\}$ *is an unbounded subset of field*$(<)$.

Also we add to the resulting sentence, as a conjunct, the sentence $\forall \mathbf{y}(\chi \vee \vartheta)$, where ϑ means that

$\lambda x f_{\mathbf{y}}(x)$ *is injective on* $\{x \mid \psi(x, \mathbf{y})\}$ *and* $\{f_{\mathbf{y}}(x) \mid \psi(x, \mathbf{y})\}$ *is a bounded subset of field*$(<)$.

Repeating this process until all occurrences of Q_1 are eliminated, we arrive at some $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-sentence $\tilde{\varphi}$ in some vocabulary $\tilde{\tau} \supseteq \tau$ such that

$$\mathrm{Mod}^{\tau}_{\mathscr{L}_{\omega\omega}(Q_1)}(\varphi) = \{\mathfrak{A} \upharpoonright \tau \mid \mathfrak{A} \in \mathrm{Mod}^{\tilde{\tau}}_{\mathscr{L}_{\omega\omega}(\mathfrak{K})}(\tilde{\varphi})\}. \quad \square$$

3.1.8 Corollary. *$\mathscr{L}_{\omega\omega}(Q_1)$ is $\aleph_0$-compact.*

Proof. Let $\Phi \subseteq \mathscr{L}_{\omega\omega}(Q_1)[\tau]$ be countable such that every finite subset of Φ has an $\mathscr{L}_{\omega\omega}(Q_1)$-model. We may suppose $< \notin \tau$. Then every finite subset of $\tilde{\Phi}$ has an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model, where $\tilde{\Phi} = \{\tilde{\varphi} \mid \varphi \in \Phi\}$ and all the function symbols used in the construction of the sentences $\tilde{\varphi}$ are chosen to be different. By $\aleph_0$-compactness of $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ (see Theorem 2.6.1) $\tilde{\Phi}$ has an $\mathscr{L}_{\omega\omega}(\mathfrak{K})$-model, and hence Φ has an $\mathscr{L}_{\omega\omega}(Q_1)$-model. $\square$

When we analyze the preceding argument, we see that it is essentially based on the ordering $\mathscr{L}_{\omega\omega}(Q_1) \leq_{\mathrm{RPC}} \mathscr{L}_{\omega\omega}(\mathfrak{K})$. Generalizing, we obtain the first part of:

3.1.9 Proposition. *Assume $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$ and κ to be infinite. Then:*

(i) *If $\mathscr{L}^*$ is κ-compact, then so is $\mathscr{L}$. Hence, if $\mathscr{L}^*$ is compact, then so is $\mathscr{L}$.*
(ii) *If $\mathscr{L}^*$ has the Löwenheim–Skolem property down to κ, then so does $\mathscr{L}$.*

Proof. To prove part (ii) for instance in the many-sorted case, assume that $\mathscr{L}^*$ has the Löwenheim–Skolem property down to κ and that φ is a satisfiable sentence from $\mathscr{L}[\tau]$. As $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$, there is some $\tau^* \supseteq \tau$ and a sentence $\varphi^* \in \mathscr{L}^*[\tau^*]$ such that $\emptyset \neq \mathrm{Mod}^{\tau}_{\mathscr{L}}(\varphi) = \mathrm{Mod}^{\tau^*}_{\mathscr{L}^*}(\varphi^*) \upharpoonright \tau$. By assumption, φ^*, having a model, has a model $\mathfrak{A}^*$ of power $\leq \kappa$. Hence $\mathfrak{A}^* \upharpoonright \tau$ is a model of φ of power $\leq \kappa$. $\square$

Proposition 3.1.9(i) is used in numerous compactness proofs. Similar to Corollary 3.1.8, the (κ-) compactness of the logic $\mathscr{L}$ in question is reduced to the (κ-) compactness of some other logic $\mathscr{L}^*$ by showing $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$ and proving (κ-) compactness for $\mathscr{L}^*$. Often $\mathscr{L}^*$ is first-order logic with some additional restrictions (for instance $\aleph_1$-like orderings). Some further examples will be presented in Section 3.2.

The general scheme underlying all these proofs can be formulated as follows: In order to show that a logic $\mathscr{L}$ has some property P, one

(a) *first finds a logic* $\mathscr{L}^*$ *such that* $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$;
(b) *then proves that* $\mathscr{L}^*$ *has property* P;
(c) *and finally verifies that* P *descends from* $\mathscr{L}^*$ *to* $\mathscr{L}$.

If P means (κ-) compactness or the Löwenheim–Skolem property down to κ, step (c) above becomes superfluous because of Proposition 3.1.9. In later sections we will see that numerous other properties are inherited downward along $\leq_{\mathrm{RPC}}$, thus enlarging the applicability of the reduction method considerably.

In many cases, if $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$, completeness properties also descend from $\mathscr{L}^*$ to $\mathscr{L}$. Rather than give a general theorem, we will confine ourselves to examples. In order to present the first one, we again let $\mathfrak{K}$ be the class of $\aleph_1$-like orderings. In the terminology of the proofs of Proposition 3.1.7 and Corollary 3.1.8 we have for any $\Phi \subseteq \mathscr{L}_{\omega\omega}(Q_1)[\tau]$ and any $\varphi \in \mathscr{L}_{\omega\omega}(Q_1)[\tau]$, that if $< \notin \tau$, then

$$\Phi \models_{\mathscr{L}_{\omega\omega}(Q_1)} \varphi \quad \text{iff} \quad \tilde{\Phi} \models_{\mathscr{L}_{\omega\omega}(\mathfrak{K})} \tilde{\varphi}.$$

As the transition from an $\mathscr{L}_{\omega\omega}(Q_1)$-sentence ψ to $\tilde{\psi}$ is effective, we obtain the following result from Corollary 2.6.2:

3.1.10 Theorem. $\mathscr{L}_{\omega\omega}(Q_1)$ *is recursively enumerable for consequence.* □

3.2. A Reduction Method

Many applications of the reduction scheme given in Section 3.1 can be systematized in a way first made explicit in Hutchinson [1976b]. The method applies to logics $\mathscr{L}$ that admit a nice set-theoretical description, and the corresponding logics $\mathscr{L}^*$ are based on specific models of set theory. Without exhausting its full power, we illustrate the method by some examples. (See also Section XVII.2.3.) First, we treat $\mathscr{L}_{\omega\omega}(Q_1)$. Then we sketch a similar procedure for $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ and for $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$. Besides Corollary 3.1.8 and Theorem 3.1.10 ($\aleph_0$-compactness and recursive enumerability for consequence) we show that $\mathscr{L}_{\omega\omega}(Q_1)$ has the Löwenheim–Skolem property down to $\aleph_1$. The reader is urged to compare the following proofs with those given in Section 3.1.

We set $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_1)$. Our first considerations aim at a suitable logic $\mathscr{L}^*$ based on models of set theory which is $\geq_{\mathrm{RPC}} \mathscr{L}$, $\aleph_0$-compact and has the Löwenheim–Skolem property down to $\aleph_1$. For our purposes it will be sufficient to have an intuitive description of $\mathscr{L}^*$. A precise definition is left to the reader.

Let τ be a countable vocabulary, which is kept fixed for the argument to follow, and let $\sigma = \{\varepsilon, c_0\} \cup \{c^{\S} \mid \S \in \tau\}$, where ε is a new binary relation symbol for the $\in$-relation between sets, and c_0 and the $c^{\S}$ are new constants. Next we define a set

$\Gamma = \Gamma(\boldsymbol{\tau})$ of $\mathscr{L}_{\omega\omega}[\boldsymbol{\sigma}]$-sentences that provides us with a set-theoretical description of $\boldsymbol{\tau}$-structures. In fact, we set

$$\Gamma = \{\psi_0\} \cup \{\psi^{\S} \mid \S \in \boldsymbol{\tau}\},$$

where

ψ_0 is $c_0 \neq \emptyset$, i.e. $\psi_0 = \exists x\ x\varepsilon c_0$, and

$$\psi^{\S} = \begin{cases} c^{\S}\ \varepsilon\ c_0, & \text{if } \S \text{ is a constant;} \\ c^{\S} \subseteq c_0^n, & \text{if } \S \text{ is an } n\text{-ary relation symbol;} \\ c^{\S}\colon c^n \to c_0, & \text{if } \S \text{ is an } n\text{-ary function symbol.} \end{cases}$$

If $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$, let φ^* be a natural set-theoretic translation of φ. For example, if φ is

$$\exists z\, Q_1 x(Rzx \wedge \neg f(x) = d)$$

put φ^* equal to

$$\exists z\varepsilon c_0 \mid \{x\varepsilon c_0 \mid (z, x)\varepsilon c^R \wedge \neg c^f(x) = c^d\} \mid \geq \aleph_1.$$

The transition from φ to φ^* enables us to treat $\mathscr{L}$-satisfaction in models of ZFC (Zermelo–Fraenkel set theory with the axiom of choice). For technical reasons, we consider a system (ZFC) that differs from ZFC in having only finitely many instances of the axiom scheme of replacement, but that is strong enough to yield all set-theoretical facts we need. The reader should think of (ZFC) as ZFC and verify that at the end we have needed only finitely many axioms of replacement. The main reason for introducing (ZFC) is the following: In contrast to the situation with ZFC, one can prove that for (ZFC) there are cofinally many ordinals α for which $(V_\alpha, \in_{V_\alpha})$ is a model of (ZFC). (V_α denotes the set of all sets of rank $<\alpha$.)

Next, we call a structure $\mathfrak{M}$ *good*, if $\varepsilon \in \boldsymbol{\tau}_{\mathfrak{M}}$, $(M, \varepsilon^{\mathfrak{M}}) \models$ (ZFC), and

$$(\aleph_1^{\mathfrak{M}}, \varepsilon_{\aleph_1}^{\mathfrak{M}}) = (\{a \in M \mid \mathfrak{M} \models a\varepsilon\aleph_1\}, \{(a, b) \in M \times M \mid \mathfrak{M} \models a\varepsilon b \wedge b\varepsilon\aleph_1\})$$

is an $\aleph_1$-like ordering. For good models $\mathfrak{M}$ (un-)countability in $\mathfrak{M}$ means (un-) countability in the real universe. This can be made precise in the following way. If $\mathfrak{A}$ is a $\boldsymbol{\tau}$-structure, there is a minimal ordinal $\alpha > \omega_1$ such that $\mathfrak{A} \in V_\alpha$ and $(V_\alpha, \in_{V_\alpha}) \models$ (ZFC). We expand $(V_\alpha, \in_{V_\alpha})$ to a good $\boldsymbol{\sigma}$-model $\mathfrak{M}(\mathfrak{A})$ of Γ such that c_0 and the $c^{\S}$ describe $\mathfrak{A}$ in $\mathfrak{M}(\mathfrak{A})$; that is,

$$A = \{a \in M = V_\alpha \mid \mathfrak{M}(\mathfrak{A}) \models a\varepsilon c_0\}$$

and, say, for unary $f \in \boldsymbol{\tau}$,

$$f^{\mathfrak{A}} = \{(a, b) \in M \times M \mid \mathfrak{M}(\mathfrak{A}) \models (a, b)\varepsilon c^f\}.$$

Conversely, any good $\boldsymbol{\sigma}$-model $\mathfrak{M}$ of Γ yields a $\boldsymbol{\tau}$-structure $\mathfrak{A}(\mathfrak{M})$ such that c_0 and the c^s describe $\mathfrak{A}(\mathfrak{M})$ in $\mathfrak{M}$. Using these notations, we have:

Lemma A. *For any* $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$, $\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\tau}]$ *and* $\mathfrak{M} \in \mathrm{Str}[\boldsymbol{\sigma}]$,

(i) *if* $\mathfrak{A} \models_{\mathscr{L}} \Phi$, *then* $\mathfrak{M}(\mathfrak{A})$ *is a good model of* $\Phi^* \cup \Gamma$; *and*
(ii) *if* $\mathfrak{M}$ *is a good model of* $\Phi^* \cup \Gamma$, *then* $\mathfrak{A}(\mathfrak{M}) \models \Phi$.

Proof. By induction one gets that for any $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ and any $\boldsymbol{\tau}$-structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ iff $\mathfrak{M}(\mathfrak{A}) \models \varphi^*$. Part (ii) is proved similarly. □

As stated above, we leave it to the reader to define a logic $\mathscr{L}^*$ that has as a standard part the class of good $\{\varepsilon\}$-structures and to show $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}^*$ (for ε-free sentences).

The next lemma yields $\aleph_0$-compactness of $\mathscr{L}^*$.

Lemma B. *For* $\Psi \subseteq \mathscr{L}[\boldsymbol{\sigma}]$, *the following are equivalent:*

(i) $\Psi \cup$ (ZFC) *has a model.*
(ii) $\Psi \cup$ (ZFC) *has a good model of power* $\aleph_1$. □

The direction from (ii) to (i) is trivial. For the other direction we invoke the so-called Keisler–Morley lemma (see Theorem IV.3.2.5(ii)), which is here stated for its own interest:

3.2.1 Lemma (Keisler, Morley). *Let* $\mathfrak{M}$ *be a countable* $\{\varepsilon\}$*-model of* (ZFC). *Then there exists a countable* $\{\varepsilon\}$*-structure* $\mathfrak{M}' \succ \mathfrak{M}$ *such that* $(\aleph_1^{\mathfrak{M}'}, \varepsilon_{\aleph_1}^{\mathfrak{M}'})$ *is a proper end extension of* $(\aleph_1^{\mathfrak{M}}, \varepsilon_{\aleph_1}^{\mathfrak{M}})$.

Now, to prove the other implication in Lemma B, we start with a countable model $\mathfrak{M}$ of $\Psi \cup$ (ZFC) and build an elementary chain $(\mathfrak{M}_\alpha)_{\alpha \leq \aleph_1}$, taking unions at limit points and setting $\mathfrak{M}_0 = \mathfrak{M}$ and $\mathfrak{M}_{\alpha+1} = \mathfrak{M}'_\alpha$ in the sense of Lemma 3.2.1. (The additional constants in $\boldsymbol{\sigma}$ are not essential.) Then $\mathfrak{M}_{\aleph_1}$ satisfies (ii) of Lemma B.

We can now show that $\mathscr{L}$ is $\aleph_0$-compact. Assume $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$, and every finite subset of Φ has a model. Then, by part (i) of Lemma A, every finite subset of $\Phi^* \cup \Gamma$ has a model, and hence so does $\Phi^* \cup \Gamma$. Using Lemma B and part (ii) of Lemma A, we see that Φ has a model of power $\leq \aleph_1$. In particular, we also obtain the conclusion that $\mathscr{L}$ has the Löwenheim–Skolem property down to $\aleph_1$. Finally, to show that $\mathscr{L}$ is recursively enumerable for consequence, we observe that for any $\Phi \cup \{\varphi\} \subseteq \mathscr{L}[\boldsymbol{\tau}]$,

$$\Phi \models_{\mathscr{L}} \varphi \quad \text{iff} \quad \Phi^* \cup \Gamma \cup (\mathrm{ZFC}) \models_{\mathscr{L}_{\omega\omega}} \varphi^*,$$

and that the operation * is effective.

In concluding this subsection, we digress to take a brief look at $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ for structures of power $\aleph_1$ (see Section IV.4.2 for the general case) as well as at $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\omega})$.

3.2.2 Theorem. *$\mathscr{L}_{\omega\omega}(\mathrm{aa})$, restricted to structures of power $\aleph_1$, is $\aleph_0$-compact and recursively enumerable for consequence.*

Proof. We proceed in a manner similar to that for $\mathscr{L}_{\omega\omega}(Q_1)$. A structure $\mathfrak{M}$ with $\varepsilon \in \mathfrak{t}_{\mathfrak{M}}$ is called *good* if $\mathfrak{M} \models$ (ZFC), $(\aleph_1^{\mathfrak{M}}, \varepsilon_{\aleph_1}^{\mathfrak{M}})$ is an $\aleph_1$-like ordering that admits a continuous embedding π of the real $\aleph_1$, and further, for every $s \in M$ such that $\mathfrak{M} \models$ *s is a stationary subset of* $\aleph_1$, the set $\{a \in M \mid a\varepsilon^{\mathfrak{M}}s\} \cap \mathrm{rg}(\pi)$ is stationary in $\mathrm{rg}(\pi)$. The analogue of Lemma A is an exercise on closed unbounded subsets of $\aleph_1$. The analogue of Lemma B uses a stronger form of the Keisler–Morley lemma due to Hutchinson [1976a], according to which, given some $s \in M$ which is a stationary subset of $\aleph_1$ in $\mathfrak{M}$, the structure $\mathfrak{M}'$ can be chosen such that $\aleph_1^{\mathfrak{M}'}$ has a least new element, say p, and $p\varepsilon^{\mathfrak{M}'}s$.

Now, to obtain a good elementary extension of some countable model $\mathfrak{M}$ of (ZFC), one splits the real $\aleph_1$ into $\aleph_1$ disjoint stationary subsets S_α (for $\alpha < \aleph_1$) and builds an elementary chain $(\mathfrak{M}_\alpha)_{\alpha \le \aleph_1}$ over $\mathfrak{M} = \mathfrak{M}_0$ by Hutchinson's lemma such that for each $s \in M_{\aleph_1}$ which is a stationary subset of $\aleph_1$ in $\mathfrak{M}_{\aleph_1}$, there is some $\alpha < \aleph_1$ with $\pi(\beta) = p_\beta \varepsilon^{\mathfrak{M}_{\beta+1}} s$ for all sufficiently large $\beta \in S_\alpha$. We describe the successor step. For simplicity we assume that all M_α are chosen as subsets of some fixed set $\{a_\alpha \mid \alpha < \aleph_1\}$, where $a_\alpha \ne a_\beta$ for $\alpha < \beta < \aleph_1$. Suppose that $\beta < \aleph_1$ and $\mathfrak{M}_\beta$ has already been constructed and is a countable elementary extension of $\mathfrak{M}$. Let $\beta \in S_\alpha$. Define s to be a_α, if a_α is a stationary subset of $\aleph_1$ in $\mathfrak{M}_\beta$ and to be $\aleph_1^{\mathfrak{M}_\beta}$ else. Then choose $\mathfrak{M}_{\beta+1}$ according to Hutchinson's lemma with a least new countable ordinal $\pi(\beta) = p_\beta$, $p_\beta \varepsilon^{\mathfrak{M}_{\beta+1}} s$. □

3.2.3 Theorem. *$\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\omega})$ is compact, recursively enumerable for consequence and has the Löwenheim–Skolem property down to $\aleph_1$.*

In this case $\mathfrak{t}$ need not be countable. We call a structure $\mathfrak{M}$ with $\varepsilon \in \mathfrak{t}_{\mathfrak{M}}$ *good*, if $\mathfrak{M} \models$ (ZFC), $(\omega^{\mathfrak{M}}, \varepsilon_\omega^{\mathfrak{M}})$ has cofinality ω, and for all $b \in M$ that are uncountable regular cardinals in $\mathfrak{M}$, $(b, \varepsilon_b^{\mathfrak{M}})$ has cofinality $\ge \omega_1$. Then the analogue of Lemma A is routine. In the analogue of Lemma B, we have to cancel the limitation of power in part (ii), if $|\mathfrak{t}| \ge \aleph_2$. The role of the Keisler–Morley lemma is taken over by:

3.2.4 Lemma. *Every* (ZFC)*-model $\mathfrak{M} = (M, \varepsilon^{\mathfrak{M}})$ has a good elementary extension.*

Proof. We start with a suitable chain construction that yields a structure $\mathfrak{M}' \succ \mathfrak{M}$ such that for all $b' \in M'$ that are uncountable regular cardinals in $\mathfrak{M}'$, $(b', \varepsilon_{b'}^{\mathfrak{M}'})$ has cofinality ω_1. A good extension $\mathfrak{M}'' \succ \mathfrak{M}'$ can now be constructed as the union of an elementary chain of length ω, where $\mathfrak{M}_0 = \mathfrak{M}'$ and for each i, $\mathfrak{M}_{i+1} \succ \mathfrak{M}_i$, $\omega^{\mathfrak{M}_i}$ gets longer in $\mathfrak{M}_{i+1}$ and no regular uncountable cardinal of $\mathfrak{M}_i$ gets longer in $\mathfrak{M}_{i+1}$. To obtain $\mathfrak{M}_{i+1}$ from $\mathfrak{M}_i$, one defines inside $\mathfrak{M}_i$ an ultrapower

$$\mathfrak{M}_i^{\omega^{\mathfrak{M}_i}}/\mathscr{U} = \mathfrak{M}_i',$$

where $\mathcal{U}$ is some free ultrafilter on $\omega^{\mathfrak{M}_i}$ in $\mathfrak{M}_i$. Then $\mathfrak{M}_{i+1}$ is chosen such that $\mathfrak{M}_{i+1} \cong \mathfrak{M}'_i$ via an extension of the canonical embedding of $\mathfrak{M}_i$ into $\mathfrak{M}'_i$ (in the real universe).

When checking the details, one sees that the proof of the ultrafilter theorem for $\mathfrak{M}'_i$ requires the instances

$$\forall \mathbf{z}(\forall x \in \omega\, \exists y\, \varphi(x, y, \mathbf{z}) \rightarrow \exists u\, \forall x \in \omega\, \exists y \in u \varphi(x, y, \mathbf{z}))$$

of the collection scheme. These can be added to (ZFC), since they are satisfied in $(V_\alpha, \in_{V_\alpha})$, if α is a limit ordinal of cofinality $\neq \omega$, and there are cofinally many such α, for which $(V_\alpha, \in_{V_\alpha})$ is a model of (ZFC). □

4. Lindström Quantifiers

Let $\mathfrak{K}$ be a class of structures of some fixed (finite) vocabulary closed under isomorphism. For a given logic $\mathcal{L}$, is there an extension of $\mathcal{L}$ more natural than $\mathcal{L}(\mathfrak{K})$, in which $\mathfrak{K}$ is characterizable? In the first part of this section, we will give an affirmative answer that uses the notion of a *Lindström quantifier* as developed by Lindström [1966a]. At the same time this notion enables us to systematize—at least to a certain extent—the variety of specific logics that we have considered up to now. The systematization not only assists in the representation of logics but can also be helpful from a methodological point of view. In the second part of this section, we will illustrate the latter aspect by proving a generalization of the back-and-forth characterization of elementary equivalence for logics with monotone quantifiers that covers several of the Ehrenfeucht–Fraissé type theorems for stronger logics. In order to avoid any cumbersome notation, we will confine ourselves to the one-sorted case and treat logics with free variables in the sense of 1.1.2.

4.1. Definitions and Examples

Let $\boldsymbol{\sigma}$ be a *finite* vocabulary and Q a quantifier symbol suitable for $\boldsymbol{\sigma}$ (in a sense that will become clear from Definition 4.1.1). Furthermore, let $\mathfrak{K}$ be a class of $\boldsymbol{\sigma}$-structures closed under isomorphism. We confine ourselves to the special case $\boldsymbol{\sigma} = \{R, f, c\}$ with binary R and unary f.

4.1.1 Definition. For any logic $\mathcal{L}$, the expanded logic $\mathcal{L}(Q_{\mathfrak{K}})$ is obtained as follows:

$\mathrm{Form}_{\mathcal{L}(Q_{\mathfrak{K}})}[\boldsymbol{\tau}]$ is taken as the smallest class containing $\mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ which is closed under boolean operations and particularizations (see Definition 1.2.1) and that with each φ, ψ, χ and for any variables $x_0 \neq x_1$, $y_0 \neq y_1$, z_0 also contains the new formula

$$\vartheta = Qx_0x_1y_0y_1z_0\varphi\psi\chi.$$

A variable u is *free* in ϑ, if it is free in φ or ψ or χ and different from x_0, x_1 or y_0, y_1 or z_0, respectively.

$\mathrm{Sent}_{\mathscr{L}(Q_{\mathfrak{K}})}[\boldsymbol{\tau}]$ is the class of sentences from $\mathrm{Form}_{\mathscr{L}(Q_{\mathfrak{K}})}[\boldsymbol{\tau}]$.

Finally, the meaning of Q is determined by the satisfaction condition:

$$\mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} Qx_0x_1y_0y_1z_0\,\varphi(x_0, x_1)\,\psi(y_0, y_1)\,\chi(z_0)$$

iff there is a $\boldsymbol{\sigma}$-structure $\mathfrak{C} \in \mathfrak{K}$ such that $C = A$,

$R^{\mathfrak{C}} = \{(a, b) \in C \times C \mid \mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} \varphi[a, b]\}$,

graph of $f^{\mathfrak{C}} = \{(a, b) \in C \times C \mid \mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} \psi[a, b]\}$,

and $\mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}})} \chi[a]$ exactly for $a = c^{\mathfrak{C}}$.

The quantifier Q with the interpretation by $\mathfrak{K}$ (for short, $Q_{\mathfrak{K}}$) is called a *Lindström quantifier.*

Let $\mathscr{L}$ be regular. As it is clear that

$$\mathfrak{K} = \mathrm{Mod}^{\boldsymbol{\sigma}}_{\mathscr{L}(Q_{\mathfrak{K}})}(Qx_0x_1y_0y_1z_0Rx_0x_1f(y_0) = y_1z_0 = c),$$

we see that $\mathfrak{K}$ is EC in $\mathscr{L}(Q_{\mathfrak{K}})$, even in $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$. On the other hand, if $\mathfrak{K}$ is EC in $\mathscr{L}$, then $\mathscr{L}(Q_{\mathfrak{K}}) \leq \mathscr{L}$ and hence $\mathscr{L}(Q_{\mathfrak{K}}) \equiv \mathscr{L}$. To see the key fact, assume that $\mathfrak{K} = \mathrm{Mod}^{\boldsymbol{\sigma}}_{\mathscr{L}}(\xi)$. Then the $\mathscr{L}(Q_{\mathfrak{K}})$-formula $Qx_0x_1y_0y_1z_0\,\varphi(x_0, x_1)\,\psi(y_0, y_1)\,\chi(z_0)$ (with $\mathscr{L}$-formulas φ, ψ, χ) has the same meaning in $\mathscr{L}(Q_{\mathfrak{K}})$ as the formula $\xi[R/\lambda x_0x_1\varphi(x_0, x_1), f/\lambda y_0y_1\psi(y_0, y_1)\; c/\lambda z_0\chi(z_0)]$ has in $\mathscr{L}$.

The definition of $\mathscr{L}(Q_{\mathfrak{K}})$ can easily be generalized to the case of more than one Lindström quantifier, and it is not difficult to see that for regular $\mathscr{L}$ the logic $\mathscr{L}(Q_{\mathfrak{K}_\iota} \mid \iota \in I)$ with Lindström quantifiers $Q_{\mathfrak{K}_\iota}$ is regular, possibly up to the relativization and the substitution property. However, the latter property holds, for example, in case $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$. A counter-example to relativization is provided by $\mathscr{L}_{\omega\omega}(Q^C)$ which is defined below. In Definition 4.1.4 we describe a variant of Lindström quantifiers that also guarantees the relativization property.

The following list demonstrates that it is possible to model numerous quantifiers on Lindström quantifiers and thus illustrates the scope of this notion.

4.1.2 Examples. In each of the following, a well-known quantifier becomes $Q_{\mathfrak{K}}$ for the class indicated:

(i) $\exists$ for $\mathfrak{K} = \{(A, C) \mid \varnothing \neq C \subseteq A\}$.
(ii) Q^n_α for $\mathfrak{K} = \{(A, M) \mid M \subseteq A^n$, there is $C \subseteq A$, $|C| \geq \aleph_\alpha$ and $C^n \subseteq M\}$;
(iii) $Q^{\mathrm{cf}\,\omega}$ for $\mathfrak{K} = \{(A, <^{\mathfrak{A}}) \mid <^{\mathfrak{A}}$ is a linear ordering relation $\subseteq A \times A$ of cofinality $\omega\}$;
(iv) Q^{WO}, the so-called *well-ordering quantifier*, for $\mathfrak{K} = \{(A, <^{\mathfrak{A}}) \mid <^{\mathfrak{A}}$ is a well-ordering relation $\subset A \times A\}$;
(v) Q^C, the so-called *Chang quantifier*, a specialization of the equicardinality quantifier I, for $\mathfrak{K} = \{(A, C) \mid C \subseteq A, |C| = |A|\}$.

In order to model higher-order quantifiers, one could introduce Lindström quantifiers of higher order. In principle, however, the present framework is universal in broad sense:

4.1.3 Theorem. *Let $\mathscr{L}$ be a regular logic which is finitary, that is for any τ,*

$$\mathrm{Form}_{\mathscr{L}}[\tau] = \bigcup_{\substack{\tau_0 \subseteq \tau \\ \tau_0 \text{ finite}}} \mathrm{Form}_{\mathscr{L}}[\tau_0].$$

Then, $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}_\iota} \mid \iota \in I)$, where the $\mathfrak{K}_\iota$ run over all classes of finite vocabulary that are EC *in $\mathscr{L}$.*

Proof. For "$\leq$" note that each $\mathfrak{K}_\iota$ is EC in $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}_\iota})$. As for the other direction, use the fact that $\mathscr{L}(Q_{\mathfrak{K}_\iota}) \equiv \mathscr{L}$ for every $\iota \in I$. ◻

In particular, second-order logic $\mathscr{L}^2$ has a representation as in Theorem 4.1.3. Any such representation requires I to be infinite, that is, $\mathscr{L}^2$ is *not finitely generated*; for otherwise, according to a consideration in Section 7.3, we would get a contradiction, since (the one-sorted version of) $\mathscr{L}^2$ has the Beth property.

Returning now to the relativization property, we introduce a variant of $\mathscr{L}(Q_{\mathfrak{K}})$.

4.1.4 Definition. The logic $\mathscr{L}(Q_{\mathfrak{K}}^*)$ is defined as follows. We change the definition of $\mathscr{L}(Q_{\mathfrak{K}})$ given in Definition 4.1.1 by allowing predicates for the domains of structures in $\mathfrak{K}$. Using a quantifier symbol Q^* instead of Q, we replace the quantifier clause for Q in Definition 4.1.1 by

$$\vartheta^* = Q^* u_0 x_0 x_1 y_0 y_1 z_0 \xi \varphi \psi \chi,$$

where the meaning of Q^* is now determined by

$$\mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}}^*)} Q^* u_0 x_0 x_1 y_0 y_1 z_0\, \xi(u_0)\, \varphi(x_0, x_1)\, \psi(y_0, y_1)\, \chi(z_0)$$

iff there is a σ-structure $\mathfrak{C} \in \mathfrak{K}$ such that $C = \{a \in A \mid \mathfrak{A} \models_{\mathscr{L}(Q_{\mathfrak{K}}^*)} \xi[a]\}$ and $R^{\mathfrak{C}}, f^{\mathfrak{C}}$ and $c^{\mathfrak{C}}$ are as in Definition 4.1.1.

For regular $\mathscr{L}$, the logic $\mathscr{L}(Q_{\mathfrak{K}}^*)$ is regular, possibly up to substitution, and really regular for instance in case $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$. Intuitively, relativization to some predicate P can be defined by induction on formulas with the essential clause for the relativization of a Q^*-formula being:

$$Q^* u_0 \ldots z_0 (Pu_0 \wedge \xi^P)(Px_0 \wedge Px_1 \wedge \varphi^P)(Py_0 \wedge Py_1 \wedge \psi^P)(Pz_0 \wedge \chi^P).$$

It is obvious that $\mathscr{L}(Q_{\mathfrak{K}}) \leq \mathscr{L}(Q_{\mathfrak{K}}^*)$. For instance, the Q-formula ϑ from Definition 4.1.1 has the same meaning in $\mathscr{L}(Q_{\mathfrak{K}})$ as the Q^*-formula ϑ^* from above has in

$\mathscr{L}(Q_{\mathfrak{K}}^*)$, if one takes $u_0 = u_0$ for ξ. Concerning the other direction we have the following fact:

4.1.5 Proposition. *With new unary U set*

$$\mathfrak{K}^* := \{\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\sigma} \cup \{U\}] \mid U^{\mathfrak{A}}\ \boldsymbol{\sigma}\text{-closed and } (\mathfrak{A} \upharpoonright \boldsymbol{\sigma}) | U^{\mathfrak{A}} \in \mathfrak{K}\}.$$

Then $\mathscr{L}(Q_{\mathfrak{K}}^*) \equiv \mathscr{L}(Q_{\mathfrak{K}^*})$.

Proof. The argument for " $\geq$ " is trivial. For " $\leq$ " observe for instance that

$$Q_{\mathfrak{K}}^* u_0 x_0 x_1 y_0 y_1 z_0\, \xi(u_0)\, \varphi(x_0, x_1)\, \psi(y_0, y_1)\, \chi(z_0)$$

has the same meaning as

$$\begin{aligned} Q_{\mathfrak{K}^*} u_0 \ldots z_0\, \xi(u_0)(\xi(x_0) \wedge \xi(x_1) \wedge \varphi(x_0, x_1)) \\ ((\xi(y_0) \wedge \xi(y_1) \wedge \psi(y_0, y_1)) \vee (\neg \xi(y_0) \wedge y_1 = y_0)) \\ (\xi(z_0) \wedge \chi(z_0)), \end{aligned}$$

where $\xi(u_0)$ represents U. $\square$

Taking Proposition 4.1.5 into consideration it is not difficult to extend results about logics $\mathscr{L}(Q_{\mathfrak{K}})$ to logics $\mathscr{L}(Q_{\mathfrak{K}}^*)$—at least in many cases (for example, Theorem 4.1.3 and the results in Section 4.2).

Let us now return to our introductory question. For numerous logics $\mathscr{L}$ such as $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$ or $\mathscr{L} = \mathscr{L}_{\kappa\lambda}(Q_{\mathfrak{K}_\iota}^* | \iota \in I)$, the logic $\mathscr{L}(Q_{\mathfrak{K}}^*)$ is, with respect to elementary classes, the smallest regular extension of $\mathscr{L}$ in which $\mathfrak{K}$ is EC. In this sense the transition from $\mathscr{L}$ to $\mathscr{L}(Q_{\mathfrak{K}}^*)$ is a natural closure operation. What can we say about the relationship to $\mathscr{L}(\mathfrak{K})$ as defined in Section 2.6? If, for instance, $\mathfrak{K} = \{(A, <^{\mathfrak{A}}) | \mathfrak{A} \cong (\omega, <)\}$, then, of course, we have

$$(*) \qquad \mathscr{L}_{\omega\omega}(\mathfrak{K}) = \omega - \text{logic} \leq \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}^*}) \equiv \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}}^*).$$

Using a method like that in the proof of Proposition 3.1.7 one obtains for the other direction

$$(**) \qquad \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}}^*) \leq_{\mathrm{PC}} \mathscr{L}_{\omega\omega}(\mathfrak{K}) \text{ for vocabularies not containing } U, <.$$

Whereas the analogue of $(*)$ is true in general, the analogue of $(**)$ may fail. For instance, if $\mathfrak{K}$ is the class of all fields of characteristic zero, $\mathscr{L}_{\omega\omega}(\mathfrak{K})$ is compact, but $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}}^*)$ is not.

4.2. Partial Isomorphisms and a Characterization of $\mathscr{L}$-Equivalence

The characterization of elementary equivalence in terms of partial isomorphisms or games by Fraissé and Ehrenfeucht (cf. Section IX.4 for a thorough treatment) has been extended to various stronger logics such as $\mathscr{L}_{\omega\omega}(Q_1)$, $\mathscr{L}_{\omega\omega}(Q_1^n)$, $\mathscr{L}_{\omega\omega}(\mathrm{aa})$. A generalization to extensions of $\mathscr{L}_{\omega\omega}$ by arbitrary Lindström quantifiers is given in Caicedo [1979]. The characterization becomes very natural for quantifiers $Q_{\mathfrak{K}}$ and $Q_{\mathfrak{K}}^*$, where $\mathfrak{K}$ is of finite relational vocabulary $\boldsymbol{\sigma}$ and monotone (Krawczyk-Krynicki [1976], Weese [1980]). The following considerations are devoted to this case. For reasons of readability we fix a relational vocabulary $\boldsymbol{\sigma} = \{S\}$, S l-ary, and a class $\mathfrak{K}$ of $\boldsymbol{\sigma}$-structures, $\mathfrak{K}$ closed under isomorphisms. We treat the quantifier $Q_{\mathfrak{K}}$.

4.2.1 Definition. For $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\boldsymbol{\tau}]$, p is a *partial isomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$, if p is a bijection from $\mathrm{dom}(p) \subseteq A$ onto $\mathrm{rg}(p) \subseteq B$ such that the following hold:

(i) for all $n \geq 1$, n-ary $R \in \boldsymbol{\tau}$ and $a_0, \ldots, a_{n-1} \in \mathrm{dom}(p)$: $R^{\mathfrak{A}}\mathbf{a}$ iff $R^{\mathfrak{B}}p(\mathbf{a})$, where $p(\mathbf{a})$ stands for $(p(a_0), \ldots, p(a_{n-1}))$;
(ii) for all $n \geq 1$, n-ary $f \in \boldsymbol{\tau}$ and $a_0, \ldots, a_{n-1}, a \in \mathrm{dom}(p)$: $f^{\mathfrak{A}}(\mathbf{a}) = a$ iff $f^{\mathfrak{B}}(p(\mathbf{a})) = p(a)$;
(iii) for all $c \in \boldsymbol{\tau}$ and $a \in \mathrm{dom}(p)$: $c^{\mathfrak{A}} = a$ iff $c^{\mathfrak{B}} = p(a)$.

Part $(\mathfrak{A}, \mathfrak{B})$ denotes the set of partial isomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$.

Sometimes, one demands in addition that the domain of a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ be $\boldsymbol{\tau}$-closed in $\mathfrak{A}$ (or empty). However, the difference between the two variants involves only minor technicalities.

4.2.2 Definition. Let $\mathfrak{A}, \mathfrak{B}$ be $\boldsymbol{\tau}$-structures, $0 \leq \alpha \leq \omega$, and $I = (I_\beta)_{\beta \leq \alpha}$ a sequence of subsets of $\mathrm{Part}(\mathfrak{A}, \mathfrak{B})$.

We say that I has the *$\exists$-forth property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $a \in A$ there exists $q \in I_m$ such that $p \subseteq q$ and $a \in \mathrm{dom}(q)$.

Similarly, we say that I has the *$\exists$-back property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $b \in B$ there exists $q \in I_m$ such that $p \subseteq q$ and $b \in \mathrm{rg}(q)$.

Likewise I has the *$Q_{\mathfrak{K}}$-forth property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $\mathfrak{C} \in \mathfrak{K}$ with $C = A$ there is $\mathfrak{D} \in \mathfrak{K}$ with $D = B$ such that for all $\mathbf{d} \in S^{\mathfrak{D}}$ there exists $q \in I_m$ with $p \subseteq q$, $d_0, \ldots, d_{l-1} \in \mathrm{rg}(q)$ and $q^{-1}(\mathbf{d}) \in S^{\mathfrak{C}}$.

Similarly, we say that I has the *$Q_{\mathfrak{K}}$-back property* iff for all $m < \alpha$, $p \in I_{m+1}$ and $\mathfrak{D} \in \mathfrak{K}$ with $D = B$ there is $\mathfrak{C} \in \mathfrak{K}$ with $C = A$ such that for all $\mathbf{c} \in S^{\mathfrak{C}}$ there exists $q \in I_m$ with $p \subseteq q$, $c_0, \ldots, c_{l-1} \in \mathrm{dom}(q)$ and $q(\mathbf{c}) \in S^{\mathfrak{D}}$.

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *α-isomorphic via I*, written $I\colon \mathfrak{A} \cong_\alpha \mathfrak{B}$, iff $I = (I_m)_{m \leq \alpha}$ is a sequence of length $(\alpha + 1)$ of non-empty subsets of $\mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ having the $\exists$-back and the $\exists$-forth property. $\mathfrak{A}$ and $\mathfrak{B}$ are *α-isomorphic*, written $\mathfrak{A} \cong_\alpha \mathfrak{B}$, iff there exists an I such that $I\colon \mathfrak{A} \cong_\alpha \mathfrak{B}$.

The notion of α, $\mathfrak{K}$-*isomorphic structures* is defined similarly, demanding in addition that the partial isomorphisms in question also meet the $Q_{\mathfrak{K}}$-back and the $Q_{\mathfrak{K}}$-forth property.

We call the class $\mathfrak{K}$ and also $Q_{\mathfrak{K}}$ *monotone*, if for all A, M, M' such that $(A, M) \in \mathfrak{K}$ and $M \subseteq M' \subseteq A^l$, we have $(A, M') \in \mathfrak{K}$.

The main result in this section can now be formulated as:

4.2.3. Theorem. *Let $\mathfrak{K}$ of finite relational vocabulary be monotone. Then for finite τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$ the following are equivalent:*

(i) $\mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})} \mathfrak{B}$;
(ii) $\mathfrak{A} \cong_{n, \mathfrak{K}} \mathfrak{B}$ *for all* n;
(iii) $\mathfrak{A} \cong_{\omega, \mathfrak{K}} \mathfrak{B}$.

If we dispense with $Q_{\mathfrak{K}}$, the proof below will yield the analogous result for $\mathscr{L}_{\omega\omega}$, that is, the Ehrenfeucht–Fraissé characterization of elementary equivalence:

4.2.4 Corollary. *For finite τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$ the following are equivalent:*

(i) $\mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}} \mathfrak{B}$;
(ii) $\mathfrak{A} \cong_{n} \mathfrak{B}$ *for all* n;
(iii) $\mathfrak{A} \cong_{\omega} \mathfrak{B}$. $\square$

Proof of Theorem 4.2.3. Let $\mathfrak{K}$ be as above. We set $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$ and fix some finite vocabulary τ. By $\varphi, \psi, \ldots$ we denote formulas from $\mathscr{L}[\tau]$. Each φ is equivalent to a so-called *term-reduced* formula—a formula where all atomic subformulas are of kinds $Rx_0 \ldots x_{n-1}$, $x = y$, $c = y$, or $f(x_0, \ldots, x_{n-1}) = y$. We can obviously confine ourselves to such formulas, which we do for technical convenience.

The implication from (iii) to (ii) is trivial. To prove that (ii) implies (i), we define the so-called *quantifier rank* of φ, $\mathrm{qrk}(\varphi)$, inductively by the following clauses:

$$\begin{aligned} \mathrm{qrk}(\varphi) &= 0, \quad \text{if } \varphi \text{ is atomic;} \\ \mathrm{qrk}(\neg\varphi) &= \mathrm{qrk}(\varphi); \\ \mathrm{qrk}(\varphi \wedge \psi) &= \max\{\mathrm{qrk}(\varphi), \mathrm{qrk}(\psi)\}; \\ \mathrm{qrk}(\exists x\varphi) &= \mathrm{qrk}(Qx\varphi) = 1 + \mathrm{qrk}(\varphi). \end{aligned}$$

Next we write

$$\mathfrak{A} \equiv_{n, \mathfrak{K}} \mathfrak{B} \quad \text{iff} \quad \text{for all (term-reduced) sentences } \varphi \text{ with } \mathrm{qrk}(\varphi) \leq n, \text{ we have } \mathfrak{A} \models \varphi \text{ iff } \mathfrak{B} \models \varphi.$$

Then the implication we want follows from:

$(*)$ For all n, if $\mathfrak{A} \cong_{n, \mathfrak{K}} \mathfrak{B}$, then $\mathfrak{A} \equiv_{n, \mathfrak{K}} \mathfrak{B}$.

To prove (*), let $I: \mathfrak{A} \cong_{n,\mathfrak{K}} \mathfrak{B}$ be given. One shows by induction on $\mathrm{qrk}(\varphi)$ that for all $m \le n$, $p \in I_m$, $\varphi(x_0, \ldots, x_{k-1})$ with $\mathrm{qrk}(\varphi) \le m$, and $a_0, \ldots, a_{k-1} \in \mathrm{dom}(p)$, $\mathfrak{A} \models \varphi[\mathbf{a}]$ iff $\mathfrak{B} \models \varphi[p(\mathbf{a})]$. For atomic φ one uses that φ is term-reduced. For the Q-step, let $m \le n$, $p \in I_m$, and $a_0, \ldots, a_{k-1} \in \mathrm{dom}(p)$ be given and assume $\varphi = Qy_0 \ldots y_{l-1}\, \psi(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{l-1})$, $\mathrm{qrk}(\varphi) \le m$. If for instance $\mathfrak{A} \models \varphi[\mathbf{a}]$, then

$$\mathfrak{C} = (A, \{\mathbf{c} \in A^l \mid \mathfrak{A} \models \psi[\mathbf{a}, \mathbf{c}]\}) \in \mathfrak{K}.$$

For $\mathfrak{C}$ and p we take $\mathfrak{D} \in \mathfrak{K}$ with $D = B$ as guaranteed by the $Q_{\mathfrak{K}}$-forth property and define $\mathfrak{D}'$ to be the structure

$$\mathfrak{D}' = (B, \{\mathbf{d} \in B^l \mid \mathfrak{B} \models \psi[p(\mathbf{a}), \mathbf{d}]\}).$$

As $\mathfrak{K}$ is monotone, we get $\mathfrak{B} \models \varphi[p(\mathbf{a})]$, if we have proved

$$(**) \qquad S^{\mathfrak{D}'} \supseteq S^{\mathfrak{D}}.$$

To see (**), let $\mathbf{d} \in S^{\mathfrak{D}}$ be given. Choose $q \in I_{m-1}$, $q \supseteq p$, such that $d_0, \ldots, d_{l-1} \in \mathrm{rg}(q)$ and $q^{-1}(\mathbf{d}) \in S^{\mathfrak{C}}$. As $\mathrm{qrk}(\psi) \le m - 1$, the induction hypothesis yields $\mathfrak{A} \models \psi[\mathbf{a}, q^{-1}(\mathbf{d})]$ iff $\mathfrak{B} \models \psi[p(\mathbf{a}), \mathbf{d}]$, and hence $\mathbf{d} \in S^{\mathfrak{D}'}$.

Finally, we come to the implication from (i) to (iii). This is the only point where we need the finiteness of τ. To give a more systematic treatment, we insert a general definition which is modelled on the extension properties of partial isomorphisms that we want to realize.

4.2.5 Definition. For $\mathfrak{A} \in \mathrm{Str}[\tau]$, $\mathbf{a} = (a_0, \ldots, a_{k-1}) \in A^k$ and $\mathbf{x} = (x_0, \ldots, x_{k-1})$ the formulas $\psi^m_{\mathfrak{K},\mathfrak{A},\mathbf{a}}(\mathbf{x})$ (or, shorter, $\psi^m_{\mathbf{a}}$) are given as follows:

(i) $$\psi^0_{\mathbf{a}} = \bigwedge \{\varphi(\mathbf{x}) \mid \varphi \text{ term-reduced, atomic or negated atomic, } \mathfrak{A} \models \varphi[\mathbf{a}]\};$$

(ii) $$\begin{aligned} \psi^{m+1}_{\mathbf{a}} = {} & \bigwedge_{c \in A} \exists y \psi^m_{\mathbf{a},c}(\mathbf{x}, y) \wedge \forall y \bigvee_{c \in A} \psi^m_{\mathbf{a},c}(\mathbf{x}, y) \\ & \wedge \bigwedge_{\substack{M \subseteq A^l \\ (A,M) \in \mathfrak{K}}} Q\mathbf{y} \bigvee_{\mathbf{c} \in M} \psi^m_{\mathbf{a},\mathbf{c}}(\mathbf{x}, \mathbf{y}) \\ & \wedge \bigwedge_{\substack{M \subseteq A^l \\ (A,M) \notin \mathfrak{K}}} \neg Q\mathbf{y} \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi^m_{\mathbf{a},\mathbf{c}}(\mathbf{x}, \mathbf{y}). \end{aligned}$$

As τ is finite, it can immediately be seen that in the definition of $\psi^m_{\mathbf{a}}$ all conjunctions and disjunctions can be chosen finite. Hence $\psi^m_{\mathbf{a}} \in \mathscr{L}[\tau]$. The following facts can easily be proved by induction on m.

4.2.6 Lemma. *For* $\mathfrak{A} \in \mathrm{Str}[\tau]$ *and* $a_0, \ldots, a_{k-1} \in A$ *we have*:

(i) $\mathrm{qrk}(\psi^m_{\mathbf{a}}) = m$;
(ii) $\mathfrak{A} \models \psi^m_{\mathbf{a}}[\mathbf{a}]$;
(iii) $\psi^m_{\mathbf{a},a} \models \psi^m_{\mathbf{a}}$ *for all* $a \in A$; *and hence*
(iv) $\psi^{m+1}_{\mathbf{a}} \models \psi^m_{\mathbf{a}}$. $\square$

The Proof of 4.2.3 *Concluded.* Assume $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ and define

$$I_m = \{p \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B}) \mid \mathrm{dom}(p) = \{a_0, \ldots, a_{k-1}\} \text{ for distinct } a_i \text{ and } \mathfrak{B} \models \psi^m_{\mathbf{a}}[p(\mathbf{a})]\}, \quad \text{and}$$

$$I_\omega = \{\varnothing\}.$$

Then the assertion follows from

$$(+) \qquad (I_\alpha)_{\alpha \le \omega} \colon \mathfrak{A} \cong_{\omega, \mathfrak{K}} \mathfrak{B}.$$

We now argue for $(+)$. First, because of $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ and Lemma 4.2.6(ii), we have $\varnothing \in I_m$ for all m. Let us, for example, check the $Q_{\mathfrak{K}}$-back property. Assume $p \in I_{m+1}$, $\mathrm{dom}(p) = \{a_0, \ldots, a_{k-1}\}$, and $(B, N) \in \mathfrak{K}$. We have to find $M \subseteq A^l$ such that $(A, M) \in \mathfrak{K}$ and (A, M) meets the further requirements of the $Q_{\mathfrak{K}}$-back property. We set

$$M = \left\{\mathbf{c} \in A^l \mid \mathfrak{B} \models \bigvee_{\mathbf{d} \in N} \psi^m_{\mathbf{a},\mathbf{c}}[p(\mathbf{a}), \mathbf{d}]\right\}.$$

First, we see that for each $\mathbf{c} \in M$ there is $\mathbf{d} \in N$ such that $\mathfrak{B} = \psi^m_{\mathbf{a},\mathbf{c}}[p(\mathbf{a}), \mathbf{d}]$. Hence, by definition of I_m and Lemma 4.2.6(iii), if $\mathbf{c}$ is given, we can choose

$$q = p \cup \{(c_0, d_0), \ldots, (c_{l-1}, d_{l-1})\} \in I_m.$$

Obviously $q \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$, because by 4.2.6(iv) we have $\mathfrak{B} = \psi^0_{\mathbf{a},\mathbf{c}}[p(\mathbf{a}), \mathbf{d}]$.

It remains to show that $(A, M) \in \mathfrak{K}$. By definition of M,

$$N' = \left\{\mathbf{d} \in B^l \mid \mathfrak{B} \models \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi^m_{\mathbf{a},\mathbf{c}}[p(\mathbf{a}), \mathbf{d}]\right\} \supseteq N,$$

and as $\mathfrak{K}$ is monotone, we obtain that $(B, N') \in \mathfrak{K}$; that is,

$$\mathfrak{B} \models Q\mathbf{y} \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi^m_{\mathbf{a},\mathbf{c}}(p(\mathbf{a}), \mathbf{y}).$$

As $\mathfrak{B} \models \psi^{m+1}_{\mathbf{a}}[p(\mathbf{a})]$, the formula

$$\neg Q\mathbf{y} \neg \bigvee_{\mathbf{c} \in A^l \setminus M} \psi^m_{\mathbf{a},\mathbf{c}}(\mathbf{x}, \mathbf{y})$$

cannot be a conjunct of $\psi^{m+1}_{\mathbf{a}}(\mathbf{x})$. Hence, $(A, M) \in \mathfrak{K}$. $\square$

Remarks. (a) In the preceding proof one can avoid the restriction to term-reduced formulas if one replaces the quantifier rank by a notion of rank that also takes into consideration the complexity of terms.

(b) Theorem 4.2.3 can be extended without difficulty to the case of finitely many monotone Lindström quantifiers.

(c) As for first-order logic, the algebraic characterization of $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$-equivalence can be reformulated in terms of game-theoretical notions; see, for example, Weese [1980]. If we translate Theorem 4.2.3, say for $\mathscr{L}_{\omega\omega}(Q_1)$—note that Q_1 is monotone!—into the game-theoretical version, we get the following characterization of $\mathscr{L}_{\omega\omega}(Q_1)$-equivalence:

For any finite τ, two τ-structures $\mathfrak{A}$, $\mathfrak{B}$ are $\mathscr{L}_{\omega\omega}(Q_1)$-equivalent iff player II *has a winning strategy in the game $G_n(\mathfrak{A}, \mathfrak{B})$ for all $n \in \omega$.*

The game $G_n(\mathfrak{A}, \mathfrak{B})$ is defined as follows: A play in $G_n(\mathfrak{A}, \mathfrak{B})$ takes place between two players I, II and consists of n consecutive moves which are either $\exists$-moves or Q_1-moves. Furthermore, at the beginning of each move player I is free to choose the kind of move he wants. The moves run as follows: $\exists$-move: Player I chooses an element $a \in A$ or an element $b \in B$. This done, player II then chooses some $b \in B$ or some $a \in A$ respectively. Q_1-move: Player I chooses a subset $M \subseteq A$ (or a subset $N \subseteq B$) of power $\geq \aleph_1$. Player II then chooses some $N \subseteq B$ (or some $M \subseteq A$) of power $\geq \aleph_1$. Subsequently, player I chooses some $b \in N$ (or some $a \in M$), and finally player II chooses some $a \in M$ (or some $b \in N$, respectively). Player II wins the play iff the set $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ of pairs from $A \times B$ chosen in the play is a partial isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.

4.2.7 Application. As an easy application of Theorem 4.2.3 we complete the argument for Keisler's *counterexample to interpolation in $\mathscr{L}_{\omega\omega}(Q_1)$* from Example 1 of Section 2.2. For $i = 0, 1$, let $\mathfrak{A}_i = (A_i, E^{\mathfrak{A}_i})$, where $E^{\mathfrak{A}_i}$ is an equivalence relation with only uncountable equivalence classes and $A_i/E^{\mathfrak{A}_i}$ is countably infinite for $i = 0$ and uncountable for $i = 1$. It is easy to see that $(I_\alpha)_{\alpha \leq \omega}$: $\mathfrak{A}_0 \cong_{\omega, Q_1} \mathfrak{A}_1$, where for $\alpha \leq \omega$ the set I_α consists of all partial isomorphisms from $\mathfrak{A}_0$ into $\mathfrak{A}_1$ which have a finite domain. By Theorem 4.2.3, $\mathfrak{A}_0 \equiv_{\mathscr{L}_{\omega\omega}(Q_1)} \mathfrak{A}_1$, and hence by Proposition 3.1.3 interpolation fails for $(*)$ in Example 1. (As $\mathfrak{A}_i \in \mathrm{Mod}(\exists R\ \varphi_i(E, R))$ and $\mathfrak{A}_0 \equiv_{\mathscr{L}_{\omega\omega}(Q_1)} \mathfrak{A}_1$, the classes $\mathrm{Mod}(\exists R\ \varphi_0(E, R))$ and $\mathrm{Mod}(\exists R\ \varphi_1(E, R))$ cannot be separated by a class EC in $\mathscr{L}_{\omega\omega}(Q_1)$.) $\square$

4.3. Partially Isomorphic Structures

In the last paragraph $\mathfrak{A}_0$ and $\mathfrak{A}_1$ were seen to be ω, Q_1-isomorphic in a strong sense, as all I_α are equal: they are *ω, Q_1-partially isomorphic.* To give a definition, let $\mathfrak{A}$, $\mathfrak{B}$ be τ-structures and $I \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$. We say that I has the *$\exists$-forth* (*$\exists$-back*) *property*, if for all $p \in I$ and $a \in A (b \in B)$ there is $q \in I$, $q \supseteq p$ with $a \in \mathrm{def}(q)$ (or $b \in \mathrm{rg}(q)$, respectively). $\mathfrak{A}$ and $\mathfrak{B}$ are called *partially isomorphic*, $\mathfrak{A} \cong_p \mathfrak{B}$, if there is I such that $I: \mathfrak{A} \cong_p \mathfrak{B}$, that is, if $I \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$, I is not empty and has the $\exists$-forth and the $\exists$-back property. The notions $I: \mathfrak{A} \cong_{p,\mathfrak{K}} \mathfrak{B}$ and $\mathfrak{A} \cong_{p,\mathfrak{K}} \mathfrak{B}$ are defined similarly, also incorporating the $Q_{\mathfrak{K}}$-forth and the $Q_{\mathfrak{K}}$-back property into the definition.

Looking first at the $Q_{\mathfrak{K}}$-free version, *a fortiori*, the structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ given in the argument of 4.2.7 are partially isomorphic. Furthermore, any two dense open orderings are partially isomorphic—also via the set of partial isomorphisms with finite domain.

The relation $\cong_\omega$ can be considered as a finite approximation of the isomorphism relation. In good accordance with this view, ω-isomorphic structures are isomorphic in case they are finite. Similarly, the stronger notion of $\cong_p$ embodies countable approximations of isomorphisms:

4.3.1 Theorem. *Countable partially isomorphic structures are isomorphic.*

Proof. Assume $I: \mathfrak{A} \cong_p \mathfrak{B}$, $A = \{a_i | i \in \omega\}$, and $B = \{b_i | i \in \omega\}$. By induction on i one can define $p_i \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ such that for all i: $p_i \subseteq p_{i+1}$, $a_i \in \mathrm{dom}(p_{2i})$, $b_i \in \mathrm{rg}(p_{2i+1})$. Then $\bigcup_i p_i: \mathfrak{A} \cong \mathfrak{B}$. □

The theorem generalizes a well-known result of Cantor according to which any two countable dense open orderings are isomorphic. However, it is not valid for uncountable structures: As mentioned above, any two dense open orderings are partially isomorphic, and there are easy examples of non-isomorphic dense open orderings even of the same cardinality $\aleph_\alpha$, for every $\alpha \geq 1$. Take, for instance, $\aleph_\alpha$ many copies of the rationals and order them either according to $\aleph_\alpha$ or inversely. Moreover, any two infinite sets or any two algebraically closed fields of infinite degree of transcendence (so-called universal domains) of the same characteristic are partially isomorphic.

We see from Theorem 4.3.1 that $\cong_p$ is strictly stronger than elementary equivalence. Hence, from a model-theoretical point of view, we may ask whether there is some logic $\mathscr{L}$ (necessarily) stronger than first-order logic, such that $\cong_p$ equals $\mathscr{L}$-equivalence. The answer is affirmative.

4.3.2 Theorem (Karp [1965]). *For all structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \cong_p \mathfrak{B}$ iff $\mathfrak{A} \equiv_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$.*

From an algebraic point of view, any two universal domains of the same characteristic—even if they are not isomorphic—are not essentially different. The fact that they are partially isomorphic demonstrates that $\cong_p$ can be considered as a methodologically interesting weakening of the isomorphism relation (see also Barwise [1973b]).

The direction from right to left in Theorem 4.3.2 tells us that $\mathscr{L}_{\infty\omega}$ is weak enough not to distinguish between structures that are "weakly identical" in the sense of being partially isomorphic. This feature leads us to a new notion: For any logic $\mathscr{L}$, define $\mathscr{L}$ to have the *Karp property* iff any two partially isomorphic structures are $\mathscr{L}$-equivalent. The direction from right to left in Theorem 4.3.2 now yields that $\mathscr{L}_{\infty\omega}$ is a strongest logic with this property, in the sense that if a logic $\mathscr{L}$ has the Karp property, then any two $\mathscr{L}_{\infty\omega}$-equivalent structures are also $\mathscr{L}$-equivalent (that is, $\mathscr{L} \leq_{\equiv} \mathscr{L}_{\infty\omega}$).

A proof of Theorem 4.3.2 (see Theorem IX.4.3.1 or Barwise [1973b, 1975]) can be given as a suitable "infinitary" version of the corresponding proof for $\mathscr{L}_{\omega\omega}$ and $\cong_{\omega}$, that is, for Corollary 4.2.4. Returning now to partial isomorphisms including Lindström quantifiers, we can proceed similarly with the proof of Theorem 4.2.3, thus verifying the following generalization of Theorem 4.3.2.

4.3.3 Theorem. *Let $Q_{\mathfrak{K}_\iota}$, for $\iota \in I$, be monotone relational Lindström quantifiers. Then for any τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}[\tau]$ we have:*

$$\mathfrak{A} \cong_{p,(\mathfrak{K}_\iota \mid \iota \in I)} \mathfrak{B} \quad \textit{iff} \quad \mathfrak{A} \equiv_{\mathscr{L}_{\infty\omega}(Q_{\mathfrak{K}_\iota} \mid \iota \in I)} \mathfrak{B}. \qquad \square$$

5. Compactness and Its Neighbourhood

Up to now we have described important examples in the framework of general logics and we have tried to isolate some systematizing aspects such as Lindström quantifiers and (R)PC-reducibility. In this and the concluding sections we will try to provide an insight into some basic features of essential model-theoretic notions. Our considerations are grouped around compactness, Löwenheim-Skolem properties and interpolation. Later chapters will exhibit interesting bridges between these concepts which constitute some of the main achievements of abstract model theory. For the remainder of this chapter, we will assume that the logics under consideration are regular.

5.1. Notions of Compactness

In Definition 1.2.4 we introduced the notions of compactness and κ-compactness. The following generalization, which deprives finiteness of its designated role, is important for instance, with infinitary languages.

5.1.1 Definition. For $\kappa \geq \lambda \geq \aleph_0$, $\mathscr{L}$ is (κ, λ)-*compact* iff for all τ and $\Phi \subseteq \mathscr{L}[\tau]$ of power $\leq \kappa$, if each subset of Φ of power $< \lambda$ has a model, then Φ has a model.

The notion "compact" stems from a connection with topology. Given $\mathscr{L}$ and τ, where $\mathscr{L}[\tau]$ is a set, define a topological space $\mathfrak{X}_{\mathscr{L}}[\tau]$ in the following way. The domain $X_{\mathscr{L}}[\tau]$ of $\mathfrak{X}_{\mathscr{L}}[\tau]$ forms a set of representatives of $\mathrm{Str}[\tau]$ modulo $\mathscr{L}$-equivalence, and a basis of (clopen) sets is given by the sets $\mathrm{Mod}_{\mathscr{L}}^{\tau}(\varphi) \cap X_{\mathscr{L}}[\tau]$ for $\varphi \in \mathscr{L}[\tau]$. $\mathfrak{X}_{\mathscr{L}}[\tau]$ is a Hausdorff space, and it is easy to prove

(*) $\quad \mathscr{L}$ is compact iff all $\mathfrak{X}_{\mathscr{L}}[\tau]$ are compact.

Call a topological space $\mathfrak{X}$ (κ, λ)-compact if for all sets C of closed subsets of $\mathfrak{X}$ with $|C| \leq \kappa$ and $\bigcap C = \emptyset$ there exists $C' \subseteq C$ with $|C'| < \lambda$ and $\bigcap C' = \emptyset$. Then, according to an observation of Mannila [1983], topological (κ, λ)-compactness does not correspond—in the sense of $(*)$—to (κ, λ)-compactness of logics, but to a stronger compactness property, the so-called $(\kappa, \lambda)^*$-compactness, which will play a central role in Chapter XVIII.

Compactness properties have an influence on the number of symbols in a sentence φ that are essential for the meaning of φ. We make this precise by use of the following notion. Let φ be from $\mathscr{L}[\boldsymbol{\tau}]$ and $\boldsymbol{\sigma} \subseteq \boldsymbol{\tau}$. We say that φ *depends only on the symbols in* $\boldsymbol{\sigma}$, if for all $\boldsymbol{\tau}$-structures $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \upharpoonright \boldsymbol{\sigma} \cong \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$ we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. For $\mathscr{L}_{\infty\omega}$ there does not exist a uniform bound for the number of symbols that are essential for the meaning of a sentence. According to the following proposition compactness properties lead to a dual situation.

5.1.2 Proposition. *If $\mathscr{L}$ is (κ, λ)-compact and $|\boldsymbol{\tau}| \leq \kappa$, then any $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$ depends on less than λ symbols. Hence, any sentence of a compact logic depends only on finitely many symbols.*

Proof. Assume $|\boldsymbol{\tau}| \leq \kappa$ and $\varphi \in \mathscr{L}[\boldsymbol{\tau}]$. We take a renaming $\rho: \boldsymbol{\tau} \to \boldsymbol{\tau}'$, where $\boldsymbol{\tau}' \cap \boldsymbol{\tau} = \emptyset$, and set

$$\begin{aligned}\Phi = {} & \{\forall \mathrm{x}(R\mathrm{x} \leftrightarrow \rho(R)\mathrm{x}) \mid R \in \boldsymbol{\tau}\} \\ & \cup \{\forall \mathrm{x}\, f(\mathrm{x}) = \rho(f)(\mathrm{x}) \mid f \in \boldsymbol{\tau}\} \cup \{c = \rho(c) \mid c \in \boldsymbol{\tau}\}.\end{aligned}$$

Then $\Phi \models \varphi \leftrightarrow \varphi^{\rho}$. As $|\Phi| \leq \kappa$, (κ, λ)-compactness yields a subset $\Phi_0 \subseteq \Phi$ with $|\Phi|_0 < \lambda$ and $\Phi_0 \models \varphi \leftrightarrow \varphi^{\rho}$. Let $\boldsymbol{\sigma}$ be the set of symbols of $\boldsymbol{\tau}$ which occur in Φ_0. Then $|\boldsymbol{\sigma}| < \lambda$, and if $\mathfrak{A}$, $\mathfrak{B}$ are $\boldsymbol{\tau}$-structures with $\mathfrak{A} \upharpoonright \boldsymbol{\sigma} \cong \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$, say $\mathfrak{A} \upharpoonright \boldsymbol{\sigma} = \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$, we have $(\mathfrak{A}, (\rho(\S)^{\mathfrak{B}^{\rho}})_{\S \in \boldsymbol{\tau}}) \models \Phi_0$ and therefore $\mathfrak{A} \models \varphi$ iff $\mathfrak{B}^{\rho} \models \varphi^{\rho}$ iff $\mathfrak{B} \models \varphi$. □

5.2. *Well-Ordering Numbers*

Compactness properties provide a powerful tool for constructing non-standard models. For instance, $\aleph_0$-compactness implies the non-characterizability of infinite well-orderings. On the other hand, the logic $\mathscr{L}_{\omega_1\omega}$, which is not $\aleph_0$-compact, admits characterizations of all countable well-orderings. By the following definitions we create the appropriate terminology to exhibit precise relations between compactness properties and the characterizability of well-orderings. For technical convenience we introduce a number ∞ with $\alpha < \infty$ for all ordinals α.

5.2.1 Definition. Let be $< \in \boldsymbol{\tau}$ and $\Phi \subseteq \mathscr{L}[\boldsymbol{\tau}]$. We say that Φ *pins down the ordinal* α *(via* $<$), if

(i) for all models $\mathfrak{A}$ of Φ, $<^{\mathfrak{A}}$ is a well-ordering of its field;
(ii) there is a model $\mathfrak{A}$ of Φ such that $<^{\mathfrak{A}}$ is a well-ordering of order type α.

We define $w_\kappa(\mathscr{L})$ to be the supremum of all ordinals that can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$ and call $w(\mathscr{L}) = w_1(\mathscr{L})$ the *well-ordering number* of $\mathscr{L}$. A logic $\mathscr{L}$ is *bounded*, if there is no sentence that pins down arbitrarily large ordinals.

By regularity of $\mathscr{L}$ we have $w(\mathscr{L}) \geq \omega$. If Φ pins down α via $<$, then any $\beta \leq \alpha$ is pinned down by $\Phi \cup \{\prec$ *is an initial segment* of $<\}$ via $\prec$, and $\alpha + 1$ is pinned down via $\prec$ by Φ together with $\prec$ *equals* $<$ *with the least element put at the end* (assumed $\alpha \geq \omega$). Hence $w_\kappa(\mathscr{L}) = \infty$ or $w_\kappa(\mathscr{L})$ is a limit ordinal, and an ordinal α can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$ iff $\alpha < w_\kappa(\mathscr{L})$. Similar arguments yield that $w_\kappa(\mathscr{L})$ is closed under the ordinal operations of addition, multiplication and exponentiation.

There is a useful characterization of well-ordering numbers:

5.2.2 Proposition. *Suppose $\kappa \geq 1$ and $w_\kappa(\mathscr{L}) < \infty$. Then $w_\kappa(\mathscr{L})$ is the least ordinal α such that for all $\Phi \subseteq \mathscr{L}[\tau]$ with $< \in \tau$ and $|\Phi| \leq \kappa$ it is the case that if for arbitrarily large $\beta < \alpha$, Φ has a model $\mathfrak{A}$ where $<^{\mathfrak{A}}$ is a well-ordering of order type β, then Φ has a model $\mathfrak{B}$, where $<^{\mathfrak{B}}$ is not a well-ordering.*

Proof. Assume $w_\kappa(\mathscr{L}) < \infty$ and let α be the ordinal in question. By constructions such as in the preceding paragraph one can easily see that $w_\kappa(\mathscr{L}) \leq \alpha$. For the other direction, it is sufficient to show: If $< \in \tau$, $\Phi \subseteq \mathscr{L}[\tau]$, $|\Phi| \leq \kappa$, and if for arbitrarily large $\beta < w_\kappa(\mathscr{L})$, Φ has a model $\mathfrak{A}$ with $<^{\mathfrak{A}}$ a well-ordering of order type β, then Φ does not pin down ordinals via $<$. In order to establish this, let Φ be given such that Φ satisfies the hypothesis and pins down ordinals via $<$. As $\mathscr{L}$ allows elimination of function symbols, we may assume that τ is relational. With new binary R, $\prec$, and f let Ψ consist of the following sentences:

(1) $\prec$ *is a linear ordering* $\wedge\ \forall x \in$ *field*$(\prec)\ \exists z\ Rxz$;

(2) $\forall x \in$ *field*$(\prec)\ \varphi^{\{z \mid Rxz\}}$ *for* $\varphi \in \Phi$;

(3) $\forall y \in$ *field*$(\prec)\ \exists x \succ y$: $\lambda z f(x, z)$ *is an isomorphism from* (*field*$(< \restriction \{z \mid Rxz\}), < \restriction \{z \mid Rxz\})$ *onto* $(\{z \mid z \prec x\}, \prec \restriction \{z \restriction z \prec x\})$.

Then Ψ pins down $w_\kappa(\mathscr{L})$ and is of power $\leq \kappa$—a contradiction. $\square$

5.2.3 Examples. (a) $w_\kappa(\mathscr{L}_{\omega\omega}) = \omega$ for all $\kappa \geq 1$.

(b) For $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_1)$ we have $w(\mathscr{L}) = w_{\aleph_0}(\mathscr{L}) = \omega$, but for instance $w_{2^{\aleph_0}}(\mathscr{L}) \geq (2^{\aleph_0})^+$. (Note that for any well-ordering $\prec$ of the reals the structure $(\mathbb{R}, +, \cdot, <, \mathbb{Q}, \prec, (r)_{r \in \mathbb{R}})$ is characterizable up to isomorphism by its $\mathscr{L}$-theory, because $(\mathbb{R}, +, \cdot, <, \mathbb{Q}, (r)_{r \in \mathbb{R}})$ is $\mathscr{L}$-maximal, that is, it has no strict extension in the sense of $\prec_{\mathscr{L}}$ (Exercise!).) For further results see Fuhrken [1965].

(c) $w(\mathscr{L}_{\omega_1\omega}) = \omega_1$. We have $w(\mathscr{L}_{\omega_1\omega}) \geq \omega_1$, because a countable ordinal $\alpha \neq 0$ is pinned down by the $\mathscr{L}_{\omega_1\omega}$-sentence

$$\text{“} < \textit{is a linear ordering}\text{”} \wedge \forall x \bigvee \{\mu_\beta(x) \mid \beta < \alpha\},$$

where μ_β is defined inductively by

$$\mu_\beta(x) = \forall y(y < x \leftrightarrow \bigvee \{\mu_\gamma(y) \mid \gamma < \beta\}).$$

A similar argument works for all admissible fragments $\mathscr{L}_{\mathscr{A}}$, showing us that $w(\mathscr{L}_{\mathscr{A}}) \geq o(\mathscr{A})$, the least ordinal not in $\mathscr{A}$. The converse inequality is true for countable $\mathscr{A}$ and yields $w(\mathscr{L}_{\omega_1\omega}) \leq \omega_1$.

(d) If $\mathscr{L} \leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}^*$, then $w_\kappa(\mathscr{L}) \leq w_\kappa(\mathscr{L}^*)$. Using this fact and the remark on countable admissible sets in (c), one can deduce that

$$w(\mathscr{L}^{w2}) = w(\mathscr{L}(Q_0)) = w(\mathscr{L}(\omega, <)) = \omega_1^{\mathrm{CK}},$$

the least non-recursive ordinal (the "Church–Kleene ω_1").

(e) The argument from (c) can be extended to arbitrary ordinals α, if we admit sentences from $\mathscr{L}_{\infty\omega}$. Hence, $w(\mathscr{L}_{\infty\omega}) = \infty$. On the other hand, $\mathscr{L}_{\infty\omega}$ is bounded (López-Escobar [1966]).

(f) The logics $\mathscr{L}^2$, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{R}})$, $\mathscr{L}_{\omega\omega}(I)$, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{H}})$, $\mathscr{L}_{\omega\omega}(Q^{\mathrm{WO}})$, $\mathscr{L}_{\omega_1\omega_1}$ are not bounded as they admit a definition of well-orderings, at least as a projective or a relativized projective class (see Sections 2.3, 2.5 and Example 4.1.2(iv)).

We now return to our introductory remark and state a precise relation between compactness and the characterizability of well-orderings. A stronger form is implicit in Theorem III.2.1.4 in the equivalence of (i) and (iii).

5.2.4 Proposition. *$\mathscr{L}$ is $\aleph_0$-compact iff $w_{\aleph_0}(\mathscr{L}) = \omega$.*

Proof. For the interesting direction, assume $\mathscr{L}$ to be not $\aleph_0$-compact and $\Phi = \{\varphi_n \mid n \in \omega\}$ to be a countable set of sentences of some vocabulary τ such that any finite subset of Φ has a model, but Φ itself does not. Since $\mathscr{L}$ allows elimination of function symbols, we can assume that τ is relational. Then, with new binary relation symbols R and $<$, the set Φ' pins down ω, where Φ' consists of

(1) $<$ *is a linear ordering*;

(2) $\forall x \in \mathit{field}(<)\, \exists z\, Rxz$;

(3) $\forall x \in \mathit{field}(<)(|\{y \mid y \leq x\}| \geq n \to \varphi_n^{\{z \mid Rxz\}})$ for $n \in \omega$. □

At this point we can make another idea precise. Often compactness of a logic can be proved by defining a calculus and showing its completeness. In the framework of our precise notions we can extract the following general fact:

5.2.5 Theorem. *Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}_0}, \ldots, Q^*_{\mathfrak{K}_{n-1}})$ be a logic with Lindström quantifiers (in the sense of Definition 4.1.4), where $\mathscr{L}$ is recursively enumerable for validity. Then, for any $\tau \in \mathrm{HF}$, $\mathscr{L}$ satisfies the compactness property for recursive sets of sentences from $\mathscr{L}[\tau]$.*

Proof. First, we treat the special case where $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^*_{\aleph_i} | i < n)$ is recursively enumerable for consequence. Let $\Phi \subseteq \mathscr{L}[\tau]$ be a recursive set of sentences such that any finite subset has a model. If Φ had no model, we could pass from Φ to a recursive (!) set Φ' as defined in the preceding proof. Adding recursive definitions of addition and multiplication on field($<$) to Φ' would lead to a recursive set Φ'' characterizing the set of natural numbers with addition and multiplication. Hence, the consequences of Φ'' could not be recursively enumerable. Contradiction. By a technique that goes back to Kleene (see Craig–Vaught [1958]) one can give a *finite* axiomatization of Φ'' by use of additional predicates. Hence, the assumption that $\mathscr{L}$ is recursively enumerable for validity is sufficient for the preceding argument. ☐

5.3. Substitutes

There are extensions of first-order logic—and $\mathscr{L}_{\omega_1\omega}$ is one of the best examples—that admit an interesting model theory despite the fact that essential properties such as compactness fail. They illustrate that the value of a logical system should not only be measured by the number of significant properties of first-order logic that are preserved. For instance, $\mathscr{L}_{\omega_1\omega}$ compensates missing compactness by other properties that are well adapted to its specific syntax and its expressive power, such as that of having the "small" well-ordering number ω_1, or the interpolation property. Guided by such experience and moreover by results such as Proposition 5.2.4, we may arrive at the idea of considering compactness not only in the "crude" sense of κ-compactness or its variants, but of measuring it, for instance, by the size of the well-ordering number. In this sense, the logic $\mathscr{L}_{\infty\omega}$, having well-ordering number ∞, but being bounded, has preserved a vestige of compactness.

Taking these aspects seriously, we are led to the following way of exploring the value of some logic $\mathscr{L}$. Instead of asking for the preservation of properties of $\mathscr{L}_{\omega\omega}$, we try to isolate properties of $\mathscr{L}$ that are able to *replace* missing properties of $\mathscr{L}_{\omega\omega}$ or are useful in connection with the special features of $\mathscr{L}$. Properties of the first kind could be called *substitutes* (for the corresponding properties of $\mathscr{L}_{\omega\omega}$). Adhering to compactness we try to give an illustration by some examples. When doing so, however, we should bear in mind that we are not searching for some technical means, but rather are on the trace of some kind of "methodological ferment".

Example 1. *Barwise compactness*, based on a suitable generalization of finiteness, may be considered as the most convincing example. (For details see Barwise [1975] or Chapter VIII.)

Example 2. *Small well-ordering numbers and boundedness.* We have already mentioned $\mathscr{L}_{\omega_1\omega}$ and the role of its well-ordering number being ω_1 (see also Flum [1975b]). A further illustration will be treated in Theorem III.3.6: If we combine boundedness as a substitute for compactness with the so-called *countable* *approximation property* (see Kueker [1977]) as a substitute for the Löwenheim–Skolem property down to $\aleph_0$, we get a "substitute" for Lindström's first theorem with $\mathscr{L}_{\infty\omega}$ as a "substitute" for $\mathscr{L}_{\omega\omega}$.

The reader who watches carefully for methodological aspects, will meet further examples at various points. Certainly he will do so when he recognizes the role of *indiscernibles* (instead of compactness properties) as a means of obtaining upper bounds for Hanf numbers ("stretching method", see the examples following Theorem 6.1.6).

6. Löwenheim–Skolem Properties

The well-ordering number $w(\mathscr{L})$ and its generalizations $w_\kappa(\mathscr{L})$ center around the characterization of well-orderings. Löwenheim–Skolem phenomena refer to analogous questions concerning the cardinality of models. There are two dual aspects: one deals with Hanf numbers (as a counterpart of well-ordering numbers), the other one with Löwenheim numbers.

The following definitions and results can be restated for the many-sorted case, if one defines the cardinality of a many-sorted τ-structure $\mathfrak{A}$ as $\sum_{s\in\tau} |A_s|$ (see Definition 1.2.4(vii)).

6.1. Hanf Numbers

For any logic $\mathscr{L}$, compactness yields the upward Löwenheim–Skolem theorem in the following form: If Φ is a set of sentences of $\mathscr{L}$ of power $\leq \kappa$ that has an infinite model, then Φ has models of arbitrarily high cardinality. In the terminology to come this means that $h_\kappa(\mathscr{L}) = \aleph_0$ for all κ.

6.1.1. Definition. We say that $\Phi \subseteq \mathscr{L}[\tau]$ *pins down the cardinal* κ iff Φ has a model of cardinality κ, but Φ does not have models of arbitrarily high cardinalities. We let $h_\kappa(\mathscr{L})$ be the supremum of all cardinals that can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$ and call $h(\mathscr{L}) := h_1(\mathscr{L})$ the *Hanf number* of $\mathscr{L}$.

By regularity, $h(\mathscr{L}) \geq \aleph_0$. To get more information, let $\Phi \subseteq \mathscr{L}[\tau]$ pin down arbitrarily high cardinals below μ, $\mu \geq \aleph_0$. Assume without loss of generality that τ is relational. Then Ψ pins down μ, where Ψ consists of

(1) $<$ *is a linear ordering of the universe*;

(2) $\forall x\ \varphi^{\{z \mid Rxz\}}$ *for* $\varphi \in \Phi$;

(3) $\forall x\ \lambda u f(x, u) \restriction \{y \mid y \leq x\}$ *is an injection into* $\{z \mid Rxz\}$.

From this we see (taking μ^+ instead of μ) that $h_\kappa(\mathscr{L}) = \infty$ or $h_\kappa(\mathscr{L})$ is a limit cardinal that cannot be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$. Hence,

$h_\kappa(\mathscr{L}) = \infty$ or $h_\kappa(\mathscr{L})$ is the least cardinal μ such that every set of $\mathscr{L}$-sentences of power $\leq \kappa$ that has a model of cardinality μ has arbitrarily large models. Moreover, we obtain as a weak analogue of Proposition 5.2.2:

6.1.2 Proposition. *If $\Phi \subseteq \mathscr{L}[\tau]$, $|\Phi| \leq \kappa$, and Φ has models of arbitrarily high cardinality below $h_\kappa(\mathscr{L})$, then Φ has models of arbitrarily high cardinality.* $\square$

We have $h(\mathscr{L}_{\infty\omega}) = \infty$ even if we restrict ourselves to finite vocabularies (for instance to $\{<\}$, as can be obtained from Examples 5.2.3(c), (e)). On the other hand, logics with "few" sentences should have Hanf numbers $< \infty$. To make this precise, we introduce a new notion.

6.1.3 Definition. Occ($\mathscr{L}$), *the occurrence number of* $\mathscr{L}$, is the least cardinal μ such that for all τ,

$$\mathscr{L}[\tau] = \bigcup_{\substack{\tau_0 \subseteq \tau \\ |\tau_0| < \mu}} \mathscr{L}[\tau_0],$$

if such a cardinal exists; otherwise Occ($\mathscr{L}$) $= \infty$.[3]

The following theorem can be considered as one of the earliest results of what is now called abstract model theory.

6.1.4 Theorem (Hanf [1960]). *Let $\mathscr{L}$ be small (that is, for all τ, $\mathscr{L}[\tau]$ is a set) and assume that* Occ($\mathscr{L}$) $< \infty$. *Then for all κ, $h_\kappa(\mathscr{L}) < \infty$.*

Proof. Set $\mu = \kappa \cdot \mathrm{Occ}(\mathscr{L})$ and let τ be a "universal" vocabulary of power μ; that is, τ contains μ many relation and function symbols of each arity and μ many constants. In order to investigate $h_\kappa(\mathscr{L})$, we can confine ourselves to τ-sentences of $\mathscr{L}$. As $\mathscr{L}[\tau]$ is a set, we have

$$h_\kappa(\mathscr{L}) = \sup\{|A| \mid \mathfrak{A} \models \Phi, \Phi \subseteq \mathscr{L}[\tau], |\Phi| \leq \kappa \text{ and } \Phi \text{ does not have arbitrarily large models}\} < \infty$$

(Axiom of Replacement!). $\square$

The use of the Axiom of Replacement in the argument above is quite essential. This can already be illustrated in case $\mathscr{L} = \mathscr{L}^2$ (see Barwise [1972b]).

[3] Just as one defines the occurrence number Occ($\mathscr{L}$) one can introduce a so-called *dependence number* $\mathbf{o}(\mathscr{L})$, as is done in Chapter XVIII, 2.1.4: $\mathbf{o}(\mathscr{L})$ is the smallest cardinal κ such that for all τ and $\varphi \in \mathscr{L}[\tau]$ there is a vocabulary $\sigma \subset \tau$ of cardinality $< \kappa$ such that φ depends only on σ, and $\mathbf{o}(\mathscr{L}) = \infty$ if no such κ exists. Intuitively, the dependence number is the semantic side and the occurrence number the syntactic side of one and the same coin. Indeed, using the substitution property to remove dummy relation symbols, function symbols, and constants, one can easily see that $\mathbf{o}(\mathscr{L})$ and Occ($\mathscr{L}$) can play the same role in the one-sorted case. In the many-sorted case this may not be true because the substitution property as we have stated it in 1.2.3 does not enable us to remove dummy sort symbols; however, it can be guaranteed by a suitable reformulation of 1.2.3 which we leave to the reader.

Compactness properties yield small Hanf numbers. For example, if $\mathscr{L}$ is (κ, λ)-compact for all κ, then $h_\kappa(\mathscr{L}) \leq \lambda$ for all κ. On the other hand, compactness fades away with growing well-ordering numbers. Hence the question: Do large well-ordering numbers come along with large Hanf numbers? For a precise answer we introduce the *beth numbers* from classical set theory:

6.1.5 Definition. We define by recursion:

(i) $\beth_0(\kappa) = \kappa$;
(ii) $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$;
(iii) $\beth_\beta(\kappa) = \sup\{\beth_\alpha(\kappa) \mid \alpha < \beta\}$ for limit β.

To illustrate the size of beth numbers, let A be a set of power κ and define $V_\alpha^*(A)$, a variant of the von Neumann hierarchy over A, by the following equations:

(i′) $V_0^*(A) = A$;
(ii′) $V_{\alpha+1}^*(A) =$ power set of $V_\alpha^*(A)$,
(iii′) $V_\beta^*(A) = \bigcup \{V_\alpha^*(A) \mid \alpha < \beta\}$ for limit β.

Then for all α we have $|V_\alpha^*(A)| = \beth_\alpha(\kappa)$.

Now assume that $\lambda < h_\kappa(\mathscr{L})$ is pinned down by a set $\Phi \subseteq \mathscr{L}[\tau]$ of power $\leq \kappa$, where τ can be chosen relational ($\mathscr{L}$ allows elimination of function symbols!). With new binary relation symbols V, ε and new constants c_0, c_1 let Φ' consist of

(1) $\exists z\, Vc_0 z \wedge \forall z(Vc_0 z \vee Vc_1 z)$;

(2) $\varphi^{\{z \mid Vc_0 z\}}$ for $\varphi \in \Phi$;

(3) $\forall xy(\forall z(z\varepsilon x \leftrightarrow z\varepsilon y) \rightarrow x = y)$; that is, "$\varepsilon$ *is extensional*";

(4) $\forall z(Vc_1 z \leftrightarrow \forall u\varepsilon z\, Vc_0 z)$.

Then for any model $\mathfrak{A}$ of Φ' we have with $\mu_i = |\{a \in A \mid (c_i^{\mathfrak{A}}, a) \in V^{\mathfrak{A}}\}|$ that $|A| \leq \mu_0 + \mu_1$, where $\mu_0 < h_\kappa(\mathscr{L})$ and $\mu_1 \leq \beth_1(\mu_0)$. Hence Φ' pins down cardinals, and obviously $h_\kappa(\mathscr{L}) > \beth_1(\lambda)$.

Φ' can be considered as a description of the first two steps of the modified von Neumann hierarchy over the domain of models of Φ, where Φ pins down λ. The construction can be easily generalized in a natural way to describe the hierarchy along well-orderings that can be pinned down in $\mathscr{L}$. Thus, one can prove:

6.1.6 Theorem. *Assume that each ordinal $\alpha < w_\kappa(\mathscr{L})$ can be pinned down by a set Ψ_α of sentences, $|\Psi_\alpha| \leq \kappa$, having a model $\mathfrak{A}$ of power $< h_\kappa(\mathscr{L})$ where $<^{\mathfrak{A}}$ is of order type α. Then for every $\lambda < h_\kappa(\mathscr{L})$, $h_\kappa(\mathscr{L}) \geq \beth_{w_\kappa(\mathscr{L})}(\lambda)$.*

As an application we obtain, for instance, that

$$h(\mathscr{L}_{\omega\omega}(Q_1)) \geq \beth_\omega(\aleph_0) = \beth_\omega(\aleph_1);$$

$$h(\mathscr{L}_{\omega_1\omega}) \geq \beth_{\omega_1}(\aleph_0);$$

$$h(\mathscr{L}) \geq \beth_{\omega_1^{CK}}(\aleph_0) \quad \text{for } \mathscr{L} = \mathscr{L}^{w2}, \mathscr{L}_{\omega\omega}(Q_0), \mathscr{L}_{\omega\omega}(\omega, <).$$

What about the other direction in these examples? It is valid, too. Thus, in each case we get equality. The corresponding proofs are based on partition theorems and indiscernibles. These techniques can also be used to get further strong results in the same direction (see, for example, Barwise [1975]).

If a logic is weak in pinning down ordinals, it may happen that we are unable to give satisfactory information about Hanf numbers. For example, for $\mathscr{L} = \mathscr{L}_{\omega\omega}(I)$, the size of $h(\mathscr{L})$ depends on set theory: If $V = L$, then $h(\mathscr{L}) = h(\mathscr{L}^2)$. On the other hand, $h(\mathscr{L})$ may be smaller than the Löwenheim number $l(\mathscr{L})$ as defined below, which may itself be smaller than $2^{\aleph_0}$ (see Section VI.2.1 and Väänänen [1982a]).

Warning. We have become accustomed to numerous preservation facts for (R)PC-reducibility. For instance, we obviously have

$$(*) \qquad \text{If } \mathscr{L} \le_{\text{PC}} \mathscr{L}^*, \quad \text{then for all } \kappa, \quad h_\kappa(\mathscr{L}) \le h_\kappa(\mathscr{L}^*).$$

However, it is plausible that we would meet difficulties if we were to try to prove $(*)$ for $\le_{\text{RPC}}$. Indeed, in the remark preceding Proposition 7.2.5 we will see that there are counterexamples.

6.2. *Löwenheim Numbers*

Löwenheim numbers measure the strength of downward Löwenheim–Skolem theorems.

6.2.1 Definition. $l_\kappa(\mathscr{L})$ is the least cardinal μ such that any satisfiable set of $\mathscr{L}$-sentences of power $\le \kappa$ has a model of power $\le \mu$, provided there is such a cardinal; otherwise, $l_\kappa(\mathscr{L}) = \infty$. We call $l(\mathscr{L}) := l_1(\mathscr{L})$ the *Löwenheim number* of $\mathscr{L}$.

Obviously, $\mathscr{L}$ has the Löwenheim–Skolem property down to λ iff $l(\mathscr{L}) \le \lambda$. By taking inequalities between κ many constants we see that $l_\kappa(\mathscr{L}) \ge \max\{\kappa, \aleph_0\}$. The proof of the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega\omega}(Q_1)$ as mentioned in Example 1 of Section 2.2 can be generalized and yields $l(\mathscr{L}_{\omega\omega}(Q_\alpha)) = l_{\aleph_\alpha}(\mathscr{L}_{\omega\omega}(Q_\alpha)) = \aleph_\alpha$. Clearly, $l(\mathscr{L}_{\infty\omega}) = \infty$. But if $\mathscr{L}$ is small (that is, if all $\mathscr{L}[\tau]$ are sets) and $\text{Occ}(\mathscr{L}) < \infty$, then by an argument like that for Hanf's theorem (6.1.4), we have $l_\kappa(\mathscr{L}) < \infty$ for all κ.

Numerous results such as $l_{\aleph_\alpha}(\mathscr{L}_{\omega\omega}(Q_\alpha)) = \aleph_\alpha$ can be strengthened by showing that structures possess small elementary substructures; however, this possibility may fail already with familiar logics. For instance, $l(\mathscr{L}_{\omega\omega}(\text{aa})) = \aleph_1$, but the existence of $\mathscr{L}_{\omega\omega}(\text{aa})$-elementary substructures of power $\le \aleph_1$ is independent from ZFC (see remark after IV.4.2.5). For a closer look at Löwenheim–Skolem properties and substitutes the reader is referred to Section III.3.

7. *Interpolation and Definability*

In this final section we return to central notions of a more "logical" character. The main topics we shall touch concern interpolation and a generalization of Robinson's consistency theorem in Section 7.1, Δ-interpolation in Section 7.2 and variations of Beth's definability theorem in Section 7.3. Again we confine ourselves to regular logics. *However, we explicitly include the many-sorted case.* As the reformulation of the usual interpolation property given in Definition 1.2.4(viii) by separability of projective classes as in Proposition 3.1.3 splits into cases—referring to "PC" in the one-sorted version and to "RPC" in the many sorted version—we use "(R)PC" to stand for "PC" in the first and for "RPC" in the second case.

7.1. *Interpolation and the Robinson Property*

As a generalization of the interpolation property, we state

7.1.1 Definition. Let $\mathscr{L}$, $\mathscr{L}^*$ be logics. *$\mathscr{L}^*$ has the interpolation property for $\mathscr{L}$* or *$\mathscr{L}^*$ allows interpolation for $\mathscr{L}$* iff any two disjoint classes of the same vocabulary that are (R)PC in $\mathscr{L}$ can be separated by a class EC in $\mathscr{L}^*$.

Interpolation is indeed rare. The positive examples among the logics we have mentioned up to now can be listed very quickly:

7.1.2 Examples. (a) $\mathscr{L}_{\omega\omega}$. The one-sorted case is due to Craig [1957a], the many-sorted one is proved in Feferman [1968a]. The one-sorted version follows from the many-sorted one, even in the stronger form with "RPC" instead of "PC", because relativized reducts can be rewritten as simple reducts of many-sorted structures (see Barwise [1973a]). It is especially with interpolation that many-sortedness pays. As seen in Feferman [1974a], the many-sorted version of the interpolation theorem together with its possible refinements is a powerful tool even for one-sorted model theory, offering for instance elegant proofs of various preservation theorems. For a proof of a strong version of $\mathscr{L}_{\omega\omega}$-interpolation the reader is referred to Theorem X.2.2.9.

(b) $\mathscr{L}_{\omega_1\omega}$ (Lopez-Escobar [1965b]) and countable admissible fragments (Barwise [1969b]).

Interpolation properties seem to indicate some kind of balance between syntax and semantics. This can be seen, for instance, from the work of Zucker [1978] or from the fact that interpolation implies Beth's definability theorem, according to which implicit definitions can be made explicit. Last but not least it is illustrated by a result of Feferman [1974a] according to which Δ-interpolation is equivalent to truth maximality (see Corollary XVII.1.1.17). Hence we may expect that interpolation properties (or definability properties, see Section 7.3) fail if syntax and semantics are not in an equilibrium. The counterexamples to interpolation that we have mentioned up to now (such as $\mathscr{L}^{w2}$, being able to code its own truth, or $\mathscr{L}_{\omega\omega}(Q_1)$, being able to characterize uncountability) are not astonishing if seen in the light of these heuristics.

7.1.3 Further Counterexamples. (a) In the case of *large infinitary languages*, the main fact is that $\mathscr{L}_{\infty\omega}$ *does not allow interpolation for* $\mathscr{L}_{\omega_2\omega}$. For a proof we consider the classes

$$\mathfrak{K}_{\aleph_0} = \{A \mid A \neq \varnothing, |A| \leq \aleph_0\}, \qquad \mathfrak{K}^{\aleph_1} = \{A \mid |A| \geq \aleph_1\}.$$

$\mathfrak{K}_{\aleph_0}$ and $\mathfrak{K}^{\aleph_1}$ are PC in $\mathscr{L}_{\omega_2\omega}$ (for $\mathfrak{K}^{\aleph_1}$ we can use the sentence

$$\bigwedge \{c_\alpha \neq c_\beta \mid \alpha < \beta < \aleph_1\}).$$

But $\mathfrak{K}_{\aleph_0}$ and $\mathfrak{K}^{\aleph_1}$ cannot be separated by a class EC in $\mathscr{L}_{\infty\omega}$, as all infinite sets are partially isomorphic and, hence, $\mathscr{L}_{\infty\omega}$-equivalent by Karp's theorem (4.3.2). (For further results see Example IX.2.3.1 and Theorem IX.2.3.2.)

(b) For extensions of $\mathscr{L}_{\omega\omega}(Q_1)$, we find that $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ does not allow interpolation for $\mathscr{L}_{\omega\omega}(Q_1)$, and $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ does not allow interpolation for $\mathscr{L}_{\omega\omega}(Q_1)$. Hence, none of the logics $\mathscr{L}_{\omega\omega}(Q_1^n)$ for $n \geq 1$, $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ or $\mathscr{L}_{\omega\omega}(\mathrm{pos})$ has the interpolation property.

To argue for the first assertion, let $\mathfrak{K}_{\mathrm{cf}\,\omega}$, $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ be the classes of orderings of cofinality ω, ω_1, respectively. Both are PC in $\mathscr{L}_{\omega\omega}(Q_1)$: $\mathfrak{K}_{\mathrm{cf}\,\omega}$ via a sentence $\varphi_0(<, U_0)$ saying that $<$ is an ordering of the universe without last element and U_0 of power $\leq \aleph_0$ a cofinal subset, and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ via a sentence $\varphi_1(<, U_1)$ saying that $<$ is an ordering of the universe and U_1 a cofinal subset such that $< \restriction U_1 \times U_1$ is an $\aleph_1$-like ordering. $\mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ cannot be separated by a class of orderings EC in $\mathscr{L}_{\omega\omega}(Q^n \mid n \geq 1)$. For let $\mathfrak{A} = (\mathbb{R}, <^{\mathbb{R}})$ be the ordering of the reals and $\mathfrak{B} = (\mathbf{B}, <^{\mathfrak{B}})$ the result of replacing each ordinal in $\aleph_1$ by a copy of $\mathfrak{A}$. Then $\mathfrak{A} \in \mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{B} \in \mathfrak{K}_{\mathrm{cf}\,\omega_1}$. On the other hand, we have $\mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}(Q_1^n)} \mathfrak{B}$ for all $n \geq 1$, as $(I)_{\alpha \leq \omega}: \mathfrak{A} \cong_{\omega, Q_1^n} \mathfrak{B}$, where I is the set of partial isomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$ with finite domain. (For the second assertion and further material, see Section IV.6.3).

(c) $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$, the fully compact extension of $\mathscr{L}_{\omega\omega}$, does not have the interpolation property (alas!). To sketch a counterexample, call a tree $(T, <, E)$ with an equivalence relation E on T whose equivalence classes are maximal antichains ("levelled tree") *rankable* by a linear ordering $(R, \prec)$, if there exists a homomorphism π from $(T, <)$ onto $(R, \prec)$ such that the equivalence classes of E are the preimages of π. Define $\mathfrak{K}_0, \mathfrak{K}_1$ to be the class of levelled trees rankable by some ordering of cofinality ω, $\geq \omega_1$, respectively. Then $\mathfrak{K}_0$ and $\mathfrak{K}_1$ are disjoint and PC in $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$. Define $\mathfrak{T}_0$ to be the set $\{t \mid t: \{a \in \mathbb{Q} \mid a <^{\mathbb{Q}} b\} \to \{0, 1\}, b \in \mathbb{Q}\}$ ordered by inclusion where two points are equivalent if they have the same domain, and define $\mathfrak{T}_1$ similarly, using a dense $\aleph_1$-like end extension of $(\mathbb{Q}, <^{\mathbb{Q}})$. Then $\mathfrak{T}_i \in \mathfrak{K}_i$ $(i = 0, 1)$, but $\mathfrak{T}_0 \prec_{\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})} \mathfrak{T}_1$. See also Mekler–Shelah [1983, Theorem 3.5].

(d) For a general class of counterexamples the reader can refer to Proposition VI.2.3.1.

In first order logic there is access to interpolation via Robinson's consistency theorem. This possibility can be generalized.

7.1.4 Definition. $\mathscr{L}$ has the *Robinson property* iff for any vocabularies τ_0, τ_1 and $\tau = \tau_0 \cap \tau_1$ and for all classes (!) $\Phi \subseteq \mathscr{L}[\tau]$ and $\Phi_i \subseteq \mathscr{L}[\tau_i]$ $(i = 0, 1)$, if Φ is complete (i.e. all τ-models of Φ are $\mathscr{L}$-equivalent) and if $\Phi \cup \Phi_i$ has a model for $i = 0, 1$, then $\Phi \cup \Phi_0 \cup \Phi_1$ has a model.

7.1.5 Proposition. *Let $\mathscr{L}$ be small (i.e. all $\mathscr{L}[\tau]$ are sets). Then, if $\mathscr{L}$ is compact, $\mathscr{L}$ has the interpolation property iff $\mathscr{L}$ has the Robinson property.*

Proof. Let $\mathscr{L}$ be compact and τ_0, τ_1 and τ be given as in Definition 7.1.4. Since $\mathscr{L}$ is small, all classes of sentences defined below are sets so that the compactness property is applicable. Assume first that $\mathscr{L}$ has the Robinson property and let $\varphi_i \in \mathscr{L}[\tau_i]$ $(i = 0, 1)$ be given such that

$$(*) \qquad \varphi_0 \models \varphi_1.$$

Setting $\Phi' = \{\varphi \in \mathscr{L}[\tau] \mid \varphi_0 \models \varphi\}$, we have $\Phi' \models \varphi_1$. (Otherwise, if $\mathfrak{B} \in \mathrm{Str}[\tau]$ has an expansion satisfying $\Phi' \cup \{\neg \varphi_1\}$, then $\mathrm{Th}_{\mathscr{L}}(\mathfrak{B}) \cup \{\neg \varphi_1\}$ has a model, and by a compactness argument, so does $\mathrm{Th}_{\mathscr{L}}(\mathfrak{B}) \cup \{\varphi_0\}$. Hence the Robinson property yields a model of $\{\varphi_0, \neg \varphi_1\}$—a contradiction to $(*)$.)

Now, by compactness, there is some finite subset of Φ', say Φ'', such that $\Phi'' \models \varphi_1$. Obviously, $\bigwedge \Phi''$ is an interpolant for $(*)$.

For the other direction let Φ, Φ_0, Φ_1 be given as in Definition 7.1.4, Φ complete, $\Phi \cup \Phi_i$ satisfiable for $i = 0, 1$ and without loss of generality $\Phi \subseteq \Phi_0$. As $\mathscr{L}$ is compact it suffices to show that for any finite conjunction φ_i over Φ_i $(i = 0, 1)$ the set $\{\varphi_0, \varphi_1\}$ is satisfiable.

Assume for contradiction that $\{\varphi_0, \varphi_1\}$ has no model. Then the interpolation property yields a sentence $\varphi \in \mathscr{L}[\tau]$ such that $\varphi_0 \models \varphi$ and $\varphi \models \neg \varphi_1$. As $\Phi \cup \{\varphi_0\}$ has a model and Φ is complete, we have $\Phi \models \varphi$. But then $\Phi \cup \{\varphi_1\}$ has no model, a contradiction to the satisfiability of $\Phi \cup \Phi_1$. $\square$

Proposition 7.1.5 can be strengthened considerably: For logics with sufficiently small occurrence number, the Robinson property yields compactness (see Theorem XIX.1.3 and Chapter XVIII).

7.2. Δ-interpolation and Δ-closure

The following notions have proved to be very fruitful.

7.2.1 Definition. A class $\mathfrak{K}$ of τ-structures is said to be Δ in $\mathscr{L}$ (in symbols $\mathfrak{K} \in \Delta_{\mathscr{L}}$) iff $\mathfrak{K}$ and $\bar{\mathfrak{K}} = \mathrm{Str}[\tau] \setminus \mathfrak{K}$ are (R)PC in $\mathscr{L}$. A logic $\mathscr{L}$ has the *Δ-interpolation property* iff every Δ class of $\mathscr{L}$ is EC in $\mathscr{L}$. A logic $\mathscr{L}^*$ has the *Δ-interpolation property for $\mathscr{L}$* (or *$\mathscr{L}^*$ allows Δ-interpolation for $\mathscr{L}$*) iff every Δ class of $\mathscr{L}$ is EC in $\mathscr{L}^*$.

As we have already observed in Section 3.1, Δ-interpolation is a weakening of interpolation. Moreover, Theorem 7.2.6 will show us that it is a strict one. For several reasons, however, it is an interesting one, one that is able to compete seriously with the perhaps too strong notion of interpolation:

(1) According to a remark after Example 7.1.2(b), Δ-interpolation is equivalent to truth-maximality and thus, in a precise sense, embodying a balance between syntax and semantics.
(2) Δ-interpolation is equivalent to a certain variant of Beth's definability theorem, see Proposition 7.3.3.
(3) Δ-interpolation is by far not as rare as interpolation. This will become clear from the notion of Δ-closure given below.

7.2.2 Examples and Counterexamples. (a) $\mathscr{L}_{\omega\omega}(Q_1)$ does not allow Δ-interpolation as the classes corresponding to Keisler's counterexample to interpolation (see $(*)$ in Example 1 of Section 2.2) are Δ in $\mathscr{L}_{\omega\omega}(Q_1)$.

(b) Even sharper: $\mathscr{L}_{\omega\omega}(Q_1^n \mid n \geq 1)$ does not allow Δ-interpolation for $\mathscr{L}_{\omega\omega}(Q_1)$. (For a proof see Theorem IV.6.3.3.)

(c) Similar to (a), the counterexample to interpolation for $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}\,\omega})$ as given in 7.1.3(c) is also a counterexample to Δ-interpolation.

In contrast to the interpolation property, the Δ-interpolation property guarantees the existence only of such elementary classes as are uniquely determined. Hence, unlike interpolation, Δ-interpolation leads to a natural closure operation which we now examine.

7.2.3 Definition. The *Δ-closure* of $\mathscr{L}$, $\Delta(\mathscr{L})$, is the logic that has as elementary classes just the classes that are Δ in $\mathscr{L}$. To develop a more precise description, let $\Delta(\mathscr{L})[\tau]$ consist of all pairs

$$\varphi = (\exists_{\tau_0\setminus\tau}\varphi_0, \exists_{\tau_1\setminus\tau}\varphi_1),$$

where $\tau_i \supseteq \tau$, $\varphi_i \in \mathscr{L}[\tau_i]$ $(i = 0, 1)$, and $\mathrm{Mod}_{\mathscr{L}}^{\tau}(\exists_{\tau_0\setminus\tau}\varphi_0)$ and $\mathrm{Mod}_{\mathscr{L}}^{\tau}(\exists_{\tau_1\setminus\tau}\varphi_1)$ are complementary, and set

$$\mathrm{Mod}_{\Delta(\mathscr{L})}^{\tau}(\varphi) = \mathrm{Mod}_{\mathscr{L}}^{\tau}(\exists_{\tau_0\setminus\tau}\varphi_0).$$

7.2.4 Theorem (Properties of the Δ-Closure). *Assume that* $\mathrm{Occ}(\mathscr{L}) = \aleph_0$. *Then*

(i) *$\Delta(\mathscr{L})$ is a regular logic with occurrence number $\aleph_0$.*
(ii) *Δ is a closure operation on the logics under consideration, that is,*
 (1) $\mathscr{L} \leq \Delta(\mathscr{L})$;
 (2) *If $\mathscr{L} \leq \mathscr{L}^*$, then $\Delta(\mathscr{L}) \leq \Delta(\mathscr{L}^*)$;*
 (3) $\Delta(\Delta(\mathscr{L})) \equiv \Delta(\mathscr{L})$.
(iii) *$\Delta(\mathscr{L})$ has the Δ-interpolation property and $\mathscr{L} \equiv_{(\mathrm{R})\mathrm{PC}} \Delta(\mathscr{L})$.*
(iv) *$\Delta(\mathscr{L})$ is modulo equality via elementary classes the strongest logic $\leq_{(\mathrm{R})\mathrm{PC}} \mathscr{L}$ and the smallest $\leq_{(\mathrm{R})\mathrm{PC}}$-extension of $\mathscr{L}$ having the Δ-interpolation property.*

Remarks. The first statement of (iv) says that if $\mathscr{L}^* \leq_{(R)PC} \mathscr{L}$, then $\mathscr{L}^* \leq \Delta(\mathscr{L})$. Thus it makes precise the range of (R)PC-reducibility: There is a unique borderline realized by $\Delta(\mathscr{L})$.

As the proof below will show the condition on $\mathrm{Occ}(\mathscr{L})$ is used for instance to formulate τ-closedness of predicates. In infinitary languages this can be done even for infinite vocabularies. Hence Theorem 7.2.4 is also valid for logics such as $\mathscr{L}_{\omega_1\omega}$ or $\mathscr{L}_{\infty\omega}$.

Sketch of Proof of Theorem 7.2.4. We show some parts of (i) and (ii)(3), confining ourselves to the one-sorted case and considering typical examples. If S is, say, unary and $\varphi(c) = (\exists R\varphi_0(R, c), \exists S\varphi_1(S, c))$, then one can take $(\exists S\varphi_1(S, c), \exists R\varphi_0(R, c))$ for $\neg\varphi(c)$ and $(\exists R\, \exists c\varphi_0(R, c), \exists S'\, \forall c\varphi_1(\lambda z S'cz, c))$ for $\exists c\varphi(c)$ where S' is a new binary relation symbol. To show the relativization property, let $\chi = (\exists_{\boldsymbol{\sigma}_0\setminus\boldsymbol{\sigma}}\chi_0, \exists_{\boldsymbol{\sigma}_1\setminus\boldsymbol{\sigma}}\chi_1) \in \Delta(\mathscr{L})[\boldsymbol{\sigma}]$ (with $\boldsymbol{\sigma} \subseteq \boldsymbol{\sigma}_i$, $\boldsymbol{\sigma}_i$ finite, $\chi_i \in \mathscr{L}[\boldsymbol{\sigma}_i]$ and, say, $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_1$) and let $\vartheta = (\vartheta_0, \vartheta_1)$ be some $\Delta(\mathscr{L})$-sentence of meaning $\forall c(Uc \leftrightarrow \varphi(c))$, U new and, say, $\vartheta \in \Delta(\mathscr{L})[\boldsymbol{\tau} \cup \{U\}]$. Then one can obtain a $\Delta(\mathscr{L})$-sentence of meaning $\chi^{\{c \mid \varphi(c)\}}$ by suitably rephrasing the equivalent statement

$$(\exists U\, \exists_{\boldsymbol{\sigma}_0\setminus\boldsymbol{\sigma}}(\vartheta_0 \wedge U\boldsymbol{\sigma}_0\text{-}closed \wedge \chi_0^U), \exists U\, \exists_{\boldsymbol{\sigma}_0\setminus\boldsymbol{\sigma}}(\vartheta_0 \wedge (\neg U\boldsymbol{\sigma}_0\text{-}closed \vee \chi_1^U))).$$

In order to prove $\Delta(\Delta(\mathscr{L})) \equiv \Delta(\mathscr{L})$ one observes that a typical sentence of $\Delta(\Delta(\mathscr{L}))$ such as

$$(\exists R(\exists S\varphi_0(R, S, \ldots), \ldots), \exists R'(\exists S'\varphi_0'(R', S', \ldots), \ldots))$$

has the same meaning as $(\exists RS\varphi_0(R, S, \ldots), \exists R'S'\varphi_0'(R', S', \ldots))$. $\square$

Remark. Properties that are transferred from $\mathscr{L}$ to $\Delta(\mathscr{L})$ include those which are inherited by $\equiv_{(R)PC}$. Therefore the Δ-closure preserves (κ, λ)-compactness, well-ordering numbers, Löwenheim numbers and boundedness. On the other hand it does not necessarily preserve Hanf numbers in the many-sorted case. For example, as shown in Väänänen [1983], it is consistent to assume in this case that $h(\mathscr{L}_{\omega\omega}(I)) < h(\Delta(\mathscr{L}_{\omega\omega}(I)))$. However, if Δ classes and the Δ-closure are defined via PC as in the one-sorted case, Hanf numbers are preserved also.

7.2.5 Proposition. *The Δ-closure does not preserve the Karp property.*

Proof. $\mathscr{L}_{\omega_2\omega}$ has the Karp property (see the remarks following Theorem 4.3.2). According to Counterexample 7.1.3(a) the classes $\mathfrak{K}_{\aleph_0} = \{A \neq \emptyset \mid |A| \leq \aleph_0\}$ and $\mathfrak{K}^{\aleph_1} = \{A \mid |A| \geq \aleph_1\}$ are Δ in $\mathscr{L}_{\omega_2\omega}$ and so are EC in $\Delta(\mathscr{L}_{\omega_2\omega})$. Therefore, $\aleph_0 \not\equiv_{\Delta(\mathscr{L}_{\omega_2\omega})} \aleph_1$. But on the other hand we have $\aleph_0 \cong_p \aleph_1$. Thus, the Karp property fails for $\Delta(\mathscr{L}_{\omega_2\omega})$. $\square$

We can say even more. By the proof of Proposition 7.2.5 we have $\mathscr{L}_{\omega\omega}(Q_1) \leq \Delta(\mathscr{L}_{\omega_2\omega})$. Hence the classes $\mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$ of orderings of cofinality ω, ω_1, respectively, which are PC in $\mathscr{L}_{\omega\omega}(Q_1)$ (compare the proof of Counterexample 7.1.3(b)), are PC in $\Delta(\mathscr{L}_{\omega_2\omega})$. Assume that there is some class $\mathfrak{K}$, $\mathfrak{K}$ EC in $\Delta(\mathscr{L}_{\omega_2\omega})$, that separates $\mathfrak{K}_{\mathrm{cf}\,\omega}$ and $\mathfrak{K}_{\mathrm{cf}\,\omega_1}$. Then for suitable $\tau \supseteq \{<\}$ and $\varphi \in \mathscr{L}_{\omega_2\omega}[\tau]$, we have $\mathfrak{K} = \mathrm{Mod}(\varphi) \upharpoonright \{<\}$. Let $\mathscr{L}$ be the smallest fragment of $\mathscr{L}_{\omega_2\omega}$ containing φ. We take some $\mathfrak{A} \in \mathrm{Str}[\tau]$ with $(A, <^{\mathfrak{A}})$ an ordering of cofinality ω_2 and build into $\mathfrak{A}$ a chain $(\mathfrak{A}_\alpha)_{\alpha \leq \aleph_1}$, forming unions at limit points, such that for all $\alpha < \aleph_1$, $\mathfrak{A}_\alpha \prec_{\mathscr{L}} \mathfrak{A}_{\alpha+1} \prec_{\mathscr{L}} \mathfrak{A}$, and $<^{\mathfrak{A}_\alpha}$ is not cofinal in $<^{\mathfrak{A}_{\alpha+1}}$. Then $\mathfrak{A}_\omega \upharpoonright \{<\} \in \mathfrak{K}$ iff $\mathfrak{A}_{\aleph_1} \upharpoonright \{<\} \in \mathfrak{K}$. This, however, is a contradiction. Thus we have proved

7.2.6 Theorem (H. Friedman). *Δ-interpolation is strictly weaker than interpolation. For instance, $\Delta(\mathscr{L}_{\omega_2\omega})$ does not allow interpolation.* □

The reader should consult Theorem IV.6.3.5 for another example.

Concluding Remarks. (a) Our definition of $\Delta(\mathscr{L})$ as sketched in Definition 7.2.3 is useful for technical purposes. But it does have a remarkable disadvantage: even the $\mathscr{L}_{\omega\omega}$-part of $\Delta(\mathscr{L}_{\omega\omega})(\equiv \mathscr{L}_{\omega\omega}(!))$ is not effective, since for sufficiently rich τ the $\Delta(\mathscr{L}_{\omega\omega})$-sentences of the form $(\varphi, \exists x\, x \neq x)$ (that is, those with $\varphi \in \mathscr{L}_{\omega\omega}[\tau]$ and $\models \varphi$) do not form a recursive set. A more significant example illustrating the task of giving an informative description of Δ-closures is due to Barwise [1974a] (see Theorem XVII.3.2.2):

$$\Delta(\mathscr{L}^{w2}) \equiv \Delta(\mathscr{L}_{\omega\omega}(Q_0)) \equiv \mathscr{L}_{\omega_1^{\mathrm{CK}}\omega} \quad \text{(for finite vocabularies).}$$

(b) If $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^*_{\mathfrak{K}_0}, \ldots, Q^*_{\mathfrak{K}_n})$ with Lindström quantifiers $Q^*_{\mathfrak{K}_i}$ and if $\mathbb{K}$ consists of those classes which are Δ in $\mathscr{L}$ and of finite vocabulary, then obviously $\Delta(\mathscr{L}) = \mathscr{L}(Q^*_{\mathfrak{K}} \mid \mathfrak{K} \in \mathbb{K})$. Now, if $\mathbb{K}_0$ is a finite subset of $\mathbb{K}$, then one can prove by a slight variation of the technique used in the proof of Proposition 3.1.7 that, if $\mathscr{L}$ is recursively enumerable for consequence then so is $\mathscr{L}(Q^*_{\mathfrak{K}} \mid \mathfrak{K} \in \mathbb{K}_0)$. It is in this sense that the Δ-closure preserves axiomatizability *locally*.

7.3. Definability Properties

In Definition 1.2.4(ix) we formulated the *Beth property* by a natural translation of Beth's definability theorem into the framework of abstract model theory. The following definitions introduce some variants.

7.3.1 Definition. Assume $\S \in \tau$ and $\varphi \in \mathscr{L}[\tau]$. We say that φ *defines* $\S$ *strongly implicitly* iff for each $\mathfrak{A} \in \mathrm{Str}[\tau \backslash \{\S\}]$ there is *exactly one* expansion $(\mathfrak{A}, \S^{\mathfrak{A}})$ of $\mathfrak{A}$ which is a model of φ. The logic $\mathscr{L}$ has the *weak Beth property* iff for each τ, $\S$ and φ as above, if φ defines $\S$ strongly implicitly, then $\S$ is explicitly definable relative to φ.

Like Δ-interpolation which guarantees the existence of uniquely determined elementary classes, the weak Beth property ensures the existence of uniquely determined explicit definitions. Hence it induces a natural closure operation on logics yielding the so-called weak Beth closure WB($\mathscr{L}$) of a logic $\mathscr{L}$ (it is treated, for example, in Sections XVII.1.2 and 4.1). As can be shown by examples (see Chapter XVIII, 4.2.2) the weak Beth property is strictly weaker than the Beth property. According to H. Friedman [1973] and Badger [1980], $\mathscr{L}_{\omega\omega}(Q_1^n)$ does not have the Beth property for $n \geq 1$. It is open as to whether or not it has the weak Beth property. (For $n = 1$ see also Mekler–Shelah [198?].)

Failure of the Weak Beth Property. We have already mentioned after Definition 2.1.2 that there is a fairly general method of disproving the (weak) Beth property by a codification of truth. The method goes back to Craig [1965] and is explicitly used in Mostowski [1968] and Lindström [1969]. It applies to logics such as $\mathscr{L}_{\omega\omega}(Q_0)$, $\mathscr{L}^{w2}$, or $\mathscr{L}_{\omega\omega}$ enlarged with finitely many Lindström quantifiers in which, for example, the standard model of arithmetic is characterizable and which allow an arithmetization of their semantics. A systematic treatment can be found in Section XVII.11.2. For illustration we give an example for the one-sorted case. We assume that $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$ with $\mathfrak{K}$ of vocabulary $\{R\}$, R binary, and that for some finite $\tau \supseteq \{+, \cdot, <, 0, 1\}$ and some $\varphi \in \mathrm{Sent}_{\mathscr{L}}[\tau]$ the sentence $\exists_{\tau \setminus \{+, \cdot, <, 0, 1\}} \varphi$ characterizes the standard model of arithmetic.

For our procedure we use an effective Gödel numbering

$$\gamma\colon \mathrm{Form}_{\mathscr{L}}[\tau] \xrightarrow[\text{onto}]{1-1} \omega,$$

and we code assignments of finitely many variables over ω—the case we are interested in—by elements of ω in some natural manner, identifying variables with natural numbers. Then, with a binary relation symbol Sat, we construct a ($\tau \cup \{\mathrm{Sat}\}$)-sentence σ of $\mathscr{L}$ such that (abbreviating $\underbrace{1 + \cdots + 1}_{m\text{-times}}$ by $\underline{m}$)

(∗) Sat is defined strongly implicitly by the sentence

$$\hat{\sigma} = (\varphi \wedge \sigma) \vee (\neg \varphi \wedge \forall xy \neg \mathrm{Sat}\, xy).$$

(∗∗) If $(\mathfrak{A}, \mathrm{Sat}^{\mathfrak{A}}) \models \varphi \wedge \sigma$, then, for all $m, n \in \omega$, we have

$\mathfrak{A} \models \mathrm{Sat}\, \underline{m}^{\mathfrak{A}} \underline{n}^{\mathfrak{A}}$ iff $\underline{m}^{\mathfrak{A}}$ codes an assignment π over A the domain of which contains all variables occurring free in $\bar{\gamma}^1(n)$ such that $\bar{\gamma}^1(n)$ is true under π in $\mathfrak{A}$.

To obtain σ, one describes the inductive definition of satisfaction for $\mathscr{L}[\tau]$-formulas. For example, the Q-step can be treated as follows: Let $f\colon \omega^3 \to \omega$ be a recursive function such that for all $l, m, n \in \omega$,

$$f(l, m, n) = \gamma(Qv_l v_m \bar{\gamma}^1(n)).$$

Then, writing "Sat(x, y)" for "Sat xy", the following sentence becomes a conjunct of σ:

$$\forall x\ \forall uvw\ (\mathrm{Sat}(w, \text{``}f(u, v, x)\text{''}) \leftrightarrow (\text{``}w \textit{ assignment for } f(u, v, x)\text{''} \\ \wedge\ Qyz\ \mathrm{Sat}(\text{``}w \upharpoonright (\mathrm{dom}(w)\setminus\{u, v\}) \cup \{(u, y), (v, z)\}\text{''}, x))),$$

where the parts in quotation marks have to be replaced by an arithmetical definition.

Proof of the Failure. Now, assume $\mathscr{L}$ to have the weak Beth property. Then, by (*), there is $\psi(v_0, v_1) \in \mathrm{Form}_{\mathscr{L}}[\boldsymbol{\tau}]$ defining Sat explicitly relative to $\hat{\sigma}$. Let n be the Gödel number of $\neg\psi(\text{``}\{(0, v_0)\}\text{''}, v_0)$ and assume $\mathfrak{A}$ to be a model of $\varphi \wedge \sigma$. Then, by (**), we obtain

$$\mathfrak{A} \models \psi(\text{``}\{(0, \underline{n})\}\text{''}, \underline{n}) \text{ iff } \mathfrak{A} \models \neg\psi(\text{``}\{(0, \underline{n})\}\text{''}, \underline{n}).$$

This is a contradiction. ☐

As the interpolation property yields the definability property, counterexamples to the latter are, in effect, counterexamples to the former. Positive results concerning the other direction are described in Chapter XVIII.4.

We conclude with a link between interpolation and definability which goes back to Feferman [1974a]. For this purpose, we strengthen the weak Beth property in a new direction.

7.3.2 Definition. The logic $\mathscr{L}$ has the *projective weak Beth property* iff for all $\boldsymbol{\tau}, \boldsymbol{\tau}'$ with $\boldsymbol{\tau} \subseteq \boldsymbol{\tau}'$ and for all $\S \in \boldsymbol{\tau}$ and $\varphi \in \mathscr{L}[\boldsymbol{\tau}']$, if $\exists_{\boldsymbol{\tau}'\setminus\boldsymbol{\tau}}\varphi$ defines $\S$ strongly implicitly, then $\S$ is explicitly definable relative to $\exists_{\boldsymbol{\tau}'\setminus\boldsymbol{\tau}}\varphi$.

7.3.3 Proposition. *$\mathscr{L}$ allows Δ-interpolation iff $\mathscr{L}$ has the projective weak Beth property.*

Proof. Assume first that $\mathscr{L}$ has the projective weak Beth property, and let $\mathfrak{K}_i = \mathrm{Mod}(\exists_{\boldsymbol{\tau}_i\setminus\boldsymbol{\tau}}\varphi_i)$ $(i = 0, 1)$ be two disjoint complementary classes of $\boldsymbol{\tau}$-structures. With new unary P (in the many-sorted case this P will be equipped with some sort symbol $s \in \boldsymbol{\tau}$) we set

$$\chi = (\exists_{\boldsymbol{\tau}_0\setminus\boldsymbol{\tau}}\varphi_0 \wedge \forall x P x) \vee (\exists_{\boldsymbol{\tau}_1\setminus\boldsymbol{\tau}}\varphi_1 \wedge \forall x \neg P x).$$

Obviously, χ strongly implicitly defines P and the projective weak Beth property applies. Let ψ be an explicit definition of P relative to χ. Then

$$\{(\mathfrak{A}, a) \mid \mathfrak{A} \in \mathrm{Str}[\boldsymbol{\tau}], \mathfrak{A} \models \exists_{\boldsymbol{\tau}_0\setminus\boldsymbol{\tau}}\varphi_0, a \in A_{(s)}\} = \mathrm{Mod}(\psi)$$

is EC in $\mathscr{L}$ and hence, by particularization, so is $\mathfrak{K}_0$.

For the other direction, assume $\mathscr{L}$ to allow Δ-interpolation and $\exists_{\tau'\setminus\tau}\varphi$ to strongly implicitly define $\S \in \tau$, say $\S = P$, a unary relation symbol. Then

$$\mathfrak{K} = \{(\mathfrak{A} \upharpoonright (\tau\setminus\{P\}), a) \mid \mathfrak{A} \models \exists_{\tau'\setminus\tau}\varphi, a \in P^{\mathfrak{A}}\}$$

is (R)PC in $\mathscr{L}$. The complementary class $\overline{\mathfrak{K}}$ can be written as

$$\overline{\mathfrak{K}} = \{(\mathfrak{A} \upharpoonright (\tau\setminus\{P\}), a) \mid \mathfrak{A} \models \exists_{\tau'\setminus\tau}\varphi, a \notin P^{\mathfrak{A}}\}$$

and hence is (R)PC in $\mathscr{L}$, too. Therefore, $\mathfrak{K}$ is EC in $\mathscr{L}$. Now take ψ such that $\mathfrak{K} = \mathrm{Mod}(\psi)$. Then ψ is an explicit definition of P relative to $\exists_{\tau'\setminus\tau}\varphi$. $\square$

Following the pattern of Definition 7.3.2, the reader may define the so-called *projective* Beth property. For instance, suppose a sentence $\varphi(\bar{R}, \bar{S}, P)$ defines P implicitly relative to $\bar{S}$ in the sense that, with new symbols $\bar{R}'$, P',

$$\varphi(\bar{R}, \bar{S}, P) \wedge \varphi(\bar{R}', \bar{S}, P') \models \forall \bar{x}\,(P\bar{x} \leftrightarrow P'\bar{x}).$$

Then the projective Beth property implies that $\varphi(\bar{R}, \bar{S}, P)$ admits an explicit definition of P relative to $\bar{S}$, that is, a formula $\psi(\bar{S}, \bar{x})$ such that

$$\varphi(\bar{R}, \bar{S}, P) \models \forall\bar{x}(P\bar{x} \leftrightarrow \psi(\bar{S}, \bar{x})).$$

The usual proofs of Beth's definability theorem—including the original proof in Beth [1953]—extend immediately to the projective Beth property. Even more: A slight modification of the preceding argument shows that the interpolation property and the projective Beth property are equivalent for all regular logics (Rowlands-Hughes [1979]). Thus we get an alternative answer to the question about the relationship between interpolation and definability as posed in the remarks following Definition 1.2.5.

Chapter III
Characterizing Logics

by J. FLUM

The model theory of first-order logic is well developed. It provides general results and methods which enable us to study and classify the models of systems of first-order axioms. Among these general results of wide applicability are the completeness theorem, the compactness theorem, and the Löwenheim–Skolem theorem. Thus, for example, the completeness theorem leads to decidability results; in many cases we obtain for a given system of axioms models with special properties using a compactness argument; finally the Löwenheim–Skolem theorem tells us that we can restrict to countable structures when classifying—with respect to its first-order properties—models of a system of axioms.

Much effort was spent in finding languages which strengthen the first-order language and which are

(i) sufficiently strong to allow the formulation of interesting systems of axioms and properties of structures which are not expressible in first-order logic, and
(ii) still simple enough to yield general principles and results which are useful in investigating and classifying models.

Taking into account the situation for first-order logic, it is not surprising that many logicians attempted to find logics satisfying the analogues of the completeness, the compactness, and the Löwenheim–Skolem theorems. That this search could not be successful was shown by the following two results, both of which are due to Lindström [1969]:

(1) First-order logic is a maximal logic with respect to expressive power satisfying the compactness theorem and the Löwenheim–Skolem theorem.
(2) First-order logic is a maximal logic satisfying the completeness theorem and the Löwenheim–Skolem theorem.

Let us point out some consequences of these results.

(a) They tell us that first-order logic is a natural logic, if one accepts the completeness (or the compactness) and the Löwenheim–Skolem property as natural properties. I suspect that most mathematicians do not accept the Löwenheim–Skolem property as natural. Quite the contrary, as Wang [1974, p. 154] remarked: "When we are interested in set theory or classical analysis, the Löwenheim theorem is usually taken as a sort of defect (often thought to be inevitable) of the first-order logic. Therefore, what is established (by Lindström's theorems) is not that first-order logic is the only possible logic but rather that it is the only possible logic when we in a sense deny reality to the concept of uncountability . . . ".

(b) Lindström's results show that it makes no sense to classify logics as either good or bad, depending on whether they are complete (compact) and have the Löwenheim–Skolem property or not. On the contrary, Lindström's result gave special emphasis to the proposal—already expressed by Kreisel in 1963—that there must be a balance between the syntax and the semantics of a logic and that the semantic properties we consider must be adapted to the expressive power and the special features of the given logic.

(c) Lindström's results were the starting point

(i) for investigations which were trying to find some order in the diversity of extensions of first-order logic, and

(ii) for a systematic study of the relationship between model-theoretic properties of logics.

In particular, these investigations have led to characterizations of other logics by means of suitable model-theoretic properties.

(d) Robinson [1973] specified the following task "... to develop topological model theory. What I have in mind is a theory which is related to algebraic-topological structures, such as topological groups and fields, as ordinary model theory is related to algebraic structures." There were some approaches to this problem which led to different logics for topological structures. However, when Ziegler [1976] proved that a certain logic $\mathscr{L}_t$ is a maximal logic—in the sense of Lindström's results—for topological structures, there was strong confidence in the fact that $\mathscr{L}_t$ is the logic for topological structures corresponding to first-order logic; and, in particular, that $\mathscr{L}_t$ should prove helpful for the investigation and classification of topological structures. It turned out that this is actually the case.

Section 1 of the present chapter is mainly devoted to a proof of Lindström's theorems. Section 2 contains some further characterizations of first-order logic by means of model-theoretic properties. In Section 3 we show that $\mathscr{L}_{\omega\omega}$ is a maximal logic satisfying properties which can be viewed as model-theoretic generalizations or substitutes for compactness and the Löwenheim–Skolem property. In Section 4 we prove that among the logics of the form $\mathscr{L}_{\omega\omega}(Q)$ with a unary monotone quantifier Q the logics $\mathscr{L}_{\omega\omega}(Q_\alpha)$, where Q_α is the quantifier "there are at least $\aleph_\alpha$-many" are the only ones with the relativization property. Finally in Section 5 an "abstract maximality theorem" is established. This result not only covers Lindström's result but it also tells us how to obtain maximal logics for other kinds of structures, such as topological structures, for example.

1. Lindström's Characterizations of First-Order Logic

We first present a proof of Lindström's first theorem ("compactness + Löwenheim–Skolem property characterize $\mathscr{L}_{\omega\omega}$"), which does not presuppose knowledge of any special model-theoretic results. We then try to minimize the assumptions and prove a lemma which, on the one hand, isolates the main step in the derivation of Lindström's first and second theorems ("recursive enumerability for validity + Löwenheim–Skolem property characterize $\mathscr{L}_{\omega\omega}$") and, on the other, makes visible the relationship between maximality and a separation property. Later, when we are characterizing $\mathscr{L}_{\infty\omega}$ and some other logics as maximal logics, we will see that a proof of the maximality along the same lines leads to a separation theorem. In this way we will obtain in a unified form some results which now appear to be scattered throughout the literature. In the second part of this section, we list some examples which show that it is not possible to strengthen Lindström's theorems in some more or less plausible ways. We will close this section by giving a characterization of the monadic part of first-order logic (and of some monadic extensions of first-order logic).

Throughout this chapter, given any vocabulary τ we denote by τ' a disjoint copy of τ. For f, R, c in τ let f', R', c' be the corresponding symbols in τ'. If $\mathscr{L}$ is a logic and ψ an $\mathscr{L}[\tau]$-sentence, then ψ' will be the $\mathscr{L}[\tau']$-sentence associated with ψ by the renaming property. Finally, for a τ-structure $\mathfrak{A}$ let $\mathfrak{A}'$ be the corresponding τ'-structure. If $\mathfrak{B} = \mathfrak{A}'$, we set $\mathfrak{B}^{-\prime} = \mathfrak{A}$.

For definiteness let us assume that all logics are one-sorted. In this section, if not explicitly stated otherwise, all logics are assumed to be closed under (finitary) boolean operations (that is, they are assumed to have the Boole property).

1.1. Lindström's First and Second Theorems

We begin by proving a version of Lindström's first theorem.

1.1.1 Theorem. *Let $\mathscr{L}$ be a logic, $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$, with the compactness property and the Löwenheim–Skolem property for countable sets of sentences. Then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*

Proof. The proof proceeds in three steps. First, we will show that each $\mathscr{L}$-sentence depends on finitely many symbols. Then, in case $\psi \in \mathscr{L}[\tau]$ is not equivalent to a first-order sentence, we will get elementarily equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$(+) \qquad \mathfrak{A} \models \psi \quad \text{and} \quad \mathfrak{B} \models \neg\psi.$$

Finally, we will see that it is even possible to obtain isomorphic $\mathfrak{A}$ and $\mathfrak{B}$ with $(+)$—a contradiction.

Let us start with the first step (see Proposition II.5.1.2).

(1) Given $\psi \in \mathscr{L}[\tau]$, there is a finite $\tau_0 \subset \tau$ such that for any τ-structures $\mathfrak{A}$ and $\mathfrak{B}$

$$\mathfrak{A} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0 \quad \text{implies} \quad (\mathfrak{A} \models \psi \text{ iff } \mathfrak{B} \models \psi).$$

To prove (1) let $\Phi \subset \mathscr{L}[\tau \cup \tau']$ be the set

$$\begin{aligned}\Phi = &\{\forall x_1 \dots \forall x_n(Rx_1 \dots x_n \leftrightarrow R'x_1 \dots x_n) \mid n \geq 1, R \in \tau \text{ } n\text{-ary}\} \\ &\cup \{\forall x_1 \dots \forall x_n f(x_1, \dots, x_n) = f'(x_1, \dots, x_n) \mid n \geq 1, f \in \tau \text{ } n\text{-ary}\} \\ &\cup \{c = c' \mid c \in \tau\}.\end{aligned}$$

Clearly $\Phi \models \psi \leftrightarrow \psi'$. Hence, by $\mathscr{L}$-compactness, there is a finite Φ_0 such that $\Phi_0 \models \psi \leftrightarrow \psi'$. But then any finite τ_1 such that $\Phi_0 \subset \mathscr{L}[\tau_1]$ leads to a finite τ_0 satisfying (1).

We now assume that the conclusion of the theorem fails; and, hence we suppose that some $\psi \in \mathscr{L}[\tau]$ is not equivalent to a first-order sentence. Choose a finite $\tau_0 \subset \tau$ according to (1). We now prove:

(2) There are τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ with

$$A = B, \qquad \mathfrak{A} \models \psi, \qquad \mathfrak{B} \models \neg\psi \quad \text{and} \quad \mathfrak{A} \upharpoonright \tau_0 \equiv \mathfrak{B} \upharpoonright \tau_0.$$

To establish (2), let $\varphi_1, \varphi_2, \dots$ be a complete list of the $\mathscr{L}_{\omega\omega}[\tau_0]$-sentences. By induction, we obtain a sequence $\psi_1, \psi_2, \dots$ such that for each n, $\psi_n = \varphi_n$ or $\psi_n = \neg\varphi_n$, and $\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ is not equivalent to a first-order sentence. Then also $\neg\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ is not equivalent to a first-order sentence. Hence, both $\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ and $\neg\psi \wedge \psi_1 \wedge \cdots \wedge \psi_n$ are satisfiable. Let $\Psi = \{\psi_n \mid n \geq 1\}$. By $\mathscr{L}$-compactness and by the assumed Löwenheim–Skolem property for $\mathscr{L}$ there are countable structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \Psi \cup \{\psi\}$ and $\mathfrak{B} \models \Psi \cup \{\neg\psi\}$. But then $\mathfrak{A} \upharpoonright \tau_0 \equiv \mathfrak{B} \upharpoonright \tau_0$ and by (1), $\mathfrak{A} \upharpoonright \tau_0 \not\cong \mathfrak{B} \upharpoonright \tau_0$. Therefore, A and B are countable and infinite. Hence, without loss of generality, $A = B$.

In the last step we obtain the desired contradiction passing in (2) to structures having isomorphic—instead of elementarily equivalent—τ_0-reducts. For this purpose, choose a disjoint copy τ' of τ and new $(2n + 1)$-ary function symbols f_n and g_n. Set $\tau^* = \tau \cup \tau' \cup \{f_n, g_n \mid n \in \omega\}$.

For each n, fix an enumeration $\langle \chi_i(x_1, \dots, x_n, x) \mid i \in \omega \rangle$ of all $\mathscr{L}_{\omega\omega}[\tau_0]$-formulas with free variables among $x_1, \dots, x_n, x$. Let Γ consist of the $\mathscr{L}[\tau^*]$-sentences

$\psi, \neg\psi'$

("the τ-reduct is a model of ψ, the τ'-reduct a model of $\neg\psi'$"),

$\varphi \leftrightarrow \varphi'$ for each $\mathscr{L}_{\omega\omega}[\tau_0]$-sentence φ

("the τ_0-reduct and the τ_0'-reduct are elementarily equivalent"),

and of the following sentences which enable us to construct in a countable model, step by step, an isomorphism of the τ_0-reduct onto the τ_0'-reduct (let $\bar{x} = x_1 \dots x_n$ and $\bar{y} = y_1 \dots y_n$)

$$(*)\qquad \begin{gathered}\forall\bar{x}\,\forall\bar{y}\,\forall x\Bigl(\exists y\Bigl(\bigwedge_{i=0}^{r}(\chi_i(\bar{x},x)\leftrightarrow\chi_i'(\bar{y},y))\Bigr)\\ \rightarrow\bigwedge_{i=0}^{r}(\chi_i(\bar{x},x)\leftrightarrow\chi_i'(\bar{y},f_n(\bar{x},\bar{y},x)))\Bigr),\\ \forall\bar{x}\,\forall\bar{y}\,\forall y\Bigl(\exists x\Bigl(\bigwedge_{i=0}^{r}(\chi_i(\bar{x},x)\leftrightarrow\chi_i'(\bar{y},y))\Bigr)\\ \rightarrow\bigwedge_{i=0}^{r}(\chi_i(\bar{x},g_n(\bar{x},\bar{y},y))\leftrightarrow\chi_i'(\bar{y},y))\Bigr),\qquad n,r\in\omega.\end{gathered}$$

Note that given a finite set Ψ_0 of sentences in $(*)$ we can expand an arbitrary $\tau_0\cup\tau_0'$-structure to a model of Ψ_0. Hence, by (2) each finite subset of Γ is satisfiable. Using the compactness and the Löwenheim–Skolem property of $\mathscr{L}$, we obtain a countable model $\mathfrak{D}$ of Γ. Let $\mathfrak{A}=\mathfrak{D}\upharpoonright\tau$ and $\mathfrak{B}=(\mathfrak{D}\upharpoonright\tau')^{-\prime}$ (where $'\colon\tau\to\tau'$ is the given renaming). Clearly, $A=B=D$, $\mathfrak{A}\models\psi$, $\mathfrak{B}\models\neg\psi$ and $\mathfrak{A}\upharpoonright\tau_0\equiv\mathfrak{B}\upharpoonright\tau_0$. We will show that $\mathfrak{A}\upharpoonright\tau_0\cong\mathfrak{B}\upharpoonright\tau_0$, which contradicts (1).

Let $d_1, d_2,\ldots$ be an enumeration of D. Since $\mathfrak{A}\upharpoonright\tau_0\equiv\mathfrak{B}\upharpoonright\tau_0$, then we have by $(*)$

$$\begin{gathered}(\mathfrak{A}\upharpoonright\tau_0,d_1)\equiv(\mathfrak{B}\upharpoonright\tau_0,f_0(d_1)),\\ (\mathfrak{A}\upharpoonright\tau_0,d_1,g_1(d_1,f_0(d_1),d_1))\equiv(\mathfrak{B}\upharpoonright\tau_0,f_0(d_1),d_1),\\ (\mathfrak{A}\upharpoonright\tau_0,d_1,g_1(d_1,f_0(d_1),d_1),d_2)\equiv(\mathfrak{B}\upharpoonright\tau_0,f_0(d_1),d_1,f_1(\ldots))\ldots\end{gathered}$$

Continuing in this way (see the proof of Theorem II.4.3.1), one obtains sequences $a_1, a_2,\ldots$ and $b_1, b_2,\ldots$ such that $A=\{a_n\mid n\in\omega\}$, $B=\{b_n\mid n\in\omega\}$ and

$$(\mathfrak{A}\upharpoonright\tau_0,a_1,a_2,\ldots)\equiv(\mathfrak{B}\upharpoonright\tau_0,b_1,b_2,\ldots).$$

But then $\pi\colon\mathfrak{A}\upharpoonright\tau_0\cong\mathfrak{B}\upharpoonright\tau_0$ for π defined by $\pi(a_n)=b_n$ for $n\in\omega$. $\square$

The following lemma contains the main step in Lindström's derivation of his theorems. We state it in the form of a "separation theorem". In this way, we will be able to obtain some further applications.

Recall that a logic $\mathscr{L}$ is said to have the finite occurrence property, if for arbitrary τ we have $\mathscr{L}[\tau]=\bigcup\{\mathscr{L}[\tau_0]\mid\tau_0\subset\tau,\ \tau_0\text{ finite}\}$.

1.1.2 Lemma. *Let $\mathscr{L}$ be a logic with the finite occurrence property. Assume $\mathscr{L}_{\omega\omega}\le\mathscr{L}$ and that $\mathscr{L}$ is closed under conjunctions and disjunctions but not necessarily under negations. Let $\mathscr{L}$ have the Löwenheim–Skolem property and suppose that there are disjoint $\mathscr{L}$-classes which cannot be separated by an elementary class, i.e. for* *some* τ_0 *there are* $\mathscr{L}[\tau_0]$*-sentences* φ *and* ψ *with* $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \varnothing$ *such that there is no* $\chi \in \mathscr{L}_{\omega\omega}[\tau_0]$ *with*

$$\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\chi) \quad \textit{and} \quad \mathrm{Mod}(\chi) \cap \mathrm{Mod}(\psi) = \varnothing.$$

Then there is for some vocabulary $\boldsymbol{\sigma}$ *containing (at least) a unary relation symbol* U *an* $\mathscr{L}[\boldsymbol{\sigma}]$*-sentence* ϑ *such that* (i) *and* (ii) *hold:*

(i) *if* $\mathfrak{A} \models \vartheta$ *then* U^A *is finite and non-empty,*
(ii) *for each* $n \geq 1$ *there is an* $\mathfrak{A} \models \vartheta$ *with* $|U^A| = n$.

Before proving this lemma, let us state some of its consequences.

1.1.3 Theorem. *Let* $\mathscr{L}$ *be a logic with the finite occurrence property,* $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$, *and assume that* $\mathscr{L}$ *is closed under conjunctions and disjunctions but not neccessarily under negations. If* $\mathscr{L}$ *has the Löwenheim–Skolem property and is countably compact, then any disjoint* $\mathscr{L}$*-classes can be separated by an elementary class.*

Proof. Otherwise, there exists an $\mathscr{L}$-sentence ϑ satisfying (i) and (ii) of Lemma 1.1.2. But then

(+) $\quad \{\vartheta\} \cup \{$"there are more than n elements x with Ux"$\mid n \geq 1\}$

is a finitely satisfiable set which has no model. ☐

Recall that in this section we assume that all logics have the Boole property, if not explicitly stated otherwise.

1.1.4 Lindström's First Theorem. *Let* $\mathscr{L}$, $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$, *be a logic with the finite occurrence property. If* $\mathscr{L}$ *has the Löwenheim–Skolem property and is countably compact, then* $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.

Proof. Given any $\mathscr{L}$-sentence φ the model classes of φ and $\neg\varphi$ are disjoint, hence, by the preceding theorem, there is a first-order sentence χ separating $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\neg\varphi)$. But then χ is equivalent to φ. ☐

1.1.5 Lindström's Second Theorem. *Assume that* $\mathscr{L}$ *is an effectively regular logic (see Chapter II for definitions). If* $\mathscr{L}$ *has the Löwenheim–Skolem property and is recursively enumerable for validity then* $\mathscr{L}_{\omega\omega}$ *effectively contains* $\mathscr{L}$.

Proof. For the sake of contradiction, suppose that some $\mathscr{L}$-sentence φ is not equivalent to a first-order sentence. Since the model classes of φ and $\neg\varphi$ cannot be separated by an elementary class there is an $\mathscr{L}$-sentence ϑ, $\vartheta \in \mathscr{L}[\boldsymbol{\sigma}]$, with properties (i) and (ii) of Lemma 1.1.2. By a theorem of Trahtenbrot [1950], for some finite $\boldsymbol{\tau}$—we can assume $\boldsymbol{\tau} \cap \boldsymbol{\sigma} = \varnothing$—the set Φ of $\mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$-sentences true in all finite models is not recursively enumerable. On the other hand, we have for $\varphi \in \mathscr{L}_{\omega\omega}[\tau]$

$$(*) \qquad \varphi \in \Phi \quad \text{iff} \quad \models \vartheta \to \varphi^U$$

where φ^U denotes the relativization of φ to U (see Definition II.1.2.2). $\mathscr{L}$ is effectively regular and recursively enumerable for validity, hence by (*), the set Φ is recursively enumerable—a contradiction. To show that $\mathscr{L}_{\omega\omega}$ *effectively* contains $\mathscr{L}$, given an $\mathscr{L}$-sentence φ, we enumerate the validities of $\mathscr{L}$ until we arrive at a formula which expresses the equivalence of φ to a first-order sentence. □

1.1.6 Remarks. (a) If $\mathscr{L}$ effectively contains $\mathscr{L}_{\omega\omega}$ then the set (+) of $\mathscr{L}$-sentences in the proof of Theorem 1.1.3 is recursive. Therefore, in this case, the assumption "$\mathscr{L}$ is countably compact" which was made in Theorems 1.1.3 and 1.1.4 can be replaced by "$\mathscr{L}$ is compact for recursive sets".

(b) In Theorems 1.1.3 and 1.1.4 we can drop the assumption "$\mathscr{L}$ has the finite occurrence property", if we assume that $\mathscr{L}$ is compact and not merely countably compact. In fact, suppose that φ and ψ are $\mathscr{L}[\tau]$ classes with disjoint model classes, then we can obtain a finite $\tau_0 \subset \tau$ such that

$$\mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{A} \upharpoonright \tau_0 \cong \mathfrak{B} \upharpoonright \tau_0 \quad \text{imply non } \mathfrak{B} \models \psi.$$

(Here we apply $\mathscr{L}$-compactness to the unsatisfiable set $\Phi \cup \{\varphi, \psi'\}$ where Φ is the set introduced in the first step of the proof of Theorem 1.1.1). Using this finite τ_0, one obtains—as in the following proof of Lemma 1.1.2—a formula ϑ with (i) and (ii).

Proof of Lemma 1.1.2. Let $\mathscr{L}$, τ_0, φ and ψ be given as in Lemma 1.1.2. Suppose by contradiction that there is no $\mathscr{L}$-sentence ϑ with the properties (i) and (ii). Then we can show:

(1) If χ is an $\mathscr{L}$-sentence not equivalent to a first-order sentence, then χ has a model of power $\aleph_0$.

In fact, given such a χ choose a finite τ such that χ is an $\mathscr{L}[\tau]$-sentence. If χ has an infinite model, then for a new unary function symbol f the $\mathscr{L}$-sentence

$$\chi \wedge \text{“}f \text{ is one-to-one but not onto”}$$

is satisfiable and by the Löwenheim–Skolem property has a countable model, which must be of power $\aleph_0$. Now suppose χ has only finite models. Since τ is finite, for each $n \in \omega$, there are only finitely many (non-isomorphic) τ-structures of size $\leq n$, and each one can be characterized by a first-order sentence. Therefore, for each n, χ must have a model with at least n elements (otherwise it would be equivalent to a first-order sentence). But in this case, $\vartheta := \chi \wedge \exists x\, Ux$ for a new unary relation symbol U is a sentence satisfying (i) and (ii). This completes the proof of (1).

By assumption, the $\mathscr{L}[\tau_0]$-sentences φ and ψ have disjoint model classes which cannot be separated by an elementary class. We can assume that τ_0 is finite. Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of the set of $\mathscr{L}_{\omega\omega}[\tau_0]$-sentences. Then:

(2) For each $n \geq 1$ there are τ_0-structures $\mathfrak{A}$ and $\mathfrak{B}$ of power $\aleph_0$ such that
$$A = B, \qquad \mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \psi \quad \text{and for } i \leq n \quad (\mathfrak{A} \models \varphi_i \text{ iff } \mathfrak{B} \models \varphi_i).$$

To establish (2), by induction choose $\psi_1, \psi_2, \ldots$ such that for each n, $\psi_n = \varphi_n$ or $\psi_n = \neg\varphi_n$ and such that the model classes of $\varphi^n := \varphi \wedge \psi_1 \wedge \ldots \wedge \psi_n$ and $\psi^n := \psi \wedge \psi_1 \wedge \ldots \wedge \psi_n$ cannot be separated by an elementary class. In particular, neither φ^n nor ψ^n is equivalent to a first-order sentence. Thus, we obtain the desired models $\mathfrak{A}$ and $\mathfrak{B}$ applying (1).

Using the notions of partial isomorphism, k-partial isomorphic, ... and the corresponding results (see Section II.4.2) we may rewrite (2) in the following form:

(2′) For each $k \in \omega$, there are τ_0-structures $\mathfrak{A}$ and $\mathfrak{B}$ of power $\aleph_0$ such that
$$A = B, \qquad \mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \psi \quad \text{and} \quad \mathfrak{A} \cong_k \mathfrak{B}.$$

In the last step, we pass in (2′) to isomorphic structures $\mathfrak{A}$ and $\mathfrak{B}$. To achieve the corresponding result in the proof of Theorem 1.1.1, we applied the Löwenheim–Skolem property to a set Γ consisting of two $\mathscr{L}$-sentences and a recursive set of $\mathscr{L}_{\omega\omega}[\tau^*]$-sentences in a vocabulary τ^* including $\tau_0 \cup \tau_0'$. By a theorem of Craig and Vaught, there is a finite set of $\mathscr{L}_{\omega\omega}$-sentences having the same $\tau_0 \cup \tau_0'$-reducts. Therefore one really needs the Löwenheim–Skolem property only for single sentences. Nevertheless, we show here explicitly how to obtain isomorphic structures in (2′), since, in this way, we can become acquainted with a proof technique which is frequently used in soft model theory in general and in this chapter in particular.

For $k \in \omega$, take $\mathfrak{A}$ and $\mathfrak{B}$ as given by (2′) and choose $(I_m)_{m \leq k}$ such that $(I_m)_{m \leq k}$: $\mathfrak{A} \cong_k \mathfrak{B}$. By the results of Section II.4.2 we can assume that $\bigcup_{m \leq k} I_m$ is countable. Moreover, suppose without loss of generality that $\{0, \ldots, k\} \subset A$. Choose a one-to-one mapping from $\bigcup_{m \leq k} I_m$ into A. In the sequel, we shall identify $p \in \bigcup_{m \leq k} I_m$ with its value under this mapping. Take new relation symbols U, P (unary), $<$, I (binary) and G (ternary) and let $\sigma = \tau_0 \cup \tau_0' \cup \{P, <, I, G\}$, where τ_0' is a disjoint copy of τ_0. Let $\mathfrak{C}$ $(= \mathfrak{C}_k)$ be the σ-structure with domain A given by

$\mathfrak{C} \upharpoonright \tau_0 = \mathfrak{A}, \qquad \mathfrak{C} \upharpoonright \tau_0' = \mathfrak{B}'$ ($\mathfrak{B}'$ denotes the τ_0'-structure corresponding to $\mathfrak{B}$),

$U^C = \{0, \ldots, k\}$,

$<^C$ is the natural ordering on $\{0, \ldots, k\}$,

$P^C p \quad$ iff $\quad p \in \bigcup_{m \leq k} I_m$,

$I^C mp \quad$ iff $\quad m \leq k$ and $p \in I_m$,

$G^C pab \quad$ iff $\quad p \in \bigcup_{m \leq k} I_m$ and $p(a) = b$.

Then $\mathfrak{C}$ is a model of the conjunction ϑ of the following $\mathscr{L}[\sigma]$-sentences

φ, ψ',

" $<$ is a discrete ordering with first and last element",

" U is the field of $<$ ",

"Each p on P is a (partial) injective mapping"

that is, $\forall p(Pp \rightarrow \forall x\, \forall y\, \forall u\, \forall v(Gpxu \wedge Gpyv \rightarrow (x = y \leftrightarrow u = v)))$,

"Each p in P preserves all symbols in τ_0"

for example, for a binary R in τ_0

$\forall p(Pp \rightarrow \forall x\, \forall y\, \forall u\, \forall v(Gpxu \wedge Gpyv \rightarrow (Rxy \leftrightarrow R'uv)))$,

"For each u in U the set I_u is non-empty"

that is, $\forall u(Uu \rightarrow \exists p(Pp \wedge Iup))$,

"The sequence of I_u's has the forth property"

that is, $\forall u\, \forall v(v < u \rightarrow \forall p(Iup \rightarrow \forall x\, \exists q\, \exists y(Ivq \wedge Gqxy \wedge \forall z\, \forall w(Gpzw \rightarrow Gqzw))))$,

"The sequence of I_u's has the back property".

Clearly, by (2′) the sentence ϑ has property (ii). In fact, $\mathfrak{C}_k$ is a model with $|U^{\mathfrak{C}_k}| = k + 1$. We show that ϑ also satisfies (i). Otherwise, ϑ has a model with infinite U-part. Let f be a new unary function symbol. Then,

$$\vartheta \wedge \text{“}f \text{ maps the } U\text{-part one-to-one onto a proper subset”}$$

has a model, and hence a countable model $\mathfrak{D}$ by the Löwenheim–Skolem property. $<^{\mathfrak{D}}$ being a discrete ordering with last element of the infinite set $U^{\mathfrak{D}}$ contains an infinite descending sequence

$$\cdots <^{\mathfrak{D}} d_2 <^{\mathfrak{D}} d_1 <^{\mathfrak{D}} d_0.$$

Let $\mathfrak{A}_0 = \mathfrak{D} \upharpoonright \tau_0$, $\mathfrak{B}_0 = (\mathfrak{D} \upharpoonright \tau_0')^{-\prime}$ and $J = \{p \mid I^{\mathfrak{D}} d_n p$ for some $n \in \omega\}$. Since $\mathfrak{D} \models \vartheta$, we can identify $p \in P^{\mathfrak{D}}$ with the partial isomorphism $\{(a, b) \mid G^{\mathfrak{D}}pab\}$ from $\mathfrak{A}_0$ to $\mathfrak{B}_0$. Moreover, by $\mathfrak{D} \models \vartheta$, we have $\mathfrak{A}_0 \models \varphi$, $\mathfrak{B}_0 \models \psi$ and $J\colon \mathfrak{A}_0 \cong_p \mathfrak{B}_0$; that is, $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are partially isomorphic via J (that J has the back and forth property can be easily seen by using the fact that the d_n's form an infinite descending sequence). But D is countable and countable partially isomorphic structures are isomorphic (see Theorem II.4.3.1). Hence, $\mathfrak{A}_0 \cong \mathfrak{B}_0$. In particular, $\mathfrak{A}_0 \models \{\varphi, \psi\}$ and therefore $\mathfrak{A} \in \mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi)$—a contradiction. $\square$

1.1.7 Examples. (a) Take as $\mathscr{L}$ in Theorem 1.1.3 the set of Σ_1^1-sentences over $\mathscr{L}_{\omega\omega}$ (that is, sentences of the form $\exists R_1 \ldots \exists R_n \varphi$, where φ is first-order). Then

Theorem 1.1.3 yields the $\mathscr{L}_{\omega\omega}$-interpolation theorem: Any two Σ_1^1-sentences with disjoint model classes can be separated by an elementary class.

(b) For $n \geq 1$ let Q_n be a quantifier binding n-ary relation variables. Fix the interpretation of Q_n by the clause

$$\mathfrak{A} \models Q_n R\varphi \quad \text{iff} \quad |\{R^A | (\mathfrak{A}, R^A) \models \varphi\}| \geq 2^{|A|}.$$

Q_n is a kind of "second-order Lindström quantifier". Call an $\mathscr{L}_{\omega\omega}(Q_n | n \geq 1)$-formula *positive*, if it is a member of the smallest set containing the first-order formulas and closed under $\wedge$, $\vee$, $\exists x$, $\forall x$ and $Q_n R$. Let $\mathscr{L}$ consist of the positive $\mathscr{L}_{\omega\omega}(Q_n | n \geq 1)$-sentences. Using the local Chang–Makkai theorem for recursively saturated structures (see Schlipf [1978]), one can show that $\mathscr{L}$ has the Löwenheim–Skolem property and is countably compact (it is even compact!). Hence, by Theorem 1.1.3, it follows that any two disjoint $\mathscr{L}$-classes can be separated by an elementary class.

The following proposition is related to a result obtained in the course of the proof of Theorem 1.1.1.

1.1.8 Proposition. *Let $\mathscr{L} \leq \mathscr{L}'$ where $\mathscr{L}$ and $\mathscr{L}'$ are logics and where $\mathscr{L}[\tau]$ is a set for any vocabulary τ. If $\mathscr{L}'$ is compact and*

$$(*) \qquad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B} \quad \textit{implies} \quad \mathfrak{A} \equiv_{\mathscr{L}'} \mathfrak{B}$$

then $\mathscr{L} \equiv \mathscr{L}'$.

Proof. For an arbitrary satisfiable $\mathscr{L}'[\tau]$-sentence φ we have by $(*)$

$$\models \varphi \leftrightarrow \bigvee_{\mathfrak{A} \models \varphi} \bigwedge_{\substack{\psi \in \mathscr{L}[\tau] \\ \mathfrak{A} \models \psi}} \psi.$$

Now standard compactness arguments will show that the disjunction and the conjunctions on the right-hand side can be replaced by finite ones. $\square$

For further reference we state:

1.1.9 Remark. The preceding proof shows that if $\mathscr{L}'$ has the finite occurrence property and $\mathscr{L}[\tau]$ is countable for finite τ, then it suffices to assume that $\mathscr{L}'$ is countably compact. Moreover, if $\mathscr{L}'$ has the Löwenheim–Skolem property down to κ for countable sets of sentences, then $(*)$ must only be required for structures of cardinality $\leq \kappa$.

1.2. Some Counterexamples

In this part we list some more or less strange examples which will show that certain strengthenings of the theorems of Lindström are not possible. A further example is contained in Section 2.3. Already in Chapter II it has been shown that $\mathscr{L}_{\omega\omega}(Q^{\omega})$, the logic with the cofinality ω quantifier, is compact, recursively enumerable for validity, and has the Löwenheim–Skolem property down to $\aleph_1$. The reader should consult Shelah [1975d] where further examples of compact extensions of first-order logics are given. In particular, there a logic $\mathscr{L}_{\omega\omega}(Q)$ is introduced, which is regular, compact, and more expressive than first-order logic even for countable structures: Let λ be the first weakly compact cardinal. The binary quantifier Q is then defined as follows:

$$\mathfrak{A} \models Qxy\varphi(x, y) \quad \text{iff} \quad \begin{array}{l}\varphi^{\mathfrak{A}} := \{(a, b) \mid \mathfrak{A} \models \varphi[a, b]\} \text{ is an ordering and} \\ \text{there is a Dedekind cut } (A_1, A_2) \text{ of } \varphi^{\mathfrak{A}} \text{ with} \\ \text{cofinalities in } \{\aleph_0, \lambda\}.\end{array}$$

A Dedekind cut of an ordering $(B, <)$ is a pair (B_1, B_2) such that $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = B$ and $b_1 < b_2$ for $b_1 \in B_1$, $b_2 \in B_2$. The cofinalities of a Dedekind cut are the cofinalities of $(B_1, <)$ and of $(B_2, >)$.

Note that $(\mathbb{Z}, <)$ and $(\mathbb{Z}, <) + (\mathbb{Z}, <)$ are not $\mathscr{L}_{\omega\omega}(Q)$-equivalent. $\mathscr{L}_{\omega\omega}(Q)$ does not have the Löwenheim–Skolem property.

1.2.1 Example. Let $\mathscr{L}$ be the logic obtained from $\mathscr{L}_{\omega\omega}$ by adding a new "atomic" sentence χ_s. Let χ_s be in $\mathscr{L}[\tau]$ for each τ and set

$$\mathfrak{A} \models \chi_s \quad \text{iff} \quad |A| \text{ is a successor cardinal.}$$

Then $\mathscr{L}$ is compact, has the interpolation property and the Löwenheim–Skolem property down to $\aleph_1$.

1.2.2 Example. Given a τ-structure $\mathfrak{A}$, say $\mathfrak{A} = (A, R_1, R_2, \ldots, f_1, f_2, \ldots, c_1, \ldots)$ denote by $\mathfrak{A}^c$ the τ-structure $(A, R_1^c, R_2^c, \ldots, f_1, f_2, \ldots, c_1, \ldots)$, where for n-ary R_i, $R_i^c = A^n \backslash R_i$. Let $\mathscr{L}^c$ be the logic with the same syntax as $\mathscr{L}_{\omega\omega}$ and with the semantics given by

$$\mathfrak{A} \models_c \varphi \quad \text{iff} \quad \begin{cases} \mathfrak{A} \models \varphi, & \text{if } A \text{ is infinite,} \\ \mathfrak{A}^c \models \varphi, & \text{if } A \text{ is finite.} \end{cases}$$

Neither $\mathscr{L}_{\omega\omega} \leq \mathscr{L}^c$ nor $\mathscr{L}^c \leq \mathscr{L}_{\omega\omega}$ holds. $\mathscr{L}^c$ is a compact logic with the Löwenheim–Skolem property and is also a maximal logic with these two properties. We leave it to the reader to adapt the preceding proofs to show that $\mathscr{L}^c$ is a maximal logic.

1.3. The Monadic Case

Throughout this part of the discussion we will restrict ourselves to monadic vocabularies; that is, we will assume that all vocabularies only contain unary relation symbols. We will give Lindström-type characterizations of monadic first-order logic and of some extensions.

When analyzing the arguments of Section 1.1 for the monadic case one should note the following differences:

(a) For a finite vocabulary, any two elementarily equivalent structures are isomorphic (thus the last step in the proof of Theorem 1.1.1 is actually redundant).
(b) Using a single monadic $\mathscr{L}_{\omega\omega}$-sentence, one cannot force a structure to be infinite (as by "f is one-to-one but not onto" in the general case).
(c) For any monadic recursive vocabulary τ, the set of $\mathscr{L}_{\omega\omega}[\tau]$-sentences valid in all finite models is recursively enumerable (it is even recursive).

In fact, while the characterization of $\mathscr{L}_{\omega\omega}$ given in Theorem 1.1.1 carries over to the monadic case (see Theorem 1.3.2 below), the following examples show that in this characterization the Löwenheim–Skolem property is really needed for countable sets (and not just for single sentences), and that Lindström's second theorem no longer holds.

1.3.1 Examples. (a) The monadic part of the logic in Example 1.2.1, with the new atomic sentence χ_s true in structures, the cardinality of which is a successor cardinal, is an example of a logic more expressive than first-order logic which is compact and has the Löwenheim–Skolem property. The Löwenheim–Skolem property follows from the fact that every satisfiable monadic $\mathscr{L}_{\omega\omega}$-sentence has a finite model.

(b) Let $\mathscr{L}$ be the logic obtained from $\mathscr{L}_{\omega\omega}$ by adding a new "atomic" sentence which is true just in the models of even finite cardinality. Then the monadic part of $\mathscr{L}$ properly extends $\mathscr{L}_{\omega\omega}$, is decidable, and has the Löwenheim–Skolem property for countable sets of sentences.

(c) Let $\mathscr{L}$ be obtained from $\mathscr{L}_{\omega\omega}$ by adding a new "atomic" sentence which is true in models of even finite or uncountable cardinality. Then the monadic part of $\mathscr{L}$ is countably compact, decidable, and each satisfiable sentence has a finite model.

It is easy to extend the above-mentioned maximality result for the monadic part of $\mathscr{L}_{\omega\omega}$ to a more general situation. For the rest of this section, however, *we restrict ourselves to monadic logics $\mathscr{L}$ with the finite occurrence property.* For an ordinal β denote by Q_β the unary quantifier "there are $\aleph_\beta$-many". Fix an ordinal α and let $\mathscr{L} = \mathscr{L}(Q_\beta \mid \beta \leq \alpha,\ \beta$ successor ordinal). Since in a structure $\mathfrak{A}$ of finite vocabulary $\tau_0 = \{R_1, \ldots, R_n\}$, the cardinalities of the boolean atoms $P_1 \cap \cdots \cap P_n$, where $P_i = R_i^A$ or $P_i = A \backslash R_i^A$, determine the isomorphism type of $\mathfrak{A}$, we have for τ_0-structures $\mathfrak{A}$ and $\mathfrak{B}$

$$(*) \qquad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B} \quad \text{and} \quad |A|, |B| \leq \aleph_\alpha \quad \text{imply} \quad \mathfrak{A} \cong \mathfrak{B}.$$

Hence, for any logic $\mathscr{L}'$ with the finite occurrence property and an arbitrary monadic vocabulary τ, we obtain from $(*)$

$$(\substack{*\\*}) \qquad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B} \quad \text{and} \quad |A|, |B| \leq \aleph_\alpha \quad \text{imply} \quad \mathfrak{A} \equiv_{\mathscr{L}'} \mathfrak{B}.$$

We will make use of $(*)$ and $(\overset{*}{*})$ in proving the following maximality theorem.

1.3.2 Theorem. *For a countable ordinal α the monadic part of $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_\beta \mid \beta \leq \alpha$, β successor ordinal) is (among the monadic logics) a maximal countably compact logic with the Löwenheim–Skolem property down to $\aleph_\alpha$ for countable sets of sentences. Furthermore, $\mathscr{L}$ has the interpolation property.*

Before undertaking the proof of Theorem 1.3.2, let us state some consequences.

1.3.3 Corollary. (a) (Tharp [1973]) *The monadic part of $\mathscr{L}_{\omega\omega}$ is a maximal logic with countable compactness and the Löwenheim–Skolem property for countable sets of sentences.*

(b) (Caicedo [1981b]) *The monadic part of $\mathscr{L}_{\omega\omega}(Q_1)$ is a maximal logic with countable compactness and the Löwenheim–Skolem property down to $\aleph_1$ for countable sets of sentences. Moreover, this logic has the interpolation property.*

(c) (Caicedo [1981b]) *The monadic part of $\mathscr{L}_{\omega\omega}(Q_1)$ and of $\mathscr{L}_{\omega\omega}$*(aa) *are equivalent. (The reader is referred to Example* 4 *of Section* 2.2, *Chapter II for the definition of $\mathscr{L}_{\omega\omega}$*(aa)).

Proof of Theorem 1.3.2. We prove the "separation property" corresponding to the maximality assertion : Let $\mathscr{L}'$, $\mathscr{L} \leq \mathscr{L}'$, be a countably compact logic with the Löwenheim–Skolem property down to $\aleph_\alpha$ for countable sets. Suppose that $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\psi)$ are disjoint $\mathscr{L}'$-classes not separable by an $\mathscr{L}$-class. Since for a finite vocabulary τ there are only countably many $\mathscr{L}$-sentences—this is the point where we need the restriction to a countable ordinal α—we obtain as in the proof of (2) in Lemma 1.1.2, using the countable compactness of $\mathscr{L}'$, structures $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\leq \aleph_\alpha$ with

$$\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}, \qquad \mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{B} \models \psi.$$

This is a contradiction in view of $(\overset{*}{*})$ above. It still remains to show that $\mathscr{L}$ is countably compact, and this will be accomplished by the next theorem. ☐

1.3.4 Theorem. *Let α be an arbitrary ordinal and $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_\beta \mid 0 < \beta \leq \alpha$, cofinality of $\beta \neq \omega)$. Then the monadic part of $\mathscr{L}$ is countably compact and has the interpolation property.*

Proof. Suppose each finite subset of $\Phi = \{\varphi_1, \varphi_2, \ldots\}$ is satisfiable. Choose $\tau = \{R_1, R_2, \ldots\}$ such that $\Phi \subset \mathscr{L}[\tau]$. We want to obtain a model $\mathfrak{A}$ of Φ, fixing step by step, the cardinalities of the boolean atoms determined by $\{R_1, \ldots, R_n\}$ in such a way that for any finite subset Φ_0 of Φ these cardinalities are realized in some model of Φ_0. For this let

$$F = \{f \mid \text{there is } k \geq 0 \text{ such that } f\colon \{1, \ldots, k\} \to \{0, 1\}\}.$$

For a τ-structure $\mathfrak{A}$ and $f \in F$ define A^f by

$$A^f := \begin{cases} A, & \text{if } f = \emptyset, \\ P_1 \cap \cdots \cap P_k, & \text{if } f\colon \{1, \ldots, k\} \to \{0, 1\}, \end{cases}$$

where

$$P_i = \begin{cases} R_i^A, & \text{if } f(i) = 1, \\ A \setminus R_i^A, & \text{if } f(i) = 0. \end{cases}$$

Suppose given pairwise distinct $f, g_1, \ldots, g_l \in F$ with $\operatorname{dom}(g_i) \subset \operatorname{dom}(f)$ for all i, and cardinals $\lambda_1, \ldots, \lambda_l$. For $n \in \omega$ *let*

$$\begin{aligned} C_n &= C(n, f, g_1, \ldots, g_l, \lambda_1, \ldots, \lambda_l) \\ &:= \{|A^f| \mid \mathfrak{A} \models \{\varphi_1, \ldots, \varphi_n\}, |A^{g_1}| = \lambda_1, \ldots, |A^{g_l}| = \lambda_l\}. \end{aligned}$$

We show that

(a) $\sup\{\kappa_m | m \in \omega\} \in C_n$, for any sequence $\kappa_1 < \kappa_2 < \cdots$ of cardinals in C_n.
(b) $C_0 \supset C_1 \supset \cdots$
(c) If $C_n \neq \emptyset$ for all $n \in \omega$, then $\bigcap \{C_n | n \in \omega\} \neq \emptyset$.

To prove (a), choose m_0 large enough such that in $\varphi_1 \wedge \cdots \wedge \varphi_n$ no quantifier Q_β with $\kappa_{m_0} \leq \kappa_\beta < \sup \kappa_m$ appears (and hence by definition of $\mathscr{L}$ no quantifier Q_β with $\kappa_{m_0} \leq \kappa_\beta \leq \sup \kappa_m$). We assume that κ_{m_0} is infinite and leave the case "κ_{m_0} finite" to the reader. Take $\mathfrak{A} \models \{\varphi_1, \ldots, \varphi_n\}$ such that $|A^{g_1}| = \lambda_1$, $\ldots$, $|A^{g_l}| = \lambda_l$ and $|A^f| = \kappa_{m_0}$. Suppose $f\colon \{1, \ldots, k\} \to \{0, 1\}$ and choose $k' \geq k$ such that $\varphi_1 \wedge \cdots \wedge \varphi_n \in \mathscr{L}[\{R_1, \ldots, R_{k'}\}]$. Since $|A^f| = \kappa_{m_0}$, there is a boolean atom determined by $R_1, \ldots, R_{k'}$, of power κ_{m_0}, which is a part of A^f. Obtain $\mathfrak{A}'$ from $\mathfrak{A}$ blowing up this boolean atom to a set of cardinality $\sup \kappa_m$. Then $\mathfrak{A}'$ shows that $\sup \kappa_m \in C_n$ (since $\kappa_m \in C_n$ for all m, we have $\sup \kappa_m \leq \lambda_i$ for any i with $g_i \subset f$). (b) is clear by definition of the C_n's, and (c) follows immediately from (a) and (b).

We now construct the desired model $\mathfrak{A} = (A, R_1^A, \ldots)$ of Φ. Let $f_0 = \emptyset$. Choose $\kappa_0 \in \bigcap \{C(n, f_0) | n \in \omega\}$ and let A be a set of cardinality κ_0. Denote by f_1 and f_2 the functions given by $f_1, f_2\colon \{1\} \to \{0, 1\}$ with $f_1(1) = 1$ and $f_2(1) = 0$. Choose $\kappa_1 \in \bigcap \{C(n, f_1, f_0, \kappa_0) | n \in \omega\}$ and $\kappa_2 \in \bigcap \{C(n, f_2, f_0, f_1, \kappa_0, \kappa_1) | n \in \omega\}$. Let R_1^A be a subset of A of cardinality κ_1 with complement of cardinality κ_2. Now one defines R_2^A choosing cardinalities for the subsets A^f, where $\operatorname{dom}(f) = \{1, 2\}$ with the help of the appropriate sets $C(\ldots)$. In this way, one can fix inductively all the R_n^A's.

We leave it to the reader to verify the interpolation property. Observe that since in any $\mathscr{L}$-sentence only finitely many Q_β occur, one can restrict to a countable sublanguage and argue as in the proof of Theorem 1.3.2). $\square$

1.3.5 Notes. Theorems 1.1.4 and 1.1.5 were proven by Lindström [1966a], [1969]; Theorem 1.1.4 was later rediscovered by Friedman, to whom assertion (1) in the proof of Theorem 1.1.1 is due. The examples listed in 1.3.1 are due to Tharp [1973]. Example 1.2.2 and the results given in Theorems 1.3.2 and 1.3.4 on monadic logics are new here, (although the countable compactness of the monadic part of $\mathscr{L}_{\omega\omega}(Q_{\alpha_1}, \ldots, Q_{\alpha_n})$ for $\alpha_1, \ldots, \alpha_n > 0$ was proved by Fajardo [1980]. *Added in proof*: Theorem 1.3.2 can be generalized to uncountable α, as will be shown elsewhere.

2. *Further Characterizations of $\mathscr{L}_{\omega\omega}$*

In this section we present some further characterizations of first-order logic, first examining those logics having the Löwenheim–Skolem property or a related property, the Karp property. In the second part we drop these assumptions. We close this section with the study of compact sublanguages of $\mathscr{L}_{\infty\omega}$.

Throughout parts 1 and 2 we will assume that all logics $\mathscr{L}$ under consideration are *regular and have the finite occurrence property* (even though many results would continue to hold under weaker assumptions). Recall that a regular logic $\mathscr{L}$ has the substitution property (and hence possesses the relativization property) and also satisfies $\mathscr{L}_{\omega\omega} \leq \mathscr{L}$.

2.1. The Löwenheim–Skolem Property and the Karp Property

By definition, a logic $\mathscr{L}$ has the *Karp property* if partially isomorphic structures are $\mathscr{L}$-equivalent, that is, if

$$\mathfrak{A} \cong_p \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}.$$

In the presence of the substitution property, we can replace in Lindström's first characterization of first-order logic the Löwenheim–Skolem property by the weaker Karp property (The reader is referred to Proposition 2.1.7 below for the relationship between the Löwenheim–Skolem and the Karp properties). Indeed, we have:

2.1.1 Theorem. *If $\mathscr{L}$ has the Karp property and is countably compact, then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Since the ordering $(\omega, <)$ cannot be defined in a countably compact logic, this theorem is a consequence of the following lemma.

2.1.2 Lemma. *If $\mathscr{L}_{\omega\omega} < \mathscr{L}$ and $\mathscr{L}$ has the Karp property, then $(\omega, <)$ is* RPC *in $\mathscr{L}$ (that is, there is a satisfiable $\mathscr{L}$-sentence $\varphi_0(U, <, \ldots)$ such that in each model $\mathfrak{A}$ of φ_0 the relativized reduct $(U^A, <^A)$ is isomorphic to $(\omega, <)$).*

Proof. Let φ be an $\mathscr{L}$-sentence not equivalent to a first-order sentence. Choose a finite τ_0 such that $\varphi \in \mathscr{L}[\tau_0]$. Then, for finitely many $\varphi_1, \ldots, \varphi_n \in \mathscr{L}_{\omega\omega}[\tau_0]$, there are $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$\mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \neg\varphi \quad \text{and} \quad (\mathfrak{A} \models \varphi_i \text{ iff } \mathfrak{B} \models \varphi_i) \quad \text{for } i \leq n.$$

Hence,

$$(*) \qquad \begin{array}{l} \text{for each } k \in \omega, \text{ there are } \mathfrak{A}_k \text{ and } \mathfrak{B}_k \text{ such that} \\ \mathfrak{A}_k \cong_k \mathfrak{B}_k, \qquad \mathfrak{A}_k \models \varphi \quad \text{and} \quad \mathfrak{B}_k \models \neg\varphi. \end{array}$$

Let U, $<$, V, W be new relation symbols, U unary, $<$, V and W binary. Coding partial isomorphisms as in the proof of Theorem 1.1.2, we obtain in a suitable vocabulary τ, an $\mathscr{L}$-sentence φ_0 expressing

> "$<$ is a discrete linear ordering of its field U with first but no last element; for each x in U the set $Vx\cdot$ (i.e. $\{y \mid Vxy\}$) is a model of φ, the set $Wx\cdot$ is a model of $\neg\varphi$, and $Vx\cdot$ and $Wx\cdot$ are x-partially isomorphic, i.e. there is a sequence, indexed by the $<$-predecessors of x of non-empty sets of partial isomorphisms with the back and forth property."

(Compare Chapter II Proposition 5.2.4 to see how we can formulate this statement by an $\mathscr{L}$-sentence). By $(*)$, φ_0 has a model $\mathfrak{A}$, where $(U^A, <^A)$ is isomorphic to $(\omega, <)$ and where attached to the k-th element a of the ordering $<^A$ are the models $\mathfrak{A}_k$ and $\mathfrak{B}_k$; that is,

$$(\{b \mid V^A ab\}, \ldots) \cong \mathfrak{A}_k \quad \text{and} \quad (\{b \mid W^A ab\}, \ldots) \cong \mathfrak{B}_k.$$

Now let $\mathfrak{B}$ be any model of φ_0; we must show that $(U^B, <^B)$ is isomorphic to $(\omega, <)$. If $(U^B, <^B) \not\cong (\omega, <)$, a "non-standard" element x in $(U^B, <^B)$, gives rise—as in the proof of Lemma 1.1.2—to partially isomorphic models $Vx\cdot$ of φ and $Wx\cdot$ of $\neg\varphi$. This is, a contradiction, however, since we assumed that $\mathscr{L}$ has the Karp property. ☐

In case $\mathscr{L}$ has the Löwenheim–Skolem property, the structures $\mathfrak{A}_k$ and $\mathfrak{B}_k$ in $(*)$ of the preceding proof can be chosen of power $\aleph_0$ and hence $(*)$ can be coded in a countable model of φ_0. Thus we can require that $<$ is an ordering of the universe of the model. Accordingly, we obtain:

2.1.3 Corollary. *If $\mathscr{L}_{\omega\omega} < \mathscr{L}$ and $\mathscr{L}$ has the Löwenheim–Skolem property, then $(\omega, <)$ is* PC *in $\mathscr{L}$.*

This corollary can also be derived from the results of the preceding section: Since $\mathscr{L}_{\omega\omega} < \mathscr{L}$ there is a sentence $\vartheta = \vartheta(U, \ldots)$ having properties (i) and (ii) of Lemma 1.1.2. Now it is not difficult (using the substitution property of $\mathscr{L}$) to write down a sentence PC-characterizing $(\omega, <)$: This sentence will express that attached to each element x of the ordering $<$ is a model of ϑ whose U-part has as many elements as the set of $<$-predecessors of x.

In the following theorem, we collect some model-theoretic properties that characterize $\mathscr{L}_{\omega\omega}$ among the logics with the Löwenheim–Skolem property. However, we state the theorem in such a way that it provides information on the expressive power of proper extensions of $\mathscr{L}_{\omega\omega}$.

2.1.4 Theorem. *For a logic $\mathscr{L}$ satisfying the Löwenheim–Skolem property the following conditions are equivalent.*

(i) $\mathscr{L}_{\omega\omega} < \mathscr{L}$.
(ii) *$\mathscr{L}$ is not countably compact.*
(iii) *(The class of structures isomorphic to) $(\omega, <)$ is* PC *in $\mathscr{L}$.*
(iv) *Each countable structure in a countable vocabulary is* PC_δ *in $\mathscr{L}$; that is, it is characterizable using additional symbols by a countable set of sentences.*
(v) $\mathscr{L}^{\mathrm{HF}} \leq_{\mathrm{RPC}} \mathscr{L}$, *where $\mathscr{L}^{\mathrm{HF}}$ is the second-order logic with quantification on hereditarily finite sets over the universe, and $\mathscr{L}_1 \leq_{\mathrm{RPC}} \mathscr{L}_2$ means that each class of relativized reducts in $\mathscr{L}_1$ is such a class in $\mathscr{L}_2$.*
(vi) *There is an $\mathscr{L}$-sentence with an infinite but no uncountable model.*
(vii) $\mathscr{L}_{\omega\omega} <_{\equiv} \mathscr{L}$; *that is, there are $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{B}$.*

Proof. Clearly each of the conditions in (ii)–(vii) implies (i). Hence, it suffices to show that (ii)–(vii) follow from (i).

(i) $\Rightarrow$ (ii). This was shown in Section 1.

(i) $\Rightarrow$ (iii). See the preceding corollary.

For the proofs of the following implications let φ_0 always denote an $\mathscr{L}[\tau_0]$-sentence PC-characterizing $(\omega, <)$.

(iii) $\Rightarrow$ (iv). Given a countable structure $\mathfrak{A}$ choose a one-to-one enumeration $\langle a_n \mid n \in \omega \rangle$ of A, write down the algebraic diagram Φ of $\mathfrak{A}$, where a_n is represented by the n-th element of an ordering $<$ of type ω. Then $\{\varphi_0\} \cup \Phi$ is a PC_δ-characterization of $\mathfrak{A}$.

(iii) $\Rightarrow$ (v). Use φ_0 (and hence $(\omega, <)$) to code the hereditarily finite sets over the universe.

(iii) $\Rightarrow$ (vi). φ_0 has no uncountable model.

(iii) $\Rightarrow$ (vii). Let $\mathfrak{A}$ be a countable model of φ_0. Then any uncountable model of $\mathrm{Th}(\mathfrak{A})$, the first-order theory of $\mathfrak{A}$, is elementarily equivalent but not $\mathscr{L}$-equivalent to $\mathfrak{A}$. □

2.1.5 Remarks. (a) In case $\mathscr{L}$ has the form $\mathscr{L}_{\omega\omega}(Q_1, \ldots, Q_n)$ where $Q_1, \ldots, Q_n$ are Lindström quantifiers we can add in Theorem 2.1.4 the condition

(viii) $\mathscr{L}$ does not have the definability property (Beth property).

The reader is referred to Chapter XVII for a proof of this result.

(b) We want to draw the reader's attention to the notion of an ω_1-securable quantifier (see Makowsky [1975b]) which captures the properties of the existential quantifier needed to prove that each structure has a countable elementary substructure. In fact, Makowsky proved the following: If $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_i | i \in I)$ is obtained from first-order logic adding ω_1-securable quantifiers, then $\mathscr{L}$ has the Löwenheim–Skolem property. Hence, for such an $\mathscr{L}$, the equivalences in Theorem 2.1.4 hold.

We use Theorem 2.1.4 to derive a further characterization of $\mathscr{L}_{\omega\omega}$. A logic $\mathscr{L}$ is said to have the *Robinson property* if the following holds: Let τ, τ_1 and τ_2 be vocabularies with $\tau = \tau_1 \cap \tau_2$. Let Φ be a set of $\mathscr{L}[\tau]$-sentences and Φ_i a set of $\mathscr{L}[\tau_i]$-sentences for $i = 1, 2$. If Φ is complete and $\Phi \cup \Phi_1$ and $\Phi \cup \Phi_2$ are satisfiable then so is $\Phi \cup \Phi_1 \cup \Phi_2$. In Chapter XIX it is shown that this is a very strong property of a logic. In fact, it is proved there that in case there are no measurable cardinals the Robinson property implies the compactness property.

2.1.6 Theorem. *If $\mathscr{L}$ has the Löwenheim–Skolem property and the Robinson property then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Proof. Since we have the general assumption that $\mathscr{L}$ has the finite occurrence property there are countable structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in a countable vocabulary τ such that

$$\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_2 \quad \text{and} \quad \mathfrak{A}_1 \not\cong \mathfrak{A}_2$$

(e.g., take non-isomorphic $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that for any finite $\tau_0 \subset \tau$ the τ_0-reducts $\mathfrak{A}_1 \upharpoonright \tau_0$ and $\mathfrak{A}_2 \upharpoonright \tau_0$ are isomorphic). Suppose by contradiction, that $\mathscr{L}_{\omega\omega} < \mathscr{L}$. Then, by the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 2.1.4, there are PC_δ-characterizations Φ_1 and Φ_2 in $\mathscr{L}$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. We use distinct additional symbols for $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Since $\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_2$, $\Phi \cup \Phi_1$ and $\Phi \cup \Phi_2$ are satisfiable; but $\Phi \cup \Phi_1 \cup \Phi_2$ has no model, as $\mathfrak{A}_1 \not\cong \mathfrak{A}_2$.

Note that in case we restrict attention to finite vocabularies, the preceding proof shows:

> If $\mathscr{L}_{\omega\omega} < \mathscr{L}$ and $\mathscr{L}$ has the Löwenheim-Skolem property and the Robinson property for countable sets of sentences (that is, countable Φ, Φ_1 and Φ_2), then $\equiv_{\mathscr{L}}$ coincides with the isomorphism relation on countable structures.

In particular, we see that weak second-order logic does not have the Robinson property; $\mathscr{L}_{\omega_1\omega}$ is a logic satisfying the hypothesis of this result.

We close this discussion with a result that clarifies the relationship between the Karp property and the Löwenheim–Skolem property.

2.1.7 Proposition. (a) *If $\mathscr{L}$ has the Löwenheim–Skolem property, then $\mathscr{L}$ has the Karp property.*

(b) *Assume $\mathscr{L}$ has both the Karp property and the interpolation property, then $\mathscr{L}$ has the Löwenheim–Skolem property.*

Proof. The proof of (a) is by contradiction. Suppose that $\mathscr{L}$ has the Löwenheim–Skolem property but that for some $\mathscr{L}$-sentence φ, we have

$$(*) \qquad \mathfrak{A} \cong_p \mathfrak{B}, \qquad \mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{B} \models \neg\varphi.$$

Coding partial isomorphisms as in the precedings proofs, we obtain an $\mathscr{L}$-sentence ψ expressing that

> "the V-part is a model of φ, the W-part is a model of $\neg\varphi$, and the V-part and the W-part are partially isomorphic".

By $(*)$, the sentence ψ has a model, and hence one of power $\leq\aleph_0$. But then we obtain countable structures $\mathfrak{A}'$ (the V-part) and $\mathfrak{B}'$ (the W-part) such that $\mathfrak{A}' \models \varphi$, $\mathfrak{B}' \models \neg\varphi$ and $\mathfrak{A}' \cong_p \mathfrak{B}'$; hence $\mathfrak{A}' \cong \mathfrak{B}'$, a contradiction.

Turning now to the proof of (b), we let $\mathscr{L}$ be given as in (b) and suppose $\mathscr{L}_{\omega\omega} < \mathscr{L}$ (if $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$, the conclusion holds). Since $\mathscr{L}$ has the Karp property, by Lemma 2.1.2 $(\omega, <)$ is RPC in $\mathscr{L}$, say $\varphi_0(U, <, \ldots)$ is an $\mathscr{L}$-sentence RPC-characterizing $(\omega, <)$. If $\mathscr{L}$ does not have the Löwenheim–Skolem property, then there is an $\mathscr{L}$-sentence φ_1 having only uncountable models. Consider the classes

$$\mathfrak{K}_0 := \{(A, U^A) \mid \text{there is } <^A, --- \text{ such that } (A, U^A, <^A, ---) \models \varphi_0\},$$
$$\mathfrak{K}_1 := \{(A, U^A) \mid \text{there is} \ldots \text{ such that } (U^A, \ldots) \models \varphi_1\}.$$

Since $(A, U^A) \in \mathfrak{K}_0$ (resp. $(A, U^A) \in \mathfrak{K}_1$) implies that U^A is countable (resp. uncountable), $\mathfrak{K}_0$ and $\mathfrak{K}_1$ are disjoint PC-classes of $\mathscr{L}$. Take an arbitrary (A, U^A) in $\mathfrak{K}_0$ and choose (B, U^B) in $\mathfrak{K}_1$ such that $|B\setminus U^B| = |A\setminus U^A|$. Then (A, U^A) and (B, U^B) are partially isomorphic. Hence, there is no $\mathscr{L}$-class separating $\mathfrak{K}_0$ and $\mathfrak{K}_1$, since $\mathscr{L}$ has the Karp property. But this contradicts the assumption that $\mathscr{L}$ has the interpolation property. $\square$

2.2. *The Tarski Union Property and the Omitting Types Property*

The following characterization of $\mathscr{L}_{\omega\omega}$ shows that an important model-theoretic tool of first-order logic, the Tarski union lemma, is not available in any proper compact extension.

First we introduce some terminology. Suppose given a logic $\mathscr{L}$. A structure $\mathfrak{B}$ is said to be an *$\mathscr{L}$-extension* of $\mathfrak{A}$, $\mathfrak{A} \prec_{\mathscr{L}} \mathfrak{B}$, if $\mathfrak{B}$ is an extension of $\mathfrak{A}$ and if for any finite $a_0, \ldots, a_{n-1} \in A$, we have $(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv_{\mathscr{L}} (\mathfrak{B}, a_0, \ldots, a_{n-1})$. (For $\mathscr{L} = \mathscr{L}_{\omega\omega}$ we say that $\mathfrak{B}$ is an elementary extension and write $\mathfrak{A} \prec \mathfrak{B}$.)

Denote by $\mathrm{Th}_{\mathscr{L}}(\mathfrak{A})$ and $\mathrm{D}_{\mathscr{L}}(\mathfrak{A})$ the *$\mathscr{L}$-theory of* $\mathfrak{A}$ and the *$\mathscr{L}$-diagram of* $\mathfrak{A}$, respectively; that is,

$$\mathrm{Th}_{\mathscr{L}}(\mathfrak{A}) := \{\varphi \mid \varphi\ \mathscr{L}\text{-sentence}, \mathfrak{A} \models \varphi\}, \qquad \mathrm{D}_{\mathscr{L}}(\mathfrak{A}) := \mathrm{Th}_{\mathscr{L}}((\mathfrak{A}, (a)_{a\in A})),$$

where in the latter case we consider the $\mathscr{L}$-theory in an expanded vocabulary containing a new constant for each $a \in A$. In case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, write $\mathrm{Th}(\mathfrak{A})$ and $\mathrm{D}(\mathfrak{A})$.

As for first-order logic, one can easily prove both $(+)$ and $(++)$ below (recall that all our logics are assumed to be regular and to have the finite occurrence property).

$(+)$ The (reducts of) models of $\mathrm{D}_{\mathscr{L}}(\mathfrak{A})$ are—up to isomorphism—the $\mathscr{L}$-extensions of $\mathfrak{A}$.

$(++)$ Assume $\mathscr{L}$ is compact. Suppose given $\mathfrak{A}$ and a set of Φ of $\mathscr{L}$-sentences If $\mathrm{Th}(\mathfrak{A}) \cup \Phi$ is satisfiable, then there is $\mathfrak{B}$ such that $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \models \Phi$.

Now we say that $\mathscr{L}$ has the *Tarski union property*, if whenever

$$\mathfrak{A}_0 \prec_{\mathscr{L}} \mathfrak{A}_1 \prec_{\mathscr{L}} \mathfrak{A}_2 \prec \cdots$$

then $\mathfrak{A}_n \prec_{\mathscr{L}} \bigcup_m \mathfrak{A}_m$ for each n.

$\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega\omega}(Q_1)$ have the Tarski union property. Moreover, we have (see Makowsky [1975b] for further examples and results):

2.2.1 Theorem. *If $\mathscr{L}$ is compact and has the Tarski union property, then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Proof. If not $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$, then there is an $\mathscr{L}$-sentence φ and structures $\mathfrak{A}$, $\mathfrak{B}$ such that

$$(1) \qquad \mathfrak{A} \equiv \mathfrak{B}, \qquad \mathfrak{A} \models \varphi \quad \text{and} \quad \mathfrak{B} \models \neg\varphi$$

(see Proposition 1.1.8). We construct by induction a sequence $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ such that

$$\begin{tikzcd}[column sep=small] \mathfrak{A} = \mathfrak{A}_0 \arrow[rr, dotted, bend right, "\prec_{\mathscr{L}}"'] & \prec & \mathfrak{A}_1 \arrow[rr, dotted, bend left, "\prec_{\mathscr{L}}"] & \prec & \mathfrak{A}_2 \arrow[rr, dotted, bend right, "\prec_{\mathscr{L}}"'] & \prec & \mathfrak{A}_3 & \prec & \cdots \end{tikzcd}$$

and $\mathfrak{A}_1 \models \neg\varphi$ as follows:

By (1), $\mathrm{Th}(\mathfrak{A}) \cup \{\neg\varphi\} = \mathrm{Th}(\mathfrak{B}) \cup \{\neg\varphi\}$ is satisfiable. Hence, by $(++)$ there is $\mathfrak{A}_1$ such that $\mathfrak{A} \prec \mathfrak{A}_1$ and $\mathfrak{A}_1 \models \neg\varphi$. Now suppose $\mathfrak{A}_n$ has already been defined. Since

$$\mathrm{Th}((\mathfrak{A}_n, (a)_{a \in A_{n-1}})) = \mathrm{Th}((\mathfrak{A}_{n-1}, (a)_{a \in A_{n-1}})) \subset \mathrm{D}_{\mathscr{L}}(\mathfrak{A}_{n-1}),$$

we have that

$$\mathrm{Th}((\mathfrak{A}_n, (a)_{a \in A_{n-1}})) \cup \mathrm{D}_{\mathscr{L}}(\mathfrak{A}_{n-1})$$

is a satisfiable set of $\mathscr{L}$-sentences. Using $(++)$ once more, we therefore obtain an elementary extension $(\mathfrak{A}_{n+1}, (a)_{a \in A_{n-1}})$ of $(\mathfrak{A}_n, (a)_{a \in A_{n-1}})$ which is a model of $\mathrm{D}_{\mathscr{L}}(\mathfrak{A}_{n-1})$. Then,

$$\begin{tikzcd} \mathfrak{A}_{n-1} \arrow[r, phantom, "\prec"] \arrow[rr, dashed, no head, bend right, "\prec_{\mathscr{L}}"'] & \mathfrak{A}_n \arrow[r, phantom, "\prec"] & \mathfrak{A}_{n+1}. \end{tikzcd}$$

Let $\mathfrak{D} = \bigcup_n \mathfrak{A}_{2n} = \bigcup_n \mathfrak{A}_{2n+1}$. By the Tarski union property, we have $\mathfrak{A}_0 \prec_{\mathscr{L}} \mathfrak{D}$ and $\mathfrak{A}_1 \prec_{\mathscr{L}} \mathfrak{D}$. But since $\mathfrak{A}_0 \models \varphi$ and $\mathfrak{A}_1 \models \neg\varphi$, we obtain the contradiction: $\mathfrak{D} \models \varphi$ and $\mathfrak{D} \models \neg\varphi$. $\square$

Lindström [1983] introduced a kind of union property for direct limits and showed by refining the previous proof, that a logic is equivalent to first-order logic if it has this generalized union property and is countably compact.

We now turn to a characterization of $\mathscr{L}_{\omega\omega}$ by means of a single property, the omitting types property for an uncountable regular cardinal.

Let κ be an infinite cardinal and $\mathscr{L}$ be a logic. Given a set Φ of $\mathscr{L}$-sentences and a set $\Gamma(x)$ of $\mathscr{L}$-formulas having at most the free variable x (see II.1.1.2), we say that $\Gamma(x)$ is a *κ-free type of* Φ, if the following hold:

> $|\Phi \cup \Gamma(x)| \leq \kappa$, Φ is satisfiable and for every set $\Psi(x)$ of $\mathscr{L}$-formulas such that $|\Psi(x)| < \kappa$, if $\Phi \cup \Psi(x)$ has a model, then for some $\chi(x) \in \Gamma(x)$ the set $\Phi \cup \Psi(x) \cup \{\neg\chi(x)\}$ has a model.

We say that $\mathscr{L}$ has the *κ-omitting types property*, if whenever $\Gamma(x)$ is a κ-free type of Φ, there is a model of Φ omitting $\Gamma(x)$.

Thus the "classical" omitting types theorem is the result that $\mathscr{L}_{\omega\omega}$ has the ω-omitting types property. In Keisler [1971] it is shown that also $\mathscr{L}_{\omega_1\omega}$ has the ω-omitting types property. A logic with the ω-omitting types property has the Löwenheim–Skolem property for countable sets of sentences: Given a countable and satisfiable set Φ, apply the ω-omitting types property to the ω-free type $\Gamma(x)$ of Φ, where $\Gamma(x) := \{\neg x = c_n \mid n \in \omega\}$ for new constants c_n.

The κ-omitting types property is strongly related to the construction method of models from constants (the reader should consult Barwise [1980] where a different notion of omitting types property—more precisely, of ω-omitting types property—is introduced, which is more sensitive to the specific features of a given logic). Using the method of construction of a model from constants, one can show that $\mathscr{L}_{\omega\omega}$ has the κ-omitting types property for all κ (see Chang–Keisler [1977]). Moreover, we have

2.2.2 Theorem. *If κ is an uncountable regular cardinal and $\mathscr{L}$ has the κ-omitting types property then $\mathscr{L}_{\omega\omega} \equiv \mathscr{L}$.*

Proof. First, we show

(*) Suppose the set Φ of $\mathscr{L}$-sentences, $|\Phi| \leq \kappa$, has a model $\mathfrak{A}$ such that $(U^{\mathfrak{A}}, \leq^{\mathfrak{A}})$ is an ordering without last element. Then Φ has a model $\mathfrak{B}$ such that $(U^{\mathfrak{B}}, \leq^{\mathfrak{B}})$ is an ordering of cofinality κ.

To prove (*), take new constants c_α, $\alpha < \kappa$; then

$$\Gamma(x) = \{Ux\} \cup \{c_\alpha \leq x \mid \alpha < \kappa\}$$

is a κ-free type of

$$\Phi_1 = \Phi \cup \{\text{“} < \text{ is an ordering of } U \text{ without last element”}\}$$
$$\cup \{c_\beta \leq c_\alpha \mid \beta < \alpha < \kappa\}.$$

In fact, if $|\Psi(x)| < \kappa$ and $\Phi_1 \cup \Psi(x) \cup \{Ux\}$ is satisfiable, then choose α sufficiently large such that c_β does not occur in $\Psi(x)$ for $\beta > \alpha$. In a model of $\Phi_1 \cup \Psi(x) \cup \{Ux\}$, all these c_β may be interpreted by a fixed element bigger than x. Thus $\Phi_1 \cup \Psi(x) \cup \{\neg c_{\alpha+1} \leq x\}$ is satisfiable. Now, since $\mathscr{L}$ has the κ-omitting types property there is a model $\mathfrak{B}$ of Φ_1 omitting $\Gamma(x)$. But then $(U^{\mathfrak{B}}, \leq^{\mathfrak{B}})$ has cofinality κ.

In particular, (*) shows that the ordering $(\omega, <)$ is not RPC in $\mathscr{L}$. Using Lemma 2.1.2, we see that in case $\mathscr{L}_{\omega\omega} < \mathscr{L}$, the logic $\mathscr{L}$ does not have the Karp property; that is, there are $\mathfrak{A}$ and $\mathfrak{B}$ such that

($\overset{*}{*}$) $\mathfrak{A} \cong_p \mathfrak{B}$ and $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{B}$.

We will code ($\overset{*}{*}$) in a model in such a way that use of the κ-omitting types property leads to isomorphic but not $\mathscr{L}$-equivalent structures—a contradiction. Choose an $\mathscr{L}$-sentence ψ such that $\mathfrak{A} \models \psi$ and $\mathfrak{B} \models \neg\psi$. Let c_α, d_α, p_α, for $\alpha < \kappa$, be new constants and V, W, I be new unary relation symbols. Let Φ be a set of $\mathscr{L}$-sentences, $|\Phi| = \kappa$, expressing the following:

“$V \cap W = \varnothing$”,

“the V-part is a model of ψ”,

“the W-part is a model of $\neg\psi$”,

“the V-part and the W-part are partially isomorphic via I”,

“Ip_α, c_α is in the domain of p_α and d_α in the range of p_α” for $\alpha < \kappa$,

“p_β is an extension of p_α” for $\alpha < \beta < \kappa$.

By ($\overset{*}{*}$), Φ is satisfiable (choose a partial isomorphism p in I where $I: \mathfrak{A} \cong_p \mathfrak{B}$, p with non-empty domain, say $a \in \mathrm{dom}(p)$, and set for all α, $p_\alpha = p$, $c_\alpha = a$, and $d_\alpha = p(a)$).

Let $\Gamma(x)$ be the type

$$\Gamma(x) = \{Vx \vee Wx\} \cup \{\neg x = c_\alpha \wedge \neg x = d_\beta \mid \alpha, \beta < \kappa\}.$$

Clearly, in a model of Φ omitting $\Gamma(x)$, the function $\bigcup_{\alpha<\kappa} p_\alpha$ is an isomorphism of the V-part onto the W-part. Therefore, it suffices to prove that $\Gamma(x)$ is a κ-free type of Φ.

Let $\Psi(x)$ be a set of $\mathscr{L}$-formulas, $|\Psi(x)| < \kappa$ and suppose $\Phi \cup \Psi(x)$ is satisfiable, say $\mathfrak{C} \models \Phi$ and $\mathfrak{C} \models \Psi[a]$. We must show that $\Phi \cup \Psi(x) \cup \{\neg\chi(x)\}$ has a model for some $\chi(x) \in \Gamma(x)$. If $a \notin V^C \cup W^C$, then $\mathfrak{C} \models \neg\chi[a]$ for $\chi = Vx \vee Wx \in \Gamma(x)$. Let $a \in V^C \cup W^C$, say $a \in V^C$. Choose $\alpha < \kappa$ large enough so that for $\beta > \alpha$, the constants p_β, c_β and d_β do not occur in $\Psi(x)$. Using the forth property, we see that there is a partial isomorphism q in the model extending p_α and with a in its domain. For $\beta > \alpha$, change the interpretation of p_β to q, of c_β to a, and of d_β to $q(a)$. This shows that $\Phi \cup \Psi(x) \cup \{x = c_{\alpha+1}\}$ is satisfiable. ☐

2.3. Compact Sublanguages of $\mathscr{L}_{\infty\omega}$

Let φ_0 be an $\mathscr{L}_{\omega_1\omega}$-sentence and denote by $\mathscr{L}_{\omega\omega}(\varphi_0)$ the smallest set of sentences containing φ_0 and closed under first-order operations. Clearly, $\mathscr{L}_{\omega\omega}(\varphi_0)$ has the Löwenheim–Skolem property. But, in general, $\mathscr{L}_{\omega\omega}(\varphi_0)$ does not have the renaming property. Therefore, in case $\mathscr{L}_{\omega\omega}(\varphi_0)$ is countably compact, we cannot apply the theorems already proven to conclude that $\mathscr{L}_{\omega\omega}(\varphi_0) \equiv \mathscr{L}_{\omega\omega}$, and hence that φ_0 is equivalent to a first-order sentence. Indeed we will show that there is a φ_0 such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ is countably compact but stronger than first-order logic. On the other hand, if $\mathscr{L}_{\omega\omega}(\varphi_0)$ is assumed to be compact (that is, is fully compact and not merely countably compact) then φ_0 already expresses a first-order property. Finally, we will see that this result does not generalize to $\mathscr{L}_{\infty\omega}$: There is $\varphi_0 \in \mathscr{L}_{\infty\omega}$ such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ properly extends $\mathscr{L}_{\omega\omega}$ and is compact.

To be precise, for an $\mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$-sentence, define $\mathscr{L} = \mathscr{L}_{\omega\omega}(\varphi_0)$ by

$$\mathscr{L}[\boldsymbol{\tau}] = \begin{cases} \varnothing, & \text{if } \boldsymbol{\sigma} \not\subset \boldsymbol{\tau}, \\ \text{smallest subset of } \mathscr{L}_{\infty\omega}[\boldsymbol{\tau}] \text{ containing } \varphi_0 \text{ and the} \\ \text{atomic } \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]\text{-formulas and closed under first-} \\ \text{order operations (say } \neg, \vee, \exists x\text{)}, & \text{if } \boldsymbol{\sigma} \subset \boldsymbol{\tau}. \end{cases}$$

Given any $\varphi_0 \in \mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$ set $\mathfrak{K}_1 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\varphi_0)$ and $\mathfrak{K}_2 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\neg\varphi_0)$. Then $\mathscr{L}_{\omega\omega}(\varphi_0)$ is compact (countably compact) if and only if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are compact (countably compact). Here a class $\mathfrak{K}$ of $\boldsymbol{\sigma}$-structures is called *compact* (*countably compact*) if the following holds: Given any set of $\mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$-sentences Φ, with $|\Phi| = \aleph_0$, where $\boldsymbol{\sigma} \subset \boldsymbol{\tau}$, if every finite subset of Φ has a model with $\boldsymbol{\sigma}$-reduct in $\mathfrak{K}$, then so has Φ.

2.3.1 Example. We will give an example of an $\mathscr{L}_{\omega_1\omega}$-sentence φ_0 such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ is a proper countably compact extension of $\mathscr{L}_{\omega\omega}$. Let each natural number code in an one-to-one and effective way a finite sequence of natural numbers. Define the binary relation $\prec$ on ω by

$n \prec m$ iff the sequence corresponding to n is an initial segment of the one corresponding to m.

There is a recursive functional T which assigns to each $X \subset \omega$ a tree $(T(X), \prec^{T(X)}) \subset (\omega, \prec)$ recursive in X with an infinite branch but with no branch hyperarithmetic in X (see Rogers [1967]). In particular, for $n \in \omega$, there is $p(n) \subset P_\omega(\omega) \times P_\omega(\omega)$, where $P_\omega(\omega)$ denotes the set of finite subsets of ω, such that for any $X \subset \omega$

(1) $\quad n \in T(X)$ iff there is $(X_1, X_2) \in p(n)$ with $X_1 \subset X$ and $X_2 \cap X = \emptyset$.

Moreover, the binary relation $\mathfrak{R}$ on $P(\omega)$, the power set of ω, given by

$$\mathfrak{R}XY \quad \text{iff} \quad Y \text{ is an infinite branch of } T(X)$$

has the property

(2)
$$\forall X \in P(\omega)\, \exists Y \in P(\omega) \mathfrak{R}XY,$$
$$\forall X \in P(\omega) \neg \exists Y \in P(\omega)\, (\mathfrak{R}XY \text{ and } Y \text{ hyperarithmetic in } X).$$

Let $\boldsymbol{\sigma} = \{R_n \mid n \in \omega\}$, where R_n are unary relation symbols and let $\mathfrak{A}_0$ be the $\boldsymbol{\sigma}$-structure $(P(\omega), (R_n^{A_0})_{n \in \omega})$, where

$$R_n^{A_0}X \quad \text{iff} \quad n \in X.$$

By (1) we have for $n \in \omega$ and $X \in P(\omega)$,

$$\mathfrak{A}_0 \models \psi_n[X] \quad \text{iff} \quad n \in T(X),$$

where $\psi_n(x)$ is the $\mathscr{L}_{\omega_1\omega}(\boldsymbol{\sigma})$-formula

$$\psi_n(x) = \bigvee_{(X_1, X_2) \in p(n)} \left(\bigwedge_{m \in X_1} R_m x \wedge \bigwedge_{m \in X_2} \neg R_m x \right).$$

Now

$$\mathfrak{A}_0 \models \varphi[X, Y] \quad \text{iff} \quad \mathfrak{R}XY$$

holds for

$$\varphi(x, y) = \left(\bigwedge_{n \in \omega} \bigvee_{\substack{m \in \omega \\ n < m}} R_m y \right) \wedge \bigwedge_{n \in \omega} (R_n y \to \psi_n(x))$$
$$\wedge \bigwedge_{n \in \omega} \left(R_n y \to \bigwedge_{\substack{m \in \omega \\ m \prec n}} (R_m y \leftrightarrow \psi_m(x) \right).$$

Moreover, one can easily verify that for any $\boldsymbol{\sigma}$-structure $\mathfrak{A}$ and $a, b \in A$,

$$\mathfrak{A} \models \psi_n[a] \quad \text{iff} \quad n \in T(\{m \mid R_m^A a\})$$

and hence, we have

$$(3) \qquad \mathfrak{A} \models \varphi[a, b] \quad \text{iff} \quad \mathfrak{R}\{n \mid R_n^A a\}\{n \mid R_n^A b\}.$$

Finally, take as φ_0 the $\mathscr{L}_{\omega_1\omega}[\boldsymbol{\sigma}]$-sentence

$$\varphi_0 = \bigwedge \mathrm{Th}(\mathfrak{A}_0) \wedge \forall x \, \exists y \, \varphi(x, y),$$

where $\mathrm{Th}(\mathfrak{A}_0)$, the theory of $\mathfrak{A}_0$, denotes the set of first-order sentences holding in $\mathfrak{A}_0$.

Clearly, φ_0 is not equivalent to a first-order sentence. But $\mathscr{L}_{\omega\omega}(\varphi_0)$ is countably compact: Set $\mathfrak{K}_1 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\varphi_0)$ and $\mathfrak{K}_2 = \mathrm{Mod}^{\boldsymbol{\sigma}}(\neg \varphi_0)$. To prove that $\mathfrak{K}_1$ is countably compact (even compact) it suffices to show that every ω-saturated model $\mathfrak{A}$ of $\mathrm{Th}(\mathfrak{A}_0)$ is a model of $\forall x \, \exists y \, \varphi(x, y)$. But for each $Y \subset \omega$, $\mathfrak{A}$ being ω-saturated contains an element a such that $Y = \{n \mid R_n^A a\}$. Then by (2) and (3),

$$\mathfrak{A} \models \forall x \, \exists y \, \varphi(x, y).$$

Toprove that $\mathfrak{K}_2 = \mathrm{Mod}^{\sigma}(\neg \varphi_0)$ is countably compact, it suffices to show that if $\Phi \cup \mathrm{Th}(\mathfrak{A}_0)$ is satisfiable, where Φ is a countable set of first-order sentences, then there is a model of $\Phi \cup \mathrm{Th}(\mathfrak{A}_0) \cup \{\neg \forall x \, \exists y \, \varphi(x, y)\}$: Take a subset $X \subset \omega$ such that $\Phi \cup \mathrm{Th}(\mathfrak{A}_0)$ is recursive in X. Inside $\mathrm{Hyp}(X)$, the smallest admissible set containing X, construct a model $\mathfrak{B}$ of

$$\Phi \cup \mathrm{Th}(\mathfrak{A}_0) \cup \{R_n c \mid n \in X\} \cup \{\neg R_n c \mid n \notin X\},$$

where c is a new constant. $\mathrm{Hyp}(X)$ only contains subsets of ω hyperarithmetic in X. Therefore by (2), $\mathfrak{B} \models \neg \exists y \, \varphi(c, y)$. ☐

On the other hand we have:

2.3.2 Theorem. *Suppose φ_0 is an $\mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$-sentence for some countable $\boldsymbol{\sigma}$. If $\mathscr{L}_{\omega\omega}(\varphi_0)$ is compact, then φ_0 is equivalent to a first-order sentence.*

Observe that for each $\mathscr{L}_{\omega_1\omega}$-sentence φ_0, there is some countable $\boldsymbol{\sigma}$ such that $\varphi_0 \in \mathscr{L}_{\omega_1\omega}[\boldsymbol{\sigma}]$.

Proof. First, we prove:

(∗) Suppose that $\mathrm{Mod}(\varphi)$ is compact, where $\varphi \in \mathscr{L}_{\infty\omega}[\boldsymbol{\sigma}]$ and $|\boldsymbol{\sigma}| \leq \aleph_0$. If $\mathfrak{A} \models \varphi$ then there is an ω-saturated $\mathfrak{A}'$ such that $\mathfrak{A}' \equiv \mathfrak{A}$ and $\mathfrak{A}' \models \varphi$.

To establish this, for each $(n+1)$-type $p \subset \mathscr{L}_{\omega\omega}[\boldsymbol{\sigma}]$,

$$p = \{\psi_m(x_1, \ldots, x_n, y) \mid m \in \omega\},$$

take a new n-ary function symbol f_p. Now, set

$$\Phi = \Bigg\{\forall x_1 \ldots \forall x_n\Bigg(\exists y \bigwedge_{i \le m} \psi_i(x_1, \ldots, x_n, y)$$

$$\to \bigwedge_{i \le m} \psi_i(x_1, \ldots, x_n, f_p(x_1, \ldots, x_n))\Bigg)$$

$$\mid m, n \in \omega, p = \{\psi_m(x_1, \ldots, x_n, y) \mid m < \omega\}(n+1)\text{-type}\Bigg\}.$$

Clearly, $\mathrm{Th}(\mathfrak{A}) \cup \Phi \cup \{\varphi\}$ is finitely satisfiable and hence satisfiable, say $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}) \cup \Phi \cup \{\varphi\}$. Let $\mathfrak{A}' = \mathfrak{B} \upharpoonright \boldsymbol{\sigma}$. Then $\mathfrak{A}' \equiv \mathfrak{A}$, $\mathfrak{A}' \models \varphi$, and $\mathfrak{A}'$ is ω-saturated since $\mathfrak{B} \models \Phi$.

Now let φ_0 and $\boldsymbol{\sigma}$ be given as in the theorem and suppose that $\mathscr{L}_{\omega\omega}(\varphi_0)$ is compact. By (the proof of) Proposition 1.1.8, it suffices to show that

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{implies} \quad \mathfrak{A} \equiv_{\mathscr{L}_{\omega\omega}(\varphi_0)} \mathfrak{B},$$

or, equivalently, that

$$\mathfrak{A} \equiv \mathfrak{B} \quad \text{implies} \quad (\mathfrak{A} \models \varphi_0 \text{ iff } \mathfrak{B} \models \varphi_0).$$

For the sake of argument, suppose that $\mathfrak{A} \models \varphi_0$ and $\mathfrak{B} \models \neg\varphi_0$. Applying (*) twice, we obtain ω-saturated $\mathfrak{A}'$ and $\mathfrak{B}'$ such that

$$\mathfrak{A}' \equiv \mathfrak{B}', \qquad \mathfrak{A}' \models \varphi_0 \quad \text{and} \quad \mathfrak{B}' \models \neg\varphi_0.$$

But this is a contradiction, since any two ω-saturated elementarily equivalent models are $\mathscr{L}_{\infty\omega}$-equivalent. □

2.3.3 Example. We will now show that Theorem 2.3.2 does not remain valid, when we drop the assumption that $\boldsymbol{\sigma}$ is countable. In fact, we can give an example of a sentence $\varphi_0 \in \mathscr{L}_{\omega_2\omega}$ such that $\mathscr{L}_{\omega\omega}(\varphi_0)$ properly extends $\mathscr{L}_{\omega\omega}$ and is compact. For $\alpha < \omega_1$, let R_α be a binary relation symbol and set $\boldsymbol{\sigma} = \{R_\alpha \mid \alpha < \omega_1\}$. Call a pair $\mathscr{F} = (\mathscr{F}_0, \mathscr{F}_1)$ of finite sets $\mathscr{F}_0$ and $\mathscr{F}_1$ of non-empty finite subsets of ω_1 *good*, if $F \not\subset E$ holds for all $E \in \mathscr{F}_0$ and $F \in \mathscr{F}_1$. (E, F, E^α, ... will always denote finite non-empty subsets of ω_1). Denote by $\varphi_{\mathscr{F}}(x)$ the formula

$$\varphi_{\mathscr{F}}(x) = \bigwedge_{E \in \mathscr{F}_0} \exists y \bigwedge_{\alpha \in E} R_\alpha xy \wedge \bigwedge_{F \in \mathscr{F}_1} \neg\exists y \bigwedge_{\alpha \in F} R_\alpha xy$$

and set $\Phi_0 = \{\exists x\, \varphi_{\mathscr{F}}(x) \mid \mathscr{F} \text{ good}\}$. Let φ_0 be the sentence

$$\varphi_0 = \bigwedge \Phi_0 \rightarrow \psi_0,$$

where

$$\psi_0 = \forall x\left(\left(\bigwedge_F \exists y \bigwedge_{\alpha \in F} R_\alpha xy\right) \rightarrow \exists y \bigwedge_{\alpha < \omega_1} R_\alpha xy\right).$$

Clearly, $\varphi_0 \in \mathscr{L}_{\omega_2\omega}$. Furthermore, (1) and (2) below show that $\mathscr{L}_{\omega\omega}(\varphi_0)$ properly extends $\mathscr{L}_{\omega\omega}$ and is compact.

(1) φ_0 is not equivalent to a first-order sentence.

To show (1), we prove that there are elementarily equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models \neg\varphi_0$ and $\mathfrak{B} \models \varphi_0$. Choose an enumeration $\langle \mathscr{F}^\beta \mid \beta < \omega_1 \rangle$ of all good pairs, say, $\mathscr{F}^\beta = (\mathscr{F}_0^\beta, \mathscr{F}_1^\beta)$ with $\mathscr{F}_0^\beta = \{E_1^\beta, \ldots, E_{m_\beta}^\beta\}$. Also let the $\boldsymbol{\sigma}$-structure $\mathfrak{A}$ be given by:

$$A = \omega_1 \cup \{\omega_1\}, \quad \text{and}$$

$$R_\alpha^A \omega_1 \gamma \quad \text{iff} \quad \alpha < \gamma < \omega_1, \qquad \text{and for } \beta < \omega_1:$$

$$R_\alpha^A \beta\gamma \quad \text{iff} \quad \gamma < m_\beta \quad \text{and} \quad \alpha \in E_{\gamma+1}^\beta.$$

Then $\mathfrak{A} \models \varphi_{\mathscr{F}^\beta}[\beta]$. Hence $\mathfrak{A} \models \Phi_0$, and $x := \omega_1$ shows that $\mathfrak{A} \not\models \psi_0$. Therefore, $\mathfrak{A} \not\models \varphi_0$. On the other hand, any ω-saturated structure $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$ is a model of ψ_0 and hence of φ_0 also.

Set $\mathfrak{K}_1 = \mathrm{Mod}^\sigma(\varphi_0)$ and $\mathfrak{K}_2 = \mathrm{Mod}^\sigma(\neg\varphi_0)$.

(2) $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are compact.

Since any ω-saturated structure is a model of ψ_0, the class $\mathfrak{K}_1$ is compact. Now, assume that $\Phi \cup \{\neg\varphi_0\}$ is finitely satisfiable, where $\Phi \subset \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$ with $\boldsymbol{\sigma} \subset \boldsymbol{\tau}$. We must show that $\Phi \cup \{\neg\varphi_0\}$ has a model. We may assume that the consistent set $\Phi \cup \Phi_0$ has built-in Skolem functions. Let $\Gamma_0(x)$ be maximal among the types $\Gamma(x)$, $\Gamma(x) \subset \mathscr{L}_{\omega\omega}[\boldsymbol{\tau}]$, with the property:

For any good $\mathscr{F}$: $\Phi \cup \Phi_0 \cup \Gamma(x) \cup \{\varphi_{\mathscr{F}}(x)\}$ is consistent.

(Note that $\Gamma(x) := \emptyset$ has this property.) By first-order compactness, there is a model $\mathfrak{A}$ and $a \in A$ such that

$$\mathfrak{A} \models \Phi \cup \Phi_0 \quad \text{and} \quad \mathfrak{A} \models \Gamma_0(x) \cup \left\{\bigwedge_F \exists y \bigwedge_{\alpha \in F} R_\alpha xy\right\}[a].$$

Let $\mathfrak{B}$ be the submodel generated by a. We will complete the proof by showing that $\mathfrak{B} \models \Phi \cup \{\neg\varphi_0\}$. Since $\mathfrak{B} \models \Phi \cup \Phi_0$, it suffices to prove that $\mathfrak{B} \not\models \neg\psi_0$.

In fact, we show $\mathfrak{B} \not\models \exists y \bigwedge_{\alpha<\omega_1} R_\alpha xy[a]$, where this assertion is obtained proving that for any unary Skolem function f

$$\text{there is some } \alpha \text{ such that } \neg R_\alpha x f(x) \in \Gamma_0(x).$$

Otherwise, for each α there is a good $\mathscr{F}^{\beta(\alpha)}$ such that

$$(*) \qquad \Phi \cup \Phi_0 \cup \Gamma_0(x) \cup \{\varphi_{\mathscr{F}^{\beta(\alpha)}}(x)\} \models R_\alpha x f(x).$$

But then by a combinatorial argument which uses a result of Erdös and Hajnal, one obtains $\alpha, \alpha' \in \omega_1$, $\alpha \neq \alpha'$ such that for $\beta := \beta(\alpha)$ and $\beta' := \beta(\alpha')$ the following hold:

$$\alpha \notin E \quad \text{for } E \in \mathscr{F}_0^{\beta'}, \qquad \alpha' \notin E \quad \text{for } E \in \mathscr{F}_0^{\beta}$$

$$E \not\subset F \quad \text{for } (E, F) \in (\mathscr{F}_0^{\beta} \times \mathscr{F}_1^{\beta'}) \cup (\mathscr{F}_0^{\beta'} \times \mathscr{F}_0^{\beta}).$$

Hence, $\mathscr{F} = (\mathscr{F}_0^{\beta} \cup \mathscr{F}_0^{\beta'}, \mathscr{F}_1^{\beta} \cup \mathscr{F}_1^{\beta'} \cup \{\alpha, \alpha'\})$ is good, and $\models \varphi_{\mathscr{F}}(x) \to (\varphi_{\mathscr{F}^{\beta}}(x) \wedge \varphi_{\mathscr{F}^{\beta'}}(x))$,

But then, using $(*)$, we obtain

$$\Phi \cup \Phi_0 \cup \Gamma_0(x) \cup \{\varphi_{\mathscr{F}}(x)\} \models R_\alpha x f(x) \wedge R_{\alpha'} x f(x).$$

But this is a contradiction, since $\models \varphi_{\mathscr{F}}(x) \to \neg \exists y (R_\alpha xy \wedge R_{\alpha'} xy)$.

2.3.4 Notes. Nearly all results of Section 2.1 are contained in Barwise [1974a] or in Lindström's papers [1966a, 1969]. The characterizations of $\mathscr{L}_{\omega\omega}$ in Section 2.2 are due to Lindström [1973a, 1974]. The reader will find a further interesting characterization of $\mathscr{L}_{\omega\omega}$ in Barwise–Moschovakis [1978]: $\mathscr{L}_{\omega\omega}$ is the unique logic with "uniformly inductive" satisfaction relation. Observe also that criteria for first-order axiomatizability of classes of structures such as

$\mathfrak{K}$ is an elementary class iff $\mathfrak{K}$ and its complement are closed under ultraproducts and isomorphisms,

may be rewritten as characterizations of first-order logic. We owe Example 2.3.1 and Theorem 2.3.2 to Gold [1978]. Example 2.3.3 is due to Ziegler (personal communication).

3. *Characterizing* $\mathscr{L}_{\infty\omega}$

This section is devoted to characterizations of $\mathscr{L}_{\infty\omega}$ by means of model-theoretic properties.

The property of a logic $\mathscr{L}$ of being *bounded* is a weakening of the compactness property ($\mathscr{L}$ is bounded, if for any $\mathscr{L}$-sentence $\varphi(<, \ldots)$ having only models with well-ordered $<$, there is an ordinal α such that the order type of $<$ is always less than α). As has been already mentioned in Chapter II this property may be regarded as a model-theoretic substitute for compactness. In fact, for some bounded logics results on non-axiomatizability, preservation thorems, upward Löwenheim–Skolem theorems and so on may be obtained in a way similar to the corresponding results for first-order logic provided one replaces compactness arguments by suitable applications of the boundedness property. This is also illustrated by the proof of Theorem 3.1 given below—a proof the reader should compare with the proof of Lemma 1.1.2.

One may regard the almost-all Löwenheim–Skolem property—the so-called *Kueker property*, which is introduced below—as a substitute for the Löwenheim–Skolem property in this model theoretic sense. Based on an interesting set-theoretical notion of countable approximations to uncountable objects, the Kueker property acts symmetrically on models and sentences. The reader should examine Kueker [1977, 1978] for a more penetrating view of the role of this property in model theory.

$\mathscr{L}_{\infty\omega}$ is bounded and has the Kueker property; and if the compactness and Löwenheim–Skolem property in Lindström's theorem are replaced by these substitutes, we obtain a characterization of $\mathscr{L}_{\infty\omega}$ as a maximal logic. We will derive this result as a consequence of Theorem 3.1, a theorem which shows that $\mathscr{L}_{\infty\omega}$ is a maximal bounded logic with the Karp property. The reader should also consult Chapter XVII, where these results are discussed from a set-theoretical point of view and where further characterizations of $\mathscr{L}_{\infty\omega}$ are obtained.

First, we define $\mathscr{L}_{\infty\omega}$-sentences which characterize the "α-isomorphism type" of a structure: Given an arbitrary τ and a τ-structure $\mathfrak{A}$, for each ordinal α, we introduce an $\mathscr{L}_{\infty\omega}[\tau]$-sentence $\varphi^\alpha_{\mathfrak{A}}$ such that for any $\mathfrak{B}$ the following are equivalent (compare Chapter VIII or Section II.4.2, in which for finite α, the corresponding formulas are introduced for the logic $\mathscr{L}_{\omega\omega}(Q_{\mathfrak{K}})$ with monotone $Q_{\mathfrak{K}}$):

(i) $\mathfrak{B} \models \varphi^\alpha_{\mathfrak{A}}$.
(ii) $\mathfrak{A} \cong_\alpha \mathfrak{B}$ (that is, $\mathfrak{A}$ and $\mathfrak{B}$ are α-isomorphic).
(iii) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same $\mathscr{L}_{\infty\omega}$-sentences of rank $\leq \alpha$.

To define $\varphi^\alpha_{\mathfrak{A}}$, we first introduce by induction on α, for each finite sequence $\mathbf{a} = a_1 \ldots a_n \in A$, an $\mathscr{L}_{\infty\omega}[\tau]$-formula $\varphi^\alpha_{\mathbf{a}}(x_1, \ldots, x_n)$:

$$\begin{aligned}\varphi^0_{\mathbf{a}} = \bigwedge \{\psi(x_1, \ldots, x_n) \mid \mathfrak{A} \models \psi[\mathbf{a}] &\text{ and } \psi \text{ has the form}\\ &(\neg)\, Rx_{i_1} \ldots x_{i_j} \text{ or } (\neg) f(x_{i_1}, \ldots, x_{i_j}) = x_i \text{ or}\\ &(\neg)\, c = x_i \text{ or } (\neg)\, x_j = x_i\},\end{aligned}$$

$$\varphi^{\alpha+1}_{\mathbf{a}} = \bigwedge_{a \in A} \exists x_{n+1}\, \varphi^\alpha_{\mathbf{a}a} \wedge \forall x_{n+1} \bigvee_{a \in A} \varphi^\alpha_{\mathbf{a}a},$$

$$\varphi^\alpha_{\mathbf{a}} = \bigwedge_{\beta < \alpha} \varphi^\beta_{\mathbf{a}} \quad \text{for a limit ordinal } \alpha.$$

Now let $\varphi^\alpha_{\mathfrak{A}}$ be the sentence $\varphi^\alpha_\varnothing$, where $\varnothing$ denotes the empty sequence. An easy induction on α shows the following: in case either $|\tau|$ or α is infinite there are not more than $\beth_{\alpha+1}(|\tau|)$ sentences (pairwise non-equivalent) of the form $\varphi^\alpha_{\mathfrak{A}}$ and each such sentence $\varphi^\alpha_{\mathfrak{A}}$ belongs to $\mathscr{L}_{\beth_\alpha(|\tau|)^+\omega}[\tau]$. Otherwise their number is finite and each is a first-order sentence. Recall that the sequence of beth cardinals $\beth_\alpha(\kappa)$, where κ is a cardinal and α an ordinal, is defined by: $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$ and $\beth_\alpha(\kappa) = \sup\{\beth_\beta(\kappa) \mid \beta < \alpha\}$, if α is a limit ordinal. Write $\beth_\alpha$ for $\beth_\alpha(0)$; in particular, $\beth_\omega = \omega$.

We adapt the proof methods used in Section 1.1 to show

3.1 Theorem. *Assume $\mathscr{L}$ is a regular logic with $\mathscr{L}_{\infty\omega} \le \mathscr{L}$. If $\mathscr{L}$ is bounded and has the Karp property, then $\mathscr{L} \equiv \mathscr{L}_{\infty\omega}$.*

Proof. By contradiction suppose that φ is an $\mathscr{L}$-sentence not equivalent to an $\mathscr{L}_{\infty\omega}$-sentence. For an ordinal α, let

$$\chi^\alpha = \bigvee \{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}.$$

Then, by the preceding remarks, χ^α is an $\mathscr{L}_{\infty\omega}$-sentence and $\models \varphi \to \chi^\alpha$. Therefore $\not\models \chi^\alpha \to \varphi$. That is, for some $\mathfrak{B}_\alpha$, $\mathfrak{B}_\alpha \models \chi^\alpha$, but $\mathfrak{B}_\alpha \models \neg\varphi$. By the definition of χ^α there exists $\mathfrak{A}_\alpha$ such that $\mathfrak{A}_\alpha \models \varphi$ and $\mathfrak{B}_\alpha \models \varphi^\alpha_{\mathfrak{A}_\alpha}$. Hence, $\mathfrak{A}_\alpha \cong_\alpha \mathfrak{B}_\alpha$. Summarizing, we thus have:

$$(*) \qquad \begin{array}{l} \text{for each ordinal } \alpha \text{ there are } \mathfrak{A}_\alpha \text{ and } \mathfrak{B}_\alpha \text{ such that} \\ \mathfrak{A}_\alpha \models \varphi, \qquad \mathfrak{B}_\alpha \models \neg\varphi \quad \text{and} \quad \mathfrak{A}_\alpha \cong_\alpha \mathfrak{B}_\alpha. \end{array}$$

Coding partial isomorphisms (as in the proof of Lemma 1.1.2), we obtain an $\mathscr{L}$-sentence which contains among others relation symbols V, W (unary) and $\le$, I (binary), and which expresses:

> "the V-part is a model of φ, the W-part a model of $\neg\varphi$; $<$ is an ordering, for each x in its field $Ix\cdot$ is a non-empty set of partial isomorphisms from the V-part to the W-part, and the sequence $Ix\cdot$ with x in the field of $<$ has the back and forth property."

By $(*)$, for each ordinal α, ψ has a model such that $<$ is well-ordered of order type $\ge\alpha$. Since $\mathscr{L}$ is bounded, ψ has a non-well-ordered model $\mathfrak{D}$. Then $V^{\mathfrak{D}}$ is a model of φ, $W^{\mathfrak{D}}$ a model of $\neg\varphi$. And, as in the preceding proofs (see Lemma 1.1.2), by choosing an element in the field of $<$ with an infinite descending sequence of predecessors, one shows that $V^{\mathfrak{D}}$ and $W^{\mathfrak{D}}$ are partially isomorphic. But this is a contradiction, since $\mathscr{L}$ was assumed to have the Karp property. ▯

Observe that in case $\mathscr{L}$ has the finite occurrence property, we can omit the hypothesis $\mathscr{L}_{\infty\omega} \le \mathscr{L}$ in the preceding theorem and obtain $\mathscr{L} \le \mathscr{L}_{\infty\omega}$ as conclusion. We state some results that are obtained by slight changes in the last proof. For $\kappa = \omega$ the following theorem is essentially the characterization of $\mathscr{L}_{\omega\omega}$ as given in Theorem 2.1.1. (Consult Section II.5.2 for the definition and properties of the well-ordering number of a logic.)

3.2 Theorem. *Suppose κ is a cardinal and $\kappa = \beth_\kappa$. Assume also that $\mathscr{L}$ with $\mathscr{L}_{\kappa\omega} \leq \mathscr{L}$ is a regular logic with occurrence number $\leq \kappa$. If the well-ordering number of $\mathscr{L}$ is $\leq \kappa$ and $\mathscr{L}$ has the Karp property, then $\mathscr{L} \equiv \mathscr{L}_{\kappa\omega}$.*

Proof. We employ the notations used in the proof of Theorem 3.1 and note that in case the well-ordering number of $\mathscr{L}$ is $\leq \kappa$ this proof shows that any $\varphi \in \mathscr{L}[\tau]$ with $|\tau| < \kappa$ is equivalent to some χ^α with $\alpha < \kappa$. For any $\beta, \lambda < \kappa$ we have $\beth_\beta(\lambda) \leq \beth_{\lambda+\beta} < \beth_\kappa = \kappa$. Hence, $\chi^\alpha \in \mathscr{L}_{\kappa\omega}$ by the above remarks on the number of non-equivalent sentences of the form $\varphi^\alpha_{\mathfrak{A}}$. $\square$

3.3 Remarks. (a) Clearly, one can generalize Theorems 3.1 and 3.2 in the spirit of the "separation theorem" 1.1.3 and, for example, derive: *Assume that $\mathscr{L}$ with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$ is a logic with the relativization property and closed under (finitary) conjunctions and disjunctions. If $\mathscr{L}$ is bounded and has the Karp property, then any two disjoint $\mathscr{L}$-classes can be separated by an $\mathscr{L}_{\infty\omega}$-class* (the reader should consult Makowsky–Shelah–Stavi [1976], where this result is stated for $\mathscr{L} = \Delta(\mathscr{L}_{\infty\omega})$).

In fact, if $\mathrm{Mod}(\varphi)$ and $\mathrm{Mod}(\psi)$ are disjoint $\mathscr{L}$-classes not separable by an $\mathscr{L}_{\infty\omega}$-class, for each α define χ^α as above. Then there are $\mathfrak{A}_\alpha$ and $\mathfrak{B}_\alpha$ such that $\mathfrak{A}_\alpha \models \varphi$, $\mathfrak{B}_\alpha \models \psi$, and $\mathfrak{B}_\alpha \models \varphi^\alpha_{\mathfrak{A}_\alpha}$, and we obtain a contradiction as in Theorem 3.1.

(b) Suppose $\mathscr{L}$ is a regular logic with the Karp property. For an $\mathscr{L}$-sentence φ and an ordinal α, let $\chi^\alpha = \bigvee\{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}$. Then $\models (\bigwedge_{\alpha\,\text{ordinal}} \chi^\alpha) \to \varphi$. In fact, suppose for the sake of argument that for all α, $\mathfrak{B} \models \chi^\alpha$ and $\mathfrak{B} \models \neg\varphi$. Let $\kappa = |B|^+$. Choose $\mathfrak{A} \models \varphi$ such that $\mathfrak{A} \cong_\kappa \mathfrak{B}$. We show that $\mathfrak{A} \cong_p \mathfrak{B}$ which—in view of $\mathfrak{B} \models \neg\varphi$ and $\mathfrak{A} \models \varphi$—contradicts the assumption "$\mathscr{L}$ has the Karp property". From $\mathfrak{A} \cong_\kappa \mathfrak{B}$, we obtain $\mathfrak{A} \equiv_{\mathscr{L}_{\kappa\omega}} \mathfrak{B}$, since each $\mathscr{L}_{\kappa\omega}$-sentence has quantifier rank $< \kappa$. Hence, $\mathfrak{A} \equiv_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$, because each $\mathscr{L}_{\infty\omega}$-sentence is equivalent in $\mathrm{Th}_{\mathscr{L}_{\kappa\omega}}(\mathfrak{B})$ to an $\mathscr{L}_{\kappa\omega}$-sentence (see Flum [1971c]). Thus $\mathfrak{A} \cong_p \mathfrak{B}$. Summarizing, we have shown: *Assume $\mathscr{L}$ is a logic with the Karp property. Then for any $\mathscr{L}$-sentence φ we have $\models \varphi \leftrightarrow \bigwedge_{\alpha\,\text{ordinal}} \chi^\alpha$, where $\chi^\alpha = \bigvee\{\varphi^\alpha_{\mathfrak{A}} \mid \mathfrak{A} \models \varphi\}$.*

Since $\mathscr{L}_{\infty G}$ is a logic with the Karp property, this result applies to $\mathscr{L}_{\infty G}$ (see Keisler [1968a] and compare with Chapter XVII for a more general version).

(c) For a generalized quantifier Q one can extend the preceding results to logics $\mathscr{L}$ of the form $\mathscr{L} = \mathscr{L}_{\infty\omega}(Q)$ or $\mathscr{L} = \mathscr{L}_{\kappa\omega}(Q)$, if there is an appropriate characterization of $\mathscr{L}$-equivalence by means of partial isomorphisms and if there are $\mathscr{L}$-sentences which play the rôle of the formulas $\varphi^\alpha_{\mathfrak{A}}$. For example, if $Q = Q_1$, that is, in case Q is the quantifier "there are uncountable many" and if we define the "$\aleph_1$-Karp property" as suggested by the corresponding back and forth notions for $\mathscr{L}_{\omega\omega}(Q_1)$ (see Section II.4.2), we then obtain (the reader is referred to Caicedo [1981b] for further results in this direction)

> If $\mathscr{L}$ with $\mathscr{L}_{\infty\omega}(Q_1) \leq \mathscr{L}$ is bounded and has the "$\aleph_1$-Karp property", then $\mathscr{L} \equiv \mathscr{L}_{\infty\omega}(Q_1)$.

From now on in this section we will assume that all logics under consideration are built up by set-theoretical principles so that their sentences are sets.

We will quickly review some definitions and results concerning the notions of approximations of sets and of the closed unbounded filter, and ask the reader to consult Barwise [1974b] for details.

We work in a universe of sets and urelements and define for any sets x and s the *approximation* x^s *of* x *in* s by $\in$-recursion:

$$p^s = p \quad \text{if } p \text{ is an urelement,}$$

$$x^s = \{y^s \mid y \in x \cap s\}, \quad \text{if } x \text{ is a set.}$$

Let M be a transitive set and let I be the set $P_{\omega_1}(M)$ of all countable subsets of M. The closed unbounded filter on M consists of all $X \subset I$ such that for some $X^0 \subset X$:

(i) every $s \in I$ is a subset of some $s' \in X^0$; and
(ii) X^0 is closed under unions of countable chains.

Let $\mathfrak{R}$ be an n-ary predicate of sets and urelements. For given $x_1, \ldots, x_n$ in a transitive set M, we say that $\mathfrak{R}x_1^s \ldots x_n^s$ *holds for almost all countable* s, if the set

$$\{s \in P_{\omega_1}(M) \mid \mathfrak{R}x_1^s \ldots x_n^s\}$$

is a member of the closed unbounded filter on M. This notion is independent of the particular transitive set M containing $x_1, \ldots, x_n$.

We say that a predicte $\mathfrak{R}$ of sets and urelements is Σ, if it is definable by a Σ-formula of set theory. Barwise [1974b] generalized Levy's Absoluteness Lemma and showed:

3.4 Proposition. *Let* $\mathfrak{R}$ *be an n-ary* Σ*-predicate. If* $\mathfrak{R}x_1 \ldots x_n$, *then* $\mathfrak{R}x_1^s \ldots x_n^s$ *for almost all countable s.* ☐

We assume that vocabularies and universes of structures consist of urelements only. Then for almost all countable s:

$$\mathfrak{A}^s \text{ is the } \tau^s\text{-substructure of } \mathfrak{A} \upharpoonright \tau^s \text{ with universe } A \cap s,$$

and for any $\mathscr{L}_{\infty\omega}$-sentence φ and almost all s,

$$\varphi^s = \varphi^{[s]},$$

where $\varphi^{[s]} = \varphi$, if φ is atomic;

$$(\neg\varphi)^{[s]} = \neg\varphi^{[s]};$$

$$(\exists x\varphi)^{[s]} = \exists x\, \varphi^{[s]};$$

and

$$(\bigvee \Phi)^{[s]} = \bigvee \{\varphi^{[s]} \mid \varphi \in \Phi \cap s\}.$$

Here—as also in the proof of Theorem 3.6 below—we assume that the operations $\varphi \mapsto \neg\varphi$, $\varphi \mapsto \exists x\varphi$, ... are "simple" operations, say Σ-operations. Thus, for example, $(\neg\varphi)^s$ and $\neg(\varphi^s)$ are equal for almost all countable s.

"$\mathfrak{A} \cong_p \mathfrak{B}$" and "$\mathfrak{A} \models \varphi$" for an $\mathscr{L}_{\infty\omega}$-sentence φ are Σ-predicates of $\mathfrak{A}$ and $\mathfrak{B}$, resp. $\mathfrak{A}$ and φ. Therefore, using Proposition 3.4, we obtain (see Barwise [1974b]):

3.5 Proposition. (a) *If $\mathfrak{A} \cong_p \mathfrak{B}$, then $\mathfrak{A}^s \cong \mathfrak{B}^s$ for almost all countable s.*

(b) *If φ is an $\mathscr{L}_{\infty\omega}$-sentence, then*

$$\mathfrak{A} \models \varphi \quad \textit{implies} \quad \mathfrak{A}^s \models \varphi^s \textit{ for almost all countable } s. \quad \square$$

We say that a logic $\mathscr{L}$ has the *Kueker property*, if for any $\mathscr{L}$-sentence φ, $\mathfrak{A} \models \varphi$ implies $\mathfrak{A}^s \models \varphi^s$ for almost all countable s. Thus, in particular, we assume that φ^s is an $\mathscr{L}$-sentence for almost all countable s.

In particular, $\mathscr{L}_{\infty\omega}$ is a logic with the Kueker property. Moreover—as was announced in the troduction to this section—this property together with the boundedness property characterize $\mathscr{L}_{\infty\omega}$.

3.6 Theorem. *Let $\mathscr{L}$ be a regular logic with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$. If $\mathscr{L}$ is bounded and has the Kueker property, then $\mathscr{L} \equiv \mathscr{L}_{\infty\omega}$.*

Proof. By Theorem 3.1 it suffices to show that $\mathscr{L}$ has the Karp property. So let φ be an $\mathscr{L}$-sentence and suppose that $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \cong_p \mathfrak{B}$. For the sake of argument suppose that $\mathfrak{B} \models \neg\varphi$. Then, by Proposition 3.5 and the Kueker property, we have for almost all countable s

$$\mathfrak{A}^s \cong \mathfrak{B}^s, \qquad \mathfrak{A}^s \models \varphi^s \quad \text{and} \quad \mathfrak{B}^s \models \neg\varphi^s,$$

a contradiction. $\square$

What is the corresponding separation property of $\mathscr{L}_{\infty\omega}$? Let φ and ψ be sentences of a logic with the Kueker property. Consider the following properties (i) and (ii) of φ and ψ:

(i) $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \varnothing$;
(ii) $\mathrm{Mod}(\varphi^s) \cap \mathrm{Mod}(\psi^s) = \varnothing$ for almost all countable s.

Clearly, (ii) implies (i). However, in general, (i) does not imply (ii); for otherwise the next theorem would show that $\mathscr{L}_{\infty\omega}$ has the interpolation property. This theorem contains the separation result corresponding to the maximality result of Theorem 3.6.

3.7 Theorem. *Suppose $\mathscr{L}$ with $\mathscr{L}_{\infty\omega} \leq \mathscr{L}$ is a logic closed under finitary conjunctions and disjunctions and has the relativization property. Assume that $\mathscr{L}$ is bounded and has the Kueker property, and let φ and ψ be $\mathscr{L}$-sentences. If* $\mathrm{Mod}(\varphi^s) \cap \mathrm{Mod}(\psi^s) = \varnothing$ *for almost all countable s then for some $\chi \in \mathscr{L}_{\infty\omega}$ and almost all countable s*

$$\mathrm{Mod}(\varphi^s) \subset \mathrm{Mod}(\chi^s) \quad \textit{and} \quad \mathrm{Mod}(\chi^s) \cap \mathrm{Mod}(\psi^s) = \varnothing,$$

and consequently,

$$\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\chi) \quad \textit{and} \quad \mathrm{Mod}(\chi) \cap \mathrm{Mod}(\psi) = \varnothing.$$

Proof. For an ordinal α, let $\chi^\alpha = \bigvee \{\varphi_{\mathfrak{A}}^\alpha \mid \mathfrak{A} \models \varphi\}$. Then χ^α is an $\mathscr{L}_{\infty\omega}$- sentence with $\models \varphi \to \chi^\alpha$. If $\mathrm{Mod}(\chi^\alpha) \cap \mathrm{Mod}(\psi) = \varnothing$ holds for some α, then we let $\chi = \chi^\alpha$. Otherwise, for each α there are structures $\mathfrak{A}_\alpha$ and $\mathfrak{B}_\alpha$ such that $\mathfrak{A}_\alpha \models \varphi$, $\mathfrak{B}_\alpha \models \psi$, and $\mathfrak{A}_\alpha \cong_\alpha \mathfrak{B}_\alpha$. Using the boundedness property of $\mathscr{L}$ and arguing as in the proof of Theorem 3.1, we obtain structures $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$\mathfrak{A} \models \varphi, \qquad \mathfrak{B} \models \psi \quad \text{and} \quad \mathfrak{A} \cong_p \mathfrak{B}.$$

Hence, $\mathfrak{A}^s \models \varphi^s$, $\mathfrak{B}^s \models \psi^s$, and $\mathfrak{A}^s \cong \mathfrak{B}^s$ for almost all countable s—a contradiction. ◻

Taking as $\mathscr{L}$ the Σ_1^1-sentences over $\mathscr{L}_{\infty\omega}$, Theorem 3.7 above is Theorem 2 in Kueker [1978].

3.8 Notes. Theorems 3.1 and 3.2 are due to Barwise [1974a]. $\mathscr{L}_{\omega_1\omega}$ is a well-behaved logic with a fruitful model theory. For the problem of characterizing $\mathscr{L}_{\omega_1\omega}$, the reader is referred to Barwise [1972a], Gostanian-Hrbacek [1980], and Harrington [1980].

4. *Characterizing Cardinality Quantifiers*

In this section we characterize the logics $\mathscr{L}_{\omega\omega}(Q_\alpha)$ with the quantifier "there are $\aleph_\alpha$-many" among the logics of the form $\mathscr{L}_{\omega\omega}(Q)$, where Q is a unary quantifier.

Given a unary Lindström quantifier Q and a non-empty set A, let $Q(A)$ be the set of "big" subsets of A,

$$Q(A) = \{X \subset A \mid (A, X) \models QyUy\}.$$

In the terminology of Chapter II, Q is the quantifier associated with the class $\mathfrak{K} = \{(A, X) \mid A \neq \varnothing, X \in Q(A)\}$. Clearly

$$\text{(1)} \qquad \text{if} \quad (A, X) \cong (B, Y), \quad \text{then} \quad (X \in Q(A) \text{ iff } Y \in Q(B)).$$

Throughout this section all quantifiers are assumed to be unary. We call a quantifier Q monotone, if

$$X \in Q(A) \quad \text{and} \quad X \subset Y \subset A \quad \text{imply} \quad Y \in Q(A).$$

We now list some examples of monotone quantifiers:

the existential quantifier $\exists$, $\exists(A) = \{X \subset A \mid X \neq \varnothing\}$,

the quantifier Q_α, $Q_\alpha(A) = \{X \subset A \mid |X| \geq \aleph_\alpha\}$,

the Chang quantifier Q_c, $Q_c(A) = \{X \subset A \mid |X| \geq \aleph_0, |X| = |A|\}$,

the "non-cofinal complement" quantifier Q_{ncc}, where

$$Q_{\text{ncc}}(A) = \{X \subset A \mid |A \backslash X| < \text{cf}(|X|)\}.$$

The dual quantifier Q^{d} of a quantifier Q is defined by

$$Q^{\text{d}}(A) = \{X \subset A \mid (A \backslash X) \notin Q(A)\}.$$

Observe that $Q^{\text{d}} y\varphi(y)$ is equivalent to $\neg Qy \neg \varphi(y)$ and that Q^{d} is monotone, if Q is monotone. Clearly, we have

(2) $$\mathscr{L}_{\omega\omega}(Q^{\text{d}}) \equiv \mathscr{L}_{\omega\omega}(Q).$$

The main result of this section is the following characterization of the logics of the form $\mathscr{L}_{\omega\omega}(Q_\alpha)$.

4.1 Theorem. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ is a regular logic where Q is a monotone quantifier. Then*

$$\mathscr{L} \equiv \mathscr{L}_{\omega\omega} \quad \textit{or} \quad \mathscr{L} \equiv \mathscr{L}_{\omega\omega}(Q_\alpha) \quad \textit{for some } \alpha.$$

As an immediate consequence of this theorem, we obtain:

4.2 Corollary. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ with $\mathscr{L}_{\omega\omega} < \mathscr{L}$ is a regular logic, where Q is a monotone quantifier. If $\mathscr{L}$ has Löwenheim number $\aleph_\alpha$, then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}(Q_\alpha)$.* □

($\mathscr{L}$ has Löwenheim number κ, if any satisfiable $\mathscr{L}$-sentence has a model of power $\leq \kappa$, and κ is the least cardinal with this property.)

To prove Theorem 4.1, we must introduce some terminology and notation.

For $n \in \omega$ let $\exists^{\geq n}$, Q_α^n, and Q_α^{cn} be the monotone quantifiers definable in $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\omega\omega}(Q_\alpha)$, respectively, by:

$\exists^{\geq n} x\varphi :\leftrightarrow$ "there are at least n elements x satisfying φ";

$$Q_\alpha^n x\varphi :\leftrightarrow (Q_\alpha x\, x = x \to Q_\alpha x\varphi) \wedge (\neg Q_\alpha x\, x = x \to \exists^{\geq n} x\varphi);$$

$$Q_\alpha^{cn} x\varphi :\leftrightarrow (Q_\alpha x\, x = x \to Q_\alpha x\varphi) \wedge (\neg Q_\alpha x\, x = x \to \exists^{< n} x \neg \varphi);$$

where $\exists^{<n}$ means "there are less than n".

Clearly, $\exists^{\geq n}$, Q_α^n, and Q_α^{cn} are monotone and

$$(3) \qquad \mathscr{L}_{\omega\omega}(\exists^{\geq n}) \equiv \mathscr{L}_{\omega\omega}; \qquad \mathscr{L}_{\omega\omega}(Q_\alpha^n) \equiv \mathscr{L}_{\omega\omega}(Q_\alpha); \qquad \mathscr{L}_{\omega\omega}(Q_\alpha^{cn}) \equiv \mathscr{L}_{\omega\omega}(Q_\alpha).$$

(For example, that $\mathscr{L}_{\omega\omega}(Q_\alpha) \leq \mathscr{L}_{\omega\omega}(Q_\alpha^n)$ holds is shown by

$$\models Q_\alpha x\varphi \leftrightarrow (Q_\alpha^n x\varphi \wedge \forall x_1 \ldots \forall x_n \neg Q_\alpha^n y\,(y = x_1 \vee \cdots \vee y = x_n)).)$$

Let Q be an arbitrary monotone quantifier. By the isomorphism condition (1) stated at the beginning of this section, whether $X \in Q(A)$ holds or not only depends on the cardinalities of the sets A, X and $A \backslash X$. We associate with Q a function g $(= g^Q)$ defined on the class of non-zero cardinals which maps each cardinal $\lambda \neq 0$ on a pair of cardinals, $g(\lambda) = (\mu, \nu)$, where for any A with $|A| = \lambda$,

$$\mu = \lambda \quad \text{and} \quad \nu = 0, \quad \text{if } Q(A) = \varnothing,$$

and otherwise

$$\mu = \inf\{|X| \mid X \in Q(A)\}, \qquad \nu = \sup\{|A\backslash X|^+ \mid X \in Q(A)\}.$$

Then, by monotonicity,

$$Q(A) = \{X \subset A \mid |X| \geq \mu, |A\backslash X| < \nu\},$$

and hence Q is uniquely determined by g. Moreover, note that $\mu \leq \lambda$, $\nu \leq \lambda^+$ and $\mu + \nu \leq \lambda^+$.

In particular,

$$g^{Q_\alpha}(\lambda) = \begin{cases} (\lambda, 0) & \text{for } \lambda < \aleph_\alpha, \\ (\aleph_\alpha, \lambda^+) & \text{for } \lambda \geq \aleph_\alpha. \end{cases}$$

Given monotone quantifiers Q and Q', we say that Q and Q' are *eventually equal*, if there is $n_0 \in \omega$ such that for all $\lambda \geq n_0$, $g^Q(\lambda) = g^{Q'}(\lambda)$. Clearly,

(4) if Q and Q' are eventually equal, then $\mathscr{L}_{\omega\omega}(Q) \equiv \mathscr{L}_{\omega\omega}(Q')$.

In view of (2)–(4), Theorem 4.1 is an immediate consequence of the following lemma.

4.3 Lemma. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ is a regular logic with a monotone quantifier Q. Then for some ordinal α and some $n \in \omega$, Q or its dual is eventually equal to*

$$\exists^{\geq n} \quad or \quad Q_\alpha^n \quad or \quad Q_\alpha^{cn}.$$

Proof. Denote by g the function g^Q. We establish the lemma by showing the following claims (i)–(v):

(i) If $\omega \leq \lambda < \mu$ and $g(\lambda) \neq (\lambda, 0)$, then $g(\mu) \neq (\mu, 0)$.
(ii) Suppose $\lambda \geq \omega$ and $n \in \omega$.

If $g(\lambda) = (\lambda, n)$ then there is $m_0 \in \omega$ such that for all $m \geq m_0$ $g(m) = (m - n + 1, n)$.

If $g(\lambda) = (n, \lambda^+)$ then there is $m_0 \in \omega$ such that for all $m \geq m_0$ $g(m) = (n, m - n + 1)$.

By (i) and (ii) we see that in case there is no $\lambda \geq \omega$ such that $g(\lambda) = (\mu, \nu)$ with infinite μ and ν, then Q or Q^d is eventually equal to $\exists^{\geq n}$.

Now, let $\lambda_0 = \inf \Lambda$ where $\Lambda = \{\lambda \mid g(\lambda) = (\mu, \nu)$ for some infinite $\mu, \nu\}$ is assumed to be non-empty.

(iii) $g(\lambda_0) = (\lambda_0, \lambda_0^+)$ or $g(\lambda_0) = (\lambda_0, \lambda_0)$.
(iv) If $\lambda_0 = \omega$ then for some m_0 and $n \in \omega$ we have
for all $m \geq m_0$, $\quad g(m) = (n, m - n + 1)$ or
for all $m \geq m_0$, $\quad g(m) = (m - n + 1, n)$.
(v) If $g(\lambda_0) = (\lambda_0, \lambda_0^+)$ then for $\lambda \geq \lambda_0$, $g(\lambda) = (\lambda_0, \lambda^+)$.

Let us show, for example, for the case $\omega < \lambda_0, g(\lambda_0) = (\lambda_0, \lambda_0)$ and $g(\omega) = (n, \omega^+)$, how we obtain from (i)–(v) the assertion of the lemma. For the dual quantifier Q^d, we have $g^d(\lambda_0) = (\lambda_0, \lambda_0^+)$ and $g^d(\omega) = (\omega, n)$. Hence, by (v)

$$g^d(\lambda) = (\lambda_0, \lambda^+) \quad \text{for } \lambda \geq \lambda_0,$$

and by (ii) there is $m_0 \in \omega$ such that

$$g^d(m) = (m - n + 1, n) \quad \text{for } m \geq m_0.$$

Thus for α with $\aleph_\alpha = \lambda_0$ we have

Q^d is eventually equal to Q_α^{cn}.

The proofs of (i)–(v) make essential use of the relativization property. We sketch the idea underlying these proofs. Suppose, for example, that $g(\lambda) = (\mu, \lambda^+)$, where $\mu = \aleph_\alpha$; that is, Q is the quantifier Q_α in models of power λ. Then each $\mathscr{L}_{\omega\omega}(Q)$-sentence is equivalent to an $\mathscr{L}_{\omega\omega}(Q_\alpha)$-sentence in models of power λ. Now for unary relations symbols U and P let φ be the relativization of $QxPx$ to U; that is, we let $\varphi = (QxPx)^U$. Then for $\mathfrak{A} = (A, U^A, P^A)$ with $U^A \supset P^A$ we have

$$(*) \qquad (A, U^A, P^A) \models \varphi \quad \text{iff} \quad P^A \in Q(U^A).$$

Let ψ be an $\mathscr{L}_{\omega\omega}(Q_\alpha)$-sentence equivalent to φ in models of power λ. By (*), we obtain the possible values of $g^Q(\rho)$ for $\rho < \lambda$—if we determine the expressive power of $\mathscr{L}_{\omega\omega}(Q_\alpha)$-sentences in structures of cardinality λ of the above form. This can be done with the back-and-forth methods of Chapter II. □

4.4 Remark. (a) One can use the idea of the preceding proof to determine the logics with the relativization property in more general cases, for example, in the cases of logics of the form $\mathscr{L}_{\omega\omega}(Q^1, \ldots, Q^n)$ with unary monotone $Q^1, \ldots, Q^n$.

(b) Since the proof of Lemma 4.3 is given in a way that only unary relation symbols are used, we see that in case we restrict to logics for monadic vocabularies the statement corresponding to Theorem 4.1 is true.

We now state yet another immediate consequence of Theorem 4.1.

4.5 Theorem. *Suppose $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q)$ with $\mathscr{L}_{\omega\omega} < \mathscr{L}$ is a regular logic with a monotone quantifier. If Q is trivial for finite sets, that is, $Q(A) = \varnothing$ for finite A, then for some α*

$$Q = Q_\alpha \quad \text{or} \quad Q = Q_\alpha^{\mathrm{d}}.$$

If, moreover,

$$X \cup Y \in Q(A) \quad \text{implies} \quad X \in Q(A) \quad \text{or} \quad Y \in Q(A),$$

then $Q = Q_\alpha$. □

Caicedo [1981b] calls a monotone quantifier a *cofilter quantifier*, if for any A and $X, Y \subset A$

$$X \cup Y \in Q(A) \quad \text{implies} \quad X \in Q(A) \quad \text{or} \quad Y \in Q(A).$$

Then for finite A we have $Q(A) = P(A)$, $Q(A) = P(A)\backslash\{\varnothing\}$ or $Q(A) = \varnothing$. Denote by Card and Card_∞ the class of non-zero cardinals and the class of infinite cardinals, respectively. If $f\colon \mathrm{Card} \to \{0, 1\} \cup \mathrm{Card}_\infty$ is a function, let Q_f be the quantifier given by

$$Q_f(A) = \{X \subset A \mid |X| \geq f(|A|)\}.$$

Clearly, Q_f is a cofilter quantifier (observe that we do not require that $\mathscr{L}_{\omega\omega}(Q_f)$ has the relativization property). Moreover, we have

4.6 Theorem. If Q is a cofilter quantifier, then $Q = Q_f$ for some $f\colon \mathrm{Card} \to \{0, 1\} \cup \mathrm{Card}_\infty$.

Proof. Note that a function $f\colon \mathrm{Card} \to \{0, 1\} \cup \mathrm{Card}_\infty$ is well defined by

$$f(|A|) = \begin{cases} \inf\{|X| \mid X \in Q(A)\} & \text{if } Q(A) \neq \varnothing, \\ \sup\{\omega, |A|^+\} & \text{if } Q(A) = \varnothing. \end{cases}$$

We show that for arbitrary A

$$(*) \qquad Q(A) = \{X \subset A \mid |X| \geq f(|A|)\},$$

that is, we show that $Q = Q_f$.

Clearly, $(*)$ holds if $Q(A) = \emptyset$. Now suppose $Q(A) \neq \emptyset$. If $f(|A|)$ is finite, then $f(|A|)$ is either 0 or 1, and $(*)$ holds by monotonicity. Let $f(|A|) = \mu$ be infinite. Then, by monotonicity $(*)$ holds, once we have established:

$(^*_*)$ There is an $X \in Q(A)$ such that $|X| = \mu$ and $|A \setminus X| \geq \mu$.

Otherwise, by definition of μ, we have $|A| = \mu$. Take any $Y \subset A$ with $|Y| = \mu$ and $|A \setminus Y| = \mu$. Since $Y \cup (A \setminus Y) = A$ and $A \in Q(A)$, we must have, by the cofilter property, $Y \in Q(A)$ or $(A \setminus Y) \in Q(A)$. But then $X := Y$ or $X := A \setminus Y$ satisfies $(^*_*)$. $\square$

4.7 Notes. Theorem 4.1 is new here. As is shown by its proof, the theorem tells us that relativization is a strong property. Theorem 4.6 is due to Caicedo [1981b].

5. *A Lindström-Type Theorem for Invariant Sentences*

Lindström's theorem tells us that for algebraic structures of the logics satisfying the compactness and the Löwenheim–Skolem theorem, first-order logic is a maximal logic. Are there maximal logics with these properties for other kinds of structures—for instance, for topological structures? By isolating the main assumptions and ideas of the proof of Lindström's theorem, we will be able to prove an abstract maximality theorem for ordinary structures. The general character of this theorem will enable us to obtain maximal logics for certain classes of structures, in particular, for the class of topological structures.

Let R be a binary relation between structures and φ a sentence of a logic $\mathscr{L}$. We say that φ is *R-invariant* if

$$\mathfrak{A} R \mathfrak{B} \quad \text{and} \quad \mathfrak{A} \models \varphi \quad \text{imply} \quad \mathfrak{B} \models \varphi.$$

Denote the class of R-invariant sentences of $\mathscr{L}$ by $\mathscr{L}^R$. In case $\mathscr{L} = \mathscr{L}^R$, we say that $\mathscr{L}$ is a *logic of R-invariant sentences.* In particular, if a logic $\mathscr{L}$ is given, then $\mathscr{L}^R$ is a logic of R-invariant sentences.

5.1. *Let $\mathscr{L}$ be a logic with the Löwenheim–Skolem property and suppose that R_1 and R_2 are binary relations between structures. If R_1 and R_2 are* PC *in $\mathscr{L}$ and agree on countable structures, then $\mathscr{L}^{R_1} = \mathscr{L}^{R_2}$.* $\square$

Let R be the relation $\cong_p$ of partial isomorphism. Logics of $\cong_p$-invariant sentences are precisely logics with the Karp property. Thus, Theorem 2.1.1 can now be stated in the following form:

5.2. *Among the logics of $\cong_p$-invariant sentences $\mathscr{L}_{\omega\omega}^{\cong_p}$ is a maximal compact logic.*

(Observe that in Theorem 2.1.1 we needed only countable compactness, since we restricted to logics with the finite occurrence property). Since $\cong_p$ and the relation $\cong$ of isomorphism agree on countable structures, we obtain from 5.2 using 5.1:

5.3. *Among the logics of $\cong$-invariant sentences $\mathscr{L}^{\cong}_{\omega\omega}$ is a maximal logic with the Löwenheim–Skolem and the compactness property.*

But $\mathscr{L}^{\cong} = \mathscr{L}$ holds for any logic, hence the result in 5.3 is precisely Lindström's first theorem.

Similarly, Theorem 3.1 can be stated in the form:

5.4. *Among the logics of $\cong_p$-invariant sentences $\mathscr{L}^{\cong_p}_{\omega\omega}$ is a maximal bounded logic.*

$\cong_p$ is a relation between structures PC in $\mathscr{L}_{\omega\omega}$. For each ordinal α, the relation $\cong_\alpha$ of α-isomorphism is an "approximation" of $\cong_p$. For finite n, $\cong_n$ is explicitly definable in $\mathscr{L}_{\omega\omega}$ in the sense that for any structure $\mathfrak{A}$, there is a sentence $\varphi^n_{\mathfrak{A}} \in \mathscr{L}_{\omega\omega}$ such that for arbitrary $\mathfrak{B}$,

$$\mathfrak{B} \models \varphi^n_{\mathfrak{A}} \quad \text{iff} \quad \mathfrak{A} \cong_n \mathfrak{B}.$$

The following "abstract maximality theorem" is obtained from 5.2 replacing $\cong_p$ by an arbitrary relation R having all the properties of $\cong_p$ and its approximations $\cong_n$ that are used in the proof of Lindström's theorem. Essentially, Theorem 5.5 tells us that in case R is itself definable by R-invariant first-order sentences and has definable approximations, then $\mathscr{L}^{R}_{\omega\omega}$ is a maximal compact logic of R-invariant sentences.

Note that Theorem 5.5 deals with many-sorted logics. For the sake of simplicity, we restrict to finite vocabularies. In the following, the term "logic" will always mean "many-sorted logic" in the sense of Chapter II. Furthermore, if it is not otherwise stated, we will always assume closure under boolean operations.

5.5 Theorem. *Suppose there is given for any vocabulary τ, a set $\Phi^\tau \subset \mathscr{L}_{\omega\omega}[\tau]$ and let $\mathfrak{K}^\tau = \mathrm{Mod}(\Phi^\tau)$. Assume that R is a binary relation between structures such that $\mathfrak{A}R\mathfrak{B}$ implies $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}^\tau$ for some τ. Suppose that*

(1) *R (restricted to τ-structures) is an equivalence relation on $\mathfrak{K}^\tau$.*
(2) *If $\rho\colon \tau \to \bar{\tau}$ is an injective renaming, then for all $\bar{\tau}$-structures $\mathfrak{A}$ and $\mathfrak{B}$*

$$\mathfrak{A}R\mathfrak{B} \quad \textit{implies} \quad \mathfrak{A}^{-\rho}R\mathfrak{B}^{-\rho}.$$

(3) *("R is invariantly definable and has definable finite approximations.") Given τ there are for some τ^*, $\tau \subset \tau^*$, $\mathscr{L}_{\omega\omega}[\tau^*]$-sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$ such that for arbitrary τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ the following hold:*

$$\mathfrak{A}R\mathfrak{B} \quad \textit{iff} \quad (\mathfrak{A}, \mathfrak{B}, \ldots) \models \{\varphi_i \mid i \in \omega\} \quad \textit{for some choice of (the universes and relations in)} \ldots,$$

and for $n \in \omega$ the relation R_n on $\mathfrak{K}^\tau$ given by

$$\mathfrak{A} R_n \mathfrak{B} \quad \textit{iff} \quad (\mathfrak{A}, \mathfrak{B}, ---) \models \{\varphi_i \mid i \leq n\} \quad \textit{for some} ---$$

has the following two properties:

(i) *R_n is an equivalence relation on $\mathfrak{K}^\tau$.*
(ii) *For $\mathfrak{A} \in \mathfrak{K}^\tau$, there is $\psi^n_\mathfrak{A} \in \mathscr{L}_{\omega\omega}[\tau]$ such that for $\mathfrak{B} \in \mathfrak{K}^\tau$*

$$\mathfrak{A} R_n \mathfrak{B} \quad \textit{iff} \quad \mathfrak{B} \models \psi^n_\mathfrak{A}.$$

Then among the logics of R-invariant sentences and semantics restricted to structures in $\bigcup \{\mathfrak{K}^\tau \mid \tau$ vocabulary$\}$ the logic $\mathscr{L}^R_{\omega\omega}$ of R-invariant first-order sentences is a maximal compact logic.

Moreover, if $\mathscr{L}$ with $\mathscr{L}^R_{\omega\omega} \leq \mathscr{L}$ is a compact logic of R-invariant sentences which is closed under conjunctions and disjunctions (but not necessarily under negations), then any two $\mathscr{L}$-classes can be separated by an $\mathscr{L}^R_{\omega\omega}$-class.

Proof. Clearly $\mathscr{L}^R_{\omega\omega}$ with semantics restricted to $\mathfrak{K} := \bigcup \{\mathfrak{K}^\tau \mid \tau$ vocabulary$\}$ is compact. Moreover, $\mathscr{L}^R_{\omega\omega}$ is closed under boolean operations (since R is an equivalence relation) and has the reduct and renaming property (by (2)). Note that for $\mathfrak{A} \in \mathfrak{K}^\tau$, the sentence $\psi^n_\mathfrak{A}$ mentioned in assumption (3) (ii) is R-invariant. In fact, let $\mathfrak{B} R \mathfrak{C}$ and $\mathfrak{B} \models \psi^n_\mathfrak{A}$. Then $\mathfrak{A} R_n \mathfrak{B}$. Since $R \subset R_n$ and R_n is an equivalence relation, we obtain $\mathfrak{A} R_n \mathfrak{C}$. Hence, $\mathfrak{C} \models \psi^n_\mathfrak{A}$.

It suffices to prove the separation claim in the theorem, since this claim implies the maximality property of $\mathscr{L}^R_{\omega\omega}$. Let $\mathscr{L}$ be as above and choose $\varphi, \psi \in \mathscr{L}[\tau]$ such that $\mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi) = \emptyset$, where Mod(...) denotes the class of models of ... in $\mathfrak{K}^\tau$.

For $n \in \omega$ we have

$$(*) \qquad \models_\mathfrak{K} \varphi \rightarrow \bigvee \{\psi^n_\mathfrak{A} \mid \mathfrak{A} \in \mathfrak{K}^\tau, \mathfrak{A} \models \varphi\}.$$

By the preceding remark, $\psi^n_\mathfrak{A}$ is R-invariant. Hence, by $\mathscr{L}$-compactness it follows that the disjunction in $(*)$ can be replaced by a finite one. That is, there is such a finite disjunction $\chi^n \in \mathscr{L}^R_{\omega\omega}$ with $\models_\mathfrak{K} \varphi \rightarrow \chi^n$.

By $\mathscr{L}$-compactness it suffices to show that $\{\chi^n \mid n \in \omega\} \cup \{\psi\}$ has no model in $\mathfrak{K}$, for it will then follow that for some $n \in \omega$, $\mathrm{Mod}(\varphi) \subset \mathrm{Mod}(\chi^0 \wedge \cdots \wedge \chi^n)$ and $\mathrm{Mod}(\chi^0 \wedge \cdots \wedge \chi^n) \cap \mathrm{Mod}(\psi) = \emptyset$. By contradiction, suppose that $\mathfrak{B}$ in $\mathfrak{K}$ is a model of $\{\chi^n \mid n \in \omega\} \cup \{\psi\}$. Then, for each n, there is $\mathfrak{A}_n \in \mathfrak{K}$ with $\mathfrak{A}_n \models \varphi$ and $\mathfrak{B} \models \psi^n_{\mathfrak{A}_n}$. Whence $\mathfrak{A}_n R_n \mathfrak{B}$. By (3), we now have

$$(\mathfrak{A}_n, \mathfrak{B}, ---) \models \{\varphi_i \mid i \leq n\}$$

for appropriate ---. By $\mathscr{L}$-compactness, there are $\overline{\mathfrak{A}}, \overline{\mathfrak{B}}$ and appropriate ... such that

$$(\overline{\mathfrak{A}}, \overline{\mathfrak{B}}, \ldots) \models \{\varphi_i \mid i \in \omega\}$$

with $\bar{\mathfrak{A}} \models \varphi$ and $\bar{\mathfrak{B}} \models \psi$. But then $\bar{\mathfrak{A}} R \bar{\mathfrak{B}}$. Hence it must be that $\bar{\mathfrak{B}} \models \varphi$, since φ is R-invariant. Therefore, $\bar{\mathfrak{B}} \in \mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi)$—a contradiction. □

5.6 Remarks. (a) Note that the preceding proof shows that each $\mathscr{L}^R_{\omega\omega}$-sentence is equivalent in $\mathfrak{K}$ to a disjunction of sentences $\psi^n_{\mathfrak{A}}$. Thus, if $\mathscr{L}$ is a compact logic of R-invariant sentences containing all first-order sentences $\psi^n_{\mathfrak{A}}$, then $\mathscr{L}^R_{\omega\omega} \equiv \mathscr{L}$.

(b) Theorem 5.5 also holds for $\mathscr{L}_{\infty\omega}$ instead of $\mathscr{L}_{\omega\omega}$, if for each ordinal α, we introduce the corresponding relations R_α and also assume that each R_α has set-many equivalence classes. The conclusions will then read as follows:

> Among the logics of R-invariant sentences, $\mathscr{L}^R_{\infty\omega}$ is a maximal bounded logic; and
>
> If $\mathscr{L}^R_{\infty\omega} \leq \mathscr{L}$ and $\mathscr{L}$ is a bounded logic of R-invariant sentences, closed under conjunctions and disjunctions, then any two disjoint $\mathscr{L}$-classes can be separated by an $\mathscr{L}^R_{\infty\omega}$-class.

(c) One can even prove a more general theorem that will cover the cases in Theorem 5.5 and in the preceding remark, replacing $\mathscr{L}_{\omega\omega}$ by an arbitrary logic $\mathscr{L}$ and explicitly using the well-ordering number of $\mathscr{L}$. This theorem would also include the corresponding results (indicated in Section 3) for the logic with the added quantifier "there are uncountably many".

We now give the applications of Theorem 5.5 to topological structures and to other types of structures as well.

A *topological structure* is a pair $(\mathfrak{A}, \mu)$ consisting of an (algebraic) structure $\mathfrak{A}$ and of a topology μ on A. Topological spaces and topological groups are examples of topological structures. Let $\underline{Top}$ denote the class of topological structures. We obtain a logic for $\underline{Top}$ which is neither compact nor has the Löwenheim–Skolem property, if we take the two-sorted first-order language corresponding to structures of the form $(\mathfrak{A}, \mu, \in)$, where $(\mathfrak{A}, \mu) \in \underline{Top}$ and where $\in$ is the membership relation between elements of A and open sets. In particular, quantified variables of the second sort range over open sets.

Now, consider arbitrary structures of the form $(\mathfrak{A}, \mu, E)$, where A and μ are the universes and E is a binary relation with $E \subset A \times \mu$. For $U \in \mu$, put

$$U_E = \{a \in A \mid aEU\} \text{ and } \mu_E = \{U_E \mid U \in \mu\}.$$

Let

$$\underline{Bas} = \{(\mathfrak{A}, \mu, E) \mid \mu_E \text{ is basis of a topology on } A\},$$

and, for $(\mathfrak{A}, \mu, E) \in \underline{Bas}$ denote by $\widetilde{(\mathfrak{A}, \mu, E)}$ the induced structure in $\underline{Top}$.

$\underline{Bas}$ consists precisely of the models of the following (two-sorted) first-order sentence

$$\varphi_{Bas} = \forall x\, \exists X\, x \in X \wedge \forall x\, \forall X\, \forall Y(x \in X \wedge x \in Y \to \exists Z(x \in Z \wedge \forall z(z \in Z \to (z \in X \wedge z \in Y)))).$$

Let $\cong^t$ be the relation of topological homeomorphism on $\underline{Bas}$. That is, $\cong^t$ is the relation given by the isomorphism relation of induced topological structures:

$$(\mathfrak{A}, \mu, E) \cong^t (\mathfrak{B}, \nu, F) \quad \text{iff} \quad \widetilde{(\mathfrak{A}, \mu, E)} \cong \widetilde{(\mathfrak{B}, \nu, F)}.$$

Observe that a sentence φ is $\cong^t$-invariant just in case

$$(1) \qquad (\mathfrak{A}, \mu, E) \models \varphi \quad \text{iff} \quad \widetilde{(\mathfrak{A}, \mu, E)} \models \varphi$$

holds for $(\mathfrak{A}, \mu, E) \in \underline{Bas}$. Therefore, we also speak of *basis-invariant* sentences instead of $\cong^t$-invariant sentences.

If $\mathscr{L}$ is any logic for topological structures, then, using (1) as definition, we obtain a logic for structures in $\underline{Bas}$ which will consist only of basis-invariant sentences. On the other hand, if $\mathscr{L}$ is a logic for structures in $\underline{Bas}$ which consists of basis-invariant sentences, then—using again (1) as definition—we obtain a logic for $\underline{Top}$. Because of this one-to-one correspondence, maximal logics for $\underline{Bas}$ "are" maximal logics for $\underline{Top}$.

We apply Theorem 5.5 to obtain maximal logics for $\underline{Bas}$—it being clear how the notion of one-sorted and many-sorted type and structure must be redefined in our case. For this, choose Φ^τ such that $\mathrm{Mod}(\Phi^\tau)$ is the class of structures in $\underline{Bas}$ of type τ, for example, $\Phi^\tau = \{\varphi_{Bas}\}$ for "one-sorted" τ. As R and R_n take the relation $\cong^t_p$ of *partial homeomorphism* and the relation $\cong^t_n$ of *n-homeomorphism*, respectively (they correspond to the relation of partial isomorphism and n-isomorphism of induced topological structures; the reader is referred to Chapter XV or to Flum–Ziegler [1980, p. 18]). By Theorem 5.5, $\mathscr{L}^{\cong^t_p}_{\omega\omega}$ is a maximal compact logic of $\cong^t_p$-invariant sentences. Since $\cong^t$ and $\cong^t_p$ are first-order definable relations which agree on countable structures, we obtain from this result and from 5.1:

5.7 Theorem. *The logic of basis-invariant first-order sentences is maximal among the logics for topological structures with the compactness property and the following Löwenheim–Skolem property: if φ has a topological model, then there is $(\mathfrak{A}, \mu) \in \underline{Top}$ such that $(\mathfrak{A}, \mu) \models \varphi$, A is countable and μ has a countable basis.* □

5.8 Remarks. (a) Since $\cong^t_p$ and $\cong^i$ agree on countable structures, one can get from the proof of Theorem 5.5, the interpolation theorem for the logic of basis-invariant first-order sentences in a way similar to that for first-order logic given in Example 1.1.7(a).

(b) By the preceding results and Remark 5.6(a), any logic containing sentences $\psi^n_{(\mathfrak{A}, \sigma)}$ characterizing the n-isomorphism type of any topological structure $(\mathfrak{A}, \sigma)$ already contains all basis-invariant first-order sentences. This result will be used in Chapter XV.

Similarly, one can obtain maximal logics for other types of structures. We will give two further examples.

A *uniform structure* is a pair $(\mathfrak{A}, \mu)$ where $\mu \subset A \times A$ is a uniformity on A. A *monotone structure* is a pair $(\mathfrak{A}, \mu)$ where $\mu \subset A$ is a monotone system on A, that is, a non-empty set of subsets of A such that $X \in \mu$ and $X \subset Y \subset A$ imply $Y \in \mu$.

Using in both cases the corresponding notions of basis and the corresponding Löwenheim–Skolem properties we obtain in the same way as for topological structures the following result:

5.9 Theorem. *Among the logics for uniform structures (monotone structures) with the compactness and the Löwenheim–Skolem property, the logic of basis-invariant first-order sentences is maximal.* □

5.10 Remarks. (a) In Chapter XV the reader can find syntactic characterizations of the basis-invariant sentences for the above cases.

(b) Observe that the result which we obtain from Remark 5.6(b) for the corresponding infinitary logics are not satisfactory. For example, Remark 5.6(b) tells us that among the logics for topological structures, $\mathscr{L}^{\cong^t_p}_{\infty\omega}$ is a maximal bounded logic of $\cong^t_p$-invariant sentences. And it is not hard to give a "syntactic" characterization of the sentences in $\mathscr{L}^{\cong^t_p}$. But is $\mathscr{L}^{\cong^t_p}_{\infty\omega} = \mathscr{L}^{\cong^t}_{\infty\omega}$? That is, is $\mathscr{L}^{\cong^t_p}_{\infty\omega}$ the class of basis-invariant $\mathscr{L}_{\infty\omega}$-sentences?

5.11 Notes. The Lindström-type results for topological structures, monotone structures, and so on are due to Ziegler [1976]. Theorem 5.5 is new here. The reader should compare our approach to maximal logics with that given by Sgro [1977b]. Sgro's main result—when translated into our terminology—reads as follows: Given a relation R between structures, the logic $\mathscr{L}^R_{\omega\omega}$ is, among the logics of R-invariant sentences, a maximal logic satisfying a "Łos ultraproduct theorem". Since the ultraproduct operation commutes with the operation which associates to each model in $\underline{Bas}$ the induced model in $\underline{Top}$, we obtain: The logic of basis-invariant sentences is a maximal logic for topological structures satisfying a Łos ultraproduct theorem.

Part B

Finitary Languages with Additional Quantifiers

Part B of the book is devoted to the study of logics with added quantifiers and the applications of such. The logics considered, for the most part, express properties of ordinary structures. Logics with additional quantifiers based on richer structures are studied in Part E.

Chapter IV begins the discussion by investigating the logic $\mathscr{L}(Q_1)$ with the quantifier "there exist uncountably many." It also discusses various extensions of $\mathscr{L}(Q_1)$ including stationary logic $\mathscr{L}(\mathrm{aa})$ and the Magidor–Malitz logic $\mathscr{L}^{<\omega}$. The primary emphasis of the chapter is on the method of constructing models of size $\aleph_1$ used by Keisler [1970] to prove his completeness theorem for $\mathscr{L}(Q_1)$, a method that has become one of the standard tools of the subject. Each of these logics comes with its own intended concepts of "small" set and "large" set. The basic idea of Keisler-type proofs is to use an elementary chain $\langle A_\alpha : \alpha < \omega_1 \rangle$ of countable non-standard or "weak" models to build a standard model, one where the quantifier has its intended interpretation. The key step is always from A_α to $A_{\alpha+1}$, constructing $A_{\alpha+1}$ so that all small definable subsets of A_α stay fixed, but where a fixed definable subset of A_α that is supposed to be large receives a new element.

Chapter V discusses the general problem of transferring results known about $\mathscr{L}(Q_\alpha)$ to some other $\mathscr{L}(Q_\beta)$, especially the problem of taking results known about $\mathscr{L}(Q_1)$, where we have powerful techniques for building models, to $\mathscr{L}(Q_{\beta+1})$ for larger β. For example, if we assume the Generalized Continuum Hypothesis, it follows that the axioms and rules that are complete for $\mathscr{L}(Q_1)$ are also complete for any logic of the form $\mathscr{L}(Q_{\beta+1})$, as long as $\aleph_\beta$ is regular. In general, this chapter depends heavily on various set-theoretical assumptions which are independent of the usual axioms of set theory, however.

Chapter VI surveys and compares the strength of a host of other logics with additional quantifiers. One of these is the class of partially ordered quantifiers like Q^{H} whose meaning is given by: $Q^{\mathrm{H}}x, y; z, w\phi(x, y, z, w)$ is true just in case for every x there is a y, and for every z there is a w, such that y depends only on x, w only on z, such that $\phi(x, y, z, w)$. Quantifiers of this kind are called partially ordered because they are often written:

$$\begin{matrix} \forall x\, \exists y \\ \forall z\, \exists w \end{matrix}\ \phi(x, y, z, w).$$

Some other quantifiers discussed in Chapter VI include:

- the Hartig quantifier I, defined so that $Ix, y[\phi(x), \psi(y)]$ means that the number of ϕ's is the same as the number of ψ's;
- the similarity quantifier S, defined so that $Sx, x'[\phi(x), \psi(x')]$ means that the substructures defined by ϕ and ψ are isomorphic; and
- the well-ordering quantifier W, defined so that $Wx, y\phi(x, y)$ means that ϕ defines a well-ordering.

The relative strengths of these logics, and their Δ-closures are discussed. For example, it is shown that $\Delta(\mathscr{L}(Q^{\mathrm{H}})) = \Delta(\mathscr{L}(S)) = \Delta(\mathscr{L}^{\mathrm{mII}})$, where $\mathscr{L}^{\mathrm{mII}}$ is monadic second-order logic. Under the assumption of the axiom of constructibility, it is also shown that $\Delta(\mathscr{L}(I)) = \Delta(\mathscr{L}^{\mathrm{mII}})$.

Chapter VII is devoted to identifying decidable and undecidable theories in logics with generalized quantifiers, especially $\mathscr{L}(Q_1)$, the Magidor–Malitz logic $\mathscr{L}^{<\omega}$, logic $\mathscr{L}(I)$ with the Hartig quantifier, and stationary logic $\mathscr{L}(\mathrm{aa})$. The chapter is organized around three main methods of proof, quantifier elimination, the method of interpretations, and the use of "dense systems." These are all well-known methods from first-order logic which have interesting extensions to stronger logics. The mathematical theories discussed include abelian groups and modules, orderings, and boolean algebras. This chapter leads into a rich literature on the decidability of theories with extra quantifiers.

Chapter IV

The Quantifier "There Exist Uncountably Many" and Some of Its Relatives

by M. KAUFMANN

The idea of adding quantifiers to first-order logic goes back at least to Mostowski [1957]. Fuhrken [1964] and Vaught [1964] were the first investigators to prove compactness and (abstract) completeness theorems for such a logic, namely the logic $\mathscr{L}(Q_1)$ obtained by adjoining the quantifier Q_1 (there exist uncountably many) to first-order logic. The first systematic study of $\mathscr{L}(Q_1)$ and, in fact, of any well-behaved logic obtained by adding a quantifier to first-order logic, appeared in Keisler's 1970 paper. By giving the completeness of a simple *explicit* set of axioms for $\mathscr{L}(Q_1)$, along with other nice features of a logic such as an omitting types theorem (with applications), Keisler's work encouraged the further study of $\mathscr{L}(Q_1)$ as well as the search for extensions of $\mathscr{L}(Q_1)$ that retain some of the nice properties of first-order logic. In this chapter we will present some of the progress in this study.

A main focus of this chapter is on the development of methods of proving completeness theorems for logics extending $\mathscr{L}(Q_1)$. (Such an approach allows compactness theorems to be derived as corollaries.) In Section 3, the proof of Keisler's concrete completeness theorem for $\mathscr{L}(Q_1)$ leads to new methods of constructing models and to a version of the omitting types theorem which differs a bit from the first-order version, and which leads to a completeness theorem for the corresponding infinitary version of $\mathscr{L}(Q_1)$. These methods, and the resulting intuition developed for $\mathscr{L}(Q_1)$, make possible the completeness proofs for the other logics that are examined in Sections 4 and 5. Although concrete completeness is a desirable feature of a logic, our main purpose here is to present the *methods* that go into the proofs of such theorems.

The basic plan for proving each of these completeness theorems is to reduce the given logic to first-order logic in some manner so that familiar tools from first-order model theory may then be applied. One such reduction is used in Section II.3 to prove that the set of validities for $\mathscr{L}(Q_1)$ is r.e. in the vocabulary; another reduction—one that is due to Fuhrken—is given in Section 1.1 below. However, in order to prove a *concrete* completeness theorem, we need a reduction that is somehow more closely tied to the logic. The notion of *weak model* is thus developed for this purpose in Section 2 although some of the details are relegated to the appendix. The general approach adopted in Section 2 enables us to give a reasonably unified treatment of the completeness theorems in Sections 3, 4, and 5.

In Section 6 we conclude our study with an investigation of interpolation and definability questions for various extensions of $\mathscr{L}(Q_1)$. The interest in these questions is largely due to the use of a variety of back-and-forth arguments for proving $\mathscr{L}$-equivalence (for various logics $\mathscr{L}$), although the original motivation was largely due to the search for well-behaved extensions of first-order logic. Several of the proofs given in Section 6 elaborate the basic model-theoretic practice of showing that certain partial isomorphisms preserve elementary equivalence.

This chapter is essentially self-contained, its only prerequisite being a reasonable familiarity with first-order model theory.

1. Introduction to $\mathscr{L}(Q_\alpha)$

Probably the simplest quantifiers which are stronger than $\exists$ and $\forall$ are the cardinality quantifiers Q_α, "there exist at least $\aleph_\alpha$" defined in Section II.2.2. When $\alpha = 1$, the subscript on Q will be omitted. In this case, Q asserts that "there exist uncountably many." The notation $\mathscr{L}(Q_\alpha)(\tau)$ denotes the set of $\mathscr{L}(Q_\alpha)$-formulas of the vocabulary τ. In the present chapter, however, we will rarely consider the case $\alpha > 1$, since it comprises part of Chapter V.

Of course, $\mathscr{L}(Q_\alpha)$ is strictly stronger than first-order logic. For example, the sentence $Q_\alpha x(x = x) \wedge \forall x \neg Q_\alpha y(y < x)$ holds in a linear order if and only if that order is $\aleph_\alpha$-like. Examples of the expressive power of $\mathscr{L}(Q_\alpha)$ tend to be rather obvious. In order to express more interesting notions in the logic, we must extend $\mathscr{L}(Q_\alpha)$. This is done in Sections 4 and 5.

As is shown in Section II.3, $\mathscr{L}(Q_1)$ is countably compact (compact for countable theories), a fact which we will again prove in this chapter, in Section 3. However, our method and emphasis are somewhat different from the one in Section II.3, as was explained in the introduction above. For now, we will begin our work by discussing the incompactness of $\mathscr{L}(Q_0)$ in subsection 1.1 and then examine some Löwenheim–Skolem properties of $\mathscr{L}(Q_\alpha)$ in Section 1.3, giving also a brief outline (with comments) of Fuhrken's original compactness proof for $\mathscr{L}(Q_1)$ in Section 1.2.

1.1. Incompactness of $\mathscr{L}(Q_0)$

The following finite theory T has only one model (up to isomorphism), namely $(\omega, <)$: $T = \{\forall x \neg Q_0 y(y < x)$, "$<$ is a linear order without last element"$\}$. It follows then that $\mathscr{L}(Q_0)$ is not countably compact. Moreover, the set of valid sentences of $\mathscr{L}(Q_0)(\tau)$ is not recursively enumerable (it is actually complete Π^1_1) if τ contains a binary relation symbol. In fact, Barwise [1974] has shown that the Δ-closure of $\mathscr{L}(Q_0)$ is equivalent to $\mathscr{L}_{\infty\omega} \cap \mathscr{L}_{\omega_1^{CK}}$ (see Section II.7.2), the latter being the hyperarithmetic fragment of $\mathscr{L}_{\infty\omega}$ (see also Theorems VI.2.3.3 and XVII.3.2.2).

Since most of the emphasis in this chapter is on logics that are countably compact, we will now turn to $\mathscr{L}(Q_i)$.

1.2. On Completeness and Compactness of $\mathscr{L}(Q_1)$

Mostowski [1957] asked whether $\mathscr{L}(Q_1)$ has a recursively enumerable set of validities. The chief result in this direction was Vaught's two-cardinal theorem (see Morley–Vaught [1962]), or, perhaps more accurately, the proof of the theorem. To be precise, Fuhrken discerned that $\mathscr{L}(Q_1)$ is countably compact by abstracting the following lemma from the proof of Vaught's theorem.

1.2.1 Lemma (Fuhrken [1964; 1.7]). *Suppose that T is a set of (first-order) sentences in a countable vocabulary τ which contains a unary relation symbol U. Let W be a new unary predicate symbol, and let Δ be the set of all sentences*

$$\forall v_0 \ldots \forall v_{n-1}[W(v_0) \wedge \cdots \wedge W(v_{n-1}) \rightarrow [\phi \leftrightarrow \phi^W]],$$

where ϕ is any τ-formula having only $v_0, \ldots, v_{n-1}$ as free variables, and ϕ^W is obtained from ϕ by relativizing all quantifiers to W. That is, W defines an elementary submodel of the universe. Then the following are equivalent:

(i) $T \cup \Delta \cup \{\forall x(U(x) \rightarrow W(x)), \exists x \neg W(x)\}$ *is consistent;*
(ii) T *has a model* $\mathfrak{A}$ *for which* $|U^{\mathfrak{A}}| < |A| = \aleph_1$;
(iii) T *has a model* $\mathfrak{A}$ *for which* $|U^{\mathfrak{A}}| < |A|$. ☐

A proof of this result is carefully worked out in Chang–Keisler [1973; §3.2, especially 3.2.12]. We will now examine the two relevant corollaries of this lemma, discussing their proofs in 1.2.4.

1.2.2 Corollary (Fuhrken [1964; Theorem 3.4]). *$\mathscr{L}(Q)$ is countably compact.* ☐

1.2.3 Corollary (Vaught [1964]). *For countable τ, the set of valid sentences of $\mathscr{L}(Q_1)(\tau)$ is recursively enumerable in τ. In fact, $\mathscr{L}(Q_1)$ is recursively enumerable for consequence (in the sense of Definition* II.1.2.4). ☐

1.2.4 Idea of Proofs of Corollaries 1.2.2 and 1.2.3. These corollaries both follow from Fuhrken [1964, Theorem 2.2]. The idea is that one can replace $\neg Qx\phi(x, \mathbf{y})$ by a statement asserting that there is a function mapping $\{x: \phi(x, \mathbf{y})\}$ one–one into U; and that one can replace $Qx\phi(x, \mathbf{y})$ by a statement asserting that there is a one–one function from the universe of the model into $\{x: \phi(x, \mathbf{y})\}$. The details of how this may be accomplished can be found in Fuhrken [1964]. However, the result is that questions about satisfiability of an $\mathscr{L}(Q_{\beta+1})$ theory Σ may be reduced to the satisfiability of a corresponding $\mathscr{L}_{\omega\omega}$ theory Σ^* in a model $\mathfrak{A}$ with $U^{\mathfrak{A}} \leq \aleph_\beta < |A|$. Setting $\beta = 0$ gives the corollaries. These ideas were expanded in Keisler [1966a] in giving an axiomatization of 2-cardinal models. The reader should also see Section V.1 for more about the method of reduction.

Comparison of Completeness Proofs and the Related Literature. As we have pointed out, Fuhrken's Lemma (1.2.1) is based largely on the proof of Vaught's

2-cardinal theorem. That is generally proved by using homogeneous models to build an appropriate elementary chain. However, the proof of Keisler's completeness theorem (see Section 3.2, also Section II.3.2) is based on the proof of Keisler's 2-cardinal theorem. That is, homogeneous models are replaced by an omitting types argument. The latter technique is what really enables Keisler to give an *explicit* set of axioms for $\mathscr{L}(Q_1)$, and to prove an omitting types theorem for $\mathscr{L}(Q_1)$. The reader should see Section 3 for more on this.

It is also interesting to compare the method of Section II.3.2 (and also of Section 3.2) to that used for the MacDowell–Specker theorem for models of arithmetic. The latter asserts that every model of Peano arithmetic (even if it is uncountable) has an elementary end extension. (See Section V.7 for a related result.) The former is more closely related to the methods used to prove an analogous theorem for models of set theory, Theorem 3.2.5 below (Keisler–Morley [1968]). The Keisler–Morley theorem does not hold for all uncountable models. However, the fact that it requires the collection schema, rather than the (stronger) induction schema does speak in its favor. The connection between the Keisler–Morley theorem and Keisler's $\mathscr{L}(Q)$ completeness theorem is made somewhat more explicit in the proof of Theorem 3.2.5 given below (the Keisler–Morley theorem), which uses the Main Lemma (3.2.1) from the proof of completeness of $\mathscr{L}(Q_1)$.

1.3. Observations on $\mathscr{L}(Q_\alpha)$

We will close this introduction by making some easy observations about $\mathscr{L}(Q_\alpha)$. The first was noticed by Mostowski, and it generalizes easily to the $\aleph_\alpha$-interpretation of $\mathscr{L}^{<\omega}$ (see Definition 5.1.3).

Before we examine the argument for this result, we should make a comment on the notation and notions involved. By $\mathfrak{B} \prec_{\mathscr{L}(Q_\alpha)} \mathfrak{A}$ we mean that $\mathfrak{B} \prec \mathfrak{A}$ and that both $\mathfrak{B}$ and $\mathfrak{A}$ satisfy the same $\mathscr{L}(Q_\alpha)$ formulas at any assignment of $\mathfrak{B}$. These ideas clear, we now turn to

1.3.1 Proposition. *If $\mathfrak{A}$ is any model, then there exists $\mathfrak{B} \prec_{\mathscr{L}(Q_\alpha)} \mathfrak{A}$ such that $|B| \leq \aleph_\alpha$.*

Sketch of Proof. For $\alpha = \omega_1$, the result follows from Fuhrken's normal form (see subsection 1.2.4) together with Lemma 1.2.1, if we only require $\mathfrak{B} \equiv_{\mathscr{L}(Q_\alpha)} \mathfrak{A}$. However, the more general statement has an even easier direct proof. Assuming that $|A| > \aleph_\alpha$ (for otherwise, the argument is done), the usual proof of the downward Löwenheim–Skolem theorem can be easily modified to provide $\aleph_\alpha$ witnesses to each $Qx\phi$ instead of only one. $\square$

On the other hand, as we will now show, the upward Löwenheim–Skolem property clearly fails. (The reader should consult Theorem II.6.1.6 and V.4.2.3 for theorems on Hanf numbers.)

1.3.2. Proposition. *For each of the conditions* (i) *through* (iv) *below, there is a sentence* ϕ *of* $\mathscr{L}(Q)$ *such that for all* α *and* β: ϕ *has a model of power* $\aleph_\beta$ *in the* α*-interpretation* (*that is, considering* ϕ *as a sentence of* $\mathscr{L}(Q_\alpha)$) *iff that condition holds.*

(i) $\beta < \alpha$.
(ii) $\beta = \alpha$.
(iii) $\beta \leq \alpha + n$, *for any* $n < \omega$.
(iv) $\aleph_\beta \leq \beth_n(\aleph_\alpha)$, *for any* $n < \omega$, *where* $\beth_0(\alpha) = \alpha$ *and* $\beth_{n+1}(\alpha) = 2^{\beth_n(\alpha)}$.

Hence, full compactness fails for all $\mathscr{L}(Q_\alpha)$.

Proof. (i) ϕ is, of course, simply $\neg Q_\alpha x(x = x)$. Thus, it follows that compactness fails for $\mathscr{L}(Q_\alpha)$: Consider the set $\{\neg Q_\alpha x(x = x)\} \cup \{c_\beta \neq c_\gamma : \beta < \gamma < \aleph_\alpha\}$.

(ii) ϕ says that $\leq$ is a (reflexive) $\aleph_\alpha$-like linear order: "$\leq$ is a linear order" $\bigwedge \forall x \neg Q_\alpha y(y \leq x) \wedge Q_\alpha x(x = x)$.

(iii) Here, such a sentence ϕ_n can be constructed by induction on n. Thus, ϕ_0 is "$\leq$ is a linear order" $\bigwedge \forall x \neg Q_\alpha y(y \leq x)$, while ϕ_{n+1} says "$<$ is a linear order and every proper initial segment can be expanded to a model of ϕ_n."

(iv) We assume that $n \geq 1$ (for, in the absence of this assumption, (iii) clearly applies). Thus, the language of ϕ includes $<$, P_0, $P_1, \ldots, P_n$, and ε. And, that much being so, we assert that each P_{i+1} is contained in the power set of P_i. (See also Theorem II.6.1.6.) ☐

This contrasts with Theorem 8 of Yasuhara [1966], which gives full compactness when one removes $=$, $\exists$, and $\forall$ from $\mathscr{L}(Q_\alpha)$, $\alpha \geq 1$.

2. *A Framework for Reducing to First-Order Logic*

Our goal in this section is to provide some means of reducing a given logic to first-order logic in order that we may develop some model theory for $\mathscr{L}(Q)$ and some of its extensions in Sections 3, 4, and 5. As we will see, when we transform a given logic into first-order logic in some manner—say, by enlarging the vocabulary—we may apply methods of first-order model theory to obtain results about the given logic. The reduction given here works for any logic that possesses some basic syntactic properties, "concrete syntax". Our notion of "concrete syntax" is neither memorable nor worthy of study in its own right. Indeed, every reasonable logic probably has this property in some sense. However, it is a notion which will enable us to prove theorems about so-called weak models, and these, in turn, will enable us to carry out the more interesting model constructions later on. In fact, we will omit the precise definition of "concrete syntax" here as well as most proofs. These are, however, included in Section 7 (the appendix) where they may be safely ignored. The reader might want to read this section with $\mathscr{L}(Q)$ in mind.

Keisler's notion of weak model is presented in Section 2.3, where it is related to the notion given here in Definition 2.1.3. That done we will then briefly touch on the logic of monotone structures.

2.1. Logics With Concrete Syntax and Weak Models

A precise definition of *concrete syntax* can be found in Definition 7.1.1. For present purposes, it suffices to say that the properties include:

- closure under $\neg$, $\vee$, $\exists$;
- possession of a notion $\vdash_{\mathscr{L}}$ of finitary proof, with a deduction theorem;
- existence of a rank function $r(\phi)$ which measures the complexity of ϕ in a reasonable way;
- existence of a function $\mathrm{frvar}(\phi)$ which gives the set of free variables of each formula ϕ, as well as a notion of substitution $\phi(f)$ for any function $f: \mathrm{frvar}(\phi) \to C$, for some set C of constants.

These properties are sufficient (when stated precisely) to prove the deduction theorem in the usual way, as in Enderton [1972].

2.1.1 Theorem (Deduction Theorem). $\Gamma \cup \{\phi\} \vdash_{\mathscr{L}(\tau)} \psi$ *iff* $\Gamma \vdash_{\mathscr{L}(\tau)} \phi \to \psi$. $\square$

Any logic with concrete syntax can be transformed into first-order logic by using extra relation symbols and "weak models" as follows in

2.1.2 Definition. Let $\mathscr{L}$ be a logic with concrete syntax. We define a map $\phi \mapsto \phi^*$ which sends $\mathscr{L}(\tau)$-formulas to $\mathscr{L}_{\omega\omega}(\tau^+)$-formulas, where $\tau^+ = \tau \cup \{R_\phi : \phi$ is an $\mathscr{L}(\tau)$-formula, neither atomic nor of the form $\neg\psi$, $\psi_1 \vee \psi_2$, or $\exists x\psi\}$. The arity of R_ϕ is $|\mathrm{frvar}(\phi)|$. The definition is by recursion on rank $r(\phi)$. If ϕ is atomic, set $\phi^* = \phi$. Also, set $(\neg\psi)^* = \neg(\psi^*)$, $(\psi_1 \vee \psi_2)^* = \psi_1^* \vee \psi_2^*$, and $(\exists x\psi)^* = \exists x(\psi^*)$. If ϕ is neither atomic nor of the form $\neg\psi$, $\psi_1 \vee \psi_2$, nor $\exists x\psi$, and if $\mathrm{frvar}(\phi) = \{v_{i_1}, \ldots, v_{i_n}\}$ with $i_1 < \cdots < i_n$, then set $\phi^* = R_\phi(v_{i_1}, \ldots, v_{i_n})$.

2.1.3 Definition (Weak Models). *A weak model for* a logic $\mathscr{L}$ with concrete syntax is a τ^+-structure $\mathfrak{A}^* = \langle \mathfrak{A}, R_\phi^{\mathfrak{A}^*} \rangle_{\phi \in \mathscr{L}(\tau)}$, for some τ, which satisfies every instance of ϕ^* for every $\vdash_{\mathscr{L}}$-axiom ϕ in $\mathscr{L}(\tau)$. For ϕ any formula of $\mathscr{L}(\tau)$, we write $\mathfrak{A}^* \models \phi[s]$ to denote $\mathfrak{A}^* \models \phi^*[s]$. Since "$*$" commutes with $\neg$, $\vee$, and $\exists$, "$\models$" obeys the usual inductive clauses for first-order satisfaction.

For weak models $\mathfrak{A}^*$ and $\mathfrak{B}^*$ of vocabulary τ^+, we write $\mathfrak{A}^* \prec^w \mathfrak{B}^*$ if $A \subseteq B$ and for all assignments s into A and all $\phi \in \mathscr{L}(\tau)$, $\mathfrak{A}^* \models \phi[s]$ iff $\mathfrak{B}^* \models \phi[s]$. Notice that this is weaker than $\mathfrak{A}^* \prec \mathfrak{B}^*$, since we restrict ourselves to formulas of the form ϕ^*.

2.2. Some Weak Model Theory

In this discussion we will present completeness (and related) theorems for weak models. The proofs, although routine, are given in the appendix. Throughout this section we assume that $\mathscr{L}$ has concrete syntax.

2.2.1 Proposition (Soundness). *Let $\mathfrak{A}^*$ be a weak model for $\mathscr{L}(\tau)$, and suppose that ϕ is an $\mathscr{L}(\tau)$-formula and $f: X \to C$, for some one–one function f, some $X \subseteq \mathrm{frvar}(\phi)$, and some set C of constants which is disjoint from τ. If $\vdash_{\mathscr{L}(\tau \cup C)} \phi(f)$ then for all $s: \mathrm{frvar}(\phi) \to A$, $\mathfrak{A}^* \models \phi[s]$.* $\square$

2.2.2 Proposition (Elementary Chain Theorem). *Let $\mathfrak{A}_\alpha^*$ be a τ_α^+-structure for all $\alpha < \gamma$, where $\alpha < \beta$ implies that $\tau_\alpha \subseteq \tau_\beta$ and $\mathfrak{A}_\alpha^* \prec^w \mathfrak{A}_\beta^* \restriction \tau_\alpha^+$. Let $\mathfrak{A}^*$ be the union of $\{\mathfrak{A}_\alpha^*: \alpha < \gamma\}$, that is, $\mathfrak{A}^*$ is a $(\bigcup_{\alpha<\gamma} \tau_\alpha^+)$-structure and for all $\alpha < \gamma$, $\mathfrak{A}^* \restriction \tau_\alpha^+ = \bigcup_{\beta \in \gamma - \alpha} \mathfrak{A}_\beta^*$. Then for all $\alpha < \gamma$, $\mathfrak{A}_\alpha^* \prec^w \mathfrak{A}_\gamma^* \restriction \tau_\alpha^+$.* $\square$

2.2.3 Theorem (Weak Completeness). *Let T be an $\mathscr{L}(\tau)$-consistent set of $\mathscr{L}(\tau)$ sentences, where τ is countable. Then T has a countable weak model, that is, there is a countable weak model $\mathfrak{A}^*$ for $L(\tau)$ such that $\mathfrak{A}^* \models \phi$ for all $\phi \in T$.*

The following extension of the weak completeness theorem will also be useful. First, however, we need a related definition which, in applications, will be equivalent to a more familiar condition.

2.2.4 Definition. Let T be an $\mathscr{L}(\tau)$-consistent set of $\mathscr{L}(\tau)$-sentences. Also let Σ be a set of $\mathscr{L}(\tau)$-formulas such that $\mathrm{frvar}(\sigma) \subseteq \mathbf{x}$ for all $\sigma \in \Sigma$; then we write $\mathrm{frvar}(\Sigma) \subseteq \mathbf{x}$. T is said to $\mathscr{L}(\tau)$-*locally omit* Σ, if for every finite set C of constant symbols, every $\mathscr{L}(\tau \cup C)$-sentence ϕ which is $\mathscr{L}(\tau \cup C)$-consistent with T, and every function f mapping $\mathbf{x}$ into the set C, there exists $\sigma \in \Sigma$ such that $\phi \wedge [\neg\sigma(f)]$ is $\mathscr{L}(\tau \cup C)$-consistent with T. Notice that range (f) may include constants of ϕ.

2.2.5 Weak Omitting Types Theorem. *Let T be an $\mathscr{L}(\tau)$-consistent set of $\mathscr{L}(\tau)$-sentences, where τ is countable. Also let $\{\Sigma_n: n < \omega\}$ be a family of countable sets of $\mathscr{L}(\tau)$-formulas with $\mathrm{frvar}(\Sigma_n) \subseteq \mathbf{x}_n$. If T $\mathscr{L}(\tau)$-locally omits Σ_n for all $n < \omega$, then T has a countable weak model omitting each Σ_n, that is, which satisfies $\bigwedge_{n<\omega} \forall \mathbf{x}_n \bigvee \{\neg\sigma: \sigma \in \Sigma_n\}$.* $\square$

The following technical lemma is used in Sections 3, 4, and 5, to extend weak models while omitting types. The exact statement can be found as Lemma 7.2.3; for the present, we will use this slightly imprecise but considerably more readable statement of it.

2.2.6 Lemma (Extension Lemma). *Suppose $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\tau)$, where τ is countable. Also let T be any consistent countable extension of the elementary diagram of $\mathfrak{A}^*$ which $\mathscr{L}(\tau)$-locally omits sets $\Sigma_n(x_n)$, each $n < \omega$. Then there exists a weak model $\mathfrak{B}^*$ of T which omits each set Σ_n, such that $\mathfrak{A}^* \prec \mathfrak{B}^* \restriction \tau^+$.*

$\square$

2.3. Connections With Monotone Structures

We will conclude this section by relating the notion of weak model as given by Keisler [1970] (and studied later by others: see Definition 2.3.3) to the notion given above. Keisler considered structures $(\mathfrak{A}, q)$, where $q \subseteq \mathscr{P}(A)$, and inductively defined satisfaction for $\mathscr{L}(Q)$ formulas in such models with the new clause

$$(\mathfrak{A}, q) \models Qx\phi[s] \quad \text{iff} \quad \{a \in A: (\mathfrak{A}, q) \models \phi[s(x, a)]\} \in q.$$

Here, $s(x, a)$ denotes $[s \restriction (\mathrm{dom}(s) - \{x\})] \cup \{\langle x, a\rangle\}$.

2.3.1 Definition. $\mathscr{L}^0(Q)$ is the logic with concrete syntax with the usual notions of substitution, $\mathrm{frvar}(\phi)$, and $r(\phi)(=\text{complexity of } \phi)$. The axioms are simply the schemas of first-order logic together with the universal closure of each formula $\forall x(\phi \leftrightarrow \psi) \rightarrow (Qx\phi \leftrightarrow Qx\psi)$, as well as of each formula $Qx\phi \leftrightarrow Qy(\phi^x_y)$ whenever y does not occur in ϕ.

Strictly speaking, $\mathscr{L}^0(Q)$ is a logic only if we give a "standard semantics", that is, a global interpretation of Q. But this is not a problem, since in this discussion we are only concerned with weak models. For a fuller explanation of this point see Remark 7.1.2.

2.3.2 Proposition. *Suppose $\mathfrak{A}^*$ is a weak model for $\mathscr{L}^0(Q)$. Let q consist of all sets of the form $\{a \in A: \mathfrak{A}^* \models \phi[s(x, a)]\}$ such that $\mathfrak{A}^* \models Qx\phi[s]$. Then for all $\phi \in \mathscr{L}^0(Q)$ and s, $\mathfrak{A}^* \models \phi[s]$ iff $(\mathfrak{A}, q) \models \phi[s]$.*

Proof. The proof is a straightforward induction on complexity. The only interesting step is that of assuming that $(\mathfrak{A}, q) \models Qv\phi[s]$ holds and showing that $\mathfrak{A}^* \models Qv\phi[s]$ must hold also. By definition, there exist $Qu\psi$ and t such that $\mathfrak{A}^* \models Qu\psi[t]$ and for all $a \in A$,

(1) $$\mathfrak{A}^* \models \psi[t(u, a)] \Leftrightarrow \mathfrak{A}^* \models \phi[s(v, a)].$$

The following two facts are easy to establish.

(2) Suppose $\mathbf{x}$ and $\mathbf{y}$ are disjoint. For every $\mathscr{L}^0(Q)$ formula $\theta(\mathbf{x})$ there is an $\mathscr{L}^0(Q)$ formula $\theta'(\mathbf{x})$ of the same vocabulary, such that no y_i from $\mathbf{y}$ occurs in θ', and $\vdash_{\mathscr{L}^0(Q)} \forall\mathbf{x}(\theta \leftrightarrow \theta')$.

(3) For any formulas θ and θ' and sequences $\mathbf{x}$ and $\mathbf{y}$ as in (2), if f maps $\mathbf{y}$ to $\mathbf{x}$, that is, $f(y_i) = x_i$, all i, then for all s, $\mathfrak{A}^* \models \theta[s]$ iff $\mathfrak{A}^* \models \theta'[s]$ iff $\mathfrak{A}^* \models \theta'^{\mathbf{x}}_{\mathbf{y}}[s \circ f]$.

For, (2) follows by induction on θ, using the axioms $Qy\alpha \leftrightarrow Qz(\alpha^y_z)$ and the theorems $\exists y\alpha \leftrightarrow \exists z(\alpha^y_z)$, while the second "iff" in (3) follows from the equality axiom $\mathbf{x} = \mathbf{y} \wedge \theta'(\mathbf{x}) \rightarrow \theta'(\mathbf{y})$. Thus, we may assume that ϕ and ψ have disjoint sets of free variables; and by changing ψ again, we may assume that u and v are the same variable. Accordingly, (1) then yields

$$\text{(4)} \qquad \mathfrak{A}^* \models (\psi \leftrightarrow \phi)[(s \cup t)(v, a)] \quad \text{for all } a \in A;$$

and hence,

$$\text{(5)} \qquad \mathfrak{A}^* \models \forall v(\psi \leftrightarrow \phi)[s \cup t].$$

By the axioms, we have

$$\mathfrak{A}^* \models (Qv\psi \leftrightarrow Qv\phi)[s \cup t].$$

Since $\mathfrak{A}^* \models Qv\psi[t]$, $\mathfrak{A}^* \models Qv\phi[s]$ and the argument is complete. ▯

We can also define the class of *monotone structures* as in

2.3.3 Definition. A structure $(\mathfrak{A}, q)$, where $q \subseteq \mathscr{P}(A)$, is said to be a *monotone structure* if for all X and Y, $X \supseteq Y \in q$ implies that $X \in q$.

For more on monotone structures, the reader should consult Makowsky-Tulipani [1977] or Ziegler [1978]. In the present volume, Chapter III, Section 4, Chapter XV, Section 6 and Section 6.4 of this chapter offer some further material along these lines.

The logic with concrete syntax $\mathscr{L}^{\mathrm{m}}(Q)$ where the "m" stands for monotone, is obtained from $\mathscr{L}^0(Q)$ by strengthening the axioms $\forall x(\phi \leftrightarrow \psi) \to (Qx\phi \leftrightarrow Qx\psi)$ to $\forall x(\phi \to \psi) \to (Qx\phi \to Qx\psi)$.

2.3.4 Proposition. *Suppose $\mathfrak{A}^*$ is a weak model for $\mathscr{L}^{\mathrm{m}}(Q)$. Let q consist of all subsets of A which contain $\{a \in A : \mathfrak{A}^* \models \phi[s(x, a)]\}$ for some ϕ and s such that $\mathfrak{A}^* \models Qx\phi[s]$. Then $(\mathfrak{A}, q)$ is a monotone structure and for all $\phi \in \mathscr{L}^{\mathrm{m}}(Q)$ and s, $\mathfrak{A}^* \models \phi[s]$ iff $(\mathfrak{A}, q) \models \phi[s]$.*

Proof. Of course, $(\mathfrak{A}, q)$ is a monotone structure. The remainder of the proof is obtained from the proof of Proposition 2.3.2 by changing "$\leftrightarrow$" to "$\to$" in (1), (4), and (5). ▯

Although our main purpose in this section has been to pave the way for completeness proofs in Sections 3, 4, and 5, we should notice that our digression here in Section 2.3 has brought us to the well-known weak completeness theorem given in

2.3.5 Corollary (Folklore Weak Completeness). *Let T be a consistent set of sentences in $\mathscr{L}^0(Q)$. Then, for all $\kappa \geq \omega$, there exists $(\mathfrak{A}, q) \models T$ such that $|A| = \kappa + |T|$. If T is in fact $\mathscr{L}^{\mathrm{m}}(Q)$-consistent, we may take $(\mathfrak{A}, q)$ to be a monotone structure. The converses (soundness) also hold, regardless of cardinalities.*

Proof. For countable T, this is immediate from the weak completeness theorem (2.2.3) together with Propositions 2.3.2 and 2.3.4. In general, we can obtain a weak model of each countable subset of T, apply first-order compactness and Löwenheim–Skolem arguments to get a weak model $\mathfrak{A}^*$ of T of the desired cardinality, and then apply Propositions 2.3.2 or 2.3.4. The argument for soundness is clear. □

2.3.6 Corollary (Compactness for Weak Models). *Let T be a set of sentences of $\mathscr{L}^0(Q)$ such that every finite subset of T has a weak model. Then T has a weak model.* The term "weak model" may have either of the two meanings from Proposition 2.3.2.

If, in fact, every finite subset of T has a weak model which is a monotone structure, then T has a weak model which is a monotone structure. The reader can find an ultraproduct proof for this in Makowsky–Tulipani [1977, §7].) □

3. $\mathscr{L}(Q_1)$ and $\mathscr{L}_{\omega_1\omega}(Q_1)$: Completeness and Omitting Types Theorems

This section consists primarily of the main results from Keisler's paper [Ke][1] on $\mathscr{L}(Q)$, where Q = "there exist uncountably many." Although we will base the proofs on the notion of weak model as presented in Section 2, the reader may prefer to use Keisler's notion (see Section 2.3) or any other notion having reasonable properties. Further applications of the completeness theorem for $\mathscr{L}(Q)$ can be found in [Ke].

3.1. The Axioms, Basic Notions, and Properties

3.1.1 Definition ([Ke]). The axioms of $\mathscr{L}(Q)$ include the universal closures of all first-order axiom schemas as well as the following axioms, all of which may have free variables other than those displayed.

(1) $\neg Qx(x = y \vee x = z)$;

(2) $\forall x(\phi \to \psi) \to (Qx\phi \to Qx\psi)$;

(3) $Qx\phi(x) \leftrightarrow Qy\phi(y)$, where $\phi(x, \ldots)$ is a formula of $\mathscr{L}(Q)$ in which y does not occur, and $\phi(y, \ldots)$ is obtained by replacing each free occurrence of x by y;

(4) $Qy\,\exists x\phi \to \exists xQy\phi \vee Qx\,\exists y\phi$.

[1] Henceforth, [Ke] will refer to Keisler [1970]. Except as otherwise noted all results in Section 3 are proved in [Ke].

The rule of inference is *modus ponens* (but universal generalization may be derived as in Enderton [1972]). Notice that the axioms are all valid (where again, $Q = Q_1$). To see that Axiom 4 is valid, consider its contrapositive $\neg Qx\,\exists y\phi \wedge \neg\exists xQy\phi \to \neg Qy\,\exists x\phi$, which asserts that a countable union of countable sets is countable. Throughout the following discussion we will assume that the axiom of choice holds. Keisler [Ke] also credits Craig and Fuhrken with the conjecture that these axioms are complete.

In order to apply the results of Section 2 (on weak models) to the problems at hand, we need the following lemma. The proof, though routine, is omitted since it lacks interest. Nevertheless, we note that the proof of (i) is similar to the $\mathscr{L}_{\omega\omega}$ case as treated in Enderton [1972].

3.1.2 Lemma. (i) *With the notion of proof as defined above, $\mathscr{L}(Q)$ has a concrete syntax (in the sense of Section 2).*

(ii) *The notion "$\mathscr{L}(Q)$-locally omits" as given in Definition 2.2.4 is equivalent to the usual notion. That is, for a fixed vocabulary τ, T $\mathscr{L}(Q)$-locally omits $\Sigma(\mathbf{x})$ iff whenever $\exists\mathbf{x}\phi$ is consistent with T, then so is $\exists\mathbf{x}(\phi \wedge \neg\sigma)$ for some $\sigma \in \Sigma$.* $\square$

For the remainder of this section, we fix a countable vocabulary τ. The proof of the completeness theorem is composed of three steps. First, the weak completeness theorem (2.2.3) is applied to obtain a countable weak model of a consistent theory T. That done, we then prove a "main lemma" which will, in effect, show how to expand "uncountable" sets while keeping "countable" sets unexpanded. Extending the given countable weak model and iterating ω_1 times using this process, we will find that the union of the structures gives the desired model of T. First, however, let us formally state the kind of extension we need.

3.1.3 Definition. Let $\mathfrak{A}^*$ and $\mathfrak{B}^*$ be countable weak models for $\mathscr{L}(Q)$. We say that $\mathfrak{B}^*$ is a *precise extension of* $\mathfrak{A}^*$ *relative to* ϕ, if $\phi(x)$ is a formula of $\mathscr{L}(Q)$ with parameters in A and

(i) $\mathfrak{A}^* \prec^w \mathfrak{B}^*$.
(ii) If $\mathfrak{A}^* \models Qx\phi$, then $\mathfrak{B}^* \models \phi(b)$ for some $b \in B - A$.
(iii) Whenever $\mathfrak{A}^* \models \neg Qx\psi$ for $Qx\psi$ a sentence with parameters in A, then $\mathfrak{B}^* \models \neg\psi(b)$ for all $b \in B - A$.

3.1.4 Remarks on Notation. Notice that the notation has become more informal than that used in Section 2. A precise definition would consider precise extensions relative to $\langle\phi, s\rangle$, where ϕ is a formula of $\mathscr{L}(Q)$, and s is an assignment into A with domain including all but at most one free variable x of ϕ. Then, for example, (ii) would be worded thus: "if $\mathfrak{A}^* \models Qx\phi[s]$ then $\mathfrak{B}^* \models \phi[s(x, b)]$ for some $b \in B - A$." The more informal notation will generally be used in the sequel.

The symbol Q^*x is an abbreviation for $\neg Qx\neg$, "for all but countably many x." Before moving to the "main lemma", we should summarize some easy consequences of the axioms. Accordingly, we have

3.1.5 Lemma. *Every formula in the following schema is a theorem of $\mathscr{L}(Q)$ and is therefore valid in every weak model for $\mathscr{L}(Q)$.*

(i) $\neg Qx\psi \leftrightarrow Q^*x \neg \psi$.

(ii) $Qx(x = x) \rightarrow Q_1x_1 \ldots Q_nx_n(\phi \wedge Q_{n+1}y_1 \ldots Q_{n+m}y_m\psi)$ $\leftrightarrow Q_1x_1 \ldots Q_nx_nQ_{n+1}y_1 \ldots Q_{n+m}\, y_m(\phi \wedge \psi)$, *whenever $y_1, \ldots, y_m$ are not free in ϕ, and each $Q_i \in \{\exists, \forall, Q, Q^*\}$.*

(iii) (Monotonicity) $\forall \mathbf{x}(\phi \rightarrow \psi) \rightarrow (q\mathbf{x}\phi \rightarrow q\mathbf{x}\psi)$, *where $q\mathbf{x}$ is any string of $\exists$, $\forall$, Q, Q^* quantifiers on $\mathbf{x}$.*

Moreover, we also have the following "Intersection principles":

(iv) $\bigwedge_{i \in I} Q^*x\psi \rightarrow Q^*x \bigwedge_{i \in I} \psi$ *(I finite).*

(v) $Qx\phi \wedge Q^*x\psi \rightarrow Qx(\phi \wedge \psi)$.

(vi) $\forall x\phi \wedge qx\psi \rightarrow qx(\phi \wedge \psi)$ (for $q = Q$ or Q^*). □

3.2. Towards a Proof of Keisler's Completeness Theorem

3.2.1 Main Lemma. *Suppose $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(Q)$, and suppose $\phi(x, \mathbf{p})$ is a formula of $\mathscr{L}(Q)$ with parameters $\mathbf{p}$ in A. Then there is a precise extension of $\mathfrak{A}^*$ relative to ϕ.*

Proof. If $\mathfrak{A}^* \models \neg Qx\phi(x, \mathbf{p})$, then we set $\mathfrak{B}^* = \mathfrak{A}^*$. So, assume that $\mathfrak{A}^* \models Qx\phi(x, \mathbf{p})$, and let $C_A = \{c_a\colon a \in A\}$ be a set of new constant symbols. Also let $D = C_A \cup \{c\}$ for yet another constant symbol c, and form the following set $T_\phi(\mathfrak{A}^*)$ of $\tau \cup D$-sentences of $\mathscr{L}(Q)$. The notation $\mathbf{c_a}$ denotes $\langle c_{a_1}, \ldots, c_{a_n}\rangle$, when $\mathbf{a} = \langle a_1, \ldots, a_n\rangle$ is any sequence of elements of A.

$$T_\phi(\mathfrak{A}^*) = \{\theta(\mathbf{c_a})\colon \mathfrak{A}^* \models \theta(\mathbf{a})\} \cup \{\phi(c, \mathbf{c_p})\}$$
$$\cup \{\neg\psi(c, \mathbf{c_a})\colon \mathfrak{A}^* \models \neg Qx\psi(x, \mathbf{a})\}.$$

For each $\psi(x, \mathbf{a})$, we define a set Σ_ψ such that

$$\Sigma_\psi = \{\psi(x, \mathbf{c_a})\} \cup \{x \neq c_b\colon \mathfrak{A}^* \models \psi(b, \mathbf{a})\}. \quad \square$$

Claim A. $T_\phi(\mathfrak{A}^*)$ is an $\mathscr{L}(Q)$-consistent theory which $\mathscr{L}(Q)$-locally omits Σ_ψ, for each $\psi(x, \mathbf{a})$, such that $\mathfrak{A}^* \models \neg Qx\psi(x, \mathbf{a})$.

Deferring the proof of Claim A for the moment, we will see how the theorem follows. Let $\mathfrak{B}^* \succ^w \mathfrak{A}^*$ be the countable weak model guaranteed by the extension lemma (2.2.6) or by Lemma 7.2.3. That is, $\mathfrak{B}^*$ omits each Σ_ψ, and there exists $e \in B$ (corresponding to c) such that for all $\theta(c, \mathbf{c_a}) \in T_\phi(\mathfrak{A}^*)$, $\mathfrak{B}^* \models \theta(e, \mathbf{a})$. Since

$\phi(c, \mathbf{c_p}) \in T_\phi(\mathfrak{A}^*)$, it follows that $\mathfrak{B}^* \models \phi(e, \mathbf{p})$. Moreover, $\neg(c = c_a) \in T_\phi(\mathfrak{A}^*)$ for all $a \in A$, since $\mathfrak{A}^* \models \neg Qx(x = a)$ by Axioms 1 and 2. Thus, $\mathfrak{B}^* \models e \neq a$ for all $a \in A$, and hence $e \notin A$. Accordingly, we see that (ii) in the definition of "precise extension relative to ϕ" is satisfied. Part (iii) holds because $\mathfrak{B}^*$ omits each necessary Σ_ψ. Thus, the proof is complete once Claim A has been proved. First, however, it is very helpful to have a useful criterion for consistency of $\tau \cup D$-sentences of $\mathscr{L}(Q)$ with $T_\phi(\mathfrak{A}^*)$.

Claim B (Consistency Criterion). For any τ-formula $\theta(y, \mathbf{z})$ of $\mathscr{L}(Q)$ and $\mathbf{a}$ in A:

(i) $\theta(c, \mathbf{c_a})$ is $\mathscr{L}(Q)$-consistent with $T_\phi(\mathfrak{A}^*)$ iff $\mathfrak{A}^* \models Qy(\phi(y, \mathbf{p}) \wedge \theta(y, \mathbf{a}))$.
(ii) $T_\phi(\mathfrak{A}^*) \vdash_{\mathscr{L}(Q)} \theta(c, \mathbf{c_a})$ iff $\mathfrak{A}^* \models Q^*y(\phi(y, \mathbf{p}) \rightarrow \theta(y, \mathbf{a}))$. (Recall $Q^* = \neg Q \neg$.)

Proof of Consistency Criterion. Using Lemma 3.1.5(i) and $\neg\theta$ for θ, it is easy to see that (i) and (ii) are equivalent. Thus, we will only prove (ii). For the ($\Leftarrow$) direction, we suppose that $\mathfrak{A}^* \models \neg Qy \neg (\phi(y, \mathbf{p}) \rightarrow \theta(y, \mathbf{a}))$. Then $\neg(\neg(\phi(c, \mathbf{c_p}) \rightarrow \theta(c, \mathbf{c_a}))) \in T_\phi(\mathfrak{A}^*)$, by definition. Thus, $T_\phi(\mathfrak{A}^*) \vdash_{\mathscr{L}(Q)} \phi(c, \mathbf{c_p}) \rightarrow \theta(c, \mathbf{c_a})$. And, since $\phi(c, \mathbf{c_p}) \in T_\phi(\mathfrak{A}^*)$, we have that $T_\phi(\mathfrak{A}^*) \vdash_{\mathscr{L}(Q)} \theta(c, \mathbf{c_a})$.

Conversely, suppose $T_\phi(\mathfrak{A}^*) \vdash \theta(c, \mathbf{c_a})$. Since proofs are finite, there exist formulas $\delta_i(\mathbf{c}_{\mathbf{a}_i})$ for $i \in I$ and $\psi_j(y, \mathbf{c}_{\mathbf{a}_j})$ with $j \in J$, where both I and J are finite, such that

(1) $\qquad \mathfrak{A}^* \models \delta_i(\mathbf{a}_i), \quad$ all $\quad i \in I$;

(2) $\qquad \mathfrak{A}^* \models \neg Qy\psi_j(y, \mathbf{a}_j), \quad$ all $\quad j \in J$;

(3) $\qquad \{\phi(c, \mathbf{c_p})\} \cup \{\delta_i(\mathbf{c}_{\mathbf{a}_i}): i \in I\} \cup \{\neg\psi_j(c, \mathbf{c}_{\mathbf{a}_j}): j \in J\} \vdash_{\mathscr{L}(Q)} \theta(c, \mathbf{c_a})$.

By repeated application of the deduction theorem (2.1.1), we see that (3) implies that

$$\vdash_{\mathscr{L}(Q)} \left(\bigwedge_{i \in I} \delta_i(\mathbf{c}_{\mathbf{a}_i})\right) \rightarrow \left[\phi(c, \mathbf{c_p}) \wedge \left(\bigwedge_{j \in J} \neg\psi_j(c, \mathbf{c}_{\mathbf{a}_j})\right) \rightarrow \theta(c, \mathbf{c_a})\right].$$

By soundness (see Proposition 2.2.1), since $\mathfrak{A}^* \models \bigwedge_{i \in I} \delta_i(\mathbf{a}_i)$ by (1) this yields

$$(4) \qquad \mathfrak{A}^* \models \forall y\left[\phi(y, \mathbf{p}) \wedge \left(\bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j)\right) \rightarrow \theta(y, \mathbf{a})\right].$$

We now make use of the "intersection principles" of Lemma 3.1.5. Applying Lemma 3.1.5(iv) and (i) to (2) above, we obtain $\mathfrak{A}^* \models Q^*y \bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j)$. Combining this with (4) above, the "intersection principle" given in Lemma 3.1.5(vi) shows that

$$\mathfrak{A}^* \models Q^*y\left[\left[\phi(y, \mathbf{p}) \wedge \left(\bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j)\right) \rightarrow \theta(y, \mathbf{a})\right] \wedge \bigwedge_{j \in J} \neg\psi_j(y, \mathbf{a}_j).\right]$$

By Lemma 3.1.5(iii) (monotonicity) this implies $\mathfrak{A}^* \models Q^*y[\phi(y, \mathbf{p}) \to \theta(y, \mathbf{a})]$, which concludes the proof of Claim B, the "consistency criterion".

It now remains to prove Claim A. First of all, the consistency criterion implies that $T_\phi(\mathfrak{A}^*)$ is $\mathscr{L}(Q)$-consistent. Now suppose that $\mathfrak{A}^* \models \neg Qx\psi(x, \mathbf{a})$. We must show that $T_\phi(\mathfrak{A}^*)$ $\mathscr{L}(Q)$-locally omits $\Sigma_\psi = \{\psi(x, \mathbf{c_a}) \cup \{x \neq c_b : \mathfrak{A}^* \models \psi(b, \mathbf{a})\}$, in the sense of Lemma 3.1.2(ii). Thus, suppose $\exists x\theta(x, c, \mathbf{c_d})$ is consistent with $T_\phi(\mathfrak{A}^*)$, where $\mathbf{d}$ is from A. By the consistency criterion and Lemma 3.1.5(ii), we have

(5) $$\mathfrak{A}^* \models Qy\, \exists x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d})].$$

If $\mathfrak{A}^* \models Qy\, \exists x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \neg\psi(x, \mathbf{a})]$, then by the consistency criterion, $\exists x[\theta(x, c, \mathbf{c_d}) \wedge \neg\psi(x, \mathbf{c_a})]$ is consistent with $T_\phi(\mathfrak{A}^*)$, and we're done. Otherwise, $\mathfrak{A}^* \models Q^*y\forall x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \to \psi(x, \mathbf{a})]$. Then, by the intersection principle Lemma 3.1.5(v) and its analogue for $\mathscr{L}_{\omega\omega}$, this combines with (5) to yield

(6) $$\mathfrak{A}^* \models Qy\, \exists x[[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d})] \wedge [\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \to \psi(x, \mathbf{a})]].$$

Applying the monotonicity principle (Lemma 3.1.5(iii)) to (6), we have

(7) $$\mathfrak{A}^* \models Qy\, \exists x[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \psi(x, \mathbf{a})].$$

Now is the time to apply the main axiom of $\mathscr{L}(Q)$, namely Axiom 4. Applied to (7) this gives

(8) $$\mathfrak{A}^* \models Qx\, \exists y[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \psi(x, \mathbf{a})],$$

or

(9) $$\mathfrak{A}^* \models \exists x Qy[\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \psi(x, \mathbf{a})].$$

But (8) is impossible, since it implies that $\mathfrak{A}^* \models Qx\psi(x, \mathbf{a})$—a contradiction of the assumption. Thus, there exists a witness $e \in A$ for (9) above. Then $\mathfrak{A}^* \models Qy[\phi(y, \mathbf{p}) \wedge \theta(e, y, \mathbf{d})]$ which further implies $\mathfrak{A}^* \models Qy\, \exists x(\phi(y, \mathbf{p}) \wedge \theta(x, y, \mathbf{d}) \wedge \neg x \neq e)$, by monotonicity. But applying the consistency criterion we see that $\exists x(\theta(x, c, \mathbf{c_d}) \wedge \neg x \neq c_e)$ is $\mathscr{L}(Q)$-consistent with $T_\phi(\mathfrak{A}^*)$, as desired. ☐

Remark. In [Ke], $\mathfrak{B}^*$ is defined to be a precise extension of $\mathfrak{A}^*$ if it is a precise extension relative to every formula. By iterating the Main Lemma ω times in an appropriate manner, we may construct such an extension. Although this would slightly simplify the proof of the completeness theorem (3.2.3), such a notion of extension is not as useful for $\mathscr{L}(\text{aa})$ in Section 4 and for $\mathscr{L}(Q^2)$ in Section 5.

The final lemma needed for the proof of the completeness theorem tells us that a careful iteration of the Main Lemma produces the desired model.

3.2.2 Lemma (Union of Chain Lemma). *Assume that $\langle \mathfrak{A}_\alpha^*: \alpha < \omega_1 \rangle$ is a chain of countable weak models for $\mathscr{L}(Q)$, with the following properties.*

(i) *For all $\alpha < \omega_1$, $\mathfrak{A}_{\alpha+1}^*$ is a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, for some ϕ.*
(ii) *For each formula $\phi(x)$ with parameters in some A_α, $\{\beta < \omega_1: \mathfrak{A}_{\beta+1}^*$ is a precise extension of $\mathfrak{A}_\beta^*$ relative to $\phi\}$ is uncountable.*
(iii) *The chain is continuous, that is, $\mathfrak{A}_\lambda^* = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha^*$ for limit $\lambda < \omega_1$.*

Then by setting $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$, we have $\mathfrak{A} \models \phi$ iff $\mathfrak{A}_\alpha^* \models \phi$ for all sentences ϕ with parameters in A_α, where $\alpha < \omega_1$ is arbitrary.

Proof. The proof is by induction on the length of ϕ. For atomic ϕ, it is clear, and both the $\neg$ and $\vee$ steps are trivial. Now notice that $\alpha < \beta < \omega_1$ implies $\mathfrak{A}_\alpha^* \prec^w \mathfrak{A}_\beta^*$, by Proposition 2.2.2 (the elementary chain theorem for weak models). The case $\phi = \exists y \psi(y)$ then follows in the usual way.

Finally, suppose that ϕ is $Qx\psi$. If $\mathfrak{A}_\alpha^* \models Qx\psi$, then by (ii) above, there exists an uncountable set $X \subseteq \omega_1$ such that for all $\beta \in X$, $\mathfrak{A}_{\beta+1}^* \models \psi(a)$ for some $a \in A_{\beta+1} - A_\beta$. This implies that $\mathfrak{A} \models \psi(a)$ for some $a \in A_{\beta+1} - A_\beta$. Thus, $\mathfrak{A} \models Qx\psi$. Conversely, if $\mathfrak{A}_\alpha^* \models \neg Qx\psi$, then by (i), it follows by induction on β (and the definition of precise extension relative to a formula) that $\mathfrak{A}_\beta^* \models \psi(a)$ implies $a \in A_\alpha$. By the inductive hypothesis, this translates into: $\mathfrak{A} \models \psi(a)$ implies $a \in A_\alpha$. Since A_α is countable, we must have that $\mathfrak{A} \models \neg Qx\psi$. ☐

3.2.3 Theorem (Completeness Theorem for $\mathscr{L}(Q)$). *Suppose T is a set of τ-sentences of $\mathscr{L}(Q)$, where τ is a countable vocabulary. Then T is $\mathscr{L}(Q)$-consistent iff T has a model.*

Proof. We have already shown soundness. For the other direction, we suppose T is $\mathscr{L}(Q)$-consistent. We wish to define a chain $\langle \mathfrak{A}_\alpha^*: \alpha < \omega_1 \rangle$ which satisfies the hypotheses of Lemma 3.2.2, the "union of chain lemma." It will be convenient to require $A_\alpha \subseteq \omega_1$ for all $\alpha < \omega_1$. For then we will have that $\bigcup_{\alpha<\omega_1} A_\alpha \subseteq \omega_1$, and the following construction will indeed witness each $Qx\phi$ uncountably many times.

More precisely, we start with any partition of ω_1 into uncountable sets X_ϕ, where ϕ ranges over formulas $\phi(x)$ with parameters in ω_1. Let us define $\mathfrak{A}_\alpha^*$ by induction on α. First, let $\mathfrak{A}_0^*$ be a countable weak model for $\mathscr{L}(Q)$ which satisfies T, by the weak completeness theorem (2.2.3). We may require $A_0 = \omega$. For successor stages $\alpha + 1$, we apply the Main Lemma (3.2.1). Let $\mathfrak{A}_{\alpha+1}^*$ be a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, where $\alpha \in X_\phi$ (unless the parameters of ϕ do not lie inside A_α, in which case set $\mathfrak{A}_{\alpha+1}^* = \mathfrak{A}_\alpha^*$). Finally, set $\mathfrak{A}_\lambda^* = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha^*$ for limit $\lambda < \omega_1$. Also set $\mathfrak{A} = \bigcup \{\mathfrak{A}_\alpha: \alpha < \omega_1\}$. By Lemma 3.2.2, we have that $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$, for all $\alpha < \omega_1$ and $\mathbf{a}$ in A_α. In particular, since $\mathfrak{A}_0^* \models \phi$ for all $\phi \in T$, we have that $\mathfrak{A}$ is a model of T. ☐

3.2.4 Corollary. *$\mathscr{L}(Q)$ is countably compact.* ☐

Before continuing with an extension of the completeness theorem to $\mathscr{L}_{\omega_1\omega}(Q)$ and omitting types in $\mathscr{L}(Q)$, we will examine a corollary to the Main Lemma, as was promised in Section 1. This result appears as Corollary 3.6.1 of [Ke].

3.2.5 Theorem (Essentially due to Keisler–Morley [1968:4.2, 2.2]). *Let $\mathfrak{A} = (A, E)$ be a countable model of* ZF, *possibly excepting power set. For all $a \in A$, set $a_E = \{b: \langle b, a\rangle \in E\}$.*

(i) *There exists $\mathfrak{B} = (B, F) \succ \mathfrak{A}$ such that for all $a \in A$, $a_F = a_E$, and the ordinals of $\mathfrak{B}$ are ω_1-like.*
(ii) *For every regular cardinal a of $\mathfrak{A}$, there exists $\mathfrak{B} = (B, F) \succ \mathfrak{A}$ such that $b_E = b_F$ for all bEa, but $\langle a_F, F \restriction a_F\rangle$ is ω_1-like.*

In fact, for (ii) *it is not necessary that $\mathfrak{A}$ satisfy the collection schema.*

Proof. (i) We expand $\mathfrak{A}$ to a weak model $\mathfrak{A}^*$ for $\mathscr{L}(Q)$ by interpreting Q as "for unboundedly many." For every ε-formula ϕ of $\mathscr{L}(Q)$, let ϕ^+ be the result of replacing each quantifier of the form "Qx" by "there exist arbitrarily large x", that is, $\forall y\, \exists x(x \notin y \wedge \cdots)$ (where y is chosen not to conflict with other variables of ϕ). We then set $R^{\mathfrak{A}^*}_{Qx\phi(x,\mathbf{y})} = \{\mathbf{a}: \mathfrak{A} \models (Qx\phi)^+(\mathbf{a})\}$. As in the proof of Proposition 2.3.2, an easy induction on complexity of $\phi \in \mathscr{L}(Q)$ shows that $\mathfrak{A}^* \models \phi(\mathbf{a})$ iff $\mathfrak{A} \models \phi^+(\mathbf{a})$ for all ϕ and $\mathbf{a}$. It is then easy to check that $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(Q)$: the axiom of collection is used to verify Axiom 4.

By the Main Lemma (3.2.1), $\mathfrak{A}^*$ has a precise extension relative to "x is an ordinal". Iterating, we thus obtain a chain $\langle \mathfrak{A}^*_\alpha : \alpha < \omega_1\rangle$ with $\mathfrak{A}^*_0 = \mathfrak{A}^*$, such that $\mathfrak{A}^*_{\alpha+1}$ is a precise extension of $\mathfrak{A}^*_\alpha$ relative to "x is an ordinal" for all $\alpha < \omega_1$, and $\mathfrak{A}^*_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{A}^*_\alpha$ for all limit $\lambda < \omega_1$. Set $\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$; then $\mathfrak{B}$ is the desired model.

(ii) The proof here is the same, except for two changes. This time, ϕ^+ is obtained by replacing each quantifier Qx by $\forall y \in a\, \exists x \in a\, (y \in x \wedge \cdots)$, and the expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ is defined accordingly. Also, in this situation we require that $\mathfrak{A}^*_{\alpha+1}$ be a precise extension of $\mathfrak{A}^*_\alpha$ relative to $x \in a$. These changes made, the proof of (i) goes through. □

A rather similar development concerning linear orders appears in Jervell [1975].

3.3. Omitting Types in $\mathscr{L}(Q)$

The next goal in this section is to get an omitting types theorem. Further on, in Section 3.4 we will discuss applications.

3.3.1 Definition ([Ke]). Let T be a set of τ-sentences, and $\Sigma(\mathbf{x})$ a set of τ-formulas (with free variables contained in the finite sequence $\mathbf{x}$), of $\mathscr{L}(Q)$. T is said to *strongly omit* Σ if the following condition is met. Let $\overline{Qy}$ be an arbitrary quantifier string of the form $Q_1 y_1 \ldots Q_n y_n$, where $Q_i \in \{\exists, Q\}$ for $1 \le i \le n$. We call such a $\overline{Qy}$ *a quexistential string*. Then, for every sentence of the form $\overline{Qy}\, \exists \mathbf{x}\phi$ which is consistent with T, there exists $\sigma \in \Sigma$ such that $\overline{Qy}\, \exists \mathbf{x}(\phi \wedge \neg\sigma)$ is consistent with T.

A weak model $\mathfrak{A}^*$ is said to *strongly omit* $\Sigma(\mathbf{x})$, where Σ may have parameters in A, if whenever $\mathfrak{A}^* \models \overline{Qy}\, \exists\mathbf{x}\phi$ with $\overline{Qy}$ a quexistential string, where ϕ may have parameters in A, then $\mathfrak{A}^* \models \overline{Qy}\, \exists\mathbf{x}(\phi \wedge \neg\sigma)$ for some $\sigma \in \Sigma$.

For applications to logics such as $\mathscr{L}^{<\omega}$ (in Section 5), it is helpful to consider certain extensions of $\mathscr{L}(Q)$.

3.3.2 Definition. A logic $\mathscr{L}$ with concrete syntax is a *reasonable extension* of $\mathscr{L}(Q)$ if it meets the following criteria.

(i) $\mathscr{L}$ is closed under Q: if $\phi \in \mathscr{L}(\tau)$ then $Qx\phi \in \mathscr{L}(\tau)$.
(ii) Every formula ϕ of $\mathscr{L}(\tau \cup C)$ with $C \cap \tau = \varnothing$, is $\psi(f)$ for some $\psi \in \mathscr{L}(\tau)$ and some f. (This is needed for the proof of the Main Lemma (3.2.1); it enables the proof of Proposition 3.1.2(ii) to go forward.)
(iii) The notions of free variable, substitution, and rank—frvar(ϕ), $\phi(f)$, $r(\phi)$ from Section 2.1—obey the obvious inductive clauses for Q.
(iv) Every axiom schema (1–4) of $\mathscr{L}(Q)$ is an axiom schema of $\mathscr{L}$. In particular, there is a notion of change of free variable to which Axiom 3 applies, as does $\exists x\phi(x) \leftrightarrow \exists y\phi(y)$.

3.3.3 Remark. The notion of "precise extension relative to ϕ", Lemma 3.1.2(ii), the quantifier manipulations of Lemma 3.1.5, and the Main Lemma (3.2.1) with Claims A and B, extend in the natural way to any reasonable extension of $\mathscr{L}(Q)$. That this is actually the case can be verified in a routine way. Accordingly, we will use these extended versions.

3.3.4 Lemma. *Fix a countable vocabulary τ. Let $\mathfrak{A}^*$ be a countable weak model for any reasonable extension $\mathscr{L}$ of $\mathscr{L}(Q)$. Suppose that $\Sigma(\mathbf{x})$ is any set of formulas in the finite sequence $\mathbf{x}$ of free variables, where Σ may have parameters in A. For every formula $\delta = \overline{Qy}\, \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})$, where ψ is parameter-free, such that $\mathbf{u}$ is disjoint from $\mathbf{x}$ and $\mathbf{y}$ and $\overline{Qy}$ is a quexistential string, let*

$$\Sigma^\delta(\mathbf{u}) = \{\overline{Qy}\, \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})\} \cup \{\neg\overline{Qy}\, \exists\mathbf{x}[\psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(x)] : \sigma \in \Sigma\}.$$

If A^* omits each such $\Sigma^\delta(\mathbf{u})$, then A^* strongly omits Σ.

Proof. The proof of this result follows immediately from the definitions. ☐

To prove the omitting types theorem we will follow the pattern of the completeness theorem proof. That is, we will obtain a weak model, iterate a "main lemma" ω_1 times, and then take the union. Hence, we will need:

3.3.5 Lemma ("Main Lemma" for Omitting Types). *Suppose $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\tau)$, τ countable, where $\mathscr{L}$ is a reasonable extension of $\mathscr{L}(Q)$. Let $\{\Sigma_n : n < \omega\}$ be a countable family of sets of τ-formulas of $\mathscr{L}$, possibly with parameters in A, each in a finite sequence $\mathbf{x}_n$ of free variables. Assume that $\mathfrak{A}^*$ strongly omits Σ_n for all $n < \omega$. Then for all $\phi(x, \mathbf{p})$, there is a precise extension of $\mathfrak{A}^*$ relative to ϕ which strongly omits each Σ_n.*

Proof. The proof is an extension of the proof of the Main Lemma (3.2.1), and we refer to that argument below. Form the theory $T_\phi(\mathfrak{A}^*)$ and the sets Σ_ψ, as before. By Claim A (from the proof of Lemma 3.2.1), $T_\phi(\mathfrak{A}^*)$ locally omits each set Σ_ψ. Suppose for the moment that $T_\phi(\mathfrak{A}^*)$ also locally omits each set Σ_n^δ, as defined in Lemma 3.3.4. Then as before, we apply the Extension Lemma (2.2.6) to obtain a precise extension $\mathfrak{B}^*$ of $\mathfrak{A}^*$ relative to ϕ, which omits each Σ_n^δ. By Lemma 3.3.4, we see that $\mathfrak{B}^*$ strongly omits each Σ_n.

It now remains to show that $T_\phi(\mathfrak{A}^*)$ locally omits each Σ_n^δ, say δ is $\overline{Qy}\ \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})$, where $\overline{Qy}$ is a quexistential string and $\mathbf{u}$ is disjoint from $\mathbf{x}$ and $\mathbf{y}$. Suppose that $\exists\mathbf{u}\theta(\mathbf{u}, \mathbf{c_a}, c)$ is consistent with $T_\phi(\mathfrak{A}^*)$. If

$$\exists\mathbf{u}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \neg\overline{Qy}\ \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]$$

is consistent with $T_\phi(\mathfrak{A}^*)$, our argument is done. Otherwise, $\exists\mathbf{u}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \overline{Qy}\ \exists\mathbf{x}\psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]$ is consistent with $T_\phi(\mathfrak{A}^*)$. By "quantifier shuffling" as discussed in Lemma 3.1.5(ii), $\exists\mathbf{u}\ \overline{Qy}\ \exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]$ is consistent with $T_\phi(\mathfrak{A}^*)$. We now apply the consistency criterion (that is, Claim B in the proof of Lemma 3.2.1) to obtain

$$\mathfrak{A}^* \models Qz[\phi(z, \mathbf{p}) \wedge \exists\mathbf{u}\ \overline{Qy}\ \exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u})]].$$

By using the last part of Definition 3.3.2(iv), we may replace $\mathbf{u}$, $\mathbf{y}$, $\mathbf{x}$ if necessary so that these are disjoint from the free variables of ϕ. Then, by using quantifier shuffling again, we have that

$$\mathfrak{A}^* \models Qz\ \exists\mathbf{u}\ \overline{Qy}\ \exists\mathbf{x}[\phi(z, \mathbf{p}) \wedge \theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u})].$$

But $Qz\ \exists\mathbf{u}\ \overline{Qy}$ is also a quexistential string; and so, since $\mathfrak{A}^*$ strongly omits Σ_n, there exists $\sigma \in \Sigma_n$ such that

$$\mathfrak{A}^* \models Qz\ \exists\mathbf{u}\ \overline{Qy}\ \exists\mathbf{x}[\phi(z, \mathbf{p}) \wedge \theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})].$$

By using quantifier shuffling again, we obtain

$$\mathfrak{A}^* \models Qz[\phi(z, \mathbf{p}) \wedge \exists\mathbf{u}\ \overline{Qy}\ \exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{a}, z) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})]].$$

And applying the consistency criterion once more, we see that

$$\exists\mathbf{u}\ \overline{Qy}\ \exists\mathbf{x}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})]$$

is consistent with $T_\phi(\mathfrak{A}^*)$. Again using quantifier shuffling, we have that

$$\exists\mathbf{u}[\theta(\mathbf{u}, \mathbf{c_a}, c) \wedge \overline{Qy}\ \exists\mathbf{x}[\psi(\mathbf{x}, \mathbf{y}, \mathbf{u}) \wedge \neg\sigma(\mathbf{x})]]$$

is consistent with $T_\phi(\mathfrak{A}^*)$, and the proof is complete. $\square$

3.3.6 Theorem (Omitting Types Theorem for $\mathscr{L}(Q)$). *Suppose that $\mathscr{L}$ is a reasonable extension of $\mathscr{L}(Q)$ and that τ is countable. Suppose also that T is a consistent*

τ-theory of $\mathscr{L}(Q)$ which strongly omits sets $\Sigma_n(\mathbf{x}_n)$ ($n < \omega$) from $\mathscr{L}(\tau)$. Then T has a model which omits each Σ_n.

Proof. As was shown in the proof of Lemma 3.3.5, it follows that T $\mathscr{L}(Q)$-locally omits the sets Σ_n^δ of Lemma 3.3.4. By the weak omitting types theorem (2.2.5), there is a countable weak model $\mathfrak{A}^*$ for $\mathscr{L}(Q)$ which omits each Σ_n^δ. Thus, by Lemma 3.3.4, $\mathfrak{A}^*$ strongly omits each Σ_n.

We now partition ω_1 into disjoint uncountable sets X_ϕ, where ϕ ranges over formulas with parameters in ω_1. We proceed, as in the proof of the completeness theorem (3.2.3), to construct a chain $\langle \mathfrak{A}_\alpha^* : \alpha < \omega_1 \rangle$, with the additional requirement that each $\mathfrak{A}_\alpha^*$ strongly omits each Σ_n. Set $\mathfrak{A}_0^* = \mathfrak{A}^*$, where we may assume that $A_0 \subseteq \omega_1$; and, in fact, each $A_\alpha \subseteq \omega_1$. For limit λ, set $\mathfrak{A}_\lambda^* = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha^*$; then it is clear from the elementary chain theorem for weak models (2.2.2) that $\mathfrak{A}_\lambda^*$ is still a weak model for $\mathscr{L}(Q)$ which strongly omits each Σ_n. For successor stages $a + 1$, we choose ϕ so that $\alpha \in X_\phi$. We may thus apply the main lemma for omitting types, Lemma 3.3.5, to obtain $\mathfrak{A}_{\alpha+1}^*$ as a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, which still strongly omits each Σ_n.

Set $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. Using the Union of Chain Lemma 3.2.2, we see that $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$ for all ϕ and for all $\mathbf{a}$ in A_α (all $\alpha < \omega_1$). Since $\mathfrak{A}_0^* \models T$, then we must also have that $\mathfrak{A} \models T$. In order to see that $\mathfrak{A}$ omits Σ_n, we suppose that $\mathbf{a}$ is a sequence from A with $|\mathbf{a}| = |\mathbf{x}_n|$. We may, of course, choose $\alpha < \omega_1$ so that $\mathbf{a} \in A_\alpha^{<\omega}$. Then, since $\mathfrak{A}_\alpha^* \models \exists \mathbf{x}(\mathbf{x} = \mathbf{a})$ (that is, $\exists \mathbf{x} \bigwedge_i x_i = a_i$), and $\mathfrak{A}_\alpha^*$ strongly omits Σ_n, we may then choose $\sigma \in \Sigma_n$ such that $\mathfrak{A}_\alpha^* \models \exists \mathbf{x}(\mathbf{x} = \mathbf{a} \wedge \neg\sigma(\mathbf{x}))$. That is to say, $\mathfrak{A}_\alpha^* \models \neg\sigma(\mathbf{a})$. Then, $\mathfrak{A} \models \neg\sigma(\mathbf{a})$ and the argument is done. $\square$

3.3.7 Remarks. At this point we should make a few remarks on some of the developments we have examined.

(i) The converse of Theorem 3.3.6 also holds for complete theories T, as the reader may verify. Hint: Use the fact that $\exists$ and Q commute with countable disjunctions.

(ii) Bruce [1978b] has improved the omitting types theorem for $\mathscr{L}(Q)$ by showing that the notion of strong omitting may be replaced by an equivalent notion, a notion in which the quexistential string $\overline{Qy}$ may be required to consist only of quantifiers Qy_i (not $\exists y_i$). His proof is a direct one which uses forcing for $\mathscr{L}(Q)$. An alternate syntactic argument can be found in Kaufmann [1979], where there is also an extension of Theorem 3.3.6 which produces models of $\bigwedge_n \neg \overline{Qx_n} \bigwedge \Sigma_n(\mathbf{x}_n)$ in which $\overline{Qx_n}$ may have Q quantifiers in addition to $\exists$ quantifiers. Finally, we remark that these results extend, in fact, to families of $< 2^\omega$ sets of formulas, by a corresponding result for first-order logic by Shelah [1978a; Conclusion 5.17B, p. 208]. In this connection the reader should also see Lemma VIII.8.2.2.

3.4. Other Topics

3.4.1 The Infinitary Case. Before we undertake the exposition of the topics to which this section is devoted, we will observe that the reader should also consult Chapter VIII for a discussion of $\mathscr{L}_{\omega_1\omega}$ without Q. That said, we will begin our formal discussion by noting that the logic $\mathscr{L}_{\omega_1\omega}(Q)$ is formed from $\mathscr{L}(Q)$ by allowing the new rule of forming countably infinite conjunctions as long as the resulting formula has only finitely many free variables. In our development we will take $\bigvee$ as a defined symbol. The axioms and rules of inference include those of $\mathscr{L}(Q)$, together with the universal closures of all formulas of the form

$$(\bigwedge) \qquad \bigwedge \Phi \to \phi \quad \text{for all} \quad \phi \in \Phi.$$

The added infinitary rule of inference is

$$\frac{\Gamma \vdash \overline{Q^*y}(\phi \to \theta) \quad \text{all} \quad \theta \in \theta}{\Gamma \vdash \overline{Q^*y}(\phi \to \bigwedge \theta)}$$

for any quexistential string $\overline{Qy}$, where $\overline{Q^*y}$ is formed by replacing Q by Q^* and $\exists$ by $\forall$, in $\overline{Qy}$.

A *fragment* is a set of formulas of $\mathscr{L}_{\omega_1\omega}(Q)$ which is closed under the finitary formula-building operations. In [Ke], these axioms and rules are proved complete for $\mathscr{L}_{\omega_1\omega}(Q)$ and its countable admissible fragments. Observe that for the latter, we show by induction on proofs that if $T \vdash \phi$, then there is a proof *in the fragment* of ϕ from T. Keisler's argument has been abstracted in Barwise [1981] and, roughly speaking, it asserts that for many logics, the omitting types theorem implies a completeness and omitting types theorem for a corresponding infinitary logic. For the details on this, the reader should see Section VIII.6.6. Furthermore, the reader who wishes to examine Keisler's argument in *this* chapter may find it for $\mathscr{L}(\mathrm{aa})$ in the proof of Theorem 4.3.4.

The following theorem is interesting even for first-order logic, and a well-written proof of it can be found in Section 5 of [Ke], as well as (in its essentials) in Keisler [1971a, Theorem 45]. As an exercise the reader should prove the analog of this result for $\mathscr{L}_{\omega_1\omega}(\mathrm{aa})$ as defined in Section 4 of this chapter.

3.4.2 Theorem. *Let T be a consistent set of sentences of the countable fragment $\mathscr{L}_{\mathscr{A}}(Q)$. Suppose that T has an uncountable model which realizes uncountably many complete $\mathscr{L}_{\mathscr{A}}(Q)$-types in k variables, some $k < \omega$. Then there is a family $\{\mathfrak{A}_f : f \in {}^{(\omega_1)}2\}$ of non-isomorphic models of T. In fact, if $f \neq g$, then $\mathfrak{A}_f$ realizes an $\mathscr{L}_{\mathscr{A}}(Q)$-type which is omitted in $\mathfrak{A}_g$. In particular, a consistent countable theory of $\mathscr{L}_{\omega\omega}$ with uncountably many complete types has 2^{ω_1} models of power ω_1.* □

The next theorem is quite striking and its proof is beyond the scope of this chapter. For extensions of this result see Section XX.3.

3.4.3 Theorem (Shelah [1975c, Theorem 5.7]). *Assume $\diamondsuit_{\omega_1}$, or even (as in later work) $2^\omega < 2^{\omega_1}$. If T is a countable consistent theory of $\mathscr{L}_{\omega_1\omega}(Q)$ containing $Qx(x = x)$ with fewer than 2^{ω_1} models of power ω_1, then T has a model of power ω_2.* □

This result stands in contrast to the situation for $\mathscr{L}$(aa). For more on this the reader should see Remark 4.1.2(v).

The study of admissible fragments $\mathscr{L}_{\mathscr{A}}(Q)$ has been advanced by the work of Harnik–Makkai [1979], and these advances were based on the earlier work of Gregory [1973] and Ressayre [1977]. As concerns Gregory [1973], the reader should consult Section VIII.7.3 of the present volume. The idea is to provide an axiomatization of $\mathscr{L}_{\mathscr{A}}(Q)$ based on the notion "if ϕ holds then ψ is countable." Proofs in this direction involve $\Sigma_{\mathscr{A}}$-saturated models.

Another direction that the study of $\mathscr{L}_{\mathscr{A}}(Q)$ has taken is that of the Robinson-style forcing of Krivine–McAloon [1973] and Bruce [1978b]. Extra predicates are used in the former development, while the latter requires no extra predicates at all. In Bruce–Keisler [1979] one can find applications to the study of "decidable" weak models for $\mathscr{L}_{\mathscr{A}}(Q)$, where the model has domain α (with $\mathscr{A} = L_\alpha$) and Q means "for unboundedly many." This idea of using L_α has been extended in Wimmers [1982] to $\mathscr{L}$(aa) and $\mathscr{L}^{<\omega}$ (see Sections 4 and 5).

In the next two sections some countably compact extensions of $\mathscr{L}(Q)$ are considered.

4. *Filter Quantifiers Stronger Than Q_1: Completeness, Compactness, and Omitting Types*

In this section we will examine extensions of $\mathscr{L}(Q_1)$ that are formed by adding "filter quantifiers" over $P_{\omega_1}(A)$ = the set of countable subsets of A. We will mainly concentrate on $\mathscr{L}$(aa), or "stationary logic". Just as Q_1 refers to the family of uncountable sets, the aa quantifier ("almost all") refers to the family of *closed unbounded* subsets of ω_1, a basic family of study in set theory. For a discussion of closed unbounded sets and their largeness properties, the reader should see Kunen [1980]. This logic was introduced in a slightly different form in Shelah [1975d], where countable compactness and abstract completeness (recursive enumerability for theories) are proved. These properties are also implicit in Schmerl [1976] and, later, in Dubiel [1977a]. The proofs of these properties are related to the argument for $\mathscr{L}$(aa) in Section II.3.2. In a manner analogous to that of Keisler's 1970 paper (see Section 3) as compared to that of Fuhrken [1964] and Vaught [1964], Barwise–Makkai [1976] introduced an explicit set of axioms for $\mathscr{L}$(aa). Their completeness proof and an omitting types theorem can be found in Barwise–Kaufmann–Makkai [1978][2] and Kaufmann [1978a]. These notions form the main part of the present section. We will conclude our exposition with a discussion of some extensions of $\mathscr{L}$(aa).

[2] Henceforth referred to as [BKM].

4.1. Preliminaries

4.1.1 Definition (Stationary Logic ($\mathscr{L}$(aa)) and the Closed Unbounded (cub) Filter). Let τ be any vocabulary. A *τ-formula* of $\mathscr{L}$(aa) is a formula which is built up from atomic τ-formulas and formulas $s_i(x_j)$, by using first-order formation rules and the following rule: If ϕ is a formula so is aa s_i ϕ. The defined quantifier stat is also useful, and, formally stat s ϕ is $\neg$aa s $\neg\phi$.

To define satisfaction, we interpret aa by the cub filter $D(A)$ on $P_{\omega_1}(A)$, an interpretation that is due to Kueker [1972] and Jech [1973]. A collection X of countable subsets of A is *cub* if X is *closed* under unions of countable chains and *unbounded* in $P_{\omega_1}(A)$; that is to say, $(\forall s \in P_{\omega_1}(A))(\exists s' \in X)(s \subseteq s')$. Then $D(A)$ is the filter generated by the cub subfamilies of $P_{\omega_1}(A)$. Satisfaction may now be defined by induction on formulas, with the new clause:

$$\mathfrak{A} \models \text{aa } s\, \phi(s) \quad \text{iff} \quad \{s \in P_{\omega_1}(A) \colon \mathfrak{A} \models \phi(s)\} \in D(A).$$

A sublogic of $\mathscr{L}$(aa) is $\mathscr{L}_{\text{pos}}$ or "positive logic", where one forms aa s ϕ only if s occurs only positively in ϕ and $\phi \in \mathscr{L}_{\text{pos}}$. For more on this the reader should see Example 3 of Section II.2.2 and Remark 4.1.2(iii) below.

4.1.2 Remarks. We will now gather some facts which serve to clarify the definition just given.

(i) Suppose $|A| = \omega_1$ and $\langle A_\alpha \colon \alpha < \omega_1 \rangle$ is a *filtration* of A, that is, we have $A_\alpha = \bigcup \{A_{\beta+1} \colon \beta < \alpha\}$ and $A = \bigcup_\alpha A_\alpha$. Then, for all $X \subseteq P_{\omega_1}(A)$, $X \in D(A)$ iff $\{\alpha < \omega_1 \colon A_\alpha \in X\}$ contains a closed unbounded subset of ω_1. It follows then that if A has domain A, then $\mathfrak{A} \models$ aa s ϕ iff $\{\alpha \colon \mathfrak{A} \models \phi(A_\alpha)\}$ contains a cub subset of ω_1, and $\mathfrak{A} \models$ stat s ϕ iff $\{\alpha \colon \mathfrak{A} \models \phi(A_\alpha)\}$ is stationary in ω_1.

(ii) Exercise: For all A, the cub filter on $P_{\omega_1}(A)$ is closed under countable intersections. In fact, even more than this is true, as the reader can confirm by examining the proof of Proposition 4.1.4.

(iii) If s occurs only positively in $\phi(s, \ldots)$ and $\mathfrak{A} \models \phi(t, \mathbf{p})$ for some $t \in P_{\omega_1}(A)$, then $\mathfrak{A} \models \phi(t', \mathbf{p})$ for all $t' \supseteq t$; and, hence, $\mathfrak{A} \models$ aa s ϕ. Hence, $\mathscr{L}_{\text{pos}}$ can be defined using $\exists s$ in place of aa s.

(iv) $\mathscr{L}_{\text{pos}}$ contains $\mathscr{L}(Q_1)$, since $Qx\phi \leftrightarrow \neg\text{aa } s\, \forall x(\phi(x) \to s(x))$.

(v) The class of ω_1-like linear orders which continuously embed ω_1, whose members are sometimes called *strongly ω_1-like*, is axiomatized in $\mathscr{L}$(aa) by: "$<$ is a linear order" $\wedge$ $Qx(x = x)$ $\wedge$ aa $s\, \exists x$("$s = \{y \colon y < x\}$"). This is easy to see using (i) above. Hence, Shelah's non-categoricity theorem for $\mathscr{L}(Q_1)$ (Theorem 3.4.3) fails for $\mathscr{L}$(aa). In fact, we just add "$<$ is dense with least element" to get a categorical sentence. The class is not $\mathscr{L}_{\text{pos}}$-axiomatizable: a back-and-forth argument such as is used in Example 6.1.2 shows that all ω_1-like dense linear orders with first element are $\mathscr{L}_{\text{pos}}$-$\equiv$. This example naturally suggests that one could restrict to strongly ω_1-like linear orders and then obtain a first-order version of $\mathscr{L}$(aa). The reader should also see Section II.3.2 for more on this.

Other properties of linear orders can be expressed in $\mathscr{L}$(aa). The following offer two interesting examples in $\mathscr{L}_{\text{pos}}$. $\mathfrak{A}$ is separable iff $\mathfrak{A} \models$ aa s (s is dense), that is to say $\mathfrak{A} \models$ aa $s\, \forall x\, \forall y\, (x < y \to \exists z(s(z) \wedge x < z \wedge z < y))$, which belongs to $\mathscr{L}_{\text{pos}}$. $\mathfrak{A}$ has cofinality ω iff $\mathfrak{A} \models \text{aa}\ s$ (s is cofinal); that is, $\mathfrak{A} \models \text{aa}\ s\, \forall x\, \exists y \in s\, (x < y)$. In fact, Shelah [1975d] has proved full compactness for such a cofinality quantifier; see Section XVIII.1.3 and Theorem II.3.2.3. None of these classes is axiomatizable in $\mathscr{L}(Q_1)$: see Theorem 6.3.3, Proposition II.7.2.5, and Theorem II.7.2.6.

(vi) Keisler's original counterexample to interpolation in $\mathscr{L}(Q_1)$ shows that the following class $\mathscr{K}$ of models is not $\mathscr{L}(Q_1)$-axiomatizable (see also Section VI.3.1 and II.4.2.8): $\mathscr{K} = \{\mathfrak{A}: \mathfrak{A} = (A, E)$, where E is an equivalence relation on A with countably many equivalence classes$\}$. However, $\mathscr{K}$ is axiomatizable in $\mathscr{L}_{\text{pos}}$ by the sentence "E is an equivalence relation" $\wedge\ \text{aa}\ s\, \forall x\, \exists y(s(y) \wedge E(x, y))$.

(vii) It is shown in [BKM] that $\mathscr{L}(\text{aa}) \not\leq \mathscr{L}_{\infty\infty}$. In fact, there does not exist κ such that $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B} \Rightarrow \mathfrak{A} \equiv_{\mathscr{L}(\text{aa})} \mathfrak{B}$ (Kaufmann [1984]). This should come as no surprise, given Kueker's game-theoretic description of the aa quantifier: If $X \subseteq P_{\omega_1}(A)$, then $X \in D(A)$ iff $\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \ldots \forall x_n\, \exists y_n \ldots (\{x_i: i < \omega\} \cup \{y_i: i < \omega\} \in X)$. (See also Chapter X of the present volume for a discussion of game quantification.)

Another sense in which $\mathscr{L}(\text{aa})$ is strictly stronger than $\mathscr{L}(Q)$ is the sense of Shelah's theorem which asserts that its Hanf number exceeds $\beth_\omega$, the Hanf number for $\mathscr{L}(Q)$; see Theorem V.3.3.11. More on this can be found in Shelah–Kaufmann [198?]. The idea is that, in a sense, $\mathscr{L}(\text{aa})$ can express well-ordering for sufficiently large structures. Notice that the constructions to follow produce models of power at most ω_1.

Alternatively, one can define $D(A)$ to be the set of subsets of κ of power less than κ. This idea was successfully applied to abelian group theory in Eklof–Mekler [1981].

4.1.3 Axioms of $\mathscr{L}$(aa). For any formula ϕ, call ψ a *quasi-universal closure* of ϕ, if ψ has no free first- or second-order variables, and ψ results by prefixing ϕ with quantifiers of the form aa s and $\forall x$.

The axioms of $\mathscr{L}(\text{aa})$ consist of the quasi-universal closures of the following.

(FO) All axioms and axiom schemas of first-order logic.

(0) $\text{aa}\ s_i \phi(s_i, \ldots) \leftrightarrow \text{aa}\ s_j \phi(s_j, \ldots)$ (s_j not occurring in $\phi(s_i)$).

(1) $\neg\ \text{aa}\ s\, (x \neq x)$.

(2) $\text{aa}\ s\, (x \in s)$.

$\text{aa}\ s_j\, (s_i \subseteq s_j)$ for $i \neq j$.

(3) $\text{aa}\ s\, \phi \wedge \text{aa}\ s\, \psi \rightarrow \text{aa}\ s\, (\phi \wedge \psi)$.

(4) $\text{aa}\ s\, (\phi \rightarrow \psi) \rightarrow (\text{aa}\ s\, \phi \rightarrow \text{aa}\ s\, \psi)$.

(5) $\forall x\ \text{aa}\ s\, \phi(x, s, \ldots) \rightarrow \text{aa}\ s\, \forall x\, (s(x) \rightarrow \phi(x, s, \ldots))$

(6) $\phi \rightarrow \text{aa}\ s\, \phi$, if s is not free in ϕ.

The only rule of inference is *modus ponens*. The reader will observe the similarity here to Keisler's axioms for $\mathscr{L}(Q_1)$ in Definition 3.1.1. In [BKM] there is a rule of aa-generalization, a rule which we do not need because we have taken quasi-universal closures in forming the axioms.

4.1.4 Proposition (Soundness). *If* $T \vdash \phi$ *in* $\mathscr{L}(\mathrm{aa})$, *then* $\mathfrak{A} \models \phi$ *for all* $\mathfrak{A}$.

Proof. It suffices to verify the validity of axioms (1)–(5), since all the others are obviously valid. Axiom 1 says $\emptyset \notin D(A)$; and Axiom 2 is equally clear since $\{t \in P_{\omega_1}(A): s \subseteq t\} \in D(A)$ for all $s \in P_{\omega_1}(A)$. Axioms 3 and 4 are valid because $D(A)$ is a filter. Finally, Axiom 5 is valid because $D(A)$ is closed under *diagonal intersections*, that is, we have that if $\{X_a: a \in A\} \subseteq D(A)$, then $\Delta\{X_a: a \in A\} = \{s \in P_{\omega_1}(A): (\forall a \in s)\, s \in X_a\} \in D(A)$. In fact, the diagonal intersection of cub families is cub, as the reader may verify. ☐

In order to apply the results of Section 2 on weak models to our development, we may now state the following proposition by way of analogy to Proposition 3.1.2 for $\mathscr{L}(Q_1)$. The proof of this result is routine and will therefore be omitted.

4.1.5 Definition. (i) *The logic* $\mathscr{L}(\mathrm{aa})$ *with the above notion of proof is a logic with concrete syntax in the sense of Section* 2.1, *when we are restricted to formulas in which no second-order variable* s_i *occurs free.*

(ii) *The notion of* "$\mathscr{L}(\mathrm{aa})$*-locally omits*" *in Definition* 2.2.4 *is equivalent to the usual notion. That is, whenever* $\exists x\phi$ *is* $\mathscr{L}(\mathrm{aa})$*-consistent with* T, *so is* $\exists x(\phi \wedge \neg\sigma)$ *for some* $\sigma \in \Sigma$. ☐

In light of the above, we may speak of weak models $\mathfrak{A}^*$ for $\mathscr{L}(\mathrm{aa})(\tau)$ when τ is a countable vocabulary. That is to say, we have $\mathfrak{A}^* = \langle \mathfrak{A}, R^{\mathfrak{A}^*}_{aa\,s\phi}\rangle_{\phi \in \mathscr{L}(\mathrm{aa})(\tau)}$. Recall now that τ^+ refers to the vocabulary of $\mathfrak{A}^*$. The reader may have guessed our strategy by now. We will require a main lemma which will show how to witness formulas stat $s\ \phi$ (recall that this means $\neg \mathrm{aa}\ s \neg \phi$), much as we witnessed formulas $Qx\phi$ in the $\mathscr{L}(Q_1)$ case. Since s is a second-order variable, we propose to witness stat $s\ \phi(s)$ by having $\phi(A)$ hold. This approach differs slightly from the one in [BKM], where 2-sorted structures are used with interpretations for first-order *and* second-order variables. Instead, we add a predicate symbol for A.

4.2. *Proving the Completeness Theorem for* $\mathscr{L}(\mathrm{aa})$

We begin this section with

4.2.1 Definition. Suppose that τ is any vocabulary and that $\mathfrak{A}^*$ is a countable weak model for $\mathscr{L}(\mathrm{aa})(\tau)$. Let P_A be a unary relation symbol not in τ. We say that

$\mathfrak{B}^*$ is a *precise extension of* $\mathfrak{A}^*$ *relative to* ϕ if $\phi(s)$ is a formula of $\mathscr{L}(\mathrm{aa})(\tau)$ with parameters in A, $\mathfrak{B}^*$ *is a* $(\tau \cup P_A)^+$-structure, and

(i) $\mathfrak{A}^* \prec^w \mathfrak{B}^* \upharpoonright \tau^+$;
(ii) if $\mathfrak{A}^* \models \text{stat } s\ \phi$, then $\mathfrak{B}^* \models \phi(P_A)$; that is to say, $\mathfrak{B}^* \models (\phi(P_A))^*$ (see Section 2);
(iii) whenever $\mathfrak{A}^* \models \text{aa } s\ \psi(s)$ for $\text{aa } s\ \psi$ a sentence with parameters in $\mathfrak{A}^*$, then $\mathfrak{B}^* \models \psi(P_A)$;
(iv) $(P_A)^{\mathfrak{B}^*} = A$.

4.2.2 Main Lemma ([BKM, 3.4]). *Suppose that A^* is a countable weak model for $\mathscr{L}(\mathrm{aa})(\tau)$ and that $\phi(s, \mathbf{p})$ is a formula of $\mathscr{L}(\mathrm{aa})(\tau)$ with parameters $\mathbf{p}$ in A. Then there is a precise extension of $\mathfrak{A}^*$ relative to ϕ.*

Proof. We may assume that $\mathfrak{A}^* \models \text{stat } s\ \phi(s, \mathbf{p})$, or else we may replace ϕ by $\forall x(x = x)$. Let $C_A = \{c_a: a \in A\}$ be a set of new constant symbols, and set

$$T_\phi(\mathfrak{A}^*) = \{\theta(\mathbf{c_a}): \mathfrak{A}^* \models \theta(\mathbf{a})\} \cup \{\phi(P_A, \mathbf{c_p})\} \cup \{\psi(P_A, \mathbf{c_a}): \mathfrak{A}^* \models \text{aa } s\ \psi(s, \mathbf{c_a})\},$$

where $\mathbf{c_a} = \langle c_{a_1} \dots c_{a_n} \rangle$ if $\mathbf{a} = \langle a_1 \dots a_n \rangle$. Also set

$$\Sigma_A(x) = \{P_A(x)\} \cup \{x \neq c_a: a \in A\}.$$

Claim A. $T_\phi(\mathfrak{A}^*)$ is an $\mathscr{L}(\mathrm{aa})(\tau \cup P_A)$-consistent theory which $\mathscr{L}(\mathrm{aa})(\tau \cup \{P_A\})$-locally omits $\Sigma_A(x)$.

As in the proof of Lemma 3.2.1, the Main Lemma for $\mathscr{L}(Q_1)$, let us see how the result follows from Claim A. Now the Extension Lemma (2.2.6) (or formally, Lemma 7.2.3) gives us a countable weak model $\mathfrak{B}^*$ for $\mathscr{L}(\mathrm{aa})(\tau \cup \{P_A\})$ such that $\mathfrak{A}^* \prec^w \mathfrak{B}^* \upharpoonright \tau^+$, $\mathfrak{B}^* \models \theta(\mathbf{a})$ whenever $\mathfrak{A}^* \models \theta(\mathbf{a})$, $\mathfrak{B}^* \models \phi(P_A, \mathbf{p})$, and $\mathfrak{B}^* \models \psi(P_A, \mathbf{a})$ whenever $\mathfrak{A}^* \models \text{aa } s\ \psi(s, \mathbf{a})$. So (i) through (iii) hold in the definition of precise extension relative to ϕ (Definition 4.2.1). Now Lemma 2.2.6 also allows us to choose $\mathfrak{B}^*$ so that it omits Σ_A, and this guarantees $(P_A)^{\mathfrak{B}^*} \subseteq A$. Since $\mathfrak{A}^* \models \text{aa } s(a \in s)$ for all $a \in A$ (Axiom 2), we have that $P(c_a) \in T_\phi(\mathfrak{A}^*)$. So $\mathfrak{B}^* \models P_A(a)$; and hence $A \subseteq (P_A)^{\mathfrak{B}^*}$. Thus, (iv) holds, and $\mathfrak{B}^*$ is the desired precise extension of $\mathfrak{A}^*$ relative to ϕ.

In order to prove Claim A we will use the analogue of Claim B in the proof of Lemma 3.2.1. The proof is essentially the same once we observe that, for every τ-formula $\theta(s, \dots)$ and every set Γ of $\mathscr{L}(\mathrm{aa})(\tau)$-sentences, if $\Gamma \vdash \theta(P_A, \dots)$ in $\mathscr{L}(\mathrm{aa})(\tau \cup \{P_A\})$, then $\Gamma \vdash \text{aa } s\ \theta(s, \dots)$ in $\mathscr{L}(\mathrm{aa})(\tau)$. This follows from an induction, using Axiom 4.

Claim B (Consistency Criterion). For any formula $\theta(s, \mathbf{z})$ of $\mathscr{L}(\mathrm{aa})(\tau)$ and $\mathbf{a}$ in A,

(i) $\theta(P_A, \mathbf{c_a})$ is $(\tau \cup \{P_A\})$-consistent with $T_\phi(\mathfrak{A}^*)$ iff $\mathfrak{A}^* \models \text{stat } s(\phi(s, \mathbf{c_p}) \wedge \theta(s, \mathbf{c_a}))$.
(ii) $T_\phi(\mathfrak{A}^*) \vdash \theta(P_A, \mathbf{c_a})$ in $\mathscr{L}(\mathrm{aa})(\tau \cup \{P_A\})$ iff $\mathfrak{A}^* \models \text{aa } s(\phi(s, \mathbf{p}) \to \theta(s, \mathbf{a}))$.

It remains to prove Claim A. The consistency criterion implies that $T_\phi(\mathfrak{A}^*)$ is consistent. Now suppose $\exists x\theta(x, P_A, \mathbf{c_a})$ is consistent with $T_\phi(\mathfrak{A}^*)$. If $\exists x(\theta(x, P_A, \mathbf{c_a}) \wedge \neg P_A(x))$ is consistent with $T_\phi(\mathfrak{A}^*)$, then our work is done. Otherwise, we have that $\exists x(\theta(x, P_A, \mathbf{c_a}) \wedge P_A(x))$ is consistent with $T_\phi(\mathfrak{A}^*)$. Thus,

$$\mathfrak{A}^* \models \text{stat } s\, \exists x[\phi(s, \mathbf{p}) \wedge \theta(x, s, \mathbf{a}) \wedge s(x)]$$

by Claim B. Rewriting this as

$$\mathfrak{A}^* \models \neg \text{aa } s\, \forall x \in s \neg [\phi(s, \mathbf{p}) \wedge \theta(x, s, \mathbf{a})],$$

we see that Axiom 5 implies that $\mathfrak{A}^* \models \neg \forall x \text{ aa } s \neg[\phi(s, \mathbf{p}) \wedge \theta(x, s, \mathbf{a})]$. That is to say, we have that

$$\mathfrak{A}^* \models \text{stat } s[\phi(s, \mathbf{p}) \wedge \theta(e, s, \mathbf{a})]$$

for some $e \in A$. Then, by using the consistency criterion again, we have that $\theta(c_e, P_A, \mathbf{c_a})$ is consistent with $T_\phi(\mathfrak{A}^*)$. And, hence, $\exists x(\theta(x, P_A, \mathbf{c_a}) \wedge \neg x \neq c_e)$ is also. ☐

4.2.3 Lemma (Union of Chain Lemma). *Assume that $\langle \mathfrak{A}_\alpha^*: \alpha < \omega_1 \rangle$ is a chain of countable weak models for $\mathscr{L}(\text{aa})$ with the following properties.*

(i) *$\mathfrak{A}_\alpha^*$ is a countable weak model for $\mathscr{L}(\text{aa})(\boldsymbol{\tau}_\alpha)$ for all $\alpha < \omega_1$, where $\boldsymbol{\tau}_\alpha = \boldsymbol{\tau} \cup \{P_{A_\beta}: \beta < \alpha\}$.*
(ii) *For all $\alpha < \omega_1$, $\mathfrak{A}_{\alpha+1}^*$ is a precise extension of $\mathfrak{A}_\alpha^*$ relative to ϕ, for some ϕ.*
(iii) *For all $\alpha < \omega_1$ and for every formula $\phi(s, \mathbf{x})$ of $\mathscr{L}(\text{aa})(\boldsymbol{\tau}_\alpha)$ and for all parameters $\mathbf{a}$ from A_α, the set $\{\beta < \omega_1: \mathfrak{A}_{\beta+1}^*$ is a precise extension of $\mathfrak{A}_\beta^*$ relative to $\phi(s, \mathbf{a})\}$ is stationary in ω_1.*
(iv) *The chain is continuous: For all limit $\lambda < \omega_1$ and $\alpha < \lambda$, $\mathfrak{A}_\lambda^* \upharpoonright \boldsymbol{\tau}_\alpha^+ = \bigcup \{\mathfrak{A}_\beta^* \upharpoonright \boldsymbol{\tau}_\alpha^+ : \alpha \le \beta < \lambda\}$.*

Set $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. That is, for all $\delta < \omega_1$, $\mathfrak{A} \upharpoonright \boldsymbol{\tau}_\delta = \bigcup \{\mathfrak{A}_\alpha \upharpoonright \boldsymbol{\tau}_\delta: \delta \le \alpha < \omega_1\}$. Then, for all $\alpha < \omega_1$, $\mathscr{L}(\text{aa})(\boldsymbol{\tau}_\alpha)$-formulas $\phi(\mathbf{x})$, and $\mathbf{a}$ in A_α, $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$.

Proof. The argument is by induction on the length of ϕ. Notice that $\alpha < \beta < \omega_1$ implies that $\mathfrak{A}_\alpha^* \prec^w \mathfrak{A}_\beta^* \upharpoonright \boldsymbol{\tau}_\alpha^+$, by Proposition 2.2.2. We first show that $\alpha < \beta < \omega_1$ implies that $(P_{A_\alpha})^{\mathfrak{A}_\beta^*} = A_\alpha$, by induction on β. For $\beta = \alpha + 1$, this is part of the definition of precise extension. Now, $\mathfrak{A}_\alpha^* \models \text{aa } s \text{ aa } t(s \subseteq t)$ by Axiom 2. So $\mathfrak{A}_{\alpha+1}^* \models \text{aa } t(P_{A_\alpha} \subseteq t)$; and, hence, $\mathfrak{A}_\gamma^* \models \text{aa } t(P_{A_\alpha} \subseteq t)$ for all $\gamma \ge \alpha + 1$. Then $\mathfrak{A}_{\gamma+1}^* \models P_{A_\alpha} \subseteq P_{A_\gamma}$ so $(P_{A_\alpha})^{\mathfrak{A}_{\gamma+1}^*} = (P_{A_\alpha})^{\mathfrak{A}_\gamma^*}$, which is A_α by the inductive hypothesis. Limit stages of the induction are clear and we have verified that $(P_{A_\alpha})^{\mathfrak{A}_\beta^*} = A_\alpha$ for all $\alpha < \beta < \omega_1$.

Clearly, $\mathfrak{A} \models \phi(\mathbf{a})$ iff $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$ for atomic ϕ. The $\vee$ and $\neg$ steps are trivial, while the $\exists$ step presents no problems. For $\phi(\mathbf{a}) = \text{aa } s\, \psi(s, \mathbf{a})$, suppose that $\mathfrak{A}_\alpha^* \models \phi(\mathbf{a})$. Then $\mathfrak{A}_\beta^* \models \phi(\mathbf{a})$ for all $\beta > \alpha$, so that $\mathfrak{A}_{\beta+1}^* \models \psi(P_{A_\beta}, \mathbf{a})$ for all $\beta > \alpha$. By the inductive hypothesis, we have that $\mathfrak{A} \models \psi(P_{A_\beta}, \mathbf{a})$ for all $\beta > \alpha$.

Hence, $\mathfrak{A} \models$ aa $s\, \psi(s, \mathbf{a})$. As to other direction, we suppose that $\mathfrak{A}^*_\alpha \models \neg$aa $s\, \psi(x, \mathbf{a})$. That is, we suppose that $\mathfrak{A}^*_\alpha \models$ stat $s\, \neg\psi(s, \mathbf{a})$. Then, $\mathfrak{A}^*_\beta \models$ stat $s\, \neg\psi(s, \mathbf{a})$ for all $\beta \geq \alpha$. But hypothesis (iii) implies that $\{\beta \geq \alpha\colon \mathfrak{A}^*_{\beta+1} \models \neg\psi(A_\beta, \mathbf{a})\}$ is stationary in ω_1. Since this set equals $\{\beta \geq \alpha\colon \mathfrak{A} \models \neg\psi(A_\beta, \mathbf{a})\}$ by the inductive hypothesis, we must have that $\mathfrak{A} \models$ stat $s\, \neg\psi(s, \mathbf{a})$ by Remark 4.1.2(ii). That is, $\mathfrak{A} \models \neg$aa $s\, \psi(s, \mathbf{a})$. □

4.2.4 Theorem (Completeness Theorem for $\mathscr{L}$(aa) [BKM]). *Suppose T is a set of sentences of $\mathscr{L}$(aa)(τ), where τ is a countable vocabulary. Then T is $\mathscr{L}$(aa)(τ)-consistent iff T has a model.*

Proof. The direction $\Leftarrow$ is Proposition 4.1.4 (Soundness). Now suppose that T is $\mathscr{L}$(aa)(τ)-consistent. For each $\alpha \leq \omega_1$, set $\tau_\alpha = \tau \cup \{P_{A_\beta}\colon \beta < \alpha\}$. By a theorem of Ulam [1930] (see, for instance, Kunen [1980, p. 79]), there is a partition of ω_1 into disjoint stationary sets X_ϕ, where ϕ ranges over formulas $\phi(s)$ of $\mathscr{L}$(aa)(τ_{ω_1}) with parameters in ω_1. Define $\langle \mathfrak{A}^*_\alpha\colon \alpha < \omega_1 \rangle$, each $\mathfrak{A}^*_\alpha$ a countable weak model for $\mathscr{L}$(aa)(τ_α), by induction on α as follows, where $A_\alpha \subseteq \omega_1$ for all $\alpha < \omega_1$. Let $\mathfrak{A}^*_0$ be a countable weak model for $\mathscr{L}$(aa)(τ) which satisfies T, and for each α, if $\alpha \in X_\phi$ let $\mathfrak{A}^*_{\alpha+1}$ be a precise extension of $\mathfrak{A}^*_\alpha$ relative to ϕ, by the Main Lemma (4.2.2). We take unions at limits.

We now set $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$. Since $\mathfrak{A}^*_0 \models T$, then $\mathfrak{A} \models T$ by the union of chain Lemma 4.2.3. □

4.2.5 Corollary. (i) *$\mathscr{L}$(aa) is countably compact.*

(ii) *Every consistent countable theory of $\mathscr{L}$(aa) has a model of power at most $\aleph_1$.* □

In connection with this corollary, it is interesting to observe that whether or not $\mathscr{L}$(aa)-elementary submodels must exist is independent. This fact has been proved by Harrington, Kunen, and Shelah (see [BKM], Footnote 2, p. 221).

As is true for $\mathscr{L}(Q_1)$, the study of $\mathscr{L}$(aa) was partly motivated by the study of end extension of linear orders and models of set theory. By analogy with Theorem 3.2.5(ii), we could reverse history by proving a relativized version of the Main Lemma (4.2.2) to obtain the following theorem of Hutchinson [1976a].

4.2.6 Theorem. *Let $\mathfrak{A} = (A, E)$ be a countable model of ZF, possibly excepting power set and the collection schema. For every regular cardinal a of $\mathfrak{A}$, there exists $\mathfrak{B} = (B, F) \succ \mathfrak{A}$ such that $b_E = b_F$ for all bEa, but $\langle a_F - a_E, F \restriction a_F - a_E \rangle$ has a least element. Moreover we may require $\langle a_F, F \restriction a_F \rangle$ to be ω_1-like and embed ω_1 continuously.*

Hint of Proof. Define $\mathfrak{A}^* \models$ aa $s\, \varphi$ iff $\mathfrak{A} \models \exists C$ (C is cub in a and $\forall\gamma \in C(\varphi(\gamma))$). □

4.2.7 Remark. Probably the closest known analogue of Theorem 3.2.5(i) is obtained by adding a quantifier "aa α" to the language of set theory, as studied independently by Kaufmann [1983] and Kakuda [1980]. See, for example, Kaufmann [1983, 2.16 and 5.8]. A combination of Peano arithmetic and $\mathscr{L}(Q^2)$, a topic that is discussed in Section 5, has been studied in Macintyre [1980], in Morgenstern [1982], and in Schmerl–Simpson [1982]. The reader should also see Schmerl [1982] for an extension.

4.3. Omitting Types and Infinitary Completeness

As in Section 3, we now extend the completeness theorem to obtain omitting types and infinitary completeness theorems. In fact, the proofs are direct descendents of Keisler's proofs for $\mathscr{L}(Q)$.

4.3.1 Definition ([BKM]). Let T be a set of $\mathscr{L}(\mathrm{aa})(\tau)$-sentences and $\Sigma(\mathbf{x}, \mathbf{t})$ a set of $\mathscr{L}(\mathrm{aa})(\tau)$-formulas in finitely many free variables $x_1 \dots x_m$, $t_1 \dots t_n$. Let S be any quantifier string composed of quantifiers stat s_i and stat t_i, where $i < j$ implies that stat t_i occurs only before stat t_j. T strongly omits Σ if for every such S and every formula $S\,\exists\mathbf{x}\,\phi(\mathbf{x}, \mathbf{s}, \mathbf{t})$ which is $\mathscr{L}(\mathrm{aa})(\tau)$-consistent with T, then $S\,\exists\mathbf{x}(\phi \wedge \neg\sigma)$ is consistent with T for some $\sigma \in \Sigma$. (Notice that we've fixed an ordering $\langle t_1, \dots, t_n\rangle$ of the second-order free variables of Σ.) We say that $\mathfrak{A}$ *omits* Σ *if* $\mathfrak{A} \models \mathrm{aa}\, t_1 \dots \mathrm{aa}\, t_n\, \forall\mathbf{x} \bigvee_{\delta\in\Sigma} \neg\sigma(\mathbf{x}, \mathbf{t})$.

4.3.2 Theorem (Omitting Types Theorem for $\mathscr{L}(\mathrm{aa})$ [BKM, 4.2]). *Let τ be countable and suppose that T is a consistent τ-theory of $\mathscr{L}(\mathrm{aa})$ which strongly omits sets $\Sigma_n(\mathbf{x}_n, \mathbf{t}_n)$, where $n < \omega$. Then T has a model which omits each Σ_n. The converse also holds if T is complete.*

Hint of Proof. Let us merely remark that Lemmas 3.3.4 and 3.3.5 have straightforward translations into $\mathscr{L}(\mathrm{aa})$, and the case $t_n = \varnothing$ for all n follows just as Theorem 3.3.6 follows for $\mathscr{L}(Q_1)$. $\square$

4.3.3 Definition. The *logic* $\mathscr{L}_{\omega_1\omega}(\mathrm{aa})$ is formed from $\mathscr{L}(\mathrm{aa})$ by allowing the new rule of forming countably infinite conjunctions, as long as the resulting formula has only finitely many free variables. The new axioms are the quasi-universal closures of

$$(\textstyle\bigwedge) \qquad \bigwedge\Phi \to \phi \quad \text{for all} \quad \phi \in \Phi;$$

and the new rules of inference are

$$\frac{\Gamma \vdash S^*(\phi \to \theta) \quad \text{all} \quad \theta \in \theta}{\Gamma \vdash S^*(\phi \to \bigwedge\theta)}$$

for any quantifier string S consisting only of quantifiers of the form stat s or $\exists x$. Here, S^* results from S by changing each stat to aa and each $\exists$ to $\forall$.

"Countable fragment" is defined as for $\mathscr{L}(Q_1)$ in Definition 3.4.1), as is the notion of $\mathscr{L}_{\mathscr{A}}(\mathrm{aa})$-consistency, that is, consistency with respect to proofs consisting of $\mathscr{L}_{\mathscr{A}}$(aa)-formulas. If the fragment is admissible, then the following theorem can be extended to give Barwise completeness and compactness.

4.3.4 Theorem (Completeness and Omitting Types Theorems for $\mathscr{L}_{\omega_1\omega}$(aa), [BKM, 4.6]). *If T is a consistent theory of a countable fragment $\mathscr{L}_{\mathscr{A}}$(aa), then T has a model. If in addition T strongly omits sets Σ_n,* where $n < \omega$, *then T has a model which omits each Σ_n.*

Proof. We will reduce to finitary logic in a manner analogous to that of Definition 2.1.2, except that we do not need to eliminate the "aa" quantifier, since we already have a completeness theorem for $\mathscr{L}$(aa). Rather, we will replace each infinitary conjunction by an atomic formula. Formally, we define a map "prime" (′) from formulas of $\mathscr{L}_{\mathscr{A}}$(aa) to $\mathscr{L}$(aa)($\boldsymbol{\tau}$), where

$$\boldsymbol{\tau} = [\text{vocabulary of } \mathscr{L}_{\mathscr{A}}(\text{aa})] \cup \{R_{\bigwedge\Phi}(\mathbf{x}, \mathbf{t}) \colon \bigwedge \Phi(\mathbf{x}, \mathbf{t}) \in \mathscr{L}_{\mathscr{A}}(\text{aa})\}.$$

We set $\phi' = \phi$ for atomic ϕ; and we set $(\phi \wedge \psi)' = \phi' \wedge \psi'$, $(\neg\phi)' = \neg\phi'$, $(\exists x\phi)' = \exists x\phi'$; and $(\text{aa } s\ \phi)' = \text{aa } s\ \phi'$; and, finally,

$$(\bigwedge \Phi)' = R_{\bigwedge\Phi}(x_1 \cdots x_m, t_1 \dots t_n),$$

where $x_1 \dots x_m$ (resp. $t_1 \dots t_n$) enumerates the first-order (resp. second-order) free variables of Φ in order of subscript. "Prime" almost has an inverse, "minus": namely, $\phi^- = \phi$ for atomic $\phi \in \mathscr{L}_{\mathscr{A}}$(aa), and "minus" commutes with the finitary connectives and quantifiers; and

$$[R_{\bigwedge\Phi}(\tau_1 \dots \tau_m, s_1 \dots s_n)]^- = \bigwedge \Phi\begin{pmatrix} x_1 & \dots & x_m & t_1 & \dots & t_n \\ \tau_1 & & \tau_m & s_1 & & s_n \end{pmatrix},$$

where $\Phi, x_1 \dots x_m, t_1 \dots t_n$ are as above. It is clear that $(\phi')^- = \phi$ for all $\phi \in \mathscr{L}_{\mathscr{A}}$(aa).

Let $T' = \{\phi' : T \vdash \phi \text{ and } \phi \in \mathscr{L}_{\mathscr{A}}(\text{aa})\} \cup \{S^*[(\phi(\mathbf{x}, \mathbf{t})^-)' \leftrightarrow \phi(\mathbf{x}, \mathbf{t})] : \phi \in \mathscr{L}(\text{aa})(\boldsymbol{\tau})$, S^* consists of quantifiers aa t, $\forall x\}$. ☐

Claim 1. *For every $\phi \in \mathscr{L}_{\mathscr{A}}$(aa, $T \vdash \phi$ iff $T' \vdash \phi'$.* The forward implication is clear. For the converse direction, we verify that if p is a proof from axioms of T' in $\mathscr{L}$(aa)($\boldsymbol{\tau}$) and p^- results from p by replacing each formula ψ in p by ψ^-, then p^- is a proof in $\mathscr{L}_{\mathscr{A}}$(aa) from axioms of T. We omit the details of the argument. So if $T' \vdash \phi'$, then $T \vdash (\phi')^-$. That is, $T \vdash \phi$.

Claim 2. *If T strongly omits Σ, then T' strongly omits $\Sigma' = \{\sigma' : \sigma \in \Sigma\}$.* For suppose that $S\ \exists\mathbf{x}\ \phi$ is consistent with T', for appropriate S. Then $S\ \exists\mathbf{x}\ \phi^-$ is consistent with T, by Claim 1. So for some $\sigma \in \Sigma$, $S\ \exists\mathbf{x}\ (\phi^- \wedge \neg\sigma)$ is consistent with T. And, hence, by using Claim 1 again we have that $S\ \exists\mathbf{x}\ ((\phi^-)' \wedge \neg\sigma')$ is consistent with T'. But since $S^*\ \forall\mathbf{x}\ [(\phi^-)' \leftrightarrow \phi]$ is an axiom of T', we must have that $S\ \exists\mathbf{x}\ (\phi \wedge \neg\sigma')$ is consistent with T', as desired.

Claim 3. *T strongly omits the set $\Sigma_{\bigwedge\Phi} = \{\neg \bigwedge \Phi\} \cup \Phi$, for each $\bigwedge \Phi \in \mathscr{L}_{\mathscr{A}}(\mathrm{aa})$.* To see this, suppose that we are given a sentence $S\, \exists \mathbf{x}\, \psi(\mathbf{x}, \mathbf{s}, \mathbf{t})$ which witnesses that T does not strongly omit $\Sigma_{\bigwedge\Phi}$. Then, for all $\phi \in \Phi$, $T \vdash S^* \forall \mathbf{x}\, (\psi \to \phi)$. By the infinitary rule of inference, we deduce that $T \vdash S^* \forall \mathbf{x}\, (\psi \to \bigwedge \Phi)$. But also $\neg \bigwedge \Phi \in \Sigma_{\bigwedge\Phi}$. Hence, by choice of $S\, \exists \mathbf{x}\, \psi$, $T \vdash S^* \forall \mathbf{x}\, (\psi \to \neg \bigwedge \Phi)$. It thus follows that $S\, \exists \mathbf{x}\, \psi$ is not consistent with T, which contradicts our choice of this sentence.

Now by Claim 1, T' is consistent, and by Claims 2 and 3, T' strongly omits $(\Sigma_n)'$ and each $(\Sigma_{\bigwedge\Phi})'$ for $\bigwedge \Phi \in \mathscr{L}_{\mathscr{A}}(\mathrm{aa})$. Let $\mathfrak{A}$ be a model of T' which omits each $(\Sigma_n)'$ and each $(\Sigma_{\bigwedge\Phi})'$, by Theorem 4.3.2 (the omitting types theorem for $\mathscr{L}(\mathrm{aa})$). The theorem now follows from the following claim.

Claim 4. *For all $\phi(\mathbf{x}, \mathbf{t}) \in \mathscr{L}_{\mathscr{A}}(\mathrm{aa})$, $\mathfrak{A} \models \mathrm{aa}\, \mathbf{t}\, \forall \mathbf{x}\, [\phi(\mathbf{x}, \mathbf{t}) \leftrightarrow \phi'(\mathbf{x}, \mathbf{t})]$.* The proof is by induction on ϕ. Although the details are left as an exercise, some hints are put forward in the following discussion. For the $\bigwedge$ step, one should at some point observe: $\models \bigwedge_{i \in I} \mathrm{aa}\, \mathbf{t}\, \theta_i \leftrightarrow \mathrm{aa}\, \mathbf{t} \bigwedge_{i \in I} \theta_i$ if I is countable; and

$$\mathfrak{A} \models \mathrm{aa}\, \mathbf{t}\, \forall \mathbf{x} \left[\bigwedge_{\phi \in \Phi} \phi' \to R_{\bigwedge\Phi}(\ldots) \right],$$

since $\mathfrak{A}$ omits $(\Sigma_{\bigwedge\Phi})'$. For the "aa" step, $\mathrm{aa}\, s\, \phi(\mathbf{x}, \mathbf{t}, s)$, one uses $\models \mathrm{aa}\, \mathbf{t}\, \mathrm{aa}\, s\, \forall \mathbf{x}\, \psi \to \mathrm{aa}\, \mathbf{t}\, \forall \mathbf{x}\, \mathrm{aa}\, s\, \psi$, where ψ is the formula $\phi(\mathbf{x}, \mathbf{t}, s) \leftrightarrow \phi'(\mathbf{x}, \mathbf{t}, s)$, together with

$$\models \mathrm{aa}\, s\, [\phi \leftrightarrow \phi'] \to [\mathrm{aa}\, s\, \phi \leftrightarrow \mathrm{aa}\, s\, \phi']. \quad \square$$

4.4. *Other Filters*

The completeness and compactness theorems of $\mathscr{L}(\mathrm{aa})$ were extended by Kaufmann [1981] to logics $\mathscr{L}^{\mathscr{F}}(\mathrm{aa}, \mathrm{M})$. The quantifier M (="most") is interpreted using a filter $\mathscr{F}$ on ω_1 which contains every cub subset of ω_1. That is, for $|A| = \omega_1$ and any filtration $\langle A_\alpha : \alpha < \omega_1 \rangle$ of A, $\mathfrak{A} \models \mathrm{M}\, s\, \phi(s)$ iff $\{\alpha < \omega_1 : \mathfrak{A} \models \phi(A_\alpha)\} \in \mathscr{F}$. Thus, for example, M is really just aa if $\mathscr{F}$ is just $\mathscr{F}^{\mathrm{cub}} = \{X \subseteq \omega_1 : X \supseteq Y$ for some cub $Y\}$. Some of these results are summarized in:

4.4.1 Theorem *Suppose that $\mathscr{F}$ is a countably complete filter on ω_1. Then $\mathscr{L}^{\mathscr{F}}(\mathrm{aa}, \mathrm{M})$ is countably compact and recursively enumerable for consequence. Moreover, for countably complete $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{L}^{\mathscr{F}}(\mathrm{aa}, \mathrm{M})$ and $\mathscr{L}^{\mathscr{G}}(\mathrm{aa}, \mathrm{M})$ have the same valid sentences iff either*

(i) *$\mathscr{F} = \mathscr{G} = \mathscr{F}^{\mathrm{cub}}$; or*
(ii) *$\mathscr{F} \neq \mathscr{F}^{\mathrm{cub}}$, $\mathscr{G} \neq \mathscr{F}^{\mathrm{cub}}$, but $\mathscr{F}$ and $\mathscr{G}$ are both closed under diagonal intersections; or*
(iii) *no diagonal intersection from $\mathscr{F}$ is empty, and the same holds for $\mathscr{G}$, but $\mathscr{F}$ and $\mathscr{G}$ are not closed under diagonal intersections; or*
(iv) *neither $\mathscr{F}$ nor $\mathscr{G}$ belongs to the classes described in* (i), (ii), *or* (iii) *above.* $\square$

Other filters $\mathscr{F}$ give compact logics. As an example, countable compactness holds for any regular ultrafilter $\mathscr{F}$ on ω_1 such that $\mathscr{F} \supseteq \mathscr{F}^{\text{cub}}$ (see Kaufmann [1981, 3.14]). We know of no filter $\mathscr{F} \supseteq \mathscr{F}^{\text{cub}}$, in fact, for which $\mathscr{L}^{\mathscr{F}}(\text{aa, M})$ is not countably compact. What about filters $\mathscr{F} \not\supseteq \mathscr{F}^{\text{cub}}$? In [BKM, 7.1], we find the "eventual filter" $\mathscr{F}^{\text{ev}} = \{X \subseteq P_{\omega_1}(\omega_1)$: for some s_0, we have $s \in X$ for all $s \supseteq s_0$ $(s \subseteq \omega_1)\}$. By [BKM, 7.2] the corresponding logic $\mathscr{L}^{\mathscr{F}}(\text{M})$ is not countably compact. However, if $\mathscr{H}$ is the filter generated by all collections of the form $\{s - F : s \in X, F$ is finite, $F \subseteq \omega_1\}$, then the resulting logic $\mathscr{L}^{\mathscr{H}}(\text{M})$ is countably compact and axiomatizable even though $\mathscr{L}^{\mathscr{H}}(\text{aa, M})$ is not (see Kaufmann [1981b, Example C, p. 189]).

5. *Extensions of $\mathscr{L}(Q_1)$ by Quantifiers Asserting the Existence of Certain Uncountable Sets*

5.1. *Preliminaries*

In Section 4 we considered an extension $\mathscr{L}(\text{aa})$ of $\mathscr{L}(Q_1)$ in which one could quantify over countable sets. A simple piece of $\mathscr{L}(\text{aa})$, $\mathscr{L}_{\text{pos}}$, was presented in which we can assert $\exists s\phi(s)$ when s occurs only positively in ϕ. A related logic is "negative logic" $\mathscr{L}_{\text{neg}}$, which is defined below. Now $\mathscr{L}_{\text{neg}}$ is not countably compact (Theorem 5.1.2), which is perhaps surprising, since it looks like a rather small extension of the logic $\mathscr{L}^{<\omega}$ of Magidor–Malitz [1977a][3], which *is* countably compact assuming $\diamondsuit$ (Corollary 5.2.6). That done, we will examine some related quantifiers of Malitz–Rubin [1980] and of Shelah [1978d].

5.1.1 Definition. The logic $\mathscr{L}_{\text{neg}}$ is formed from the atomic formulas by closing under $\neg$, $\vee$, $\exists x$, and a second-order quantifier $\exists X$: if X is a unary relation symbol which occurs only negatively in ϕ, where $\phi \in \mathscr{L}_{\text{neg}}$, then $\exists X\phi \in \mathscr{L}_{\text{neg}}$. Hence, we allow $\forall X\phi$ when X occurs only positively in ϕ and $\phi \in \mathscr{L}_{\text{neg}}$. The interpretation of $\exists X$ is given by: $A \models \exists X\phi$ iff $(A, X) \models \phi$ for some uncountable $X \subseteq A$. Notice that $\mathscr{L}_{\text{neg}}$ contains $\mathscr{L}(Q_1)$, since $Qx\phi(x) \leftrightarrow \exists X\, \forall x(X(x) \rightarrow \phi(x))$.

5.1.2 Theorem (Stavi and Malitz, Independently). *The class*

$$\{(A, <) : (A, <) \cong (\omega_1, \in)\}$$

is RPC *in* $\mathscr{L}_{\text{neg}}$. *Hence*, $\mathscr{L}_{\text{neg}}$ *is not countably compact.*

Proof. Let ϕ be the conjunction of the following: a sufficiently large finite amount of set theory (for the argument below); $\forall x(U(x) \leftrightarrow x \in \omega_1)$; and the sentence ψ,

[3] Henceforth we will write [M²] for Magidor–Malitz [1977].

where ψ says that U is ω_1-like in the real world and that every uncountable subset of U (in the real world) contains a subset internal to the model. Formally, ψ is

$$Qx(x \in \omega_1) \wedge \forall\alpha \in \omega_1 \neg Qx(x \in \alpha) \\ \wedge \forall X[X \subseteq \omega_1 \to \exists y(|y| = \omega_1 \wedge y \subseteq X)].$$

Now if $(A, E) \models \phi$ and some reasonable set theory holds in (A, E), then $(\omega_1^{\mathfrak{A}}, E \restriction \omega_1^{\mathfrak{A}})$ is well-ordered. For, in (A, E) we let X be any strictly increasing ω_1-sequence that is cofinal in $\omega_1^{\mathfrak{A}}$. If $y \subseteq X$, then y is also well-ordered. Now, choose $y \in A$, witnessing ψ. Then the transitive collapse of y in $\mathfrak{A}$, which must be ω_1 in $\mathfrak{A}$, is well-ordered. □

Note: Suppose that there is an "almost disjoint" family of $\aleph_2$ subsets of ω_1, that is, every pair has countable intersection; this, of course, is the case if CH holds. Then the argument above shows that although ϕ has an uncountable model, the argument above shows that ϕ has no model of power at most $\aleph_1$. Hence, the following logic is *properly* contained in $\mathscr{L}_{\text{neg}}$, as the reader can easily verify by using the exercise which immediately precedes Proposition 1.3.1.

5.1.3 Definition ([M²]). The logic $\mathscr{L}^{<\omega} = \mathscr{L}(Q, Q^2, Q^3, \ldots, Q^n, \ldots)$ is obtained by closing the atomic formulas under $\neg$, $\vee$, $\exists x$, and the quantifiers Q^n: If ϕ is a formula of $\mathscr{L}^{<\omega}$ so is $Q^n x_1 x_2 \ldots x_n \phi$. The semantics are defined with the new rule: $\mathfrak{A} \models Q^n x_1 \ldots x_n \phi(\mathbf{x})$ iff for some uncountable $X \subseteq A$, $\mathfrak{A} \models \phi(a_1 \ldots a_n)$ for all distinct $a_1, \ldots, a_n \in X$; that is, "there is an uncountable homogeneous set for ϕ". This is really a definition of $\mathscr{L}(Q_1, Q_1^2, \ldots, Q_1^n, \ldots)$. A compactness theorem for the $\aleph_\alpha$-interpretation is proved in Shelah [1981a] for $\alpha = \lambda^+$, assuming $\diamondsuit_\alpha$ and $\diamondsuit_\lambda$; see Section V.8. See also Remark 4.2.7 and VII.1, 2, and 5 for "applied" results on the $\mathscr{L}(Q_\alpha^n)$. Notice that if $i < j$ then Q^i is definable in terms of Q^j, that is:

$$Q^i x_1 \ldots x_i \phi(x_1 \ldots x_i) \leftrightarrow Q^j x_1 \ldots x_j \phi(x_1 \ldots x_i).$$

Let $\mathscr{L}(Q^n)$ denote the restriction of $\mathscr{L}^{<\omega}$ to the quantifiers $Q, Q^2, \ldots, Q^n$.

Recall that in $\mathscr{L}(\text{aa})$ we may axiomatize the class of models $\mathfrak{A} = (A, E)$ such that E is an equivalence relation on A with only countably many equivalence classes. This is also possible in $\mathscr{L}(Q^2)$ using the sentence $\neg Q^2 xy \neg E(x, y)$. Hence, $\mathscr{L}(Q^2)$ is also a proper extension of $\mathscr{L}(Q)$. In fact, Garavaglia [1978b] has shown in ZFC that $\mathscr{L}(Q^n)$-equivalence does not imply $\mathscr{L}(Q^{n+1})$-equivalence, and it is shown in Rubin–Shelah [1983] that $\{\mathfrak{A}: \mathfrak{A} \models \neg Q^{n+1} x_1 \ldots x_{n+1} R(x_1 \ldots x_{n+1})\}$ is not the class of reducts of models of a countable $\mathscr{L}(Q^n)$-theory, assuming $\diamondsuit_{\omega_1}$. However, while satisfiability is absolute in $\mathscr{L}(\text{aa})$ (by the completeness theorem), this is not the case for $\mathscr{L}(Q^2)$:

5.1.4 Example ([M²]). A *Suslin-like tree* is an ω_1-tree $(T, <, \leq)$ (also see Section V.3.3) such that:

(i) there is no branch; that is, $\neg Q^2 xy(x < y \vee y < x)$, and
(ii) there is no uncountable antichain; that is, $\neg Q^2 xy(\neg x < y \wedge \neg y < x)$.

It is easy to see that there exists a Suslin-like tree iff there exists a Suslin tree. However, the latter is independent of ZFC. Therefore, satisfiability of $\mathscr{L}(Q^2)$ sentences is not absolute for models of ZFC. $\square$

5.2. The Magidor–Malitz Completeness Theorem

The next goal is to prove completeness for $\mathscr{L}(Q^2)$. The same idea works for $\mathscr{L}^{<\omega}$, although the notation there is more involved. That being so, we will only indicate how the argument for $\mathscr{L}(Q^2)$ extends to $\mathscr{L}(Q^3)$ (in Section 5.2.5), rather than to all of $\mathscr{L}^{<\omega}$. Sections 5.2.1 through 5.2.6 are adapted from [M²].

5.2.1 Axioms of $\mathscr{L}(Q^2)$. An *acceptable* vocabulary τ is a vocabulary which contains a $(|\mathbf{z}| + 1)$-ary predicate symbol $P_{\phi,x,y,\mathbf{z}}$ for all formulas ϕ which do not contain constants, and for all distinct x, y, $\mathbf{z}$. We will feel free to write P_ϕ for $P_{\phi,x,y,\mathbf{z}}$, when the variables are understood. The axioms for $\mathscr{L}(Q^2)$ include the universal closures of [0]–[6] below. Notice that [0]–[4] are exactly the $\mathscr{L}(Q)$ schemas (see Definition 3.1.1). Fix an acceptable vocabulary.

[0] All first-order axiom schemas.

[1] $\neg Qx(x = y \vee x = z)$.

[2] $\forall x[\phi \to \psi] \to (Qx\phi \to Qx\psi)$.

[3] $Qx\phi(x) \leftrightarrow Qy\phi(y)$, where $\phi(x)$ is a formula in which y does not occur.

[4] $Qy\, \exists x\phi \to \exists x Qy\phi \vee Qx\, \exists y\phi$.

[5] "Witnessing schema": this axiom schema says that $P_{\theta,x_1,x_2,\mathbf{z}}(x, \mathbf{a})$ provides a witness to $Q^2x_1x_2\theta(x_1, x_2, \mathbf{a})$:

$$[Q^2x_1x_2\theta(x_1, x_2, \mathbf{y}) \to QxP_{\theta,x_1,x_2,\mathbf{y}}(x, \mathbf{y})]$$
$$\wedge\ [QxP_{\theta,x_1,x_2,\mathbf{y}}(x, \mathbf{y}) \to \forall x_1\, \forall x_2[P_{\theta,x_1,x_2,\mathbf{y}}(x_1, \mathbf{y})$$
$$\wedge\ P_{\theta,x_1,x_2,\mathbf{y}}(x_2, \mathbf{y}) \wedge x_1 \neq x_2 \to \theta(x_1, x_2, \mathbf{y})]].$$

And, finally, there is the following schema, a schema that is both difficult to describe and hard to look at (hence the name "Medusan"). For now, think of it as saying that ψ produces a homogeneous set for θ. What this actually means will become clearer in the proof of soundness which follows. Moreover, the origin of these axioms will be explained in the proof of completeness.

[6] "Medusan axioms": Let $\overline{Q\mathbf{y}}$ be a quexistential string, that is, a string of quantifiers of the form Qy_i or $\exists y_i$. Also let $\overline{Q^*\mathbf{y}}$ be the result of replacing each Qy_i and $\exists y_i$ by Q^*y_i and $\forall y_i$, respectively. Then

$$\overline{Q\mathbf{y}}\, \exists x\psi(x, \mathbf{y}) \wedge \overline{Q^*\mathbf{y}}\, \forall x[\psi(x, \mathbf{y}) \to \overline{Q^*\mathbf{y}'}\, \forall x'(\psi(x', \mathbf{y}')$$
$$\to x' \neq x \wedge \theta(x', x) \wedge \theta(x, x'))] \to Q^2x_1x_2\theta(x_1, x_2)$$

is an axiom, whenever all variables in the list x, $\mathbf{y}$, x', $\mathbf{y}'$ are distinct.

The rule of inference is *modus ponens*, and, as usual, we can check that universal generalization is a derived rule.

Clearly every τ-structure may be expanded to a σ-structure for some acceptable $\sigma \supseteq \tau$ so that schema [5] holds, where we may assume that no P_θ is in τ. Hence, soundness follows from

5.2.2 Proposition. *The "Medusan axioms" [6] are valid.*

Proof. Suppose $\overline{Qy}$ is quexistential and

$$\text{(1)} \qquad \mathfrak{A} \models \overline{Qy}\, \exists x \psi(x, \mathbf{y}),$$

$$\text{(2)} \qquad \mathfrak{A} \models \overline{Q^*y}\, \forall x[\psi(x, \mathbf{y}) \to \eta(x)],$$

where $\eta(x)$ is $\overline{Q^*y'}\, \forall x'[\psi(x', \mathbf{y}') \to \theta(x, x') \wedge \theta(x', x) \wedge x \neq x']$. We will construct a homogeneous set $\{x_\alpha : \alpha < \omega_1\}$ for θ by induction on α with inductive hypotheses (a) $\mathfrak{A} \models \eta(x_\beta)$, (b) $\mathfrak{A} \models \theta(x_\beta, x_\gamma) \wedge \theta(x_\gamma, x_\beta) \wedge x_\beta \neq x_\gamma$, all $\gamma < \beta \leq \alpha$. To define x_0, notice that (1) and (2), together with an appropriate intersection principle (Lemma 3.1.5), combine to yield $\mathfrak{A} \models \exists \mathbf{y}\, \exists x[(\psi(x, \mathbf{y}) \to \eta(x)) \wedge \psi(x, \mathbf{y})]$. It follows then that $\mathfrak{A} \models \exists x \eta(x)$; choose x_0 such that $\mathfrak{A} \models \eta(x_0)$.

Now, suppose that we have x_β for all $\beta < \alpha$, where the inductive hypotheses hold for all $\beta < \alpha$. Then (a) implies that $\mathfrak{A} \models \eta(x_\beta)$ for all $\beta < \alpha$; that is, for all $\beta < \alpha$, we have

$$\text{(3)} \qquad \mathfrak{A} \models \overline{Q^*y}\, \forall x[\psi(x, \mathbf{y}) \to \theta(x, x_\beta) \wedge \theta(x_\beta, x) \wedge x \neq x_\beta].$$

(1) through (3) yield, by "intersecting",

$$\mathfrak{A} \models \exists \mathbf{y}\, \exists x[\psi(x, \mathbf{y}) \wedge [\psi(x, \mathbf{y}) \to \eta(x)]$$
$$\wedge \bigwedge_{\beta<\alpha} [\psi(x, \mathbf{y}) \to \theta(x, x_\beta) \wedge \theta(x_\beta, x) \wedge x \neq x_\beta]].$$

This implies that

$$\text{(4)} \qquad \mathfrak{A} \models \exists x(\eta(x) \wedge \bigwedge_{\beta<\alpha} [\theta(x, x_\beta) \wedge \theta(x_\beta, x) \wedge x \neq x_\beta]).$$

Pick any witness to (4) and call it x_α. Then the inductive hypotheses are preserved.

Inductive hypothesis (b) guarantees that $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable homogeneous set for θ. $\square$

As for $\mathscr{L}(Q_1)$ and $\mathscr{L}(\text{aa})$, it is convenient to observe:

5.2.3 Lemma. *$\mathscr{L}(Q^2)$ is a reasonable extension of $\mathscr{L}(Q)$ (see Definition 3.3.2) if we are restricted to acceptable vocabularies (defined in Section 5.2.1).* $\square$

Example 5.1.4 shows that the set of valid sentences of $\mathscr{L}(Q^2)$ is not absolute. Therefore, we will need an added set-theoretic hypothesis in order to prove that the axioms, already proved sound in Proposition 5.2.2, are also complete. The following well-known principal of Jensen is a consequence of $V = L$.

> $\diamondsuit$: There is a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$, such that for all $X \subseteq \omega_1$, $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary.

We will call such a sequence a *$\diamondsuit$-sequence.*

5.2.4 Theorem (Completeness Theorem for $\mathscr{L}(Q^2)$). *Assume $\diamondsuit$. Let τ be a countable acceptable vocabulary (see Section 5.2.1) and suppose that T is an $\mathscr{L}(Q^2)(\tau)$-consistent set of τ-sentences of $\mathscr{L}(Q^2)$. Then T has a model.*

To prepare for the proof of this theorem, we will give a fairly detailed outline in the following discussion. We will build an ω_1-chain of weak models, much as we did for $\mathscr{L}(Q)$. The "witnessing schema" [5] will guarantee that sentences $Qx_1x_2\theta(x_1, x_2)$ which hold in some $\mathfrak{A}_\alpha^*$ will also hold in $\mathfrak{A}$. The key problem is to guarantee that when some $\mathfrak{A}_\alpha^*$ satisfies $\neg Q^2x_1x_2\theta$, then this also holds in $\mathfrak{A}$. So, we must in a sense "kill off" all potential uncountable homogeneous sets for such θ. The following diagram summarizes the plan of the proof, as explained further below. Notice the similarity to Jensen's construction of a Suslin tree from $\diamondsuit$. An arrow indicates that the lower box is intended to make the upper box true.

Goal: To "kill" all potential uncountable homogeneous sets for θ.

↑

Keep each S_α from growing into an uncountable homogeneous set for θ.

↑

Suffices to omit (in $\mathfrak{A}$) a type $\Sigma_{\theta,\alpha}^0(x)$ which says that $S_\alpha \cup \{x\}$ is a homogeneous set for θ.

Instead, it suffices to omit a slightly bigger type $\Sigma_{\theta,\alpha}(x)$ (as we will see).

↑

It suffices that $\mathfrak{A}_\gamma^*$ strongly omit $\Sigma_{\theta,\alpha}(x)$ for all $\gamma \geq \alpha$.

↑

Suffices that $\mathfrak{A}_\alpha^*$ strongly omit $\Sigma_{\theta,\alpha}$, which follows from the Medusan Axioms.

As before, the interesting stages of the construction are the successor stages. Suppose we already have $\mathfrak{A}^*_\beta$ and want to get $\mathfrak{A}^*_{\beta+1}$. We form essentially the same theory $T_\phi(\mathfrak{A}^*_\beta)$ as in the proof of the Main Lemma 3.2.1, for appropriate ϕ. The consistency criterion still holds. Keeping countable sets from expanding can be accomplished just as before, by omitting certain types. It makes sense that we *also* omit types to keep homogeneous sets countable, as follows.

Suppose that a set S_β is a homogeneous set for a formula $\theta(x_1, x_2)$, where $\mathfrak{A}^*_\beta \models \neg Q x_1 x_2 \theta(x_1, x_2)$. Here, S_β is the βth member of a fixed $\diamond$-sequence. How can we keep S_β from expanding to an uncountable homogeneous set for θ? We would like to omit the type

$$\Sigma^0_{\theta,\beta}(x) = \{x \neq a \wedge \theta(x, a) : a \in S_\beta\},$$

where, for the sake of simplicity, we will suppose that θ is symmetric; that is, $\models \theta(x_1, x_2) \leftrightarrow \theta(x_2, x_1)$. As before, it will suffice that $\mathfrak{A}^*_\beta$ strongly omit $\Sigma^0_{\theta,\beta}$. In this way, we can keep strongly omitting this type at later stages.

How can it be that $\mathfrak{A}^*_\beta$ does *not* strongly omit $\Sigma^0_{\theta,\beta}$? That means that there is some $\psi(x, \mathbf{y})$ for which (1) and (2) below hold in $\mathfrak{A}^*_\beta$:

$$\text{(1)} \qquad \overline{Qy}\, \exists x \psi(x, \mathbf{y})$$

$$\text{(2)} \qquad \overline{Q^* y}\, \forall x(\psi(x, \mathbf{y}) \rightarrow (x \neq a \wedge \theta(x, a)) \quad \text{for all} \quad a \in S_\beta.$$

However, this is not enough. A bigger type than $\Sigma^0_{\phi,\beta}$ might be easier to strongly omit—that is, *failure* to strongly omit a *bigger* type might have *stronger* consequences. Regard (2) as a formula $\eta(a)$—then $\eta(a)$ holds for all $a \in S_\beta$. If we had chosen $\Sigma^0_{\theta,\beta}$ so that it included $\eta(x)$, we would then have $\mathfrak{A}^*_\beta$ satisfying

$$\text{(3)} \qquad \overline{Q^* y}\, \forall x(\psi(x, \mathbf{y}) \rightarrow \eta(x)).$$

But, "(1) $\wedge$ (3) $\rightarrow Q x_1 x_2 \theta$" is an instance of the Medusan Axioms (6). Hence $\mathfrak{A}^*_\beta \models Q x_1 x_2 \theta$, a contradiction.

So $\mathfrak{A}^*_\beta$ *does* strongly omit any type containing $\Sigma^0_{\theta,\beta}$ which also contains every formula $\eta(x)$ having the property possessed by our "η" above, that is $\mathfrak{A}^*_\beta \models \eta(a)$ for all $a \in S_\beta$. So set

$$\Sigma_{\theta,\beta} = \Sigma^0_{\theta,\beta} \cup \{\delta(x) : \text{for all } a \in S_\beta, \mathfrak{A}^*_\beta \models \delta(a)\}.$$

As previously mentioned, we can continue strongly omitting this type in models $\mathfrak{A}^*_\gamma$ for $\gamma \geq \beta$. Hence, as in the previous proof for $\mathscr{L}(Q)$, $\mathfrak{A}^*$ omits $\Sigma_{\theta,\beta}$, where $\mathfrak{A}^* = \bigcup_{\alpha<\omega_1} \mathfrak{A}^*_\alpha$.

Suppose that $\mathfrak{A}^*_{\alpha_0} \models \neg Q^2 x_1 x_2 \theta$; we want to show that $\mathfrak{A} \models \neg Q^2 x_1 x_2 \theta$. To this purpose, we will suppose not, and choose $S \subseteq A$ so that $\mathfrak{A} \models \theta(a, b)$ for all distinct $a, b \in S$. $\diamond$ will give us $\alpha \geq \alpha_0$ for which $(\mathfrak{A}^*_\alpha, S_\alpha) \prec (\mathfrak{A}^*, S)$. We may check that every $a \in S - A_\alpha$ realizes $\Sigma_{\theta,\alpha}$ in $\mathfrak{A}^*$. On the other hand, since $\mathfrak{A}^*_{\alpha_0} \prec^w \mathfrak{A}^*_\alpha$, $\mathfrak{A}^*_\alpha \models \neg Q^2 x_1 x_2 \theta$, which implies (by construction) that $\mathfrak{A}^*$ *omits* $\Sigma_{\theta,\alpha}$!

Proof of Theorem 5.2.4. Partition $\omega_1 = \bigcup \{X_\phi : \phi$ is a formula of $\mathscr{L}(Q^2)$ with parameters in $\omega_1\}$. For all $\alpha < \omega_1$, choose ϕ_α so that $\alpha \in X_{\phi_\alpha}$. Fix a $\diamondsuit$-sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$. We build a chain $\langle \mathfrak{A}^*_\alpha : \alpha < \omega_1 \rangle$ of countable weak models satisfying the following inductive hypotheses on α:

(a) $\mathfrak{A}^*_0 \models T$.
(b) If $\alpha = \beta + 1$, then $\mathfrak{A}^*_\alpha$ is a precise extension of $\mathfrak{A}^*_\beta$ relative to ϕ_β.
(c) If α is a limit, then $\mathfrak{A}^*_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}^*_\beta$.
(d) $A_\alpha = \omega \cdot (1 + \alpha)$
(e) For each $\delta < \alpha$ and formula $\theta(x_1, x_2)$ with parameters in A_δ, *if* S_δ is a homogeneous set for θ in $\mathfrak{A}^*_\delta$ (that is, $\forall x_1 x_2 \in S_\delta, \mathfrak{A}^*_\delta \models \theta(x_1, x_2) \vee x_1 = x_2$), and $\mathfrak{A}^*_\delta \models \neg Q x_1 x_2 \theta$, *then* $\mathfrak{A}^*_\alpha$ strongly omits

$$\Sigma_{\theta,\delta}(x) = \{x \neq a \wedge \theta(x, a) \wedge \theta(a, x) : a \in S_\delta\}$$
$$\cup \{\eta(x) : \text{for all } a \in S_\delta, \mathfrak{A}^*_\delta \models \eta(a)\}.$$

$\mathfrak{A}^*_0$ is constructed by applying the weak completeness theorem (2.2.3). For limit α, it's easy to see that, by setting $\mathfrak{A}^*_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}^*_\beta$, we preserve the inductive hypotheses.

For the successor step $\alpha = \beta + 1$, we want to use the Main Lemma 3.3.5 from the proof of the omitting types theorem for $\mathscr{L}(Q)$. Hence, it suffices to see that $\mathfrak{A}^*_\beta$ strongly omits all of the sets given in inductive hypothesis (e) for α. For $\delta < \beta$, $\mathfrak{A}^*_\beta$ strongly omits $\Sigma_{\theta,\delta}(x)$ whenever S_δ is a homogeneous set for θ in $\mathfrak{A}^*_\delta$, by the inductive hypothesis. So we are left with the problem of showing that for any formula $\theta(x_1, x_2)$ with parameters in A_β, if S_β is a homogeneous set for θ in $\mathfrak{A}^*_\beta$ and $\mathfrak{A}^*_\beta \models \neg Q^2 x_1 x_2 \theta(x_1, x_2)$, then $\mathfrak{A}^*_\beta$ strongly omits $\Sigma_{\theta,\beta}(x)$.

To obtain a contradiction, we suppose not. Then, for some formula $\psi(x, \mathbf{y})$ with parameters in $\mathfrak{A}^*_\beta$ and some quexistential $\overline{Q\mathbf{y}}$, $\overline{Q\mathbf{y}}\, \exists x \psi$ witnesses this supposition. Hence, we have

$$\text{(1)} \qquad \mathfrak{A}^*_\beta \models \overline{Q\mathbf{y}}\, \exists x \psi(x, \mathbf{y});$$

$$\text{(2)} \qquad \mathfrak{A}^*_\beta \models \overline{Q^*\mathbf{y}}\, \forall x[\psi(x, \mathbf{y}) \rightarrow x \neq a \wedge \theta(x, a) \wedge \theta(a, x)] \quad \text{for each} \quad a \in S_\beta.$$

By (2), the formula $\eta(x) \in \Sigma_{\theta,\beta}(x)$, where

$$\text{(3)} \qquad \eta(x) \equiv \overline{Q^*\mathbf{y}'}\, \forall x'[\psi(x', \mathbf{y}') \rightarrow x' \neq x \wedge \theta(x', x) \wedge \theta(x, x')].$$

So, by choice of $\overline{Q\mathbf{y}}\, \exists x \psi$, we have

$$\text{(4)} \qquad \mathfrak{A}^*_\beta \models \overline{Q^*\mathbf{y}}\, \forall x[\psi(x, \mathbf{y}) \rightarrow \eta(x)].$$

By the Medusan axiom schema [6], together with (1), (3), and (4) above, $\mathfrak{A}^*_\beta \models Q^2 x_1 x_2 \theta$. This contradicts our assumption and the successor step is thus complete. Hence, the induction is also complete.

Set $\mathfrak{A}^* = \bigcup_{\alpha<\omega_1} \mathfrak{A}^*_\alpha$. Since $\mathfrak{A}^*_0 \models T$, the proof will be finished once the following claim has been established.

Claim. For every $\alpha < \omega_1$, sequence $\mathbf{a}$ of members of A_α, and formula $\phi(\mathbf{y})$ of $\mathscr{L}(Q^2)$,

$$\mathfrak{A}^*_\alpha \models \phi(\mathbf{a}) \quad \text{iff} \quad \mathfrak{A} \models \phi(\mathbf{a}).$$

The proof is by induction on the number of Q^2 quantifiers occurring in ϕ, and within a fixed such number, by induction on the complexity of ϕ. All of the inductive steps except Q^2 work just as in the proof of the union of chain Lemma 3.2.2. Let us therefore focus on the Q^2 step.

Using the witnessing schema [5], the direction ($\Rightarrow$) is easy. For the converse, we suppose that $\mathfrak{A} \models Q^2x_1x_2\theta(x_1, x_2, \mathbf{a})$; say $S \subseteq A$, S uncountable and for all $x_1, x_2 \in S$, $x_1 \neq x_2$ implies that $\mathfrak{A} \models \theta(x_1, x_2, \mathbf{a})$. Let C_1 and C_2 be the following cub subsets of ω_1:

$$C_1 = \{\alpha < \omega_1 : \omega \cdot (1 + \alpha) = \alpha\} = \{\alpha < \omega_1 : A_\alpha = \alpha\};$$
$$C_2 = \{\alpha < \omega_1 : (\mathfrak{A}^*_\alpha, S \cap A_\alpha) \prec (\mathfrak{A}^*, S)\}.$$

Also, define a set

$$E = \{\alpha < \omega_1 : S \cap \alpha = S_\alpha\};$$

then E is stationary by choice of $\langle S_\alpha : \alpha < \omega_1 \rangle$. Choose $\delta \in C_1 \cap C_2 \cap E$ such that $\mathbf{a} \in A_\delta^{<\omega}$ and pick $b \in S - A_\delta$.

Subclaim 1. $\mathfrak{A}^* \models \bigwedge \Sigma_{\theta(x_1, x_2, \mathbf{a}), \delta}(b)$.

Proof. Choose $\sigma \in \Sigma_{\theta(x_1, x_2, \mathbf{a})}$. If σ is $x \neq c \wedge \theta(x, c, \mathbf{a}) \wedge \theta(c, x, \mathbf{a})$ for some $c \in S_\delta$, then $c \in S$. So this follows from the choice of S, since $\mathfrak{A}^* \models \theta(b, c, \mathbf{a})$ iff, by the inductive hypothesis, $\mathfrak{A} \models \theta(b, c, \mathbf{a})$, which is true for all distinct $b, c \in S$. Otherwise, σ is $\eta(x)$ for some η holding in $\mathfrak{A}^*_\delta$ of every element of S_δ. But $S_\delta = S \cap \delta = S \cap A_\delta$, since $\delta \in E \cap C_1$. Thus, since $\delta \in C_2$, and since $S_\delta = S \cap A_\delta$ implies that $\mathfrak{A}^*_\delta \models \forall x(x \in S \rightarrow \eta(x))$, we have $\mathfrak{A}^* \models \forall x(x \in S \rightarrow \eta(x))$. Therefore, $\mathfrak{A}^* \models \eta(b)$. $\square$

Subclaim 2. $\mathfrak{A}^*_\delta \models Qx_1x_2\theta(x_1, x_2)$.

Proof. Suppose not. By inductive hypothesis (e), that would imply that $\mathfrak{A}^*_\beta$ strongly omits $\Sigma_{\theta(x_1, x_2, \mathbf{a}), \delta}$ for all $\beta > \delta$. But then $\mathfrak{A}^*$ *strongly omits* and hence *omits* $\Sigma_{\theta(x_1, x_2, \mathbf{a}), \delta}$, contradicting subclaim 1.

Now $(\mathfrak{A}^*_\beta : \beta < \omega_1)$ is an $\prec^w$-elementary chain, by construction. Thus, by subclaim 2, $\mathfrak{A}^*_\alpha \models Q^2x_1x_2\theta(x_1, x_2)$. $\square$

5.2.5 The Case $\mathscr{L}(Q^3)$. The goal here is to lift the preceding results to $\mathscr{L}(Q^3)$, so that one can believe in a corresponding set of results for $\mathscr{L}^{<\omega}$ without going through all the ghastly notation which might otherwise be required. For $\mathscr{L}(Q^3)$ we need extra witnessing axioms of the form

$$\begin{aligned}
&\forall \mathbf{y}[Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3, \mathbf{y}) \to QzP_\theta(z, \mathbf{y})] \wedge \\
&\forall \mathbf{y}[QzP_\theta(z, \mathbf{y}) \to \forall x_1 x_2 x_3 (P_\theta(x_1, \mathbf{y}) \\
&\qquad \wedge P_\theta(x_2, \mathbf{y}) \wedge P_\theta(x_3, \mathbf{y}) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \\
&\qquad \wedge x_2 \neq x_3 \to \theta(x_1, x_2, x_3, \mathbf{y}))].
\end{aligned}$$

Furthermore, we need extra Medusan axioms, and these are described below.

To prove the completeness theorem, we proceed as in the situation for $\mathscr{L}(Q^2)$. As before, we want to omit a type $\Sigma_{\theta,\alpha}$, whenever $\mathfrak{A}_\alpha^* \models \neg Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3)$. By analogy, we have

$$\begin{aligned}
\Sigma_{\theta,\alpha} = {} & \{\theta'(x, a, b) \colon a, b \in S_\alpha, a \neq b\} \\
& \cup \{\eta(x) \colon \text{for all but at most one } a \in S_\alpha, \mathfrak{A}_\alpha^* \models \eta(a)\};
\end{aligned}$$

here $\theta'(x_1, x_2, x_3)$ is $\bigwedge \{\theta(x_{i_1}, x_{i_2}, x_{i_3}) \colon i \text{ a permutation}\}$. If $\mathfrak{A}_\alpha^*$ does not strongly omit $\Sigma_{\theta,\alpha}$, then for some $\overline{Qy}\, \exists x \psi$ where $\overline{Qy}$ is quexistential, $\mathfrak{A}_\alpha^* \models \overline{Qy}\, \exists x \psi$ and $\mathfrak{A}_\alpha^* \models \overline{Q^*y}\, \forall x(\psi(x, \mathbf{y}) \to x \neq a \wedge x \neq b \wedge \theta'(x, a, b))$ for all distinct $a, b \in S_\alpha$. Thus, we may write $\mathfrak{A}_\alpha^* \models \eta_1(a, b)$, where

$$\eta_1(x, b) \equiv \overline{Q^*y}^1\, \forall x^1(\psi(x^1, \mathbf{y}^1) \to x^1 \neq x \wedge x^1 \neq b \wedge \theta'(x^1, x, b)).$$

Accordingly, $[x \neq b \wedge \eta_1(x, b)] \in \Sigma_{\theta,\alpha}$ for all $b \in S_\alpha$; then

$$\text{(1)} \qquad \mathfrak{A}_\alpha^* \models \overline{Q^*y}\, \forall x(\psi(x, \mathbf{y}) \to x \neq b \wedge \eta_1(x, b)), \quad \text{all} \quad b \in S_\alpha.$$

Now set

$$\eta_2(x) \equiv \overline{Q^*y}^2\, \forall x^2(\psi(x^2, \mathbf{y}^2) \to x^2 \neq x \wedge \eta_1(x^2, x)).$$

Then (1) says that $\mathfrak{A}_\alpha^* \models \eta_2(b)$ for all $b \in S_\alpha$. Thus, $\eta_2(x) \in \Sigma_{\theta,\alpha}$. Again, using our choice of $\overline{Qy}\, \exists x \psi$,

$$\mathfrak{A}_\alpha^* \models \overline{Q^*y}\, \forall x(\psi(x, \mathbf{y}) \to \eta_2(x)).$$

This yields $\mathfrak{A}_\alpha^* \models Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3)$—a contradiction—*if* we make the following an axiom:

$$[\overline{Qy}\, \exists x \psi \wedge \overline{Q^*y}\, \forall x(\psi(x, \mathbf{y}) \to \eta_2(x))] \to Q^3 x_1 x_2 x_3 \theta(x_1, x_2, x_3),$$

where η_2 is as defined above.

As before, at stages $\beta \geq \alpha$ we can strongly omit $\Sigma_{\theta,\alpha}(x)$, if $\mathfrak{A}_\alpha^* \models \neg Q^3 x_1 x_2 x_3 \theta$. And, as before, this does the job.

Are the new axioms valid? Suppose that $\mathfrak{A}$ is a model of the hypotheses of a new Medusan axiom. Define $\{x_\alpha : \alpha < \omega_1\}$ subject to the following inductive hypotheses on α. For all $\alpha_1 < \alpha_2 < \alpha_3 \leq \alpha$:

(i) $\mathfrak{A} \models \eta_2(x_{\alpha_3})$.
(ii) $\mathfrak{A} \models \eta_1(x_{\alpha_2}, x_{\alpha_3})$.
(iii) $\mathfrak{A} \models \theta'(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3})$.

Details are straightforward extensions of those given in Theorem 5.2.2 for $\mathscr{L}(Q^2)$. ▯

5.2.6 Corollary ([M²]). *Assume* $\diamondsuit$. *Then* $\mathscr{L}^{<\omega}$ *is countably compact and recursively enumerable for consequence.*

Proof for $\mathscr{L}(Q^2)$. This result follows from the completeness theorem, since every countable vacabulary τ can be expanded to an acceptable τ' which is still countable and is recursive in τ. ▯

There is no known *explicit* set of axioms for $\mathscr{L}(Q^2)$, that is, axioms which do not require τ to be acceptable, even assuming $\diamondsuit$. Shelah has recently shown that in a certain sense, no finite set of schema axiomatizes the set of validities of $\mathscr{L}(Q^2)$; see Shelah–Steinhorn [1982].

The following theorem bears on the sensitivity of $\mathscr{L}^{<\omega}$ to the axioms in the metatheory. We are indebted to Ken Kunen for supplying the following theorem and proofs. In this connection we note that it would also be interesting to find a complete set of axioms under MA + $\neg$CH.

5.2.7 Theorem (Kunen). (i) *One cannot prove in* ZFC + SH (*Suslin's hypothesis*) *that adding* "SH" *to the axioms for* $\mathscr{L}(Q^2)$ (*in Section* 5.2.1) *results in a complete axiomatization for* $\mathscr{L}(Q^2)$.

(ii) *One cannot prove in* ZFC + $\neg$SH *that the usual axioms of* $\mathscr{L}(Q^2)$ (*see Section* 5.2.1) *are complete.*

Proof. (i) Otherwise, satisfiability for $\mathscr{L}(Q^2)$-sentences is absolute for models of ZFC + SH. However, in $\mathscr{L}(Q^2)$ one can assert that a partial order is c.c.c. On the one hand, Con(ZFC + SH + CH) by Jensen, and CH $\rightarrow \exists \mathbb{P}$ ($\mathbb{P}$ is c.c.c and $\mathbb{P} \times \mathbb{P}$ is not c.c.c.), by Laver and Galvin. While on the other hand, MA + $\neg$CH $\rightarrow$ SH and MA + $\neg$CH $\rightarrow \forall \mathbb{P}$ ($\mathbb{P}$ is c.c.c. $\rightarrow \mathbb{P} \times \mathbb{P}$ is c.c.c.).

(ii) In Kunen–Van Douwen [1982] we find that by iterating c.c.c. forcing, we may obtain the consistency of ZFC + $\neg$CH + $\neg$SH +

(∗) Whenever $A_\alpha \subset \mathbb{Q}$ for $\alpha < \omega_1$ satisfy $\forall \alpha < \beta < \omega_1$ ($A_\beta - A_\alpha$ is bounded $\wedge\, A_\alpha - A_\beta$ is unbounded), there is an $X \subset \omega_1$ such that $|X| = \omega_1$ and $\forall \alpha, \beta \in X$ ($\alpha < \beta \rightarrow A_\beta \not\subset A_\alpha$).

Observe that here, $S \subseteq \mathbb{Q}$ is *bounded* iff $\exists q(S \subset (-\infty, q))$. However, there is a sentence ϕ of $\mathscr{L}(Q^2)$ which has a model iff $\neg(*)$. ϕ is consistent with the usual axioms because CH $\Rightarrow \neg(*)$. ▯

It is shown in [M²] that the axioms for $\mathscr{L}^{<\omega}$ remain complete in some model of $\neg\diamondsuit$, namely when one adds $\aleph_2$ Cohen reals to a model of $\diamondsuit$. Nevertheless, Shelah has recently found a model of set theory in which $\mathscr{L}(Q^2)$ is not countably compact.

5.3. Other Related Logics

An extension of $\mathscr{L}^{<\omega}$ has been proved to be countably compact (assuming $\diamondsuit$) in Malitz–Rubin [1980]. In this logic, for example, we can say that $\{\langle x, y\rangle : \phi(x, y)\}$ contains an equivalence relation with uncountably many uncountable equivalence classes as follows, where X^2 ranges over uncountable sets of uncountable sets.

$$(\exists X^2)(\forall X_1^1 \in X^2)(\forall X_2^1 \in X^2)(\forall x_1 \in X_1^1)(\forall x_2 \in X_1^1)(\forall x_3 \in X_2^1)$$
$$[\phi(x_1, x_2) \wedge \neg\phi(x_1, x_3)].$$

Here, it is understood that distinct variables are intended to represent distinct things. More generally, we allow "descending quantifier strings", which begin with $(\exists X^n)$ (some n) and contain various $(\forall X_j^i \in X_k^{i+1})$ for $i < n$, such that each such X_k^{i+1} is either X^n or else appears in an earlier quantifier $(\forall X_k^{i+1} \in X_l^{i+2})$. Here, $X_j^0 = x_j$ ranges over $K^0(A) = A$, and X_j^{i+1} ranges over

$$\{Z \subseteq K^n(A) : |Z| \geq \omega_1\} = K^{n+1}(A).$$

Perhaps a simpler logic, which is equivalent—at least if one has a pairing function—allows quantifiers $(\exists X^n)(\forall x_s : s \in T)\ \phi(\langle x_s : s \in T\rangle)$, where T is any finite subtree of ω^n (ordered by inclusion). This quantifier is interpreted as follows: X^n is a tree of height n having uncountably many elements of level 0 as well as uncountably many immediate successors of each element of level $<(n-1)$; and x_s is to be $< x_t$ whenever $s < t$, with x_s ranging over elements of level $|s| - 1$.

Another version of this quantifier is defined in Rubin–Shelah [1983]. Moreover, it is there proved that one has a strict hierarchy of these quantifiers.

The following definition gives a simplified version of the quantifier "there is a branch" from Shelah [1978d], the general version being found in Section V.8 of the present volume. Shelah's quantifier is, in fact, *fully* compact, whereas the following version is not (and this for the same reason that $\mathscr{L}(Q_1)$ is not). It is also interesting to note that this logic is a countably compact piece of $\mathscr{L}(Q^2)$ for which $\diamondsuit$ is not needed, since satisfiability is absolute. Intuitively, it seems then that one has the equation

$$\frac{\mathscr{L}(\text{"there is a branch"})}{\mathscr{L}(Q^2)} \simeq \frac{\text{Aronszajn tree}}{\text{Suslin tree}}.$$

And, of course, $\diamondsuit$ (or something at least) is needed to construct a Suslin tree, but not to construct an Aronszajn tree.

5.3.1 Definition. $\mathscr{L}(Q^B)$ is the logic formed from $\mathscr{L}(Q_1)$ by adding an additional quantifier Q^B: if $\phi(x, y)$ is a formula of $\mathscr{L}(Q^B)$ then so is $Q^B xy\phi$. Write $\eta^{\mathfrak{A}}$ for $\{\mathbf{a}: \mathfrak{A} \models \eta(\mathbf{a})\}$. Then the new inductive clause for satisfaction is

$$\mathfrak{A} \models Q^B xy\, \phi(x, y) \quad \text{iff} \quad \begin{array}{l} \langle \text{field}(\phi^{\mathfrak{A}}), \phi^{\mathfrak{A}}\rangle \text{ is a tree satisfying} \\ \forall y \neg Qx\phi(x, y), \text{ such that there is an} \\ \text{uncountable branch through this tree.} \end{array}$$

5.3.2 Theorem (Shelah [1978d]). *$\mathscr{L}(Q^B)$ is countably compact and recursively enumerable for consequence.*

Proof. An approximate idea of the proof is to place $\mathscr{L}(Q^B)$ inside $\Delta(\mathscr{L}(Q))$ (defined in Section II.7.2), more or less, in a generic extension of universe. We then may use the absoluteness of $\mathscr{L}(Q)$-satisfiability. Fix a vocabulary τ. We define maps $\phi \mapsto \phi^\exists$ and $\phi \mapsto \phi^\forall$ from $\mathscr{L}(Q^B)(\tau)$ to $\mathscr{L}(Q)(\tau')$, where $\tau' = \tau \cup S$ for some set S of new relation symbols (these relation symbols will be determined below). The approximate idea here is that if the world were perfect, then ϕ would be equivalent to $\exists \mathbf{X}\phi^\exists$ and to $\forall \mathbf{Y}\phi^\forall$, where $\mathbf{X}$ and $\mathbf{Y}$ are the new relation symbols in $\phi^\exists$ and $\phi^\forall$, respectively.

$\phi^\exists$ and $\phi^\forall$ are defined by induction on the following depth $r(\phi)$ of the quantifiers Q^B and Q in ϕ: $r(\phi) = 0$ for ϕ atomic, $r(\neg\psi) = r(\forall x\psi) = r(\psi)$, $r(\phi \vee \psi) = \max(r(\phi), r(\psi))$, $r(Qx\phi) = r(\phi) + 1$ and $r(Q^B xy\phi) = r(\phi) + 2$. Set $\phi^\exists = \phi^\forall = \phi$ for ϕ atomic. Now, suppose that $\phi^\exists$ and $\phi^\forall$ are defined for $r(\phi) < n$. We then define $\phi^\exists$ and $\phi^\forall$ for $r(\phi) = n$ by induction on ϕ. If ϕ is $\neg\psi$, then $\phi^\exists$ is $\neg(\psi^\forall)$ and $\phi^\forall$ is $\neg(\psi^\exists)$. Suppose that ϕ is $\theta \vee \psi$. Then, of course, $\phi^\exists$ is $\theta^\exists \vee \psi^\exists$. To define $\phi^\forall$, where ϕ is $\theta \vee \psi$, we first make the new relation symbols of $\theta^\forall$ disjoint from those of $\psi^\forall$, say by suffixing a "0" on those of $\theta^\forall$ and a "1" on those of $\psi^\forall$. Call these modified formulas θ' and ψ', and set $(\theta \vee \psi)^\forall = \theta' \vee \psi'$. The next case is $\phi = \forall x\psi$. Then $\phi^\forall$ is $\forall x\psi^\forall$. For $\phi^\exists$, we first consider the choice schema

$$\forall x\, \exists Y\eta(x, Y, \ldots) \to \exists Y'\, \forall x\eta(x, (Y')_x, \ldots),$$

where $\eta(x, (Y')_x, \ldots)$ denotes the result of replacing each occurrence of the form $Y(\mathbf{y})$ by an occurrence of $Y'(x, \mathbf{y})$ in η. Then $\phi^\exists = \forall x\psi^\exists((X^1)_x, \ldots, (x^n)_x)$, where $X^1, \ldots, X^n$ are the new relation symbols occurring in $\psi^\exists$. (Observe that this idea appears in the proof of Theorem II.7.2.4(a).) The next step in our development is to define $(Qx\theta)^\exists$ as $QxX(x) \wedge [\forall x(X(x) \to \theta)]^\exists$ by using the rules above. Similarly, we have that $(Qx\theta)^\forall$ is $[Q^*xX(x) \to [\exists x(X(x) \wedge \theta)]^\forall] \wedge Qx(x = x)$.

Finally, we wish to define $\eta^\exists$ and $\eta^\forall$, when η is $Q^B xy\phi(x, y)$. $\eta^\exists$ is easy to define, since it simply is ["X is an uncountable branch of the tree $\langle \text{field}(\phi), \phi\rangle$" $\wedge$ $\forall y \neg Qx\phi(x, y)]^\exists$. In order to define $\eta^\forall$, we imagine that if a given ranked tree does *not* have a branch, then there is an order-preserving map from that tree into the rationals. Thus, $\eta^\forall$ is the following formula, where R and S are binary relation symbols not in τ which do not occur in η: $\neg$["R is an order-preserving function from the tree $\{\langle x, y\rangle: \phi(x, y)\}$ to the countable linear order S" $\vee$ $\neg$"ϕ is a tree" $\vee$ $\exists yQx\phi(x, y)]^\exists$.

It is routine to verify by induction on ϕ that for all $\phi \in \mathscr{L}(Q^B)$, we have that

(1) $\models \phi^\exists \to \phi$ and $\models \phi \to \phi^\forall$.

Now we claim that for any set Γ of sentences of $\mathscr{L}(Q^B)$, if $\Gamma^\exists = \{\phi^\exists : \phi \in \Gamma\}$, then:

(2) Γ is satisfiable iff $\Gamma^\exists$ is satisfiable.

Since $\phi^\exists \in \mathscr{L}(Q)$ for all $\phi \in \mathscr{L}(Q^B)$, the theorem follows from (2) above. The direction ($\Leftarrow$) follows immediately from (1) above. For the proof of ($\Rightarrow$), we suppose that $\mathfrak{A} \models \Gamma$. Let $(U, <)$ be the disjoint sum of all trees P such that $\forall y \neg QxPxy$, with field contained in $\mathfrak{A}$, that do not contain an uncountable branch. Then $(U, <)$ does not contain an uncountable branch. By Baumgartner–Malitz–Reinhardt [1970], there is a c.c.c. partial order which generically adds an order-preserving map from $(U, <)$ into the rationals. An easy induction shows that the predicate "$\mathfrak{A} \models \phi[s]$" is absolute for the generic extension, since no new branches are added. Thus, Γ remains satisfiable. Moreover, in the generic extension: $\exists \mathbf{X}\phi^\exists \leftrightarrow \phi$, $\phi \leftrightarrow \forall \mathbf{Y}\phi^\forall$ are valid in $\mathfrak{A}$ for all $\phi \in \mathscr{L}(Q^B)$, as one can again check by induction on ϕ. Hence, we have that $\Gamma^\exists$ is satisfiable in the generic extension. Thus, $\Gamma^\exists$ is $\mathscr{L}(Q)$-consistent in the generic extension, which implies that $\Gamma^\exists$ is $\mathscr{L}(Q)$-consistent in V, since consistency is finitary. By the completeness theorem for $\mathscr{L}(Q)$, $\Gamma^\exists$ is satisfiable (in V). $\square$

6. *Interpolation and Preservation Questions*

In this section we will survey some of the results, methods, and questions that are related to definability properties of $\mathscr{L}(Q_1)$ and (to some extent, at least) its extensions $\mathscr{L}(\text{aa})$ and $\mathscr{L}^{<\omega}$. In Section 6.4 we will consider such properties for the "weak models" of Section 2.3.

6.1. *Preservation of $\mathscr{L}$-equivalence Under Products and Unions, for $\mathscr{L} = \mathscr{L}(Q_\alpha)$, $\mathscr{L}(\text{aa})$, $\mathscr{L}^{<\omega}$*

The following theorem is proved in Lipner [1970].

6.1.1 Theorem. *Suppose that $\{\mathfrak{A}_i : i \in I\}$ and $\{\mathfrak{B}_i : i \in I\}$ are finite families of τ-structures, where $\mathfrak{A}_i \equiv_{\mathscr{L}(Q_\alpha)} \mathfrak{B}_i$ for all $i \in I$. Then $\prod \{\mathfrak{A}_i : i \in I\} \equiv_{\mathscr{L}(Q_\alpha)} \prod \{\mathfrak{B}_i : i \in I\}$. And if τ has no function symbols, then the disjoint unions $\dot{\bigcup} \{\mathfrak{A}_i : i \in I\}$ and $\dot{\bigcup} \{\mathfrak{B}_i : i \in I\}$ are also $\mathscr{L}(Q_\alpha)$-equivalent.* $\square$

One method of proof is the method of *back-and-forth systems*, also known as "Ehrenfeucht games": see Section II.4.2. This method has also been used in

Vinner [1972]. Badger[1977] has also given an appropriate back-and-forth criterion for $\mathscr{L}^{<\omega}$. In that work, we also find—in spite of this criterion—that $\mathscr{L}(Q^2)$-equivalence is not preserved by finite direct products. The reader should see Definition II.4.2.2 for material on back-and-forth systems in a more general setting.

By assuming an appropriate combinatorial hypothesis on $\aleph_\alpha$, Lipner has also proved Theorem 6.1.1 for other powers of I. For $\aleph_\alpha = 2^\omega$, this theorem also holds for any index set I which is not at least as large as some measurable cardinal, and this even if we only assume $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are $\mathscr{L}_{\omega\omega}$-equivalent for all $i \in I$ (Flum [1975a, Theorem 2.10]). Related results on preservation of $\mathscr{L}(Q_\alpha)$-equivalence by reduced products, where $\aleph_\alpha = 2^\omega$ can also be found in Flum [1975a].

We will now turn to $\mathscr{L}$(aa). There are several back-and-forth criteria for $\mathscr{L}$(aa)-equivalence. These results were developed independently by Caicedo [1978], Makowsky (see Makowsky–Shelah [1981, Section 2]), Kaufmann [1978a], and Seese and Weese [1982]. Nevertheless, the following example shows that $\mathscr{L}$(aa)-equivalence is not preserved by disjoint unions. A similar argument can also be given to show that it is not preserved by finite products.

6.1.2 Example (Shelah). Let S be a stationary subset of ω_1 with stationary complement, and set $(A, <)$ equal to the ordered sum $\sum \{X_\alpha : \alpha < \omega_1\}$ where $X_\alpha = \mathbb{Q}$ if $\alpha \in S$, otherwise $X_\alpha = 1 + \mathbb{Q}$, and $<$ results from replacing each α by X_α. Similarly, let $(B, <)$ be the ordered sum $\sum \{Y_\alpha : \alpha < \omega_1\}$ where $Y_\alpha = 1 + \mathbb{Q}$ if $\alpha \in S$, otherwise $Y_\alpha = \mathbb{Q}$. A back-and-forth argument establishes that $(A, <)$ and $(B, <)$ are $\mathscr{L}_{\infty\omega}$(aa)-equivalent. [*Hint*: let $X'_\alpha = \bigcup \{X_\gamma : \gamma < \alpha\}$, and $Y'_\alpha = \bigcup \{Y_\gamma : \gamma < \alpha\}$. By induction on $\phi(\mathbf{s}, \mathbf{x})$ show that if $\alpha_0 < \cdots < \alpha_{n-1}$ and $\beta_0 < \cdots < \beta_{n-1}$, where $\alpha_i \in S$ iff $\beta_i \notin S$, and if f is a partial isomorphism from $(\mathfrak{A}, X'_{\alpha_0}, \ldots, X'_{\alpha_{n-1}})$ to $(\mathfrak{B}, Y'_{\alpha_0}, \ldots, Y'_{\alpha_{n-1}})$, then $\mathfrak{A} \models \phi(X'_{\alpha_0}, \ldots, X'_{\alpha_{n-1}}, \text{domain} f)$ iff $\mathfrak{B} \models \phi(Y'_{\alpha_0}, \ldots, Y'_{\alpha_{n-1}}, \text{range} f)$.] However, if $(A', <')$ is a disjoint copy of $(A, <)$, then $(A' \cup A, <' \cup <)$ and $(A' \cup B, <' \cup <)$ are *not* $\mathscr{L}$(aa)-equivalent. For, the following sentence θ holds in the former but not in the latter: $\theta \equiv \text{stat}\, s\, \exists x \exists y \forall z (s(z) \leftrightarrow z <' x \vee z < y)$.

To take care of this problem, Kaufmann [1978a] defines and Eklof–Mekler [1979] further studies the notion of finitely determinate structure. Roughly speaking, such a structure is one in which we do not have disjoint definable stationary sets. In fact, we might say that the aa quantifier is self-dual on such structures. More precisely, we have

6.1.3 Definition. A structure $\mathfrak{A}$ is *finitely determinate* if it satisfies all formulas of the form $\text{aa}\, s_1 \ldots \text{aa}\, s_n \forall \mathbf{x}\, [\text{stat}\, t\, \phi(\mathbf{x}, \mathbf{s}, t) \to \text{aa}\, t\, \phi(\mathbf{x}, \mathbf{s}, t)]$.

We observe that many familiar structures are finitely determinate, for example, $(\mathbb{R}, <)$ and all modules are proved to be finitely determinate in Eklof–Mekler [1979].

Using back-and-forth systems for finitely determinate structures (see Kaufmann [1978a] or Eklof–Mekler [1979]), we can prove an analogue of Theorem 6.1.1.

6.1.4 Theorem (Kaufmann [1978a]). *Suppose that $\{\mathfrak{A}_i : i \in I\}$ and $\{\mathfrak{B}_i : i \in I\}$ are families of finitely determinate τ-structures such that $\mathfrak{A}_i \equiv_{\mathscr{L}(\mathrm{aa})} \mathfrak{B}_i$ for all $i \in I$.*

(i) *If I is finite then $\prod \{\mathfrak{A}_i : i \in I\}$ and $\prod \{\mathfrak{B}_i : i \in I\}$ are $\mathscr{L}(\mathrm{aa})$-equivalent and finitely determinate.*
(ii) *If τ is relational then the disjoint unions $\bigcup \{\mathfrak{A}_i : i \in I\}$ and $\bigcup \{\mathfrak{B}_i : i \in I\}$ are $\mathscr{L}(\mathrm{aa})$-equivalent and finitely determinate.* ☐

6.1.5 Remark. Shelah has recently shown that every countable consistent theory of $\mathscr{L}(Q_1)$ has a finitely determinate model; see Mekler–Shelah [198?].

A number of variants of Theorems 6.1.4 and 6.1.1 have been proved. Aside from some obvious extensions to $\mathscr{L}_{\infty\omega}(Q_\alpha)$ and to $\mathscr{L}_{\infty\omega}(\mathrm{aa})$, we may also consider other operations on structures, such as direct sums (see Eklof–Mekler [1979] or Kaufmann [1978a, III.3.11]). Moreover, Seese [1981b] and Mekler [1984] have used ordered sums to prove theorems such as Seese's theorems that every ordinal $(\alpha, \in)$ is finitely determinate, and that the $\mathscr{L}(\mathrm{aa})$-theory of ordinals is decidable; see also Section VII.4.

6.2. *Preservation of $\mathscr{L}(Q_1)$-sentences by Extensions, and Related Problems*

Among the definability problems that might be raised for $\mathscr{L}(Q_1)$, one that has received some attention (see, for example, Bruce [1978a]) is:

6.2.1 Question. Classify those sentences ϕ of $\mathscr{L}(Q_1)$ such that whenever $\mathfrak{A} \models \phi$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{B} \models \phi$. Such ϕ are said to be *preserved by extensions.*

Of course, the Łos–Tarski theorem for first-order logic establishes that the class of existential sentences is the answer if one restricts to $\mathscr{L}_{\omega\omega}$. The natural generalization is the class of quasi-existential or "quexistential" sentences:

6.2.2 Definition (Bruce [1978a]). A formula ϕ of $\mathscr{L}(Q)$ is *quexistential* if it is in prenex form, with only Q and $\exists$ quantifiers.

One may easily verify, as Bruce has noted, that every quexistential sentence is preserved by extensions. Although Question 6.2.1 remains open, the natural conjecture was proved false in Baldwin–Miller [1982] if one restricts to the class of models of a given theory. The general result, given below, is new and due to Shelah:

6.2.3 Example (Shelah). There is a sentence ϕ of $\mathscr{L}(Q_1)$ which is preserved by extensions but which is not equivalent to a quexistential sentence.

Proof. Let ψ be the conjunction of:

(a) $<$ is a linear order of the universe such that every proper initial segment is countable.
(b) $<^*$ is a linear order of the universe.
(c) $Qx(x = x) \to \exists x[\{y: y < x\}$ is dense for $<^*]$.

Then ψ is preserved by submodels. This is not difficult to see, since every suborder of a separable linear order is separable. It suffices to show, then, that $\neg\psi$ $(=\phi)$ is not equivalent to a quexistential sentence. This, in turn, follows from the existence of models $\mathfrak{A} = (A, <_A, <_A^*)$ and $\mathfrak{B} = (B, <_B, <_B^*)$ such that $\mathfrak{A} \models \neg\psi$ and $\mathfrak{B} \models \psi$, and for every quexistential sentence θ, if $\mathfrak{A} \models \theta$ then $\mathfrak{B} \models \theta$. We will construct models $\mathfrak{A}$ and $\mathfrak{B}$ with the following properties:

$(1)_A$ $(\mathfrak{A}, <_A^*)$ is a non-separable linear order.

$(2)_A$ $(\mathfrak{A}, <_A^*)$ is ω_1-dense, that is, it satisfies

$$\forall x\, \forall y(x <^* y \to Qz(x <^* z <^* y)).$$

$(3)_A$ $(\mathfrak{A}, <_A)$ is ω_1-like.

$(1)_B$ $(\mathfrak{B}, <_B^*)$ is an ω_1-dense subset of $\mathbb{R}$ of power ω_1.

$(2)_B$ If $a <_B b$ and $c <_B^* d$ then for some e, $a <_B e <_B b$ and $c <_B^* e <_B^* d$; and there is no $<_B$-least element.

$(3)_B$ $(\mathfrak{B}, <_B)$ *is* ω_1*-like.*

Assume for the moment that such models have indeed been constructed. Then $\mathfrak{A} \models \neg\psi$ and $\mathfrak{B} \models \psi$. In fact, $\{x: x <_B b\}$ is $<_B^*$-dense for all $b \in B$, by $(2)_B$ above. We thus claim that for every quexistential formula $\theta(\mathbf{x})$ and finite partial isomorphism $\{\langle a_i, b_i\rangle : i \in I\}$, if $\mathfrak{A} \models \theta(\mathbf{a})$ then $\mathfrak{B} \models \theta(\mathbf{b})$ also. This is easy to show by induction on complexity of θ. We use $(2)_B$ for the $\exists$ step. As to the Q step, if $a_1 <_A^* a_2 <_A^* \cdots <_A^* a_{n-1}$ and $\mathfrak{A} \models Qx\theta(x, \mathbf{a})$, then for some $j < n$ there exist uncountably many x such that $\mathfrak{A} \models \theta(x, \mathbf{a}) \wedge a_j <^* x <^* a_{j+1}$ (where $a_0 = -\infty$ and $a_n = \infty$). Since almost all these x are $<_A$-greater than every a_i by $(3)_A$, then every x with $b_j <_B^* x <_B^* b_{j+1}$ which is $<_B$-greater than every b_j, will satisfy $\theta(x, \mathbf{b})$ in $\mathfrak{B}$, by the inductive hypothesis. By $(1)_B$ and $(3)_B$ above, we have that $\mathfrak{B} \models Qx\theta(x, \mathbf{b})$.

It now remains to construct such models $\mathfrak{A}$ and $\mathfrak{B}$. $(A, <_A^*)$ is any ω_1-dense cofinal subset of $\mathbb{R} \cdot \omega_1$, of power ω_1. Then $<_A$ is any ω_1-like ordering of A. Also $(B, <_B^*)$ is easy to choose so that $(1)_B$ holds. The construction of $<_B$ so that $(2)_B$ and $(3)_B$ hold is left to the reader, with the hint that it proceeds ω steps at a time, and that it suffices to consider rational numbers c and d in $(2)_B$ above. $\square$

An analogous question is raised in Bruce [1978a] for $\mathscr{L}(\text{aa})$. One might conjecture that the sentences preserved by extensions are the "generalized Σ" sentences, that is, those sentences in prenex form with no universal quantifiers. In fact such sentences are preserved by extensions (Bruce [1978a, Theorem 3.1]). But here again equality eludes us. For, Theorem 3.5 of Baldwin–Miller [1982] states that the class of separable dense linear orders is defined by some sentence ϕ for which $\neg\phi$ is not equivalent to a generalized Σ sentence, and yet $\text{Mod}(\phi)$ is closed under substructures.

Of course, there are other preservation questions we might raise, and they are all open. For example, Bruce has conjectured that by analogy to first-order logic, a sentence is preserved by unions of ω-chains iff it is of the form $Q_1^* x_1 \ldots Q_n^* x_n \phi(x)$, where each $Q_i^* \in \{\forall, Q^*\}$ and ϕ is preserved by extensions. Here again, one direction is easy. Bruce points out that interpolation properties can be useful in proving such theorems—see, for example, Section 6.4(1). Thus, let us turn next to the interpolation problem.

6.3. *The Interpolation Problem for Extensions of $\mathscr{L}(Q_1)$*

Recall that the interpolation property (even Δ-interpolation) fails for $\mathscr{L}(Q_1)$; see Remark 4.1.2 (vi). However, research has been stimulated by questions such as the following, which was raised by Feferman and others (also see Makowsky–Shelah–Stavi [1976, §3]).

6.3.1 Question. Is there an extension of $\mathscr{L}(Q_1)$ which is countably compact and satisfies the interpolation property?

Shelah has recently announced that it is relatively consistent with ZFC that the answer here should be affirmative, and he has also recently shown [1982a] that every valid implication in $\mathscr{L}^{\text{cf}\,\omega}$ (see Section II.2.4) has an interpolant in $\mathscr{L}(\text{aa})$. A topological result of Caicedo [1981b, 1.3] is that interpolation holds for the restriction of $\mathscr{L}(Q_1)$ to monadic vocabularies. Since space is limited here, we will not prove any of the (admittedly limited) number of positive results. Instead, we will indicate some obstacles to the interpolation property by way of presenting a few examples. This will also provide us with a rationale for becoming more familiar with the expressive powers of the logics we have been discussing.

6.3.2 Lemma (Badger [1977]; see also Ebbinghaus [1975b]). *Let κ be a cardinal and suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are linear orders which are κ-dense, that is, they satisfy $\forall x\, \forall y[x < y \rightarrow Q_\kappa z\,(x < z < y)]$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}_{\infty\omega}^{<\omega}$-equivalent in the κ-interpretation.*

Proof. A routine induction on formulas ϕ of $\mathscr{L}_{\infty\omega}^{<\omega}$ shows that for every partial isomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$, $\mathfrak{A} \models_\kappa \phi[s]$ iff $\mathfrak{B} \models_\kappa \phi[f \circ s]$ for every assignment s of the free variables of ϕ into domain(f). The key observation here is that if $-\infty = a_0 < a_1 < \cdots < a_k = \infty$ and $\mathfrak{A} \models_\kappa Q^n x_0 \cdots x_{n-1} \theta(\mathbf{x}, \mathbf{a})$, then for some $i < k$,

$\mathfrak{A} \models_\kappa Q^n \mathbf{x}(\bigwedge_{j<n} a_i < x_j < a_{i+1} \wedge \theta(\mathbf{x}, \mathbf{a}))$; and, hence, for all one–one $\mathbf{x}$ in A, $\mathfrak{A} \models_\kappa (\bigwedge_{j<n} a_i < x_j < a_{i+1} \to \theta(\mathbf{x}, \mathbf{a}))$. We argue similarly for $\mathfrak{B}$. □

6.3.3 *Theorem* (Based on Makowsky–Shelah–Stavi [1976, Theorem 2.15]). *$\mathscr{L}^{<\omega}_{\infty\omega}$ does not allow Δ-interpolation for $\mathscr{L}(Q_1)$.*

Proof. Let $\mathscr{K}$ be the class of separable ω_1-dense linear orders without endpoints. Clearly, $\mathscr{K}$ is $\Sigma^1_1(\mathscr{L}(Q))$. Also the complement of $\mathscr{K}$ is immediately seen to be $\Sigma^1_1(\mathscr{L}(Q))$, once we observe that a dense linear order L is non-separable iff $L \times L$ has an uncountable family of pairwise disjoint open rectangles not meeting the diagonal (Kurepa [1952]). Certainly, if L is separable, then so is $L \times L$; and, for the converse, notice that for every maximal family of pairwise disjoint rectangles not meeting the diagonal $\{(a_i, b_i) \times (c_i, d_i): i \in I\}$, $\{a_i: i \in I\}$ is dense.

However, $\mathscr{K}$ is not elementary in $\mathscr{L}^{<\omega}_{\infty\omega}$. For $\mathfrak{A} = (\mathbb{R}, <)$ is separable, while $\mathfrak{B} = (\mathbb{R} \cdot \omega_1, <)$ is not separable, yet $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}^{<\omega}_{\infty\omega}$-equivalent by Lemma 6.3.2. □

Badger [1977] has shown that $\mathscr{L}^{<\omega}_{\infty\omega}$ does not allow Δ-interpolation for $\mathscr{L}(Q^2)$, in every cardinal interpretation. In Badger [1980] one finds that the Beth property fails for $\mathscr{L}^{<\omega}_{\infty\omega}$, in every cardinal interpretation κ with κ regular. This partially generalizes a theorem (and its proof) of *H*. Friedman [1973], that the Beth property fails for every $\mathscr{L}_{\infty\omega}(Q_\alpha)$. In particular we can prove:

6.3.4 Corollary (Badger [1980]). *There is an implicitly definable relation of $\mathscr{L}(Q)$ which is not explicitly definable in $\mathscr{L}^{<\omega}_{\infty\omega}$ (in the ω_1-interpretation).*

Hint of proof. We may combine the proofs of Corollary 6.3.4 and of Theorem XVIII.4.3, which then say that under suitable conditions the Beth property implies interpolation. Roughly speaking, we may show by induction on formulas that for any two tree structures as in the proof of Theorem XVIII.4.3, every map which is a partial isomorphism from a subtree onto a subtree (and appropriately respects the tree order) actually does preserve $\mathscr{L}^{<\omega}_{\infty\omega}$-formulas. □

Following is a natural example which shows that the Δ-interpolation property does not imply the interpolation property. The original version appears in Theorem II.7.2.6 and is due to H. Friedman. Although that result involves infinitary logic, the following one, in fact, is based on it.

6.3.5 Theorem. $\Delta(\mathscr{L}^{<\omega})$ *does not allow interpolation* (*or the Robinson property*) *for $\mathscr{L}(Q)$.*

Proof. If $\kappa = \omega$ or $\kappa = \omega_1$ then the class of linear orders of cofinality κ is a PC class of $\mathscr{L}(Q)$, as we simply assert that X is a cofinal subset which is countable (if $\kappa = \omega$) or is ω_1-like (if $\kappa = \omega_1$). Since these classes are disjoint, it suffices to find two linear orders which are $\Delta(\mathscr{L}^{<\omega})$-equivalent, and whose cofinalities are ω and ω_1. Choose a structure $\mathfrak{A} = (R_\alpha, \omega_2, \in)$, where $(R_\alpha, \in)$ satisfies the same Σ_n-sentences with parameters in $(R_\alpha, \in)$ as does $(V, \in)$, by the reflection theorem; n

should be sufficiently large so that the definitions of satisfaction for $\Delta(\mathscr{L}^{<\omega})$, and of ω_2 should be, say, Σ_{n-10} (to be safe). Thus, let us say that $\omega_2 = P^A$. Choose $\mathfrak{B}_1 \prec \mathfrak{A}$ and $\mathfrak{B}_2 \prec \mathfrak{A}$ such that $\omega_1 + 1 \subseteq B_1 \cap B_2$, $(P^{\mathfrak{B}_1}, \in)$ has cofinality ω, and $(P^{\mathfrak{B}_2}, \in)$ has cofinality ω_1. That done, our proof will be complete once we have proved that for all formulas ϕ of $\Delta(\mathscr{L}^{<\omega})$:

(*) Let $\mathfrak{C} \prec \mathfrak{A}$, where $\omega_1 + 1 \subseteq C$. Then, for all **a** in $P^{\mathfrak{C}}$ and all $X_1, \ldots, X_n \in C$ $(X_i \subseteq \omega_2^{m_i})$,

$$(P^{\mathfrak{C}}, \in, X_1 \cap C, \ldots, X_n \cap C) \models \phi(\mathbf{a}) \text{ iff } (\omega_2, \in, X_1, \ldots, X_n) \models \phi(\mathbf{a}).$$

To prove (*) it suffices (see the proof of the result in Theorem II.7.2.4(i)) to show that for all Σ formulas θ of $\mathscr{L}^{<\omega}$, if $\mathbf{X} \in C$ $(\mathbf{X} \subseteq \omega_2^{<\omega})$ and $(\omega_2, \in, \mathbf{X}) \models \theta(\mathbf{a})$ then $(P^{\mathfrak{C}}, \in, \mathbf{X} \cap C) \models \theta(\mathbf{a})$. Thus, suppose that $(\omega_2, \in, \mathbf{X}) \models \exists Y \psi(\mathbf{a}, Y)$, where $\psi \in \mathscr{L}^{<\omega}$. Then, by choice of $\mathfrak{A}$, $\mathfrak{A} \models \exists Y$"$(P, \in, \mathbf{X}, Y) \models \psi(\mathbf{a}, Y)$". Since $\mathfrak{C} \prec \mathfrak{A}$ we have $\mathfrak{A} \models$ "$(P, \in, \mathbf{X}, Y) \models \psi(\mathbf{a}, Y)$" for some $Y \in C$; and then again, by choice of $\mathfrak{A}$, it follows that $(\omega_2, \in, \mathbf{X}, Y) \models \psi(\mathbf{a}, Y)$. Hence, it suffices to prove (*) for all $\phi \in \mathscr{L}^{<\omega}$. However, this is merely a straightforward induction on ϕ and is therefore left to the reader. □

For $\mathscr{L}^{<\omega}$, we have just presented counterexamples to interpolation involving separability of linear orders and countable versus uncountable cofinality, notions which are elementary in $\mathscr{L}(\mathrm{aa})$. However, the notion of whether an ω_1-like tree has a branch is elementary in $\mathscr{L}^{<\omega}$ but not in $\mathscr{L}(\mathrm{aa})$. This was observed by Shelah, and the relevant details are supplied in Makowsky–Shelah [1981]. In this connection, the reader should also see Ebbinghaus [1975b] for a related theorem. For an extension see also Example XVII.2.4.5 and Proposition XVII.2.4.6.

6.3.6 Theorem. $\mathscr{L}(\mathrm{aa})$ *does not allow interpolation for* $\mathscr{L}(Q)$. *In fact, under* MA + $\neg$CH, $\mathscr{L}(\mathrm{aa})$ *does not allow* Δ*-interpolation for* $\mathscr{L}(Q)$.

Hint of Proof. We find two ranked trees of height ω_1 (or ω_1-like) which are $\mathscr{L}(\mathrm{aa})$-equivalent, but such that one has a branch and the other does not. More precisely, we find $\mathscr{L}(\mathrm{aa})$-equivalent structures in the following disjoint PC classes of $\mathscr{L}(Q)$: the class of ω_1-like ranked trees satisfying $\exists X$ ("X is an uncountable linearly ordered subset"); and the class of models of $\exists f$ ("f is an order-preserving map into a countable linear order"). According to a theorem of Baumgartner–Malitz–Reinhardt [1970, Theorem 4], under MA + $\neg$CH, these classes are complementary in the class of ω_1-like ranked trees. □

Shelah [1982a] has announced the relative consistency of Δ-interpolation of $\mathscr{L}(\mathrm{aa})$ for $\mathscr{L}(Q)$. We should also note that by a theorem of Caicedo [1981b, 4.1], Theorem 6.3.6 implies that $\mathscr{L}_{\infty\omega}(\mathrm{aa})$ does not allow interpolation for $\mathscr{L}(Q)$.

Question 6.3.1 might be asked for the Robinson property rather than for the interpolation property. As to that case, Mundici [1981c] used uncountable vocabularies to supply a negative answer. See also Section XVIII.4.1 for a generalization.

6.4. *Interpolation and Preservation Revisited: Monotone Structures*

Consider the logic of monotone structures $(\mathfrak{A}, q)$ as defined in Section 2.3; that logic has nice properties, including not only compactness and axiomatizability (see Section 2.3), but also interpolation. Interpolation was first proved independently by Shelah (see Bruce [1978a; 3.1, 3.2]), Sgro, and Makowsky–Tulipani [1977, Corollary 3.1]. The reader should also see Chapter XV for related theorems about topological logics. A particularly straightforward way of obtaining completeness and interpolation theorems, even for countable fragments of $\mathscr{L}_{\omega_1\omega}$, is to use consistency properties: see Section VIII.3. We add the following clause: if $Qx\phi$ and $Q^*x\psi$ are in s where s belongs to a consistency property S, then for some c, $s \cup \{\phi(c), \psi(c)\} \in S$. The details involved in this development are straightforward for the reader who is familiar with consistency properties.

Some Directions Radiating from the Study of the Logic of Monotone Structures

(1) Guichard [1980] has used consistency properties to generalize Feferman's many-sorted interpolation theorem [1974a] and its application to preservation theorems, so as to obtain a preservation theorem for bounded quantifiers $Q^y x$ ("for many $x \in y$"), such as are studied in Barwise [1978b].

(2) Interpolation and countable compactness theorems can be proved for the logics $\mathscr{L}^{\mathscr{F}}(Q)$, whose structures are of the form $(A, \ldots; q_{\mathscr{F}})$, where $q_{\mathscr{F}} = \{X \subseteq \omega : \omega - X \notin \mathscr{F}\}$, with $\mathscr{F}$ being any given filter on ω which properly contains the cofinite filter; see Kaufmann [1984a]. There seems to be a connection with uniform validity (as discussed in Kueker [1978]) which has not yet been fully clarified although related work has been undertaken by S. Buechler and D. Kueker.

(3) Finally, we will mention a paper by Ebbinghaus–Ziegler [1982], a paper in which the quantifiers Q^n (as discussed in Section 5) are studied for monotone structures $(\mathfrak{A}, q)$, especially when q is an ultrafilter on A. It is proved there (Theorem 1.1) that the following are equivalent, where we write $\mathscr{L}^{\mathrm{U}}(Q^n)$ to indicate our restriction to ultrafilters:

(i) $\mathscr{L}^{\mathrm{U}}(Q^n)$ is compact;
(ii) $\mathscr{L}^{\mathrm{U}}(Q^n)$ satisfies interpolation;
(iii) $n = 1$.

The emphasis in this chapter has been on logics involving cardinality $\aleph_1$. Although monotone structures may provide some additional understanding of the area, their application to logics with cardinality (and related) quantifiers seems to be limited. Methods appropriate to higher cardinals are studied in the next chapter.

We will conclude this section with a problem: To find an extension of $\mathscr{L}(Q_1)$ which has a Lindström-type characterization (in the sense of Chapter III).

7. *Appendix* (*An Elaboration of Section 2*)

In this section we will present here the precise definitions and the proofs that were promised in section 2. We will begin by considering

7.1. *Concrete Syntax*

In the ensuing discussions, we will frequently make use of the notation $\mathscr{L} = \bigcup_\tau \mathscr{L}(\tau)$. This clear we now present

7.1.1 Definition. A logic $\mathscr{L}$ has *concrete syntax* if the following properties hold.

(i) $\mathscr{L}_{\omega\omega}(\tau) \subseteq \mathscr{L}(\tau)$ for all τ, and furthermore $\mathscr{L}$ is closed under first-order operations $\neg$, $\vee$, $\exists$ (and there is unique readability). If τ is countable, so is $\mathscr{L}(\tau)$. $\forall$, $\wedge$, $\rightarrow$, and $\leftrightarrow$ are defined symbols. Finally, $\tau_1 \subseteq \tau_2$ implies that $\mathscr{L}(\tau_1) \subseteq \mathscr{L}(\tau_2)$.

We allow the map $\tau \mapsto \mathscr{L}(\tau)$ to be a partial map, provided that $\mathscr{L}(\tau \cup C)$ exists whenever $\mathscr{L}(\tau)$ exists and C is a set of constant symbols.

(ii) There is a map *frvar* which assigns a finite set of variables to each formula ϕ of $\mathscr{L}$. Moreover, for $\phi \in \mathscr{L}_{\omega\omega}$, frvar($\phi$) is the set of free variables of ϕ. As usual, a *sentence* is a formula ϕ such that frvar(ϕ) $= \varnothing$. Finally, the map frvar obeys the obvious rules for $\neg$, $\vee$, $\exists$.

(iii) For each formula ϕ of $\mathscr{L}$ and function f mapping a finite set of variables to constants, there is a unique formula $\phi(f)$, which has the usual meaning (substitute $f(v)$ for v) if ϕ is atomic. If $\phi \in \mathscr{L}(\tau)$, then $\phi(f) \in \mathscr{L}(\tau \cup \text{range}(f))$. Moreover,

(a) $(\exists v\phi)(f) = \begin{cases} \exists v(\phi(f)), & \text{if } v \notin \text{dom}(f), \\ \exists v(\phi(f - \{\langle v, f(v)\rangle\})), & \text{if } v \in \text{dom}(f). \end{cases}$

(b) $(\neg\phi)(f) = \neg\phi(f)$, $(\phi \vee \psi)(f) = \phi(f) \vee \psi(f)$.

(c) ("Restriction rule of substitution") $\phi(f) = \phi(f \upharpoonright \text{frvar}(\phi))$.

(d) $\phi(f)(g) = \phi(f \cup g)$, if $f \cup g$ is a function.

(e) $\phi(\varnothing) = \phi$

(iv) There is a notion $\vdash_{\mathscr{L}(\tau)}$ of $\mathscr{L}(\tau)$-proof satisfying the following properties:

(a) An $\mathscr{L}(\tau)$-proof is a finite sequence of $\mathscr{L}(\tau)$-formulas. We write $\Gamma \vdash_{\mathscr{L}(\tau)} \phi$ to indicate that an $\mathscr{L}(\tau)$-proof from $\Gamma \subseteq \mathscr{L}(\tau)$ exists. That is to say, each formula in the proof is either a member of Γ, or an axiom of $\mathscr{L}(\tau)$ (where the axioms include those given below), or else follows from previous formulas in the proof by *modus ponens*. We also require that whenever $\Gamma \vdash_{\mathscr{L}(\tau \cup \{c\})} \phi(\{\langle x, c\rangle\})$ where $\Gamma \cup \{\phi\} \subseteq \mathscr{L}(\tau)$, then $\Gamma \vdash_{\mathscr{L}(\tau)} \forall x\phi$. That is, universal generalization is a derived (and not an explicit) rule.

(b) Every tautology in $\mathscr{L}(\tau)$ is an axiom of $\mathscr{L}(\tau)$, as is every $\mathscr{L}(\tau)$-formula $\forall x \neg\phi \leftrightarrow \neg\exists x\phi$.

(c) Every $\mathscr{L}(\tau)$-formula $\neg\exists y\phi(f) \rightarrow \neg\phi(f \cup \{\langle y, c\rangle\})$ is an axiom of $\mathscr{L}(\tau)$, when $y \notin \text{dom}(f)$.

(d) Every equality axiom of first-order logic which belongs to $\mathscr{L}(\tau)$, is an axiom of $\mathscr{L}(\tau)$. And, for all $\phi \in \mathscr{L}(\tau)$, and f and g such that f and g map variables to constants in τ, where $\mathrm{dom}(f) = \mathrm{dom}(g)$, we have $\vdash_{\mathscr{L}(\tau)} \phi(f) \wedge \bigwedge_{x \in \mathrm{dom}(f)} f(x) = g(x) \to \phi(g)$.
(e) If $\Gamma \cup \{\phi\} \subseteq \mathscr{L}(\tau)$ and $\tau' = \tau \cup C$ for some set C of constants, then $\Gamma \vdash_{\mathscr{L}(\tau)} \phi$ iff $\Gamma \vdash_{\mathscr{L}(\tau')} \phi$.

The final condition for a concrete syntax is:

(v) There is a "rank function" r from $\mathscr{L}$ into the ordinals such that $r(\phi)$ is less than each of $r(\exists x\phi)$, $r(\phi \vee \psi)$, $r(\neg \phi)$.

7.1.2 Remark. For the purposes of Section 2.3, we note that we may speak of a "concrete syntax" even if we omit all of the semantics of a logic. Since all of the results below rely only on the first-order semantics of weak models anyhow, they also make sense and remain true when the semantics is removed.

7.1.3 *Proof of Soundness* (Proposition 2.2.1). Assume that $\vdash_{\mathscr{L}(\tau \cup C)} \phi(f)$, where $\mathrm{range}(f) \cap \mathrm{frvar}(\phi) = \varnothing$. Using the derived rule of universal generalization, we have $\vdash_{\mathscr{L}(\tau \cup C)} \forall x_1 \ldots \forall x_n \phi$, where $\mathrm{dom}(f) = (x_1, \ldots, x_n)$. Then $\vdash_{\mathscr{L}(\tau)} \forall x_1 \cdots \forall x_n \phi$, by Definition 7.1.1 (iv) (e). Thus, we have reduced to the case $f = \varnothing$. But this is a trivial induction on the length of the proof, since *modus ponens* is the only rule of inference and every axiom is valid in $\mathfrak{A}^*$. ☐

7.2. *Proofs of the Weak Completeness Theorem and Its Extensions*

7.2.1 *Proof of Theorem* 2.2.3 (Weak Completeness Theorem). The argument here is a straightforward Henkin argument. However, it should be observed that we do not attempt to control what sentences hold in $\mathfrak{A}^*$ other than, of course, those of the form ϕ^*. Since τ is countable, so is $\mathscr{L}(\tau)$ by Definition 7.1.1(i). Now, let C be a countable set of constant symbols disjoint from τ, and let $\{\langle \phi_n, f_n \rangle : n < \omega\}$ enumerate all $\langle \phi, f \rangle$ such that ϕ is an $\mathscr{L}(\tau)$-formula and $f : \mathrm{frvar}(\phi) \to C$. By proceeding in the usual way, we may form finite theories T_n of $\mathscr{L}(\tau \cup C)$ such that $T \cup T_n$ is $\mathscr{L}(\tau \cup C)$-consistent, such that for all n:

(i) $\phi_n(f_n) \in T_{n+1}$ or $\neg \phi_n(f_n) \in T_{n+1}$;
(ii) if ϕ_n is $\exists y \psi$ and $\phi_n(f_n) \in T_n$ then $\psi(f \cup \{\langle y, c \rangle\}) \in T_{n+1}$ for some $c \in C$.

Let $T_\omega = \bigcup_{n \in \omega} T_n$. Observe that T_ω is deductively closed. Form the Henkin model from equivalence classes from C ($[c] = [d]$ iff "$c = d$" $\in T_\omega$). For *atomic $\mathscr{L}(\tau)$-formulas* ϕ, define

$$(*) \qquad \mathfrak{A}^* \models \phi[\tilde{f}] \quad \text{iff} \quad \phi(f) \in T_\omega, \quad \text{whenever} \quad f : \mathrm{frvar}(\phi) \to C,$$

where $\tilde{f}(x) = [f(x)]$. For $\mathscr{L}(\tau)$-formulas ϕ which are neither atomic, nor a negation, nor a disjunction, nor of the form $\exists x\psi$, define

$$R_\phi^{\mathfrak{A}^*} = \{\langle[c_0], \ldots, [c_{n-1}]\rangle : \phi(f) \in T_\omega,$$

where

$$\mathrm{dom}(f) = \{v_{i_0}, \ldots, v_{i_{n-1}}\} = \mathrm{frvar}(\phi) \quad (i_0 < \cdots < i_{n-1})$$

and

$$f(v_{i_k}) = c_{i_k} \quad (\text{all } k < n)\}.$$

This is well defined by the equality axioms for $\mathscr{L}$, and we see that $(*)$ holds for all such ϕ also. As usual (using the rank function $r(\phi)$ so that we can carry out the induction), $(*)$ holds for formulas $\neg\psi$ and $\psi_1 \vee \psi_2$. The latter uses the "restriction rule of substitution," which is given in Definition 7.1.1(iii)(c) and which we henceforth use implicitly. Finally, for the $\exists$ step, suppose that $\mathfrak{A}^* \models \exists y\psi[\tilde{f}]$, where $\mathrm{dom}(f) = \mathrm{frvar}(\exists y\psi)$, so $y \notin \mathrm{dom}(f)$. Choose $c \in C$ such that $\mathfrak{A}^* \models \psi[\tilde{f} \cup \{\langle y, [c]\rangle\}]$. By the inductive hypothesis (since $r(\psi) < r(\exists y\psi)$), $\psi(f \cup \{\langle y, c\rangle\}) \in T_\omega$. Hence, $(\exists y\psi)(f) \in T_\omega$. For otherwise, we would have that $\neg\exists y\psi(f) \in T_\omega$, so that by an axiom and *modus ponens*, $\neg\psi(f \cup \{\langle y, c\rangle\}) \in T_\omega$, contradicting consistency of some T_n. For the other direction of $(*)$, suppose that $(\exists y\psi)(f) \in T_\omega$, where f: $\mathrm{frvar}(\phi) \to C$. Then for some n, we have that $\langle(\exists y\psi), f\rangle = \langle\phi_n, f_n\rangle$ holds. Thus, by construction, there exists $c \in C$ such that $\psi(f \cup \{\langle y, c\rangle\}) \in T_{n+1}$. By the inductive hypothesis, $\mathfrak{A}^* \models \psi(\tilde{f} \cup \{\langle y, [c]\rangle\})$. So, $\mathfrak{A}^* \models \exists y\psi[\tilde{f}]$.

By construction, $T \subseteq T_\omega$ and every $\mathscr{L}(\tau)$ axiom belongs to T_ω. By $(*)$ it now follows that $\mathfrak{A}^*$ is a weak model of T. ☐

7.2.2 *Proof of Weak Omitting Types Theorem* (2.2.5). The proof of the weak completeness theorem given above will suffice here provided we mix in some additional steps as follows. We enumerate (in type ω) all pairs $\langle\Sigma, f\rangle$ such that $\Sigma = \Sigma_n$ for some $n < \omega$ and f maps $\mathbf{x}_n$ to C. At stage $(n + 1)$ we guarantee that $\mathfrak{A}^* \models \neg\sigma[\tilde{f}]$ for some $\sigma \in \Sigma$. By $(*)$ in the proof of Theorem 2.2.3, it suffices that $\neg\sigma(f) \in T_{n+1}$ for some $\sigma \in \Sigma$. But this can be easily achieved by using the local omitting hypothesis, since T_n is consistent with T. ☐

The following technical lemma is used in Sections 3, 4, and 5, to extend weak models. It is the precise version of Lemma 2.2.6.

7.2.3 Lemma (Extension Lemma). *Assume the following hypotheses, where $\mathscr{L}$ has concrete syntax.*

(i) τ, τ', *and D are disjoint countable vocabularies.*
(ii) $\mathfrak{A}^*$ *is a countable weak model for $\mathscr{L}(\tau)$, and $D \supseteq D_A = \{d_a : a \in A\}$.*

(iii) *T is an $\mathscr{L}(\tau \cup \tau' \cup D)$-consistent set of $\mathscr{L}(\tau \cup \tau' \cup D)$-sentences; and in addition, $T = \{\phi(f) : \langle \phi, f \rangle \in \Gamma\}$ for some Γ such that for all $\langle \phi, f \rangle \in \Gamma$, $\phi \in \mathscr{L}(\tau \cup \tau')$ and range $f \subseteq D$.*

(iv) *For all assignments s in A, set $\tilde{s} = \{\langle x, d_{s(x)} \rangle : x \in \operatorname{dom} s\}$; then*

$$\{\langle \phi, \tilde{s} \rangle : \mathfrak{A}^* \models \phi[s]\} \subseteq \Gamma.$$

(v) *T $\mathscr{L}(\tau \cup \tau' \cup D)$-locally omits sets $\Sigma_n(f_n) = \{\sigma(f_n) : \sigma \in \Sigma_n\}$ for all $n < \omega$, where Σ_n has free variables $\mathbf{x}_n$ and $\mathbf{y}_n$ (disjoint) and $f_n : \mathbf{y}_n \to D_A$. Observe that $\mathbf{y}_n$ may be infinite.*

Then there exists a countable weak model $\mathfrak{B}^*$ for $\mathscr{L}(\tau \cup \tau')$ such that $\mathfrak{A}^* \prec^w \mathfrak{B}^* \restriction \tau^+$, and moreover there exists a function $g : D \to B$ such that $g(d_a) = a$ for all $a \in A$ and $\mathfrak{B}^* \models \phi[g \circ f]$ for all $\langle \phi, f \rangle \in \Gamma$. Finally, $\mathfrak{B}^* \models \forall \mathbf{x}_n \vee \{\neg \sigma : \sigma \in \Sigma_n\} [g \circ f_n]$ for all $n < \omega$.

Proof. The proofs of the weak completeness and weak omitting types theorems show that if we add a countable set C of new constant symbols, we may obtain an $\mathscr{L}(\tau \cup \tau' \cup D \cup C)$-theory T_ω with the following properties.

(1) $T \subseteq T_\omega$.

(2) For all $\phi \in \mathscr{L}(\tau \cup \tau')$ and all maps $f : \operatorname{frvar}(\phi) \to D \cup C$, $\phi(f) \in T_\omega$ iff $\neg\phi(f) \notin T_\omega$.

(3) For all $\exists x \phi \in \mathscr{L}(\tau \cup \tau')$ and $f : \operatorname{frvar}(\exists x \phi) \to D \cup C$, if $\exists x \phi(f) \in T_\omega$ then $\phi(f \cup \{\langle x, c \rangle\}) \in T_\omega$ for some $c \in C$.

(4) For all $f : \mathbf{x}_n \to D \cup C$, $\neg \sigma(f_n \cup f) \in T_\omega$ for some $\sigma \in \Sigma_n$.

Form the Henkin model $\mathfrak{B}^*$ from $D \cup C$. As in the proof of Theorem 2.2.3, (2) and (3) together imply that

(*) $\mathfrak{B}^* \models \phi[\bar{f}]$ iff $\phi(f) \in T_\omega$

for all $\mathscr{L}(\tau \cup \tau')$-formulas ϕ and functions $f : \operatorname{frvar}(\phi) \to C \cup D$, where $\bar{f}(x) = [f(x)]$. In particular, since $T \subseteq T_\omega$ by (1) above, $\mathfrak{B}^* \models \phi[\bar{f}]$ for all $\langle \phi, f \rangle \in \Gamma$, by (iii). Now, if $\mathfrak{A}^* \models \phi[s]$, then $\langle \phi, \tilde{s} \rangle \in \Gamma$ so $\mathfrak{B}^* \models \phi[\bar{\tilde{s}}]$. Hence, by identifying $[d_a]$ and a for all $a \in A$, we obtain $\bar{\tilde{s}} = s$ and conclude that $\mathfrak{A}^* \prec^w \mathfrak{B}^* \restriction \tau^+$. Finally, if we set $g(c) = [c]$ for all $c \in D$, then, by using (4) to see that each Σ_n is appropriately omitted, we obtain the desired conclusions. $\square$

Chapter V

Transfer Theorems and Their Applications to Logics

by J. H. SCHMERL

This chapter is primarily concerned with the general problem of transferring results about one logic, say $\mathscr{L}(Q_1)$, to another logic, say $\mathscr{L}(Q_\alpha)$. A typical such property is $\aleph_0$-compactness. It is known from Chapter IV that $\mathscr{L}(Q_1)$ is $\aleph_0$-compact. Under certain set-theoretic assumptions on α discussed in this chapter, the logic $\mathscr{L}(Q_1)$ transfers to $\mathscr{L}(Q_\alpha)$. In such cases we can then conclude that $\mathscr{L}(Q_\alpha)$ is also $\aleph_0$-compact. The logics that we consider in this chapter are variants and generalizations of $\mathscr{L}(Q_1)$, and the properties of these logics which we are most concerned with are compactness and recursive enumerability for validity.

1. The Notions of Transfer and Reduction

After presenting the basic definitions that allow useful model-theoretic comparisons between logics, we present applications to compactness and recursive enumerability of logics and to two-cardinal questions.

1.1. Transfer

The substantive theme of this chapter is the notion of transfer and we will begin our explorations with

1.1.1 Definition. Suppose $\mathscr{L}_0$ and $\mathscr{L}_1$ are two logics which have exactly the same syntax but differ in their semantics. Then $\mathscr{L}_0$ *transfers to* $\mathscr{L}_1$ iff every sentence which is satisfiable relative to $\mathscr{L}_0$ is also satisfiable relative to $\mathscr{L}_1$. In symbols, we write $\mathscr{L}_0 \to \mathscr{L}_1$.

Transfer becomes quite fruitful when there is *mutual transfer*, when both $\mathscr{L}_0 \to \mathscr{L}_1$ and $\mathscr{L}_1 \to \mathscr{L}_0$ hold. For, in this situation $\mathscr{L}_0$ and $\mathscr{L}_1$ have exactly the same valid sentences, so that a syntactic property known to hold for $\mathscr{L}_0$ will also hold for $\mathscr{L}_1$. For example, if $\mathscr{L}_0$ has the Beth property, then so does $\mathscr{L}_1$. In this chapter we will generally be concerned with two properties which are especially amenable to verification using the methods of transfer. These properties are compactness and, to a lesser degree, recursive enumerability for validity. To be sure, if there is mutual transfer $\mathscr{L}_0 \to \mathscr{L}_1$ and $\mathscr{L}_1 \to \mathscr{L}_0$ and if either one of these logics is compact or recursively enumerable for validity, then so is the other. However, it often turns out that the proof of a specific transfer theorem yields a sort of self-transfer theorem of the form $\mathscr{L} \to \mathscr{L}$. And while the transfer $\mathscr{L} \to \mathscr{L}$ is evidently trivial, one nevertheless often obtains a stronger form having as a consequence the compactness and the recursive enumerability for validity of $\mathscr{L}$. This is the approach that Fuhrken and Vaught used in the original proofs of the compactness and the recursive enumerability for validity of $\mathscr{L}(Q_1)$.

To see how compactness typically obtains, we need a strengthening of the notion of transfer. For κ an infinite cardinal, we say that $\mathscr{L}_0 \to \mathscr{L}_1$ κ-compactly iff whenever a set of at most κ sentences is finitely satisfiable relative to $\mathscr{L}_0$, then it is satisfiable relative to $\mathscr{L}_1$. Fuhrken and Vaught observed that $\mathscr{L}(Q_{\alpha+1}) \to \mathscr{L}(Q_1)$ $\aleph_0$-compactly. In particular, $\mathscr{L}(Q_1) \to \mathscr{L}(Q_1)$ $\aleph_0$-compactly, which is just another way of saying that $\mathscr{L}(Q_1)$ is $\aleph_0$-compact.

For the sake of completeness, we will mention a further generalization of transfer at this point. For each $j \in J$, let $\mathscr{L}_j$ be a logic with the same syntax as $\mathscr{L}$. Then $\{\mathscr{L}_j : j \in J\} \to \mathscr{L}$ iff each sentence which is satisfiable relative to each $\mathscr{L}_j$ is also satisfiable relative to $\mathscr{L}$. Similarly, $\{\mathscr{L}_j : j \in J\} \to \mathscr{L}$ κ-compactly iff whenever a set of at most κ sentences is finitely satisfiable relative to each $\mathscr{L}_j$, then it is satisfiable relative to $\mathscr{L}$.

1.2. Reduction

Although it was noted at the outset of this Section that the notation of transfer provides the *substantive theme* of the present chapter, there is, nevertheless, a *methodological theme* appearing in this chapter: Reduction. This notion of reduction is of considerable importance in our exposition and the basic idea underlying it is to associate (usually effectively) with each sentence in some logic $\mathscr{L}$ a corresponding first-order sentence, and then reduce the study of the model theory of $\mathscr{L}$ to the study of those models of some first-order theory satisfying some additional property.

Much of what we do in this chapter will concern the logic $\mathscr{L}(Q)$ with various cardinality interpretations, which have already been discussed in Sections II.2.2 and III.2.4, and (for $\kappa = \aleph_1$), in Section IV.3. For any infinite cardinal κ, if we are defining the κ-interpretation of $\mathscr{L}(Q)$, then the key clause in the definition is that

$$\mathfrak{A} \models Qx\phi(x) \quad \text{iff} \quad |\{a \in A : \mathfrak{A} \models \phi(a)\}| \geq \kappa.$$

We will also adhere to the convention that if $\mathfrak{A}$ is a structure appropriate for $\mathscr{L}(Q)$ with the κ-interpretation, then $|A| \geq \kappa$; that is, $Qx(x = x)$ is a valid sentence. If $\kappa = \aleph_\alpha$, then $\mathscr{L}(Q_\alpha)$ simply denotes $\mathscr{L}(Q)$ with the κ-interpretation.

Fuhrken [1964] introduced the reduction of these logics to cardinal-like structures. We will consider the Fuhrken reduction in some detail, since it is quite typical of other reductions. Finally, typical applications will be given in Section 1.3.

1.2.1 Definition. A linearly ordered set $(A, <)$ is *κ-like* iff $(A, <) \models \forall x \neg Qy(y < x)$ under the κ-interpretation. A structure $\mathfrak{A} = (A, <, \ldots)$ is *κ-like* iff $(A, <)$ is *κ-like.* $\mathfrak{A}$ is *cardinal-like* iff it is κ-like for some κ. We let $K(\kappa)$ denote the class of κ-like structures.

Examples of κ-like linearly ordered sets are well-ordered sets with order type κ. If κ is uncountable, then there are linearly ordered sets which are κ-like but not well-ordered. On the other hand, $(\omega, <)$ is (up to order-isomorphism) the only $\aleph_0$-like linearly ordered set.

To begin the Fuhrken reduction, let us fix a vocabulary τ which includes neither the binary relation symbol $<$ nor the ternary relation symbol R. Consider the first-order sentence σ which is the conjunction of the universal closures of the following three formulas:

$$R(x_1, y, z) \wedge R(x_2, y, z) \to x_1 = x_2,$$

$$R(x, y_1, z) \wedge R(x, y_2, z) \to y_1 = y_2,$$

$$x_2 < x_1 \wedge R(x_1, y_1, z) \to \exists y_2 R(x_2, y_2, z).$$

The intention here is that σ should express the fact that as z varies, R encodes a set of bijections $x \mapsto y$ whose domains are (possibly improper) initial segments.

With each $\mathscr{L}(Q)(\tau)$-formula ϕ we will associate a first-order $(\tau \cup \{R, <\})$-formula ϕ^* having the same free variables as ϕ by the following inductive procedure:

$$\phi^* = \phi, \quad \text{if } \phi \text{ is atomic,}$$

$$(\neg\phi)^* = \neg\phi^*,$$

$$(\phi_1 \wedge \phi_2)^* = \phi_1^* \wedge \phi_2^*,$$

$$(\exists y\phi)^* = \exists y\phi^*,$$

$$(Qy\phi)^* = \exists z\, \forall x\, \exists y(R(x, y, z) \wedge \phi^*).$$

We will also associate with each $\mathscr{L}(Q)(\tau)$-formula ϕ a first-order $\tau \cup \{R, <\}$-sentence σ_ϕ by the following inductive procedure:

$$\sigma_\phi = \sigma, \quad \text{if } \phi \text{ is atomic,}$$

$$\sigma_{\neg\phi} = \sigma_\phi,$$

$$\sigma_{\phi_1 \wedge \phi_2} = \sigma_{\phi_1} \wedge \sigma_{\phi_2},$$

$$\sigma_{\exists y\phi} = \sigma_\phi,$$

$$\sigma_{Qy\phi} = \sigma_\phi \wedge \forall \bar{x}\, \exists z\, \forall y[\phi^*(\bar{x}, y) \leftrightarrow \exists x R(x, y, z)].$$

The following two lemmas give the essential properties of the Fuhrken reduction.

1.2.2 Lemma. *If* $(\mathfrak{A}, R, <)$ *is a* κ*-like* $(\tau \cup \{R, <\})$*-structure and* $\phi(\bar{x})$ *is an* $\mathscr{L}(Q)(\tau)$*-formula, then*

$$(\mathfrak{A}, R, <) \models \sigma_\phi \leftrightarrow \forall \bar{x}(\phi(\bar{x}) \leftrightarrow \phi^*(\bar{x}))$$

in the κ*-interpretation.* □

1.2.3 Lemma. *If* $\mathfrak{A}$ *is a* τ*-structure with* $|\tau| \leq \kappa = |A|$, *then* $\mathfrak{A}$ *can be expanded to a* κ*-like structure* $(\mathfrak{A}, R, <)$ *such that for every* $\mathscr{L}(Q)(\tau)$*-formula* ϕ, $(\mathfrak{A}, R, <) \models \sigma_\phi$. □

The proof of Lemma 1.2.2 can be obtained by a rather routine induction on formulas. In Lemma 1.2.3, the expansion of $\mathfrak{A}$ is done in the following manner. First, let $<$ be any well-ordering of A which has order type κ, and let d_ξ be the ξ-th element of A in this well-ordering. By the cardinality conditions imposed on σ and A, there are exactly κ subsets of A which are $\mathscr{L}(Q)$-definable. Let these be $\{D_\xi : \xi < \kappa\}$, and for each $\xi < \kappa$ let $f_\xi : D_\xi \to A$ be a one–one function onto an initial segment of A (which may, of course, be all of A). Now let $R \subseteq A^3$ be such that $R(a, b, c)$ holds iff there is $\xi < \kappa$ such that $c = d_\xi$, $b \in D_\xi$ and $a = f_\xi(b)$. It is now clear that $(\mathfrak{A}, R, <)$ is a κ-like model of σ. The problem of showing that $(\mathfrak{A}, R, <) \models \sigma_\phi$ involves merely another rather routine induction on formulas.

1.3. Applications of Reduction

In this subsection we will describe some applications of the specific reduction that was discussed in Subsection 1.2. We begin with the definition of transfer for cardinal-like models which is in complete analogy with the definitions of transfer given in Subsection 1.1.

1.3.1 Definition. Let λ, μ and κ_j, for $j \in J$, be infinite cardinals. Then $\{\kappa_j : j \in J\} \to \lambda$ *μ-compactly* iff every set of at most μ first-order sentences, each finite subset of which has a κ_j-like model for each $j \in J$, has a λ-like model.

We remark that by comparison with the corresponding definitions of transfer given in Subsection 1.1, the meaning of each of $\kappa \to \lambda$, $\kappa \to \lambda$ μ-compactly, and $\{\kappa_j : j \in J\} \to \lambda$ is obvious.

1.3.2 Proposition. *The following two statements are equivalent*:

(1) $\{\mathscr{L}(Q_{\alpha_j}) : j \in J\} \to \mathscr{L}(Q_\alpha)$ *μ-compactly*;
(2) $\{\aleph_{\alpha_j} : j \in J\} \to \aleph_\alpha$ *μ-compactly.*

Proof. We will first show that (1) implies (2). Actually, this is the trivial direction. Let T' be a set of at most μ first-order sentences each finite subset of which has an

$\aleph_{\alpha_j}$-like model, for each $j \in J$. Consider the set $T' \cup \{\forall x \neg Qy(y < x)\} \cup \{$"$<$ is a linear order"$\}$, and apply (1) to it.

We will now show that (2) implies (1). Clearly, if (2) holds, then we can assume $\mu < \aleph_\alpha$. Let T' be a set of at most μ $L(Q)$-sentences each finite subset of which has a model in each of the $\aleph_{\alpha_j}$-interpretations. By Lemma 1.2.3, for each finite $T'_0 \subseteq T'$ and each $j \in J$, we have that $T'_0 \cup \{\sigma_\phi : \phi \in T'_0\}$ has an $\aleph_{\alpha_j}$-like model; and by Lemma 1.2.2, this model is also a model of $\{\phi^* : \phi \in T'_0\}$. By (2), we thus have that $\{\phi^* : \phi \in T'\} \cup \{\sigma_\phi : \phi \in T'\}$ has an $\aleph_\alpha$-like model which, by Lemma 1.2.2, is also a model of T'. □

The preceding proposition and its proof remain valid even when both references to the phrase "μ-compactly" are deleted.

1.3.3 Definition. Let K be a class of structures and μ an infinite cardinal. Then K is *μ-compact* iff any set of not more than μ first-order sentences which is finitely satisfiable in K is also satisfiable in K. Moreover, K is *recursively enumerable for validity* iff for any recursive vocabulary τ the set of all first-order sentences valid in every τ-structure in K is recursively enumerable.

1.3.4 Corollary. *The following are equivalent:*

(1) *$\mathscr{L}(Q_\alpha)$ is μ-compact;*
(2) *$K(\aleph_\alpha)$ is μ-compact.*

Proof. The proof for this result follows immediately from Proposition 1.3.2 upon noting the following obvious equivalences: $\mathscr{L}(Q_\alpha)$ is μ-compact iff $\mathscr{L}(Q_\alpha) \to \mathscr{L}(Q_\alpha)$ μ-compactly; $K(\aleph_\alpha)$ is μ-compact iff $\aleph_\alpha \to \aleph_\alpha$ μ-compactly. □

It should be recognized that the Fuhrken reduction given in Subsection 1.2 is effective. That is to say, if τ is a recursive vocabulary, then both the functions $\phi \mapsto \phi^*$ and $\sigma \mapsto \sigma_\phi$ are recursive. This yields the following equivalence involving the recursive enumerability for validity of $\mathscr{L}(Q)$ under the cardinal interpretations.

1.3.5 Corollary. *The following are equivalent:*

(1) *$\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity;*
(2) *$K(\aleph_\alpha)$ is recursively enumerable for validity.*

Proof. That (1) implies (2) is trivial. For the argument that (2) implies (1), we merely note that any $\mathscr{L}(Q)$-sentence ϕ has a model in the $\aleph_\alpha$-interpretation just in case $\sigma_\phi \wedge \phi^*$ has a κ-like model. □

1.4. Two-Cardinal Models

In Subsection 1.2 we saw how to reduce $\mathscr{L}(Q)$ to cardinal-like structures. A further reduction to two-cardinal structures will be described in this subsection. The symbol U will always denote a unary relation symbol.

1.4.1 Definition. A structure $\mathfrak{A}$ is a (κ, λ)-structure if $|A| = \kappa$ and $|U| = \lambda$. Moreover, if $\kappa > \lambda$, then $\mathfrak{A}$ is a *two-cardinal* structure, and if $\kappa = \lambda^+ > \aleph_0$, then $\mathfrak{A}$ is a *gap*-1 two-cardinal structure. We will let $K(\kappa, \lambda)$ denote the class of (κ, λ)-structures.

1.4.2 Definition. $(\kappa_1, \lambda_1) \to (\kappa_2, \lambda_2)$ iff every first-order sentence which has a (κ_1, λ_1)-model also has a (κ_2, λ_2)-model.

We will leave the details of the remainder of this subsection as an easy, and yet instructive, exercise for the reader.

1.4.3 Proposition. *There is a first-order sentence σ in the vocabulary $\{<, U, S\}$, where S is a ternary relation symbol, such that*

(1) *if $\mathfrak{A} \models \sigma$ is κ-like, then for some λ, $\kappa = \lambda^+$ and $\mathfrak{A}$ is a (λ^+, λ)-structure;*
(2) *if $\mathfrak{A} \models \sigma$ is a two-cardinal (κ, λ)-structure, then $\kappa = \lambda^+$ and $\mathfrak{A}$ is κ-like;*
(3) *if τ is a vocabulary not including either $<$ or U or S, then*
   (i) *any λ^+-like $(\tau \cup \{<\})$-structure can be expanded to a model of σ, and*
   (ii) *any gap-1 two-cardinal $(\tau \cup \{U\})$-structure can be expanded to a model of σ.* ☐

Obvious consequences of Proposition 1.4.3 equate transfer for gap-1 two-cardinal models with the corresponding transfer for successor cardinal-like models. This immediately yields that for cardinals κ and μ, $K(\kappa^+, \kappa)$ is μ-compact iff $K(\kappa^+)$ is μ-compact. Similarly, $K(\kappa^+, \kappa)$ is recursively enumerable for validity iff $K(\kappa^+)$ is recursively enumerable for validity.

2. *The Classical Transfer Theorems*

This section contains what might be referred to as the classical transfer theorems. Included under this rubric is the earliest of the two-cardinal theorems—the fundamental one of Vaught. Also included are those results which are directly inspired by Vaught's result, namely the transfer theorems of Keisler, Chang, Fuhrken and R. B. Jensen. The reduction of the previous section will yield information about the logics $\mathscr{L}(Q)$ under various cardinality interpretations. Some applications and counterexamples are also included in this section. We will conclude this section with a discussion of gap-n and multi-cardinal transfer theorems.

2.1. The Gap-1 Transfer Theorems

The earliest of the gap-1 transfer theorems is the following one. This result, first proven by Vaught in Morley-Vaught [1962], has already been discussed in Chapters II and IV.

2.1.1 Theorem. *For any cardinal $\kappa \geq \aleph_0$, $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ $\aleph_0$-compactly.* □

A consequence of Theorem 2.1.1 is that $K(\aleph_1, \aleph_0)$ is $\aleph_0$-compact. Moreover, Vaught's proof of Theorem 2.1.1 shows that $K(\aleph_1, \aleph_0)$ is recursively enumerable for validity. Thus, the following corollary of Fuhrken [1964] and Vaught [1964] follows.

2.1.2 Corollary. *$\mathscr{L}(Q_1)$ is compact and recursively enumerable for validity.* □

Keisler's proof of Theorem 2.1.1 in [1966b] also yields Corollary 2.1.2. In fact, his proof results in an elegant and comprehensible axiomatization for the class $K(\aleph_1)$. Corollary 1.4.6 suggests that there should also be an axiomatization for $K(\aleph_1, \aleph_0)$. Such an axiomatization, although less elegant than that for $K(\aleph_1)$, was indeed obtained by Keisler [1966a].

Theorem 2.1.1 is equivalent to $\kappa^+ \to \aleph_1$ $\aleph_0$-compactly. Fuhrken [1965] noticed that the proof of Theorem 2.1.1 can be used to prove the following generalization.

2.1.3 Theorem. *For any regular $\kappa \geq \aleph_0$, $\kappa \to \aleph_1$ $\aleph_0$-compactly.* □

Yet another proof of Theorem 2.1.1 was given by Shelah [1978] using the method of identities. This method will be discussed in the next section.

The problem of the "converse" transfer of Theorem 2.1.1 was attacked by Chang [1965a] with notable partial success.

2.1.4 Theorem. *Assume* GCH. *For any regular cardinal κ, $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ κ-compactly.* □

One of the byproducts of Theorem 2.1.4—or of any other instance of $(\aleph_1, \aleph_0) \to (\aleph_{\alpha+1}, \aleph_\alpha)$—is that there is then a completeness theorem for $\mathscr{L}(Q_{\alpha+1})$ which is, of course, the same completeness theorem as the one for $\mathscr{L}(Q_1)$ that is given in Section IV.3.

2.1.5 Corollary. *Assume* GCH. *If $\aleph_\alpha$ is regular, then $\mathscr{L}(Q_{\alpha+1})$ is $\aleph_\alpha$-compact and recursively enumerable for validity.* □

In order to eliminate the requirement that κ be regular in the statement of Theorem 2.1.4, it is natural to replace the use of saturated models by special models. In fact, R. B. Jensen [1972] did just that, but only with an additional set-theoretic assumption which is a consequence of $V = L$.

2.1.6 Theorem. *Assume $V = L$. For any $\kappa \geq \aleph_0$, $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ κ-compactly.* □

2.1.7 Corollary. *Assume $V = L$. For any α, $\mathscr{L}(Q_{\alpha+1})$ is $\aleph_\alpha$-compact and recursively enumerable for validity.* □

We will end this subsection with an application to combinatorics. Shelah [1976a] proved the following result.

2.1.8 Theorem. *There is a linear order of power $\aleph_1$ whose square can be covered by countably many chains.* $\square$

We present the following simple exercise for the reader. Write down a first-order sentence σ with the property that for any cardinals $\kappa \geq \lambda \geq \aleph_0$, σ has a (κ, λ)-model iff there is a linear order of power κ whose square can be covered by λ chains. This done, the following, for example, becomes an immediate consequence.

2.1.9 Corollary. *Assume $V = L$. For any κ, there is a linear order of power κ^+ whose square can be covered by κ^+ chains.* $\square$

2.2. Trees: Some Applications

In this subsection an application and a counterexample, both of which are related to the previous subsection, will be presented. And both require special Aronszajn trees. Since trees will be useful at later points in this chapter, we will devote the first few paragraphs of the present discussion to the requisite definitions.

A *tree* is a partially ordered set $(A, <)$ such that the set of predecessors $\hat{a}$ of any element $a \in A$ is linearly ordered. Contrary to usual practice in set theory, we do not require that a tree be well-founded. A well-founded tree $(A, <)$ has associated with it a rank function rk, where $\mathrm{rk}(a)$ is the ordinal of the order type of $\hat{a}$. In the non-well-founded case there are no such intrinsic rank functions. However, we will overcome this deficiency by introducing *ranked trees* $(A, <, \preccurlyeq)$, where $\preccurlyeq$ is a quasi-order (that is, it is transitive, reflexive, and connected, although not necessarily anti-symmetric) on A such that $(A, <)$ is a tree and $(A, <, \preccurlyeq)$ satisfies the following two sentences:

$$x < y \rightarrow x \preccurlyeq y \wedge \neg y \preccurlyeq x,$$

$$x \preccurlyeq y \rightarrow \exists z(z \leq y \wedge x \preccurlyeq z \wedge z \preccurlyeq x).$$

A well-founded tree $(A, <)$ has a unique expansion to a ranked tree; and the rank order $\preccurlyeq$ is defined so that $a \preccurlyeq b$ iff $\mathrm{rk}(a) \leq \mathrm{rk}(b)$.

In order to make some definitions concerning ranked trees, we let $(A, <, \preccurlyeq)$ be an arbitrary ranked tree. For a regular cardinal κ, we say that $(A, <, \preccurlyeq)$ is a κ-*tree* if $|A| = \kappa$ and, for every $a \in A$. $|\{b \in A: b \preccurlyeq a\}| < \kappa$. A *branch* B of $(A, <, \preccurlyeq)$ is a maximal linearly ordered (by $<$) subset of A which has elements of arbitrarily high rank in the sense that for any $a \in A$ there is $b \in B$ such that $a \preccurlyeq b$. $(A, <, \preccurlyeq)$ is an *Aronszajn κ-tree* if it is a κ-tree which has no branches. At the other extreme, a κ-tree $(A, <, \preccurlyeq)$ is a *Kurepa κ-tree* if it has at least κ^+ branches. Suppose, now, that $(A, <, \preccurlyeq)$ is a λ^+-tree and that there is a function $f: A \rightarrow \lambda$ such that whenever

$x < y$ are elements of A, then $f(x) \neq f(y)$. Then $(A, <, \preccurlyeq)$ is an Aronszajn λ^+-tree. A λ^+-tree for which such a function exists is a *special* Aronszajn λ^+-tree.

The proof of the following result is left as an easy exercise for the reader.

2.2.1 Proposition. *There is a sentence σ of $\mathscr{L}(Q)$ such that for any regular cardinal κ the following are equivalent:*

(1) *there is a special Aronszajn κ-tree;*
(2) *there is a well-founded special Aronszajn κ-tree;*
(3) *there is a model for σ in the κ-interpretation.* ☐

The existence of an Aronszajn $\aleph_1$-tree was first established by Aronszajn. His construction actually produced a well-founded special Aronszajn $\aleph_1$-tree. The construction is well-known and can be found, for example, in Jech [1978].

2.2.2 Theorem. *There exists a special Aronszajn $\aleph_1$-tree.*

Later—although still prior to Chang's two-cardinal theorem—Specker [1949] proved the existence of special Aronszajn κ-trees, for some cardinals $\kappa > \aleph_1$. Assuming GCH, we can arrive at the same conclusion by use of Theorem 2.1.4.

2.2.3 Corollary. (1) *Assume GCH. If κ is regular, then there is a special Aronszajn κ^+-tree;*
(2) *Assume $V = L$. For any κ, there is a special Aronszajn κ^+-tree.* ☐

Special Aronszajn trees can be used to show the failure of two-cardinal transfer, or—to put it another way—the necessity of GCH in Chang's theorem (2.1.4). Mitchell [1972] proved the following consistency result concerning the non-existence of special Aronszajn trees. A different proof using iterated perfect set forcing, was developed by Baumgartner and Laver [1979].

2.2.4 Theorem. *If* Con(ZFC + "*there is a Mahlo cardinal*"), *then* Con(ZFC + "*there is no special Aronszajn $\aleph_2$-tree*"). ☐

2.2.5 Corollary. *If* Con(ZFC + "*there is a Mahlo cardinal*"), *then* Con(ZFC + "$\aleph_1 \not\to \aleph_2$"). ☐

Some further results along these lines, results which use generalizations of special Aronszajn trees, can be found in Schmerl [1974].

We will conclude this subsection with a result indicating that $V = L$ cannot be eliminated from the hypothesis of Jensen's theorem (2.1.6) unless there does not exist a certain kind of very large cardinal. This proof of Ben-David [1978a] and Shelah also makes use of trees.

2.2.6 Theorem. *If* Con(ZFC + GCH + "*there is a strongly compact cardinal*"), *then* Con(ZFC + GCH + "$\aleph_1 \nrightarrow \aleph_{\omega+1}$"). ▯

2.3. Gap-2 Transfer

The gap-1 transfer theorems of Section 2.1 suggest the possibility of "gap-2 transfer theorems", that is, theorems of the sort $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$. Rather simple versions need not be true. For example, if the continuum hypothesis fails and yet $2^\kappa = \kappa^+$, then $(\aleph_2, \aleph_0) \nrightarrow (\kappa^{++}, \kappa)$. Even the GCH is not a sufficient hypothesis, as we shall now see.

From the previous subsection recall the notion of a Kurepa κ-tree. The following straightforward proposition relates Kurepa trees with gap-2 models.

2.3.1 Proposition. *There is a sentence σ such that, for any regular cardinal κ, the following are equivalent:*

(1) *there is a Kurepa κ-tree;*
(2) *there is a well-founded Kurepa κ-tree;*
(3) *there is a (κ^{++}, κ)-model of σ.* ▯

This result can be used to find examples of failure of gap-2 transfer. This is exactly what was done by Silver [1971b] where the following is proven.

2.3.2 Theorem. *If* Con(ZFC + "*there is an inaccessible cardinal*"), *then* Con(ZFC + GCH + "*there is a Kurepa $\aleph_2$-tree but no Kurepa $\aleph_1$-tree*"). ▯

2.3.3 Corollary. *If* Con(ZFC + "*there is an inaccessible cardinal*"), *then* Con(ZFC + GCH + "$(\aleph_3, \aleph_1) \nrightarrow (\aleph_2, \aleph_0)$"). ▯

In Theorem 2.3.2 it would not be sufficient to assume the consistency of just ZFC, for Solovay has shown that if there are no Kurepa κ-trees, then κ^+ is inaccessible in the constructible universe L. A proof of this result can be found in Devlin [1973a]. In particular, if $V = L$, then for every regular κ there exists a Kurepa κ-tree. This suggests the truth of the gap-2 transfer theorem assuming that $V = L$. Indeed this was proven by R. B. Jensen. A proof of this can also be found in Devlin [1973a].

2.3.4 Theorem. *If $V = L$, then $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$ λ-compactly, for any infinite cardinals κ and λ.* ▯

The proof of Theorem 2.3.4 is quite difficult, using much of the intricate machinery of the fine structure of L. A simple proof by Burgess [1978a] yields just the consistency of gap-2 transfer relative to ZFC. A reduction of the type in Section 1 yields that $V = L$ implies, for example, that $\mathscr{L}(Q_1, Q_2)$ is $\aleph_0$-compact. The proof of Theorem 2.3.4 also shows that $V = L$ implies that $\mathscr{L}(Q_1, Q_2)$ is recursively enumerable for validity.

2.4. Gap-n and Multi-Cardinal Theorems

In order to generalize the gap-2 transfer of the previous subsection to gap-n, it will be useful to have the iterated successor function. For cardinal λ and ordinal $\alpha > 0$, let $\aleph_0(\lambda) = \lambda$ and $\aleph_\alpha(\lambda) = \sup\{(\aleph_\beta(\lambda))^+ : \beta < \alpha\}$. A gap-$n$ structure is an $(\aleph_n(\lambda), \lambda)$-structure, for some λ. We will use $U_1, U_2, U_3, \ldots$ to denote unary relations.

2.4.1 Definition. A structure $\mathfrak{A}$ is a $(\kappa_0, \kappa_1, \ldots, \kappa_n)$-structure if $|A| = \kappa_0, |U_i| = \kappa_i$ for $i = 1, 2, \ldots, n$ and (for the sake of orderliness), $\kappa_0 \geq \kappa_1 \geq \cdots \geq \kappa_n$.

$$(\kappa_0, \kappa_1, \ldots, \kappa_n) \to (\lambda_0, \lambda_1, \ldots, \lambda_n)$$

iff every sentence which has a $(\kappa_0, \kappa_1, \ldots, \kappa_n)$-model also has a $(\lambda_0, \lambda_1, \ldots, \lambda_n)$-model.

There are other notions of transfer for multi-cardinal models which are analogues of those in Definition 1.3.1.

Every gap-n theorem yields an ostensibly stronger multicardinal theorem. This is a consequence of the following observation which the reader should be able to prove.

2.4.2 Proposition. *For each $1 \leq n < \omega$ and each first-order sentence σ, there is a sentence σ' such that, for each infinite cardinal κ, the following are equivalent:*

(1) *σ has an $(\aleph_n(\kappa), \kappa)$-model;*
(2) *σ' has an $(\aleph_n(\kappa), \aleph_{n-1}(\kappa), \ldots, \kappa^+, \kappa)$-model.* □

The gap-2 theorem (2.3.4) has been extended by Jensen using techniques which are of such extreme difficulty that to date the proof remains unpublished, although it has been confirmed by rumor.

2.4.3 Theorem. *Assume $V = L$. For any $n < \omega$ and any infinite cardinals κ and λ, $(\aleph_n(\kappa), \kappa) \to (\aleph_n(\lambda), \lambda)$ λ-compactly.*

From Proposition 2.2.3 we can quite easily obtain the $\aleph_\alpha$-compactness of the logic $\mathscr{L}(Q_{\alpha+1}, Q_{\alpha+2}, \ldots, Q_{\alpha+n})$, assuming $V = L$. The proof of Theorem 2.4.3 also shows that $V = L$ implies that $\mathscr{L}(Q_1, Q_2, \ldots, Q_n)$ is recursively enumerable for validity.

At this point it is interesting to take note of the Lachlan multi-cardinal theorem for stable theories. The original proof is in Lachlan [1973] and a later, more simple proof can be found in Baldwin [1975].

2.4.4 Theorem. *Let T be a stable first-order theory which has a $(\kappa_0, \kappa_1, \ldots, \kappa_n)$-model, where $\kappa_0 > \kappa_1 > \cdots > \kappa_n$. Then T has a $(\lambda_0, \lambda_1, \ldots, \lambda_n)$-model whenever $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$.*

Some multi-cardinal theorems have applications to the calculations of Hanf numbers. This will be discussed at the end of Section 3.3.

3. Two-Cardinal Theorems and the Method of Identities

This section will examine a powerful approach to analyzing two-cardinal transfer and two-cardinal compactness. These developments will, of course, have important implications for the logics $\mathscr{L}(Q)$ and for the various cardinal interpretations via Proposition 1.3.2.

In its simplest form, this method is the familiar one of employing indiscernibles as generators in such a way that throughout a very tight control is maintained over the generated model. For example, subsequent to the original proof of Vaught's gap-ω theorem in Vaught [1965a], a result that is here formulated as Corollary 3.3.7, Keisler and Morley used indiscernibles obtained via the Erdös-Rado theorem (see Example 3.1.2 below) to give an alternate and more simple proof of that result. Generators which are only partially indiscernible can be used with nearly the same resulting tight control. Moreover, there is an added flexibility that guarantees that the distinguished subset and the model itself have the desired cardinality. It will be seen that identities are used as a sort of local description of the partition of the set of all finite subsets of a set.

3.1. Identities

We will begin this subsection with the definition of an identity and some rather closely related notions.

3.1.1 Definition. An *identity* I is an equivalence relation on $[D]^{<\omega}$, where D is a finite set, such that if $X, Y \in [D]^{<\omega}$ and XIY, then $|X| = |Y|$. The set D is the *domain* of I, and $|D|$ is the *length* of I.

In general, we will not distinguish between equivalent identities. Two identities I_1 and I_2 are *equivalent* if there is a bijection $\alpha: D_1 \to D_2$, where D_1 and D_2 are domains of I_1 and I_2, respectively, such that whenever $X, Y \in [D_1]^{<\omega}$, then XI_1Y iff $\alpha[X]I_2\alpha[Y]$. Thus, for example, we will consider there to be only countably many distinct, that is, inequivalent, identities. An identity I_1 is called a *subidentity* of I_2 if there is an injection $\alpha: D_1 \to D_2$ from the domain of I_1 to the domain of I_2 such that whenever $X, Y \in [D_1]^{<\omega}$, then XI_1Y implies $\alpha[X]I_2\alpha[Y]$.

Suppose that $f: [A]^{<\omega} \to B$ is a partition of $[A]^{<\omega}$, and suppose also that $D \in [A]^{<\omega}$. Then f *induces* the identity I with domain D if, whenever $X, Y \in [D]^{<\omega}$, then XIY iff both $f(X) = f(Y)$ and $|X| = |Y|$. The set of identities which are subidentities of those induced by f is denoted by $\mathscr{I}(f)$. For infinite cardinals $\kappa \geq \lambda$, let $\mathscr{I}(\kappa, \lambda)$ be the set of all identities I which are in $\mathscr{I}(f)$ whenever $f: [\kappa]^{<\omega} \to \lambda$.

There is an immediate simple observation to be made regarding $\mathscr{I}(\kappa, \lambda)$: These sets are monotone in κ and λ. Specifically, if $\kappa_1 \geq \kappa_2 \geq \lambda_2 \geq \lambda_1$, then $\mathscr{I}(\kappa_2, \lambda_2) \subseteq \mathscr{I}(\kappa_1, \lambda_1)$. The further apart κ and λ happen to be the larger will be $\mathscr{I}(\kappa, \lambda)$, and the closer together they are, the smaller will be $\mathscr{I}(\kappa, \lambda)$. Thus, $\mathscr{I}(\kappa, \kappa)$ is minimal.

In fact, for every κ, the set $\mathscr{I}(\kappa, \kappa)$ consists merely of the *trivial* identities. Conversely, whenever $\kappa > \lambda$, then $\mathscr{I}(\kappa, \lambda)$ contains some nontrivial identity, the simplest one being the identity I with domain 2 in which $\{0\}$ and $\{1\}$ are equivalent.

The previous example will be generalized in Example 3.1.3 by using the iterated successor function. A more instructive example is one which uses the iterated exponential function defined in the following manner:

$$\beth_0(\lambda) = \lambda,$$

$$\beth_{\alpha+1}(\lambda) = 2^{\beth_\alpha(\lambda)},$$

$$\beth_\beta(\lambda) = \bigcup_{\alpha<\beta} \beth_\alpha(\lambda),$$

where α is any ordinal and β a limit ordinal. When $\lambda = \aleph_0$ reference to λ will be surpressed, resulting in the standard $\beth_\alpha$ for $\beth_\alpha(\aleph_0)$. This example indicates how identities are to be used in place of indiscernibles when complete indiscernibility is not possible.

3.1.2 Example. If $\kappa \geq \beth_\omega(\lambda)$, then the partition theorem of Erdös and Rado (see Chang–Keisler [1977]) implies that $\mathscr{I}(\kappa, \lambda)$ is the set of all identities. More specifically, if $\kappa > \beth_n(\lambda)$, then all identities of length at most $(n + 2)$ are in $\mathscr{I}(\kappa, \lambda)$. Conversely, by the Erdös–Hajnal–Rado [1965] converse to the Erdös–Rado theorem, if $\lambda \leq \kappa \leq \beth_n(\lambda)$, then there is an identity of length $(n + 2)$ which is not in $\mathscr{I}(\kappa, \lambda)$. The missing identity is the one in which all sets of the same size are equivalent.

Finally, we note that the reader should see Subsection 2.4 for the definition of $\aleph_\alpha(\lambda)$.

3.1.3 Example. Let I_n be the identity, having domain $D_n = \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$, which is the equivalence relation that makes $X, Y \in [D_n]^{<\omega}$ equivalent iff either $X = Y$ or else, for each $i < n$, $|X \cap \{a_i, b_i\}| = |Y \cap \{a_i, b_i\}| \leq 1$. It is left as an interesting exercise to verify that $I_n \in \mathscr{I}(\kappa, \lambda)$ iff $\aleph_n(\lambda) \leq \kappa$.

Identities have a very close relationship with two-cardinal models. The proposition below indicates one direction of this relationship, the other direction being the deeper connection that is revealed in the next subsection by Theorem 3.2.1.

3.1.4 Proposition. *With each identity I one can effectively associate a first-order sentence σ_I such that whenever $\kappa \geq \lambda \geq \aleph_0$, then σ_I has a (κ, λ)-model iff $I \notin \mathscr{I}(\kappa, \lambda)$.*

Proof. Suppose that I is an identity of length n. The sentence σ_I will be in the vocabulary $\tau = \{U, f_1, f_2, \ldots, f_n\}$, where each f_i is an i-ary function symbol, and it will assert that each f_i is a function on the set of subsets of cardinality i, that the range of each f_i is included in U, and that the identity I is not a subidentity of one which is induced by the function $f_1 \cup \cdots \cup f_n$. It thus follows quite immediately from the definition of $\mathscr{I}(\kappa, \lambda)$ that the sentence σ_I has the required property. ☐

For instance, by applying Example 3.1.2 (or, respectively, Example 3.1.3) to the preceding proposition we obtain, for each $n < \omega$, an example of a sentence σ_n which has a (κ, λ)-model iff $\lambda \leq \kappa \leq \beth_n(\lambda)$ (or, respectively, $\lambda \leq \kappa \leq \aleph_n(\lambda)$).

3.2. The Two-Cardinal Compactness/Transfer Theorem

We now come to the fundamental two-cardinal compactness/transfer theorem, a result which was first enunciated by Shelah [1971d]. Some of its consequences will be given in the next subsection.

3.2.1 Theorem (The Two-Cardinal Compactness/Transfer Theorem). *Suppose that* $\kappa \geq \lambda$ *and that* $\kappa_j \geq \lambda_j$ *for each* $j \in J$. *Then each of the following is equivalent to each of the others*:

(1) $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$ $\aleph_0$-*compactly*;
(2) $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$ λ-*compactly*;
(3) *There exists a function* $f: [\kappa]^{<\omega} \to \lambda$ *such that* $\mathscr{I}(f) \subseteq \bigcup \{\mathscr{I}(\kappa_j, \lambda_j): j \in J\}$.

Proof. The implication (2) implies (1) is trivial. The implication (1) implies (3) is an easy consequence of Proposition 3.1.4. To see this, we let $\{I_i: i < \omega\}$ be the set of those identities not in each $\mathscr{I}(\kappa_j, \lambda_j)$. Let σ_{I_i} be the sentence from Proposition 3.1.4, so that each σ_{I_i} has a (κ_j, λ_j)-model, for each $j \in J$. Then, each finite subset of $\{\sigma_{I_i}: i < \omega\}$ has a (κ_j, λ_j)-model, for each $j \in J$. Thus, by (1) above, $\{\sigma_{I_i}: i < \omega\}$ has a (κ, λ)-model $(A, U, f_1, f_2, \ldots)$. Assuming that $A = \kappa$ and $U = \lambda$ both hold, we see that $f = \{f_i: i < \omega\}$ is the desired function.

The most interesting of the implications, and the one which demonstrates the real strength of identities, is the remaining one, (3) implies (2). Here, let T be a first-order theory in the vocabulary τ such that each finite subtheory $T_0 \subseteq T$ has a (κ_j, λ_j)-model for each $j \in J$. Because of the cardinality restrictions on T, it can be assumed that $|\tau| \leq \lambda$. The standard technique of adjoining Skolem functions can be used, so that we may as well assume that T is a Skolem theory. Thus, to every τ-formula $\phi(x_0, x_1, \ldots, x_{n-1}, y)$, there corresponds an n-ary term $\ell(x_0, \ldots, x_{n-1})$ in the vocabulary τ such that the sentence

$$\forall \bar{x}[\exists y \phi(\bar{x}, y) \to \phi(\bar{x}, \ell(\bar{x}))]$$

is a consequence of T.

The vocabulary τ will now be augmented by the adjunction of some constant symbols. For each $\xi < \lambda$, let b_ξ be a new individual constant; and, for each $\alpha < \kappa$, let c_α be a new individual constant, yielding the expanded vocabularies $\tau_1 = \tau \cup \{b_\xi: \xi < \lambda\}$ and $\tau_2 = \tau_1 \cup \{c_\alpha: \alpha < \kappa\}$. We will define a theory T_f in the expanded vocabulary τ_2 which depends only on the function $f: [\kappa]^{<\omega} \to \lambda$, whose existence is guaranteed by (3) and which consists of the following sentences:

(i) $b_\xi \neq b_\eta \quad (\xi < \eta < \lambda)$;
(ii) $U(b_\xi) \quad (\xi < \lambda)$;
(iii) $c_\alpha \neq c_\beta \quad (\alpha < \beta < \kappa)$;

(iv) $U(t(c_{\alpha_0}, \ldots, c_{\alpha_{m-1}})) \to t(c_{\alpha_0}, \ldots, c_{\alpha_{m-\tau}}) = t(c_{\beta_0}, \ldots, c_{\beta_{m-1}})$, where t is a τ_1-term, $\alpha_0 < \alpha_1 < \cdots < \alpha_{m-1} < \kappa$, $\beta_0 < \beta_1 < \cdots < \beta_{m-1} < \kappa$, and $f(\{\beta_0, \ldots, \beta_{m-1}\}) = f(\{\alpha_0, \ldots, \alpha_{m-1}\})$.

The key sentences are, of course, those occurring in (iv) above, and it should be noted that the terms t appearing there are τ_1-terms so that they may include some of the b_ξ.

There are two crucial facts about T_f that together will complete the proof of the theorem. The first is

Fact 1: Every minimal model of T_f is a (κ, λ)-model;

and the second is

Fact 2: $T \cup T_f$ is consistent.

By Fact 2, the theory $T \cup T_f$ has a model; and, by Fact 1, the minimal submodel of this model, which is also a model of T because T is a Skolem theory, is a (κ, λ)-model. It now remains to supply the proofs of these facts.

The proof of Fact 1 is very easy. Suppose that $\mathfrak{A} = (A, U, \ldots)$ is a minimal model of T_f. Then $|A| \geq \kappa$ holds, because of sentences (iii) above, and $|A| = \kappa$ since $\mathfrak{A}$ is a minimal model for a vocabulary τ_2, where $|\tau_2| \leq \kappa$. Thus, $|A| = \kappa$. Also, $|U| \geq \lambda$ holds, because of sentences (i) and (ii) above. Finally, to see that $|U| \leq \lambda$ holds, we observe that for each $b \in U$, there is some n-ary τ_1-term $t(x_0, \ldots, x_{n-1})$ and some $\xi < \lambda$ such that whenever $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ and $f(\{\alpha_0, \ldots, \alpha_{n-1}\}) = \xi$, then $\mathfrak{A} \models t(c_{\alpha_0}, \ldots, c_{\alpha_{n-1}}) = b$. Therefore, $|U| \leq \lambda$ must hold since $|\tau_1| \leq \lambda$ holds, thus showing that $\mathfrak{A}$ is a (κ, λ)-model.

To demonstrate that Fact 2 holds, that is, that $T \cup T_f$ is consistent, we will show that every finite subtheory $T_0 \subseteq T \cup T_f$ is consistent. Thus, let $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be the finite set consisting of those α for which c_α occurs in some sentence in T_0, where $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$. Then f induces an identity I with domain $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$. Statement (3) of the theorem implies the existence of some $j \in J$ for which $I \in \mathscr{I}(\kappa_j, \lambda_j)$. Let $\mathfrak{A}$ be a (κ_j, λ_j)-model of $T_0 \cap T$; such a model exists by the assumption on T.

Let $\xi_0, \xi_1, \ldots, \xi_s < \lambda_j$ be such that if b_ξ occurs in T_0, then ξ is among $\xi_0, \xi_1, \ldots, \xi_s$. Expand $\mathfrak{A}$ to a structure $\mathfrak{A}_1 = (\mathfrak{A}, b_{\xi_0}, b_{\xi_1}, \ldots, b_{\xi_s})$, where each of the b_{ξ_i} denote distinct elements of U. By very simple cardinality considerations, there is a function $g: [A]^{<\omega} \to \lambda_j$ such that whenever $\{a_0, a_1, \ldots, a_{n-1}\}, \{a'_0, a'_1, \ldots, a'_{n-1}\} \in [A]^n$, then $g(\{a_0, a_1, \ldots, a_{n-1}\}) = g(\{a'_0, a'_1, \ldots, a'_{n-1}\})$ iff

$$\begin{aligned}\mathfrak{A}_1 \models [U(t(a_0, \ldots, a_{n-1})) \vee U(t(a'_0, \ldots, a'_{n-1}))] &\to t(a_0, \ldots, a_{n-1}) \\ &= t(a'_0, \ldots, a'_{n-1})\end{aligned}$$

for each τ_1-term t occurring in T_0.

Recall that $I \in \mathscr{I}(\kappa_j, \lambda_j)$. Hence, there exists $D \subseteq A$ such that the injection $h: \{\alpha_0, \ldots, \alpha_{n-1}\} \to D$ demonstrates that I is a subidentity of the identity with domain D induced by g. Expand $\mathfrak{A}_1$ to the structure $\mathfrak{A}_2 = (\mathfrak{A}_1, c_{\alpha_0}, \ldots, c_{\alpha_{n-1}})$, where $c_{\alpha_i} = h(\alpha_1)$. Then $\mathfrak{A}_2$ is a model of T_0, thus demonstrating the consistency of T_0. $\square$

3.3. Some Consequences

The two-cardinal compactness/transfer theorem of the previous subsection has many consequences. This subsection will be devoted to the most interesting and important of them. One immediate consequence is that in statements (1) through (3) it always suffices to consider just some countable subset $J_0 \subseteq J$.

3.3.1 Corollary. *If* $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$ $\aleph_0$*-compactly, then for some countable* $J_0 \subseteq J$, $\{(\kappa_j, \lambda_j): j \in J_0\} \to (\kappa, \lambda)$ λ*-compactly.*

Proof. Consider statement (3) of Theorem 3.2.1. Since $\mathscr{I}(f)$ is countable, there is some countable $J_0 \subseteq J$ such that $\mathscr{I}(f) \subseteq \{\mathscr{I}(\kappa_j, \lambda_j): j \in J_0\}$. □

A function such as the one whose existence is asserted by clause (3) of Theorem 3.2.1 is called a *fundamental function for the relation* $\{(\kappa_j, \lambda_j): j \in J\} \to (\kappa, \lambda)$. If $f: [\kappa]^{<\omega} \to \lambda$ is a fundamental function for $(\kappa, \lambda) \to (\kappa, \lambda)$, then we will say simply that $f: [\kappa]^{<\omega} \to \lambda$ is *fundamental.* Thus, as is very easy to see, $f: [\kappa]^{<\omega} \to \lambda$ is fundamental iff $\mathscr{I}(f) = \mathscr{I}(\kappa, \lambda)$.

The statement that $(\kappa, \lambda) \to (\kappa, \lambda)$ μ-compactly is evidently equivalent to $K(\kappa, \lambda)$ being μ-compact. Thus, Theorem 3.2.1 yields the following corollary.

3.3.2 Corollary. *If* $\kappa \geq \lambda \geq \aleph_0$, *then each of the following is equivalent to each of the others*:

(1) $K(\kappa, \lambda)$ *is* $\aleph_0$*-compact*;
(2) $K(\kappa, \lambda)$ *is* λ*-compact*;
(3) *There is a fundamental function* $f: [\kappa]^{<\omega} \to \lambda$. □

The corollary thus characterizes compactness in terms of the purely combinatorial property of the existence of fundamental functions. In general, the question of the existence of fundamental functions remains unsolved. However, with some very mild restrictions imposed upon the cardinals, their existence can be easily demonstrated.

3.3.3 Lemma. *If* $\kappa > \lambda^{\aleph_0} = \lambda$, *then there is a fundamental function* $f: [\kappa]^{<\omega} \to \lambda$.

Proof. Let $\{I_n: n < \omega\}$ be the set of identities which are not in $\mathscr{I}(\kappa, \lambda)$. We pause at this point to observe that if this set is finite, or even empty, then things become even easier than is otherwise the case. For each n, let $f_n: [\kappa]^{<\omega} \to \lambda$ be such that $I_n \notin \mathscr{I}(f_n)$. Let $g: \lambda^\omega \to \lambda$ be a bijection. Define $f: [\kappa]^{<\omega} \to \lambda$ such that whenever $A \in [\kappa]^{<\omega}$ and $n < \omega$, then $f(A) = g(\langle f_n(A): n < \omega \rangle)$. We immediately see that $\mathscr{I}(f) = \mathscr{I}(\kappa, \lambda)$, so that f is fundamental. □

The Shelah-Fuhrken two-cardinal compactness theorem, a result which was first proven in Fuhrken [1965] with stronger hypotheses using ultraproducts and which was later improved in Shelah [1971d], is an instantaneous consequence of Lemma 3.3.3 and Corollary 3.3.2.

3.3.4 Corollary. *If $\kappa > \lambda^{\aleph_0} = \lambda$, then $K(\kappa, \lambda)$ is λ-compact.* $\square$

3.3.5 Corollary. *If $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then $\mathscr{L}(Q_{\alpha+1})$ is $\aleph_\alpha$-compact.*

Proof. See Corollaries 1.3.4 and 3.3.4. $\square$

Corollary 3.3.5 will be generalized later in Corollaries 4.2.1 and 5.1.3. There are instances of compactness of $\mathscr{L}(Q_{\alpha+1})$ not covered by Corollary 3.3.5, the most notable being $\mathscr{L}(Q_1)$ which is known to be $\aleph_0$-compact (see Chapter IV) even though $\aleph_0^{\aleph_0} > \aleph_0$. In fact, no example is known for even the consistency of the failure of $\aleph_0$-compactness of any $\mathscr{L}(Q_{\alpha+1})$. On the other hand, it is unknown whether it is a theorem of ZFC that $\mathscr{L}(Q_2)$ is $\aleph_0$-compact, although it does follow from ZFC + CH.

The two-cardinal compactness/transfer theorem has two transfer theorems as rather immediate corollaries. The first is the Chang–Keisler [1962] gap narrowing theorem, a result that was originally proven using ultrapowers, and the second is Vaught's gap-ω theorem, a result originally proven by Vaught [1965a] using self-extending models, a concept which will be discussed in Section 6.

3.3.6 Corollary. *If $\kappa \geq \mu \geq \lambda^{\aleph_0}$, then $(\kappa, \lambda) \to (\kappa, \mu)$ μ-compactly.*

Proof. Since $(\lambda^{\aleph_0})^{\aleph_0} = \lambda^{\aleph_0}$, we see from Lemma 3.3.3 that there is a fundamental function $f\colon [\kappa]^{<\omega} \to \lambda^{\aleph_0}$. Thus, we have that $\mathscr{I}(f) = \mathscr{I}(\kappa, \lambda^{\aleph_0}) \subseteq \mathscr{I}(\kappa, \lambda)$. Since $\lambda^{\aleph_0} \leq \mu$ holds, we can consider f to have range μ, so that $f\colon [\kappa]^{<\omega} \to \mu$ is fundamental for $(\kappa, \lambda) \to (\kappa, \mu)$. $\square$

3.3.7 Theorem. *If $\kappa \geq \lambda \geq \aleph_0$ and if $\kappa_n \geq \beth_n(\lambda_n)$ for each $n < \omega$, then $\{(\kappa_n, \lambda_n)\colon n < \omega\} \to (\kappa, \lambda)$ λ-compactly.*

Proof. Because of Example 3.1.3 any $f\colon [\kappa]^{<\omega} \to \lambda$ is fundamental for $\{(\kappa_n, \lambda_n)\colon n < \omega\} \to (\kappa, \lambda)$. $\square$

The following corollary can be extracted from the proof of Theorem 3.2.1.

3.3.8 Corollary. *If $K(\kappa, \lambda)$ is $\aleph_0$-compact, then $K(\kappa, \lambda)$ is recursively enumerable for validity iff $\mathscr{I}(\kappa, \lambda)$ is recursively enumerable.* $\square$

If $\kappa \geq \beth_\omega(\lambda)$, then $\mathscr{I}(\kappa, \lambda)$ is the set of all identities, and is therefore evidently recursive. This observation yields the following corollary.

3.3.9 Corollary. *If $\kappa \geq \beth_\omega(\lambda)$, then $K(\kappa, \lambda)$ is recursively enumerable for validity.* $\square$

The following three-cardinal theorem can be proven in a manner quite similar to the one that was used to prove Theorem 3.3.7.

3.3.10 Theorem. *If $\kappa \geq \aleph_1$ and if $\kappa_n \geq \beth_n(\lambda_n)$ and $\lambda_n > \mu_n$, for each $n < \omega$, then $\{(\kappa_n, \lambda_n, \mu_n)\colon n < \omega\} \to (\kappa, \aleph_1, \aleph_0)$ $\aleph_0$-compactly.* $\square$

An immediate consequence of this theorem, a consequence that can be obtained by setting each $\lambda_n = \aleph_1$ and $\mu_n = \aleph_0$, is that the Hanf number of $\mathscr{L}(Q_1)$ is $\beth_\omega$. All that was needed concerning $\mathscr{L}(Q_1)$ was the $\aleph_0$-compactness of $\mathscr{L}(Q_1)$. Thus, the more general result on Hanf numbers $h_\lambda(\mathscr{L}(Q_\alpha))$ can be proven by the same technique.

3.3.11 Theorem. *If $\mathscr{L}(Q_\alpha)$ is λ-compact, then $h_\lambda(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$.* □

Consequently, Proposition II.5.2.4 yields the following characterization.

3.3.12 Corollary. *$h_{\aleph_0}(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$ iff $\mathscr{L}(Q_\alpha)$ is $\aleph_0$-compact.* □

In particular, Corollary 3.3.5 implies some specific Hanf numbers.

3.3.13 Corollary. *If $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$, then $h_{\aleph_\alpha}(\mathscr{L}(Q_{\alpha+1})) = \beth_\omega(\aleph_\alpha)$.*

3.4. *Employing the Methodology of Identities*

The two-cardinal compactness/transfer theorem (3.2.1) suggests a method for proving specific two-cardinal transfer theorems. Suppose it is desired to prove the transfer $(\kappa_1, \lambda_1) \to (\kappa_2, \lambda_2)$ $\aleph_0$-compactly. Using the methodology of identities, we can employ the following three-step strategy:

(A) Define a set $\mathscr{I}_0$ of identities.
(B) Show that $\mathscr{I}_0 \subseteq \mathscr{I}(\kappa_1, \lambda_1)$.
(C) Show that there is a function $f\colon [\kappa_2]^{<\omega} \to \lambda_2$ such that $\mathscr{I}(f) \subseteq \mathscr{I}_0$.

This procedure has been used successfully by Shelah to prove several transfer theorems which will be discussed in this section. First, we will suggest an alternate proof of Vaught's theorem (2.1.1) that is due to Shelah [1978e]. In this proof we will only perform step (A), omitting steps (B) and (C) altogether. Second, we will discuss Shelah's transfer theorem $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$, which was proven in Shelah [1977]. We will consider only steps (A) and (C).

Vaught's Theorem. Our first task will be to define a set $\mathscr{I}_{\mathrm{Vau}}$ of identities. To do this, a method for building a new identity from an old one will now be described. Let I be an identity with domain $n \in \omega$, and let $E \subseteq n$. The *identity J obtained from I by duplicating E* is constructed as follows: The domain of J is $(n + m)$, where $m = |E|$. Let $\alpha\colon n + m \to n$ be the function such that $\alpha | n$ is the identity function on n and $\alpha | \{n, n + 1, \ldots, n + m - 1\}$ is an order-preserving bijection onto E. Now J is defined so that if X, $Y \in [n + m]^{<\omega}$, then XJY iff either $X = Y$ or each of the following three conditions is satisfied:

(1) $X \subseteq n$ or $X \cap E = \phi$;
(2) $Y \subseteq n$ or $Y \cap E = \phi$;
(3) $\alpha[X]\, I \alpha[Y]$.

Define $\mathscr{I}^*$ to be the smallest set of identities containing the identity with domain 1 and such that whenever $I \in \mathscr{I}^*$ has domain n and $k < n$, then the ordered identity obtained from I by duplicating $\{k, k+1, \ldots, n-1\}$ is in $\mathscr{I}^*$.

Then $\mathscr{I}_{\mathrm{Vau}}$ can now be defined. It is the smallest set of identities which is closed under the taking of subidentities and which also contains all identities I which are in $\mathscr{I}^*$.

3.4.1 Theorem. $\mathscr{I}(\aleph_1, \aleph_0) = \mathscr{I}_{\mathrm{Vau}}$. ☐

This approach to Vaught's theorem is interesting since it yields a description of the set $\mathscr{I}(\aleph_1, \aleph_0)$. Now Theorem 2.1.2 and Corollary 3.3.8 predict that $\mathscr{I}(\aleph_1, \aleph_0)$ is merely recursively enumerable. However, since $\mathscr{I}(\aleph_1, \aleph_0) = \mathscr{I}_{\mathrm{Vau}}$, and this latter set is evidently recursive, the following corollary results.

3.4.2 Corollary. *The set* $\mathscr{I}(\aleph_1, \aleph_0)$ *is recursive.* ☐

Shelah's Theorem. The three-step strategy is the only known method for proving the theorem of Shelah [1977] that $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$ $\aleph_0$-compactly. This theorem can be stated in a more general form for which a definition is required. For an infinite cardinal κ let $\mathrm{ded}^*(\kappa)$ be the least cardinal λ such that every (well-founded) ranked tree (see Section 2.2) of cardinality κ has fewer than λ branches. Note that $\mathrm{ded}^*(\aleph_0) = (2^{\aleph_0})^+$ and that $\kappa^+ < \mathrm{ded}^*(\kappa) \leq (2^\kappa)^+$. On the other hand, Mitchell [1972] has shown that $\mathrm{ded}^*(\aleph_1) \leq 2^{\aleph_1}$ is relatively consistent with ZFC.

3.4.3 Theorem. *If* $\mathrm{ded}^*(\lambda) > \kappa \geq \lambda$ *and if* $\kappa_n \geq \aleph_n(\lambda_n)$ *for each* $n < \omega$, *then* $\{(\kappa_n, \lambda_n)\colon n < \omega\} \to (\kappa, \lambda)$ λ*-compactly.*

In order to execute step (A), we will first define a set $\mathscr{I}^*$ of identities as the smallest set of identities containing the identity with domain 1, and such that whenever $I \in \mathscr{I}^*$ has domain n and $k < n$, then the identity obtained from I by duplicating $\{k\}$ is in $\mathscr{I}^*$. Then $\mathscr{I}_{\mathrm{She}}$ can now be defined as the smallest set of identities which is closed under the taking of subidentities and which contains all identities I which are in $\mathscr{I}^*$.

Having completed step (A), we will now proceed to develop a broad hint for Step (C). Let $(A, <)$ be a well-founded tree which has at least κ branches such that $|A| = \lambda$. Let B be a set of branches of $(A, <)$ of cardinality exactly κ. We will define a function f with domain $[B]^{<\omega}$. Suppose that $b_0, b_1, \ldots, b_n \in B$ are distinct branches. Then let α be the least ordinal such that the elements $a_0 \in b_0, a_1 \in b_1, \ldots, a_n \in b_n$ each of rank α are pairwise distinct. Finally, set $f(\{b_0, b_1, \ldots, b_n\}) = \{a_0, a\ , \ldots, a_n\}$. It is clear that the range of f has cardinality at most λ. A rather easy induction on n can be used to demonstrate that $\mathscr{I}(f) \subseteq \mathscr{I}_{\mathrm{She}}$.

The proof of step (B) can be found in Shelah [1977].

3.4.4 Corollary. *Suppose* $2^{\aleph_0} > \aleph_\omega$. *Then* $K(2^{\aleph_0}, \aleph_0)$ *is* $\aleph_0$*-compact and recursively enumerable for validity. In fact,* $\mathscr{I}(2^{\aleph_0}, \aleph_0)$ *is recursive.* ☐

It seems appropriate at this point to mention a closely related theorem that is due to Shelah [1975b], a result which is conveniently stated as a three-cardinal theorem.

3.4.5 Theorem. *For each $n < \omega$, let κ_n, p_n, q_n be cardinals such that $n^n \leq q_n^n \leq p_n < \aleph_0 \leq \kappa_n$. Also, let* $\mathrm{ded}^*(\lambda) > \kappa \geq \lambda$. *Then $\{(\kappa_n, p_n, q_n): n < \omega\} \to (\kappa, \kappa, \lambda)$ λ-compactly.* ☐

To prove this theorem, a modification of the aforementioned three-step procedure is used, steps (A) and (C) being almost exactly the same as in the proof of Theorem 3.4.3. A proof of a suitable version of step (B) can be given inside of Peano arithmetic, so the following corollary becomes a consequence of Theorem 3.4.5.

3.4.6 Corollary. *Let $\mathcal{M}$ be a model of Peano arithmetic and $I \subseteq M$ a proper initial segment closed under multiplication. Then, whenever* $\mathrm{ded}^*(\lambda) > \kappa \geq \lambda$ *there is a model $(\mathcal{N}, J) \equiv (\mathcal{M}, I)$ such that $|J| = \lambda$ yet every initial segment of $\mathcal{N}$ properly containing J has cardinality κ.* ☐

For the case in which $\kappa = 2^{\aleph_0}$ and $\lambda = \aleph_0$, this corollary was proven by Paris and Mills [1979]. Corollary 3.4.6 thus also follows from their result using Theorem 3.4.3 and some absoluteness considerations.

4. Singular Cardinal-like Structures

The topic of this section is the transfer theorem for singular cardinals which was obtained by Keisler [1968b]. This theorem and its proof have consequences concerning the compactness and recursive enumerability for validity of the language with cardinality quantifier Q_α with $\mathfrak{A}_\alpha$ a singular, strong limit cardinal.

4.1. Keisler's Transfer Theorem

In the following discussion Keisler's transfer theorem, which is the main result of this section, will be examined. To this purpose, we recall that a cardinal κ is a strong limit cardinal if $2^\lambda < \kappa$ whenever $\lambda < \kappa$. We will begin our development with a simple example limiting possible generalizations of the theorem.

4.1.1 Example. Let σ_1 be a first-order sentence in the vocabulary $\{<, R\}$, where R is a binary relation symbol, describing the fact that there is an injection of the universe in the power set of some proper initial segment. Then σ_1 has a κ-like model iff κ is not a strong limit cardinal.

4.1.2 Theorem (Keisler [1968b]). *Suppose that κ is a strong limit cardinal and that $\lambda > \mu \geq \aleph_0$, where λ is a singular cardinal. Then $\kappa \to \lambda$ μ-compactly.*

Only the initial portion of the proof will be presented here. Thus, suppose that τ is a vocabulary of cardinality at most μ which contains the n-ary Skolem function symbol f_ϕ for each $(n+1)$-ary τ-formula ϕ. Let $\tau' = \tau \cup \{c_{i,j}: i, j < \omega\}$, where the $c_{i,j}$ are new, distinct, individual constants. Define a set Γ to consist of the following τ'-sentences:

(1) $\forall \bar{x}[\exists y \phi(\bar{x}, y) \to \phi(\bar{x}, f_\phi(\bar{x}))]$, for each τ-formula ϕ;
(2) $c_{i,j} < c_{i,k}$, whenever $i < \omega$ and $j < k < \omega$;
(3) $t < c_{i,j}$, where t is any τ'-term that does not involve any constant $c_{k,n}$ with $k \geq i$;
(4) $\forall x_0, \ldots, x_{n-1}[x_0 < c_{i,r} \wedge \cdots \wedge x_{n-1} < c_{i,r}$
$$\to (\phi(\bar{x}, c_{m,j_0}, c_{m,j_1}, \ldots, c_{m,j_s}) \leftrightarrow \phi(\bar{x}, c_{m,k_0}, c_{m_{k_1}}, \ldots, c_{m,k_s}))],$$

whenever $i < m$, $j_0 < j_1 < \cdots < j_s$, $k_0 < k_1 < \cdots < k_s$ and $\phi(\bar{x}, \bar{y})$ is a τ'-formula which does not involve any $c_{p,q}$ for $p \leq m$.

There are now two crucial properties that must be verified:

(I) Every set of τ-sentences consistent with Γ has a λ-like model;
(II) Any τ-sentence which has a κ-like model is consistent with Γ.

We end with a hint that in order to prove property (II) above, it is necessary to apply the Erdös–Rado theorem several times.

4.2. Some Consequences

Theorem 4.1.2 and its proof yield some immediate consequences.

4.2.1 Corollary. *If $\aleph_\alpha$ is a singular, strong limit cardinal and $\aleph_0 \leq \lambda < \aleph_\alpha$, then $\mathscr{L}(Q_\alpha)$ is λ-compact.* ☐

By using a different approach to handle regular κ, we will see as a consequence of Theorem 5.1.3 that the requirement of singularity can be dropped in this corollary.

The upshot of (I) and (II) in the proof of Theorem 4.1.2 lies in the fact that if κ is a singular, strong limit cardinal and σ is a τ-sentence, then σ has a κ-like model iff σ is consistent with Γ. An inspection of the proof reveals that if τ is recursively enumerable, then so is Γ. Thus, the set of τ-sentences true in every κ-like model is recursively enumerable. This proves the following result.

4.2.2 Corollary. *If $\aleph_\alpha$ is a singular, strong limit cardinal, then $\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity.* ☐

As a consequence of Corollary 3.3.11, some more Hanf numbers can be computed.

4.2.3 Corollary. *If $\aleph_\alpha$ is a singular, strong limit cardinal and $\lambda < \aleph_\alpha$, then $h_\lambda(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$.* ☐

Corollaries 4.2.1 and 4.2.2 have immediate consequences with respect to the logic $\mathscr{L}(Q^C)$ involving the Chang quantifier. (See Chapter VI.) Recall that the syntax of this logic is the same as the syntax of the logic $\mathscr{L}(Q)$ with the cardinality quantifier, and its interpretation in the structure $\mathfrak{A}$ is that of $\mathscr{L}(Q)$ using the $|A|$-interpretation, with the restriction that $\mathfrak{A}$ be infinite.

4.2.4 Corollary. *Assume* GCH. *$\mathscr{L}(Q^C)$ is recursively enumerable for validity and is compact.*

Proof. $\mathscr{L}(Q_1)$ is recursively enumerable for validity according to Theorem 2.1.2, and so is $\mathscr{L}(Q_\omega)$ by Corollary 4.2.2, since by GCH $\aleph_\omega$ is a strong limit cardinal. Now, by Theorems 2.1.3 and 4.1.2, σ is valid for $\mathscr{L}(Q^C)$ iff it is valid for both $\mathscr{L}(Q_1)$ and $\mathscr{L}(Q_\omega)$. Hence, $\mathscr{L}(Q^C)$ is recursively enumerable for validity.

Let Σ be a set of κ sentences of $\mathscr{L}(Q^C)$ which is finitely consistent. Then either every finite $\Sigma_0 \subseteq \Sigma$ is consistent for $\mathscr{L}(Q_1)$, or every finite $\Sigma_0 \subseteq \Sigma$ is consistent for $\mathscr{L}(Q_\omega)$. Using Theorem 2.1.4 in the first case and Theorem 4.1.2 in the second, there is a model $\mathfrak{A}$ of Σ in the $\aleph_\alpha$-interpretation for appropriate $\aleph_\alpha > \kappa$. Since the Löwenheim number $l_\kappa(\mathscr{L}(Q_\alpha)) = \aleph_\alpha$, we can require that $|A| = \aleph_\alpha$. Thus, we have that $\mathfrak{A}$ is also a $\mathscr{L}(Q^C)$-model of Σ. $\square$

5. *Regular Cardinal-like Structures*

By means of more elaborate forms of identities, κ-like anologues of some of the results given in Section 3 can be obtained. The main interest occurs when κ is inaccessible. Some of these results will be discussed in this section.

5.1. *The Compactness/Transfer Theorem*

We will begin this discussion with the basic compactness/transfer theorem.

5.1.1 Theorem (The Regular Cardinal-like Compactness/Transfer Theorem). *Suppose that $\kappa > \aleph_0$ and that κ_j is regular for each $j \in J$. Then the following are equivalent:*

(1) $\{\kappa_j : j \in J\} \to \kappa$ *$\aleph_0$-compactly;*
(2) $\{\kappa_j : j \in J\} \to \kappa$ *λ-compactly, for each $\lambda < \kappa$.* $\square$

5.1.2 Corollary. *If $\aleph_\alpha > \lambda \geq \aleph_0$ and $\aleph_\alpha$ is regular, then*

$$\mathscr{L}(Q_\alpha) \text{ is } \aleph_0\text{-compact} \quad \text{iff} \quad \mathscr{L}(Q_\alpha) \text{ is } \lambda\text{-compact.}$$

A proof of Theorem 5.1.1 would yield the following instances of compactness as a consequence.

5.1.3 Corollary. *If $\aleph_\alpha > \lambda \geq \aleph_0$, for regular $\aleph_\alpha$, and $\aleph_\beta^{\aleph_0} < \aleph_\alpha$ for $\beta < \alpha$, then $\mathscr{L}(Q_\alpha)$ is λ-compact.* $\square$

Combining this result with Corollaries 2.1.7 and 4.2.1 yields the following general result.

5.1.4 Theorem. *Assume $V = L$. If $\aleph_\alpha > \lambda \geq \aleph_0$, then $\mathscr{L}(Q_\alpha)$ is λ-compact.* $\square$

This allows us to use Theorem 3.3.11 in computing Hanf numbers.

5.1.5 Theorem. *Assume $V = L$. If $\aleph_\alpha > \lambda \geq \aleph_0$, then $h_\lambda(\mathscr{L}(Q_\alpha)) = \beth_\omega(\aleph_\alpha)$.* $\square$

It is not known whether the $V = L$ hypothesis can be eliminated from Theorems 5.1.4 and 5.1.5.

A cardinal κ is 0-*Mahlo* iff it is inaccessible. For $\alpha > 0$, the cardinal κ is α-*Mahlo* if, whenever $\beta < \alpha$ and $C \subseteq \kappa$ is closed and unbounded, then there is a β-Mahlo cardinal in C. The cardinal κ is *strongly α-Mahlo* if it is strongly inaccessible in addition to being α-Mahlo. It is known that if κ is weakly compact, then κ is κ-Mahlo and also that there are many cardinals $\lambda < \kappa$ which are λ-Mahlo.

The following theorem was given a combinatorial proof in Schmerl [1972]. In this connection we point out that there is also the beautiful Silver–Kaufmann approach, which uses models of ZFC and which is detailed in Kaufmann [1983a].

5.1.6 Theorem. *For each $n < \omega$ there is an $\mathscr{L}(Q)$ sentence σ_n such that for each regular κ, σ_n is consistent in the κ interpretation iff κ is not strongly n-Mahlo.* $\square$

The following theorem of Schmerl and Shelah [1972] is a best possible result by Theorem 5.1.6.

5.1.7 Theorem. *For each $n < \omega$ let κ_n be strongly n-Mahlo, and let $\kappa > \lambda \geq \aleph_0$. Then $\{\kappa_n : n < \omega\} \to \kappa$ λ-compactly.* $\square$

One possible approach to proving this theorem uses generalizations of identities. For another approach, which uses self-extending models, see Theorem 6.1.3. Either approach enables us to obtain the following corollary.

5.1.8 Corollary. *If $\aleph_\alpha$ is strongly ω-Mahlo, then $\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity.* $\square$

It is not known whether the hypothesis of the corollary can be weakened. For example, whether or not $\mathscr{L}(Q_\alpha)$ is recursively enumerable for validity when $\aleph_\alpha$ is the first strongly inaccessible remains open. Indeed, it is not even known whether it is even *consistent* with ZFC that there be *any* $\alpha > 0$ for which $\mathscr{L}(Q_\alpha)$ is not recursively enumerable for validity.

5.2. Strongly Cardinal-like Structures

Suppose we consider the vocabulary having only the binary relation symbol $<$ and the sentence of stationary logic which is the conjunction of a sentence asserting that $<$ is a linear order and the sentence

$$\text{aa}\, s\, \exists x\, \forall y(y \in s \leftrightarrow y < x).$$

Then $(A, <)$ is a model of this sentence iff it is $\aleph_1$-like and there is a closed, unbounded subset of A which has order type ω_1. A well-ordered subset $X \subseteq A$ is closed and unbounded iff whenever $a \in X$ is a limit point, then a is the least upper bound of the set $\{x \in X: x < a\}$ in A. The next definition generalizes this type of ordering.

5.2.1 Definition. A linearly ordered set $(A, <)$ is *strongly κ-like*, where κ is a regular, uncountable cardinal, if it is κ-like and contains a closed, unbounded subset. A structure $\mathfrak{A} = (A, <, \ldots)$ is *strongly κ-like* if $(A, <)$ is strongly κ-like.

There is a reduction of $\mathscr{L}(\text{aa})$ to strongly $\aleph_1$-like structures.

5.2.2 Theorem. *With each sentence σ of $\mathscr{L}(\text{aa})$ we can effectively associate a first-order sentence σ^* such that the following are equivalent:*

(1) *σ is consistent;*
(2) *σ has a model of cardinality $\aleph_1$;*
(3) *σ^* has a strongly $\aleph_1$-like model.* □

In order to get the κ-interpretation, where κ is regular and uncountable, we consider the set $P_\kappa(A)$ which is the set consisting of just those subsets of A having cardinality $<\kappa$. A subset $C \subseteq P_\kappa(A)$ is *closed* if it is closed under the union of chains of length $< \kappa$, and it is *unbounded* if, for every $s \in P_\kappa(A)$, there is $t \in C$ such that $s \subseteq t$. Let $D_\kappa(A)$ be the filter generated by the closed unbounded subsets of $P_\kappa(A)$. The new clause in the definition of satisfaction in the κ-interpretation is now clear:

$$\mathfrak{A} \models \text{aa}\, s\, \phi(s) \quad \text{iff} \quad \{s \in P_\kappa(A): \mathfrak{A} \models \phi(s)\} \in D_\kappa(A).$$

Compare this definition with Definition IV.4.1.1. Stationary logic with the κ interpretation, where $\kappa = \aleph_\alpha$, will be denoted by $\mathscr{L}(\text{aa}_\alpha)$, so that $\mathscr{L}(\text{aa}_1) = \mathscr{L}(\text{aa})$.

The following transfer theorem becomes apparent upon checking that all the axioms for $\mathscr{L}(\text{aa})$ are valid in arbitrary $\mathscr{L}(\text{aa}_\alpha)$.

5.2.3 Theorem. *If $\aleph_\alpha > \aleph_0$ is regular, then $\mathscr{L}(\text{aa}_\alpha) \to \mathscr{L}(\text{aa}_1)$ $\aleph_0$-compactly.* □

Instead of proving transfer theorems of the form suggested by Theorem 5.2.3, we will concentrate on theorems concerning strongly κ-like structures. This is justified by the following two observations. The first is that in the κ-interpretation, the linearly ordered set $(A, <)$ is a model of the $\mathscr{L}(\mathrm{aa})$ sentence displayed at the beginning of this subsection iff $(A, <)$ is strongly κ-like. In the second observation we state a theorem whose proof is identical to the proof of Theorem 5.2.2.

5.2.4 Theorem. *With each sentence σ of $\mathscr{L}(\mathrm{aa})$ we can effectively associate a first-order sentence σ^* such that for each regular uncountable κ, the following are equivalent:*

(1) *in the κ-interpretation, σ has a model of cardinality κ;*
(2) *σ^* has a strongly κ-like model.* $\square$

In light of the above, the next definition is natural.

5.2.5 Definition. For regular uncountable cardinals, κ and λ, $\kappa \underset{s}{\rightarrow} \lambda$ if whenever σ is a first-order sentence which has a strongly κ-like model, then σ has a strongly λ-like model.

The customary variations on the above definition will be in force. For example, for regular uncountable κ, Theorem 5.2.3 implies that $\kappa \underset{s}{\rightarrow} \aleph_1$ $\aleph_0$-compactly.

The following theorem is the compactness/transfer theorem for strongly cardinal-like models. Its proof resembles the proofs of Theorems 3.2.1 and 5.1.1, although it does use an even more elaborate notion of identity.

5.2.6 Theorem. *Suppose that κ and κ_j are regular, uncountable cardinals, for each $j \in J$, such that for each $n < \omega$ there is some $j \in J$ for which $\kappa_j \geq \aleph_n$. Then the following are equivalent*

(1) *$\{\kappa_j : j \in J\} \underset{s}{\rightarrow} \kappa$ $\aleph_0$-compactly;*
(2) *$\{\kappa_j : j \in J\} \underset{s}{\rightarrow} \kappa$ λ-compactly for each $\lambda < \kappa$.* $\square$

Many corollaries of the same sort as those derived from Theorems 3.2.1 and 5.1.1 can be derived from this theorem. We will mention only one of them here.

5.2.7 Corollary. *If $\kappa > \lambda \geq \aleph_\omega$, κ is regular, and $\mu^{\aleph_0} < \kappa$ for each $\mu < \kappa$, then the class of strongly κ-like structures is λ-compact.* $\square$

The subtle hierarchy of cardinals was defined in Baumgartner [1975] and in Schmerl [1976]. A cardinal κ is *subtle* iff whenever $\langle S_\alpha : \alpha < \kappa \rangle$ is such that each $S_\alpha \subseteq \alpha$ and whenever $C \subseteq \kappa$ is closed and unbounded, then there are $\alpha < \beta$, both in C, such that $S_\beta \cap \alpha = S_\alpha$. Subtle cardinals are large in the sense that they are all strongly inaccessible. And yet, the first one—if it exists—is far larger than the first strongly inaccessible. For each ordinal α, we will define α-subtle cardinals with 0-subtle cardinals being regular, uncountable cardinals and 1-subtle cardinals being the same as subtle cardinals. However, we will be even more general than this by defining what is meant by a subset $X \subseteq \kappa$ being α-subtle. To this end, let us assume that κ is a regular, uncountable cardinal and $X \subseteq \kappa$. Then X is 0-*subtle* iff X is stationary. Inductively, X is $(\alpha + 1)$-*subtle* iff whenever $\langle S_\nu : \nu < \kappa \rangle$ is such that each $S_\nu \subseteq \nu$, then

$$\{\mu \in X : \{\nu \in X \cap \mu : S_\nu = \nu \cap S_\mu\} \text{ is } \alpha\text{-subtle}\}$$

is stationary. If α is a limit ordinal, then X is α-*subtle* provided it is β-subtle for each $\beta < \alpha$. The cardinal κ is α-*subtle* if it is α-subtle when considered as a subset of itself.

The following theorem was proven in Schmerl [1976] using combinatorial techniques. However, for a much easier proof which uses models of set theory, see Kaufmann [1983a].

5.2.8 Theorem. *For each $n < \omega$, there is a first-order sentence σ_n such that for each regular, uncountable κ, σ_n has a strongly κ-like model iff κ is not n-subtle.* $\square$

Can this theorem be extended, for example, by finding a sentence σ which has a strongly κ-like model iff κ itself is not ω-subtle? The answer is no because of the following theorem which is the analogue of Theorems 3.3.7 and 5.1.7. A proof of this result will be given in Section 6.

5.2.9 Theorem. *For each n let κ_n be an n-subtle cardinal and $\kappa > \lambda \geq \aleph_0$, where κ is regular. Then $\{\kappa_n : n < \omega\} \underset{s}{\to} \kappa$ λ-compactly.* $\square$

6. *Self-extending Models*

Models which have canonical, internal proper elementary extensions of themselves will be considered in this section. By iterating these extensions many times, taking unions at limit stages, we can construct models with particular properties This method will be discussed in Section 6.1 where alternate proofs of Theorems 3.3.7, 5.1.7 and 5.2.9 will be indicated. This technique will be exploited in Subsection 6.2 to prove the MacDowell–Specker–Shelah theorem.

6.1. *Self-Extending Theories*

Consider the language $\mathscr{L}(Q)$, and consider a consistent theory T in this language which has the following two properties:

(1) *T is a Skolem theory*: For every formula $\phi(x_0, \ldots, x_{n-1}, y)$ there is a term $\ell(x_0, x_1, \ldots, x_{n-1})$ such that

$$\forall \bar{x}(\exists y \phi(\bar{x}, y) \to \phi(\bar{x}, \ell(\bar{x})))$$

is in T;

(2) *Q behaves as a nonprincipal ultrafilter*: All universal closures of formulas of the following form are in T:

$$Qx\phi(x) \leftrightarrow \neg Qx \neg \phi(x),$$

$$\forall y \neg Qx(x = y),$$

$$(\phi(x) \rightarrow \psi(x)) \rightarrow (Qx\phi(x) \rightarrow Qx\psi(x)),$$

$$Qx\phi(x) \wedge Qx\psi(x) \rightarrow Qx(\phi(x) \wedge \psi(x)).$$

A model of T has the form $(\mathfrak{A}, q)$, where q is a collection of subsets of A with the obvious additional clause needed in the definition of satisfaction:

$$(\mathfrak{A}, q) \models Qx\phi(x) \quad \text{iff} \quad \{a \in A : (\mathfrak{A}, q) \models \phi(a)\} \in q.$$

A model $(\mathfrak{A}, q)$ of T is *reduced* if every set in q is definable. Since replacing q by the subset of itself which consists only of definable sets does not alter the satisfaction relation, we can always assume that models of T are reduced.

There is a canonical elementary extension of $(\mathfrak{A}, q)$ which is obtained by a modified ultrapower construction. Let B be the set of definable functions $f: A \rightarrow A$ considered modulo q. That is, two definable functions $f, g: A \rightarrow A$ are to be considered as equal if $(\mathfrak{A}, q) \models Qx(f(x) = g(x))$. There is a unique reduced structure $(\mathfrak{B}, r)$ such that for any formula $\phi(x_0, \ldots, x_{n-1})$ and all functions $f_0, f_1, \ldots, f_{n-1} \in B$,

$$(\mathfrak{B}, r) \models \phi(f_0, \ldots, f_{n-1}) \quad \text{iff} \quad (\mathfrak{A}, q) \models Qx\phi(f_0(x), \ldots, f_{n-1}(x)).$$

The set r consists of all those sets of the form

$$\{g \in B : (\mathfrak{A}, q) \models Qx\phi(f_0(x), \ldots, f_{n-1}(x), g(x))\},$$

where $(\mathfrak{A}, q) \models QxQy\phi(f_0(x), \ldots, f_{n-1}(x), y)$. The structure $(\mathfrak{B}, r)$ is an elementary extension of $(\mathfrak{A}, q)$ if the elements of A are identified with the constant functions. Thus, the following definition is appropriate.

6.1.1 Definition. A consistent theory T satisfying (1) and (2) above is called a *self-extending* theory.

One important fact about the canonical extensions of models of a self-extending theory is that "large sets become larger." To make this precise, let $\imath: A \rightarrow A$ be the identity function so that if $(\mathfrak{B}, r)$ is the canonical extension of the model $(\mathfrak{A}, q)$ of a self-extending theory, then

$$(\mathfrak{B}, r) \models Qx\phi(x, \bar{a}) \rightarrow \phi(\imath, \bar{a}),$$

for any formula ϕ and $a_0, a_1, \ldots, a_{n-1} \in A$.

These self-extending theories can be applied to give alternate proofs of Theorems 3.3.7, 5.1.6 and 5.2.9. We state the relevant results in this regard.

6.1.2 Theorem. *Let T be a first-order theory such that, for each $n < \omega$, there are cardinals κ, λ, with $\kappa > \beth_n(\lambda)$, and a (κ, λ)-model of T. Then T can be extended to a self-extending theory which contains all universal closures of formulas of the form*

$$Qx\, \exists y(\phi(x, y) \wedge U(y)) \rightarrow \exists y Qx \phi(x, y). \quad \square$$

6.1.3 Theorem. *Let T be a first-order theory such that, for each $n < \omega$, there is a strongly n-Mahlo cardinal κ and a κ-like model of T. Then T can be extended to a self-extending theory which contains all universal closures of formulas of the form*

$$\forall z Qx\, \exists y(\phi(x, y) \wedge y < z) \rightarrow \exists y Qx \phi(x, y). \quad \square$$

Actually, a theorem which was first proven in Schmerl [1976] and which is slightly stronger than Theorem 5.2.9, will be considered here. In order to state it, we need the following

6.1.4 Definition. Let κ be a regular uncountable cardinal and $X \subseteq \kappa$. A linearly ordered set $(A, <)$ is *(κ, X)-like* if it is κ-like and there is an increasing function $e: X \rightarrow A$ such that whenever $\alpha \in X$ and $\alpha = \sup(\{v \in X: v < \alpha\}) \in X$, then $e(\alpha) = \sup(\{e(v): v \in X \cap \alpha\})$. A structure $\mathfrak{A} = (A, <, \ldots)$ is *(κ, X)-like* if $(A, <)$ is (κ, X)-like.

From this definition we see that $\mathfrak{A}$ is strongly κ-like iff it is (κ, κ)-like.

6.1.5 Theorem. *Suppose κ is a regular uncountable cardinal and T is a first-order theory such that $|T| < \kappa$. Also assume that, for each $n < \omega$, there is a cardinal κ_n and an n-subtle $X \subseteq \kappa_n$ such that that T has a (κ_n, X)-like model. Then T has a strongly κ-like model.*

In order to prove this theorem using self-extending models, we need

6.1.6 Theorem. *Let T be a first-order theory such that for each $n < \omega$ there is a cardinal κ, an n-subtle $X \subseteq \kappa$, and a (κ, X)-like model of T. Then T can be extended to a self-extending theory which contains the universal closures of all formulas of the form*

$$Qx\, \exists y(\phi(x, y) \wedge y < x) \rightarrow \exists y Qx \phi(x, y).$$

To see just how Theorems 6.1.2, 6.1.3 and 6.1.6 imply the corresponding transfer theorems, let us focus attention on Theorem 6.1.6 alone as a typical example. Suppose that T is a first-order theory satisfying the hypothesis of Theorem 6.1.6. Thus, according to that theorem, T can be extended to a self-extending theory T' containing the required sentences. Without loss of generality, we can require that

$|T'| = |T| + \aleph_0$. The sentences in T' imposed by Theorem 6.1.6 guarantee that the canonical extension of any model of T' is an end-extension. Furthermore, this extension has a least new element. Thus, in order to form a strongly λ-like model of T, where $\lambda > |T| + \aleph_0$ is regular, we begin with a model $(\mathfrak{A}_0, q_0)$ of T' with $|A_0| < \lambda$. We then form an increasing chain of models $\langle(\mathfrak{A}_\nu, q_\nu): \nu \leq \lambda\rangle$ by letting $(\mathfrak{A}_{\nu+1}, q_{\nu+1})$ be the canonical extension of $(\mathfrak{A}_\nu, q_\nu)$, and by letting $(\mathfrak{A}_\nu, q_\nu)$ be the union of the previously constructed structures if ν is a limit ordinal. Then $\mathfrak{A}_\lambda$ is a λ-like model of T. In order to see that it is strongly λ-like, we let a_ν be the least new element in the extension $(\mathfrak{A}_{\nu+1}, q_{\nu+1})$ of $(\mathfrak{A}_\nu, q_\nu)$. Thus, $A_\nu = \{x \in A_\lambda : x < a_\nu\}$. Then $\{a_\nu : \nu < \lambda\}$ is a closed subset of A_λ, demonstrating that $(A_\lambda, <)$ is strongly λ-like.

In order to see how to prove Theorems 6.1.2, 6.1.3 and 6.1.6, we will again consider Theorem 6.1.6 as a typical example. Our aim here is to show that T is consistent with some theory, call it T', so by compactness we can assume that T is countable, and then consider some finite $T_0 \subseteq T'$ and show the consistency of just $T \cup T_0$. To this end, we choose an $n < \omega$ which is sufficiently large (depending on T_0) and let $\mathfrak{A}_{n+1}$ be a (κ_{n+1}, X_{n+1})-like model of T, where X_{n+1} is an $(n + 1)$-subtle subset of κ_{n+1}. Moreover, let $e: X_{n+1} \to A_{n+1}$ be the function which demonstrates that $(A_{n+1}, <)$ is (κ_{n+1}, X_{n+1})-like. Inductively, we will thus obtain structures $\mathfrak{A}_n, \mathfrak{A}_{n-1}, \ldots, \mathfrak{A}_0$ and $\mathfrak{B}_n, \mathfrak{B}_{n-1}, \ldots, \mathfrak{B}_0$. Each $\mathfrak{A}_i$ will be an expansion of $\mathfrak{B}_i$, and A_i will be an initial segment determined by an element $e(\kappa_i)$, where $\kappa_i \in X_{n+1}$; that is,

$$A_i = \{x \in A_{i+1} : x < e(\kappa_i)\}.$$

In order to get $\mathfrak{A}_n$ and $\mathfrak{B}_n$, let $\{\phi_\nu(v_0): \nu < \kappa_{n+1}\}$ be a nonrepeating list of all formulas with one free variable v_0 in the vocabulary of $\mathfrak{A}_{n+1}$ allowing parameters from A_{n+1}. There is a closed unbounded subset $C \subseteq \kappa_{n+1}$ such that whenever $\alpha \in C \cap X_{n+1}$ and $\phi_\nu(v_0)$ involves only parameters from the set $\{b \in A_{n+1} : b < e(\alpha)\}$, then $\nu < \alpha$. For each $\alpha \in C \cap X_{n+1}$, we let

$$S_\alpha = \{\nu < \alpha : \mathfrak{A}_{n+1} \models \phi_\nu(e(\alpha))\}.$$

We can also assume that if $\alpha \in C \cap X_{n+1}$, then $\mathfrak{A}_{n+1} | \{x \in A_{n+1} : x < e(\alpha)\} \prec \mathfrak{A}_{n+1}$. Using the definition of the subtle hierarchy, we find $\kappa_n \in C \cap X_{n+1}$ such that if

$$X_n = \{\nu \in X_{n+1} \cap \kappa_n : S_\nu = \nu \cap S_{\kappa_n}\},$$

then X_n is an n-subtle subset of κ_n. Let $A_n = \{b \in A_{n+1} : b < e(\kappa_n)\}$, and let $\mathfrak{B}_n = \mathfrak{A}_{n+1} | A_n$ so that $\mathfrak{B}_n \prec \mathfrak{A}_{n+1}$. The important fact to notice here is that, for any $\nu \in X_n$, both $e(\nu)$ and $e(\kappa_n)$ realize the same type over $\{b \in A_{n+1} : b < e(\nu)\}$.

Now let $\mathscr{D}$ be the collection of subsets D which are definable in $\mathfrak{A}_{n+1}$ using only parameters from A_n and for which $e(\kappa_n) \in D$. Now, expand $\mathfrak{B}_n$ to a structure $\mathfrak{B}'_n$ by adjoining a binary relation R_n so that

$$\{\{x \in A_n : \mathfrak{B}'_n \models R_n(b, x)\} : b \in A_n\} = \{D \cap A_n : D \in \mathscr{D}\}.$$

Let $\mathfrak{A}_n$ be the expansion of $\mathfrak{B}'_n$ obtained by adjoining all Skolem functions. The structure $\mathfrak{A}_n$ is (κ_n, X_n)-like for n-subtle $X_n \subseteq \kappa_n$.

The remainder of the $\mathfrak{A}_i$ and $\mathfrak{B}_i$ are constructed in exactly the same fashion. Having finally obtained $\mathfrak{A}_0$, we let $q = \{x \in A_0 : \mathfrak{A}_0 \models R_0(b, x),\ b \in A_0\}$. The structure $(\mathfrak{A}_0, q)$ is clearly a model of T and, without much difficulty, it can be shown to be a model of T_0 also. This demonstrates the consistency of $T \cup T_0$.

6.2. The MacDowell–Specker–Shelah Theorem

Our concern in this subsection is to use self-extending models to prove the following theorem.

6.2.1 Theorem. *If* $\aleph_0 \leq \mu < \lambda$, *then* $\aleph_0 \rightarrow \lambda$ *μ-compactly.* $\square$

Fuhrken [1965] observed that this theorem is a direct consequence of the well-known theorem of MacDowell and Specker [1961] which asserts that every model of Peano arithmetic has a proper, elementary end-extension. There are two features of Peano arithmetic that are used in the MacDowell–Specker theorem. One is that there is a definable pairing function which allows the coding of finite sequences. The other is that the induction scheme is true in Peano arithmetic, where by the *induction scheme* is meant the sentence

"$<$ is a linear order with a first but no last element"

together with all sentences which are universal closures of formulas of the form

$$[\exists x\phi(x) \wedge \forall x\, \exists y(\phi(x) \rightarrow \phi(y) \wedge x < y)] \rightarrow \forall x\, \exists y(\phi(y) \wedge x < y).$$

In words, this simply asserts that every nonempty definable set with no largest element is cofinal.

Shelah [1978b] showed that only the induction scheme is necessary. Notice that if we extend a theory which satisfies the induction scheme by adjoining all definable terms, then the extended theory is a Skolem theory. Thus, we will consider such theories to be already Skolem theories.

6.2.2 Theorem. *Let T be a consistent, countable first-order theory which satisfies the induction scheme. To each first-order formula $\phi(x_0, x_1, \ldots, x_{n-1}, y)$ there is associated another first-order formula $\sigma_\phi(x_0, x_1, \ldots, x_{n-1})$ such that T can be extended to a self-extending theory which contains the universal closures of all formulas of the form*

$$Qy\phi(\bar{x}, y) \leftrightarrow \sigma_\phi(\bar{x})$$

and of the form

$$\forall z Qx\, \exists y(\phi(x, y) \wedge y < z) \rightarrow \exists y Qx\phi(x, y).$$

Theorem 6.2.1 follows from this theorem. Furthermore, any model of the induction scheme in a countable vocabulary has a proper, elementary end-extension.

Proof. The first step in the proof is to observe that, for each $n < \omega$, there is a $2n$-ary formula $\psi_n(x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1})$—which we will abbreviate by $\bar{x} <_n \bar{y}$—which defines a linear order on the set of n-tuples and which satisfies the induction scheme. These formulas can be obtained inductively by letting $<_1$ be $<$ and then allowing $\bar{x} <_{n+1} \bar{y}$ to be the formula

$$\begin{aligned}&(\max(x_0, \ldots, x_n) < \max(y_0, \ldots, y_n)) \vee [\max(x_0, \ldots, x_n)\\&\quad = \max(y_0, \ldots, y_n) \wedge (x_n < y_n \vee (x_n = y_n \wedge ((x_0, \ldots, x_{n-1})\\&\quad <_n (y_0, \ldots, y_{n-1})))))].\end{aligned}$$

Now consider a sequence $\langle \phi_n(\bar{x}, y) : n < \omega \rangle$ of all formulas, where ϕ_n has its free variable among $x_0, x_1, \ldots, x_n, y$. Our object is to find formulas $\sigma_n(\bar{x})$ and at the same time formulas $\theta_n(y)$ such that the following are all consequences of T:

$$\forall w\, \exists y > w \theta_n(y),$$

$$\theta_{n+1}(y) \to \theta_n(y),$$

$$\exists w\, \forall y > w(\theta_n(y) \to (\phi_n(\bar{x}_1 y) \leftrightarrow \sigma_n(\bar{x})).$$

We will proceed by induction on n. For convenience, we will let $\theta_{-1}(y)$ be $y = y$. Having $\theta_{n-1}(y)$ and $\sigma_n(\bar{x})$, we easily find an appropriate $\theta_n(y)$. For example, let $\theta_n(y)$ be

$$\begin{aligned}&\theta_{n-1}(y) \wedge \exists z < y[\forall \bar{x} <_{n+1} z^{n+1}(\sigma_n(\bar{x}) \leftrightarrow \phi_n(\bar{x}, y)\\&\quad \wedge \forall w((\forall \bar{x} <_{n+1} z^{n+1}(\sigma_n(\bar{x}) \leftrightarrow \phi_n(\bar{x}, w) \wedge \theta_{n-1}(w))\\&\quad \to w \leq z \vee y \leq z))],\end{aligned}$$

where by z^{n+1} is meant the $(n + 1)$-tuple $(z, z, \ldots, z)$.

We have now reached the crux of the proof: To define $\sigma_n(\bar{x})$, knowing $\theta_n(y)$. Let $E(\bar{x}, y, z)$ be the formula

$$\forall \bar{w} <_{n+1} \bar{x}(\phi_n(\bar{w}, y) \leftrightarrow \phi_n(\bar{w}, z)).$$

For fixed $\bar{x}$, the formula $E(\bar{x}, y, z)$ defines an equivalence relation with only "boundedly" many equivalence classes. As $\bar{x}$ gets larger (in the sense of $<_{n+1}$), then the corresponding equivalence relation gets finer. Thus, the formula $E(\bar{x}, y, z)$ can be viewed as defining a tree, the nodes of rank $\bar{x}$ being the equivalence classes of the equivalence relation corresponding to $\bar{x}$. For each rank $\bar{x}$, there is an equivalence class containing an unbounded set of elements all of which satisfy θ_n. Call such an equivalence class *large*. Then, the following formula $L(\bar{x}, y, z)$ will assist us in selecting a canonical large equivalence class of each rank:

$$\exists \bar{w} <_{n+1} \bar{x}(\neg \phi_n(\bar{w}, y) \wedge \phi_n(\bar{w}, z) \wedge E(\bar{w}, y, z)).$$

The formula $L(\bar{x}, y, z)$ linearly orders the equivalence classes of rank $\bar{x}$. Thus, we let $S(\bar{x}, y)$ be a formula selecting the first large one. Thus, let $S(\bar{x}, y)$ be

$$\begin{aligned}&\forall w\, \exists z > w(\theta_n(z) \wedge E(\bar{x}, y, z)) \wedge \forall v(L(\bar{x}, v, y)\\ &\qquad \rightarrow \exists w\, \forall z > w(\theta_n(z) \rightarrow \neg E(\bar{x}, y', z))).\end{aligned}$$

The large classes selected in this way form a branch. That is, T implies $\bar{w} <_{n+1} \bar{x} \wedge S(\bar{x}, y) \rightarrow S(\bar{w}, y)$. It is now evident that $\sigma_n(\bar{x})$ should be $\forall \bar{w}\, \exists y(S(\bar{w}, y) \wedge \phi_n(\bar{x}, y))$. □

7. *Final Remarks*

The final section of this chapter mentions some results which would have been discussed in more detail had space allowed.

7.1. *Other Logics*

The logic of Magidor and Malitz [1977a] can be given cardinality interpretations other than the $\aleph_1$-interpretation discussed in Section IV.5. The logic $\mathscr{L}(Q, Q^2, Q^3, \ldots)$ which uses the $\aleph_\alpha$-interpretation is denoted by $\mathscr{L}(Q_\alpha, Q_\alpha^2, Q_\alpha^3, \ldots)$. The Magidor-Malitz completeness theorem (see Section IV.5.2) also proves the following transfer theorem.

7.1.1 Theorem. *Assume* $\diamond$. *If* $\kappa = \aleph_\alpha$ *is regular, then* $\mathscr{L}(Q_\alpha, Q_\alpha^2, Q_\alpha^3, \ldots) \rightarrow \mathscr{L}(Q_1, Q_1^2, Q_1^3, \ldots)$ $\aleph_0$*-compactly.* □

A converse of the previous transfer theorem has been proven by Shelah [1980].

7.1.2 Theorem. *Assume* $\diamond_{\aleph_\alpha}$ *and* $\diamond_{\aleph_{\alpha+1}}$. *Then*

$$\mathscr{L}(Q_1, Q_1^2, Q_1^3, \ldots) \rightarrow \mathscr{L}(Q_{\alpha+1}, Q_{\alpha+1}^2, Q_{\alpha+1}^3, \ldots)$$

$\aleph_\alpha$*-compactly.* □

Theorems 7.1.1 and 7.1.2 together with the the Magidor-Malitz completeness theorem imply that $\mathscr{L}(Q_\alpha, Q_\alpha^2, Q_\alpha^3, \ldots)$ is recursively enumerable for validity under the appropriate hypothesis on $\aleph_\alpha$.

The cofinality quantifier (see Section II.2.4) yields a logic which is fully compact. We denote the quantifier by Q^{cf}, and for regular cardinal κ, its κ-interpretation is defined so that $Q^{\mathrm{cf}}xy\varphi(x, y)$ holds iff $\varphi(x, y)$ defines a linear order with cofinality κ. The logic with this quantifier with the $\aleph_\alpha$-interpretation is denoted by $\mathscr{L}(Q_\alpha^{\mathrm{cf}})$. A proof of the following transfer theorem can be found in Makowsky–Shelah [1981].

7.1.3 Theorem. *Let $\aleph_\alpha$ and $\aleph_\beta$ be regular cardinals. Then $\mathscr{L}(Q_\alpha^{\mathrm{cf}}) \to \mathscr{L}(Q_\beta^{\mathrm{cf}})$ λ-compactly for any cardinal λ.* $\square$

Consequently, $\mathscr{L}(Q_0^{\mathrm{cf}})$ is fully compact. The proof also yields that $\mathscr{L}(Q_0^{\mathrm{cf}})$ is recursively enumerable for validity.

7.2. *Infinitary Languages*

Some of the transfer theorems we have discussed have extensions to infinitary languages. For example, the proof of Keisler [1966b] of Theorem 2.1.3 yields an $\mathscr{L}_{\omega_1,\omega}$ version.

7.2.1 Theorem. *If $\aleph_\alpha$ is regular, then $\mathscr{L}_{\omega_1,\omega}(Q_\alpha) \to \mathscr{L}_{\omega_1,\omega}(Q_1)$.* $\square$

Some theorems of Section 5 also have infinitary versions which can be proven by the techniques of that section or those of Section 6. The reader should refer to Definition II.5.2.1 for the notion of the well-ordering number $w(\mathscr{L})$ of a logic and to Chapter VIII for $\mathscr{L}_A$, where A is an admissible set. If A is countable, then $w(\mathscr{L}_A) = A \cap \mathrm{Ord}$.

7.2.2 Theorem. *Let A be an admissible set and φ a sentence of $\mathscr{L}_A$.*

(1) *Suppose that for each $\alpha < w(\mathscr{L}_A)$, there is a strongly α-Mahlo cardinal κ and a κ-like model of φ. Then, for each $\lambda > |A|$, φ has a λ-like model.*
(2) *Suppose that, for each $\alpha < w(\mathscr{L}_A)$, there is an α-subtle cardinal κ and a strongly κ-like model of φ. Then, for each $\lambda > |A|$, φ has a strongly λ-like model.* $\square$

Similarly, the Hanf numbers of admissible fragments can be computed.

7.2.3 Theorem. *Let A be admissible and $\omega < \alpha = w(\mathscr{L}_A)$. Then $h(\mathscr{L}_A) = \beth_\alpha$.* $\square$

Chapter VI
Other Quantifiers: An Overview

by D. MUNDICI

Generalized quantifiers were introduced by Mostowski [1957] as a means of generating new logics. In the meantime, their study has greatly developed, so that today there are more quantifiers in the literature than there are abstract model theorists under the sun. In any logic $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q^i)_{i\in I}$ one does not need to introduce specific formation rules for renaming and substitution; for, upon adding to the finite set of logical symbols of $\mathscr{L}_{\omega\omega}$ one new symbol for each Q^i, all sentences in $\mathscr{L}$ are obtainable by an induction procedure on strings of symbols, pretty much as in $\mathscr{L}_{\omega\omega}$. One can gödelize sentences and start studying the axiomatizability and decidability of theories in $\mathscr{L}$. One might even go as far as to write down the proof of a theorem in $\mathscr{L}$ and then have it published in some mathematical journal. For infinitary logics this all seems to be a bit more problematic.

There are several ways to introduce quantifiers. For instance, nonlinear prefixes of existentially and universally quantified variables may be regarded as quantifiers as is discussed in Section 1. Quantifiers are also used for transforming concepts such as isomorphism, well-order, cardinality, continuity, metric completeness, and the "almost all" notion into primitive logical notions such as $=$ (see Sections 2 and 3).

There is no reason why quantifiers introduced via the above definability criteria should also preserve the nice algebraic properties of $\mathscr{L}_{\omega\omega}$. Indeed, in many cases they do not. However, in a final section of this chapter we will briefly describe a novel approach to quantifiers, an approach that is based on the fact that every separable Robinson equivalence relation $\sim$ on structures is canonically representable as $\mathscr{L}$-equivalence, $\equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}_{\omega\omega}\{Q \mid \equiv_{\mathscr{L}(Q)}$ is coarser than $\sim\}$. In addition to this, $\mathscr{L}$ turns out to have compactness and interpolation: The open, interior quantifiers and their n-dimensional variants can be introduced in this way, starting from a suitable approximation of homeomorphism.

We do not aim at an encyclopedic coverage here. Rather, we only aim to present an anthology of the most significant facts and techniques in the variegated realm of quantifiers. In line with this, highly developed quantifiers or special topics are discussed in detail in Chapters IV, V, VII, and XV.

Throughout this chapter $\mathscr{L}^{\mathrm{mII}}$ will be taken to mean second-order logic with universal and existential quantifiers over *unary* relations. Moreover, we will also write $\mathscr{L}(Q^i)_{i\in I}$ instead of $\mathscr{L}_{\omega\omega}(Q^i)_{i\in I}$.

1. Quantifiers from Partially Ordered Prefixes

In this section we will present the logic $\mathscr{L}^{\circledast}$ with quantifiers which arise from nonlinear prefixes (see Section 1.1). The logic $\mathscr{L}(Q^{\mathrm{H}})$ with the smallest such quantifier often gives a full account of the whole $\mathscr{L}^{\circledast}$ (see Section 1.2). Further topics on $\mathscr{L}(Q^{\mathrm{H}})$ are discussed in Section 1.3.

1.1. Partially Ordered Quantifiers

Let φ be a first-order formula in prenex normal form. Each existentially quantified variable x in the prefix of φ only depends on the universally quantified variables which precede x. We can naturally consider formulas with nonlinearly ordered prefixes such as, for example,

$$(1) \qquad \left\{\begin{matrix}\forall x, x' \ \exists y & \\ \forall t & \exists z, z'\end{matrix}\right\} \psi(x, x', t, y, z, z'),$$

which is equivalent to $\exists f, g, g' \, \forall x, x', t \, \psi(x, x', t, f(x, x'), g(t), g'(t))$. Another example is Henkin's prefix (see also Chapter II):

$$(2) \qquad \left\{\begin{matrix}\forall x & \exists y \\ \forall x' & \exists y'\end{matrix}\right\} \psi(x, x', y, y'), \quad \text{viz.,} \quad \exists f, f' \, \forall x, x' \, \psi(x, x', f(x), f'(x')).$$

The smallest logic which is closed under this prefix is $\mathscr{L}(Q^{\mathrm{H}})$, where $Q^{\mathrm{H}} = \{\langle A, R\rangle \mid R \subseteq A^4 \text{ and } R \supseteq f \times g \text{ for some } f, g\colon A \to A\}$ = Henkin's quantifier. Similarly, the prefix in (1) results in a quantifier Q which is given by

$$Q = \{\langle A, R\rangle \mid R \subseteq A^6 \text{ and } R \supseteq f \times g \text{ for some } f\colon A^2 \to A \text{ and } g\colon A \to A^2\}.$$

We will agree to say that the (variable binding) pattern of Q^{H} is $\{\langle 1, 1\rangle, \langle 1, 1\rangle\}$, and that the pattern of Q in the discussion above is $\{\langle 2, 1\rangle, \langle 1, 2\rangle\}$. More generally, we set

1.1.1 Definition. Let $\pi = \{\langle n_1, m_1\rangle, \ldots, \langle n_r, m_r\rangle\}$ be a sequence of pairs of natural numbers ≥ 1. Then the *partially ordered quantifier Q_π with pattern π* is given by

$$Q_\pi = \{\langle A, R\rangle \mid R \subseteq A^s \text{ and } R \supseteq f_1 \times \cdots \times f_r \text{ for some } f_1\colon A^{n_1} \to A^{m_1}, \ldots, f_r\colon A^{n_r} \to A^{m_r}\},$$

where $s = n_1 + m_1 + \cdots + n_r + m_r$. We will also say that Q_π has *r rows*.

Partially ordered quantifiers do express some genuine mathematical notion, namely, uniformization. As a matter of fact, the quantifier $\forall x \, \exists y Rxy$ expresses the fact that the binary relation R can be uniformized, just as the quantifier

$$\begin{Bmatrix} \forall x \ \exists y \\ \forall x' \ \exists y' \end{Bmatrix} S(x, x', y, y')$$

expresses the fact that the 4-ary relation S contains the product of two binary uniformizable relations. Similar considerations hold for every partially ordered quantifier.

The syntactical rules for forming formulas in $\mathscr{L}(Q)$, with $Q = Q_\pi$, are naturally obtained by generalizing the rules for $\mathscr{L}(Q^{\mathrm{H}})$. Thus, Q binds s distinct variables, and if we display Q as

$$(3) \qquad Q = \begin{Bmatrix} \forall x_1^1 \cdots x_{n_1}^1 \ \exists y_1^1 \cdots y_{m_1}^1 \\ \cdots \quad \cdots \quad \cdots \\ \forall x_1^r \ldots x_{n_r}^r \ \exists y_1^r \ldots y_{m_r}^r \end{Bmatrix},$$

then we immediately obtain the semantics of Q. In this development, the existentially quantified variables in a row are thought of as only depending on the universally quantified variables in the same row. Let us denote by $\mathscr{L}^{\circledast}$ the smallest logic in which all partially ordered prefixes of the form (3) are allowed. If this is done, we then have:

1.1.2 Theorem. *For an arbitrary class K, if K is* PC *in $\mathscr{L}_{\omega\omega}$ then K is* EC *in $\mathscr{L}^{\circledast}$. Indeed, $K = \mathrm{Mod}_{\mathscr{L}^{\circledast}}\,\psi$, for some ψ of the form $Q\chi$ where Q is a partially ordered quantifier as in* (3) *above, and $\chi \in \mathscr{L}_{\omega\omega}$ is quantifier-free.*

Proof. Upon replacing relations by their characteristic functions, $K = \mathrm{Mod}\ \exists g_1 \ldots g_j \theta$, where θ is a first-order formula in prenex normal form. Using Skolem functions, θ becomes equivalent to $\exists f_1 \ldots f_n \forall x_1 \ldots x_m \alpha$, where α is quantifier-free. The terms in α can be safely assumed to have the form $f(y_1 \ldots y_k)$, where $y_1, \ldots, y_k$ are variable symbols, so that no function symbol occurs in the argument of f. Indeed, one might use the equivalence between, for example, $\forall y, z\beta(f(g(y, z), h(y)))$ and $\forall y, z, t, u[t = g(y, z) \wedge u = h(y) \to \beta(f(t, u))]$. By similarly adding new universally quantified variables, we can also assume, without loss of generality, that in the argument of any two different functions there are no common variables and also that the n variables occurring in the argument of each n-ary function are all distinct. We finally make sure that a function symbol does not occur in two different terms. Thus, we replace, for example, $\exists f \forall x, y, z\varphi(f(x, y), f(y, z))$ by writing $\exists f, g\ \forall x, y, t, z\{[x = t \wedge y = z \to f(x, y) = g(t, z)] \wedge [t = y \to \varphi(f(x, y), g(t, z))]\}$. Now, K is reduced to the desired form. ☐

1.2. *The Relationship Between $\mathscr{L}^{\mathrm{mII}}$, $\mathscr{L}^{\circledast}$ and $\mathscr{L}(Q^{\mathrm{H}})$*

Walkoe [1970] observed that if Q is any partially ordered quantifier such that $\mathscr{L}(Q) \not\equiv \mathscr{L}_{\omega\omega}$, then $\mathscr{L}(Q^{\mathrm{H}}) \leq \mathscr{L}(Q)$. Thus, Q^{H} is the weakest partially ordered quantifier. The two theorems of this subsection tell us to which extent Q^{H} alone can replace the denumerable set of all partially ordered quantifiers. We shall also investigate the relationship between partially ordered quantifiers and second-order logic. In this latter respect, Väänänen [1977c] proved that there is no generalized quantifier Q such that $\mathscr{L}(Q) \equiv$ full second-order logic.

1.2.1 Theorem. *$\mathscr{L}^{\circledast}$ is equivalent to $\mathscr{L}(Q^{\mathrm{H}})$ in first-order Peano arithmetic. That is, for every φ in $\mathscr{L}^{\circledast}$, there is a ψ in $\mathscr{L}(Q^{\mathrm{H}})$ having the same models as φ among the models of Peano arithmetic.*

Proof. By making repeated use of pairing functions (say, by using formula $\chi(x, y, z)$ in the language of Peano arithmetic, which defines a bijection from M^2 onto M in each model $\mathfrak{M}$ of Peano arithmetic), we can safely assume that every quantifier Q in φ has only one universally quantified variable and only one existentially quantified variable in each row. Moreover, it is no loss of generality to assume that Q has only two rows. As a matter of fact, we have the equivalence between

$$\begin{Bmatrix} \forall x_1 \; \exists z_1 \\ \cdots \; \cdots \\ \forall x_n \; \exists z_n \end{Bmatrix} \theta \quad \text{and} \quad \begin{Bmatrix} \forall x_1 \cdots x_n \, \exists z_1 \cdots z_n \\ \forall x'_1 \cdots x'_n \, \exists z'_1 \cdots z'_n \end{Bmatrix} \left[\theta \wedge \bigwedge_{i=1}^{n} (x_i = x'_i \rightarrow z_i = z'_i) \right].$$

We can now use pairing functions again to contract the latter prefix into Q^{H}. This concludes the proof of the theorem. □

Remark. Theorem 1.2.1 can be generalized (without altering the proof) to any arbitrary first-order theory where a definable pairing function is available.

Recall the definitions of $\leq_{\mathrm{RPC}}$ and of the Δ-closure $\Delta\mathscr{L}$ of a logic $\mathscr{L}$ from Chapter II. Intuitive notions stemming from first-order logic might suggest that $\Delta\mathscr{L}^{\mathrm{mII}} = \mathscr{L}^{\mathrm{mII}}$. However, this is not the case. Indeed, recall that in the definition of $\leq_{\mathrm{RPC}}$, extra universes are allowed which, in settings where Löwenheim–Skolem fails, cannot be coded as extra relations on some given universe.

1.2.2 Theorem. $\Delta\mathscr{L}(Q^{\mathrm{H}}) = \Delta\mathscr{L}^{\circledast} = \Delta\mathscr{L}^{\mathrm{mII}}$.

The proof proceeds through the following two claims:

Claim 1. $\mathscr{L}^{\circledast} \leq_{\mathrm{RPC}} \mathscr{L}^{\mathrm{mII}}$.

Proof. It suffices to show that for every $\varphi \in \mathscr{L}^{\circledast}(\tau)$, Mod φ is in $\mathrm{RPC}_{\mathscr{L}^{\mathrm{mII}}}$. For the moment, assume that τ has just one sort s, and that only Q^{H} occurs in φ. Now, $\neg Q^{\mathrm{H}}$ asserts the nonexistence of functions, while $\mathscr{L}^{\mathrm{mII}}$ can only express the nonexistence of sets. To overcome this difficulty, we add a binary function symbol J to τ, and let the first-order sentence α assert that J maps the set of all pairs in s one-one onto a new sort s'. For X any set-variable of $\mathscr{L}^{\mathrm{mII}}$, let $\beta(X)$ assert that X represents via J (that is, $J^{-1}[X]$ is) a function: namely, $\beta(X)$ is $\forall x \, \exists! y \, \exists z(z \in X \wedge z = J(x, y))$. If X represents a function $\hat{X}$, then the fact that $\hat{X}$ maps x into y, for short $X(x) = y$, is simply expressed by the $\mathscr{L}^{\mathrm{mII}}$-formula

$J(x, y) \in X$. Now, let $\varphi' \in \mathscr{L}^{\mathrm{mII}}$ be obtained from φ via the following inductive procedure: $\psi' = \psi$ if ψ is atomic, $(\neg\psi)' = \neg(\psi')$, $(\psi \wedge \chi)' = \psi' \wedge \chi'$, $(\exists x\psi)' = \exists x(\psi')$. For the crucial Q^{H}-clause, where ψ is given by

$$\left\{\begin{matrix}\forall x & \exists y \\ \forall x' & \exists y'\end{matrix}\right\} \theta(x, x', y, y'), \quad \text{viz.,} \quad \exists g, g' \, \forall x, x' \, \theta(x, x', g(x), g'(x')),$$

we let ψ' be given by $\exists X, X'[\beta(X) \wedge \beta(X') \wedge \forall x, x', y, y'(y = X(x) \wedge y' = X'(x') \rightarrow \theta')]$. Clearly, the τ-reducts of the models of $\alpha \wedge \varphi'$ are exactly the models of φ so that Mod $\varphi \in \mathrm{RPC}_{\mathscr{L}^{\mathrm{mII}}}$ as required. If φ has many sorts, or if φ has a p.o. quantifier $Q \neq Q^{\mathrm{H}}$, then we proceed similarly, using maps $J_Q \colon A_s^{n+1} \rightarrow A_{s''}$ to code into subsets of a new sort s'' each n-ary function asserted to exist by Q.

Claim 2. $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(Q^{\mathrm{H}})$.

Proof. If $A \neq \varnothing$ and $\{\varnothing, A\} \subseteq S \subseteq P(A)$, where P denotes power set, then $S \neq P(A)$ iff $\exists f \colon A \rightarrow \{0, 1\}$ such that $\forall r \in S$, $r \neq f^{-1}(1)$; that is to say, iff $\exists f \colon A \rightarrow \{0, 1\}$ and $\exists g \colon S \rightarrow A$ such that $\forall r \in S$, $\forall x \in A$ $[x = g(r) \rightarrow (g(r) \in r \leftrightarrow f(x) = 0)]$. Using relativized Q^{H} we can equivalently say the following:

$$(1) \qquad \left\{\begin{matrix}\forall x \in A & \exists y \in \{0, 1\} \\ \forall r \in S & \exists t \in A\end{matrix}\right\} \qquad t = x \rightarrow (t \in r \leftrightarrow y = 0).$$

Now, to prove our claim, it is enough to show that for every $\varphi \in \mathscr{L}^{\mathrm{mII}}(\tau)$, Mod $\varphi \in \mathrm{RPC}_{\mathscr{L}(Q^{\mathrm{H}})}$. To this purpose, add to τ new unary relations A and S, as well as one binary relation E and the constants 0 and 1. Let the roles of S, A, 0, 1, E be described by sentence α which is given by the conjunction of the following formulas: $\forall x((Sx \vee Ax) \wedge \neg(Sx \wedge Ax))$, $S0 \wedge S1$, $\forall x(Ax \rightarrow Ex1)$, $\neg\exists x(Ax \wedge Ex0)$, $\forall^S r, \forall^S r' \; [r = r' \leftrightarrow \forall^A x(Exr \leftrightarrow Exr')]$, *where* $\forall^Z x\theta$ as usual means $\forall x(Zx \rightarrow \theta)$. Let β be a reformulation of (1) without relativizations, that is,

$$\left\{\begin{matrix}\forall x & \exists y \\ \forall r & \exists t\end{matrix}\right\} \{Ax \wedge Sr \rightarrow [At \wedge (y = 0 \vee y = 1) \wedge (t = x \rightarrow (Etr \leftrightarrow y = 0))]\}.$$

Let φ' be obtained from φ by relativizing to A (that is, to $\{x \mid Ax\}$) each quantified individual variable in φ, and by relativizing to S each quantified set variable in φ (we can add more A's and S's if more sorts occur in φ), and finally by replacing $y \in X$ throughout by Eyx_X, where x_X is an individual variable. By the above discussion, the τ-reducts of models of $\alpha \wedge \neg\beta \wedge \varphi'$, upon restriction to $\{x \mid Ax\}$, are exactly the models of φ. As a matter of fact, $\alpha \wedge \neg\beta$ ensures that in our transcription of second-order variables as variables ranging over S we are missing no subset of A. Thus, we have proved that $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(Q^{\mathrm{H}})$. Those who do care to relativize classes may add one more sort s'' as well as a function symbol f and assert that f is an isomorphic embedding of the structure on sort s'' onto the restriction to $\{x \mid Ax\}$ of τ-reducts of models of $\alpha \wedge \neg\beta \wedge \varphi'$. $\square$

1.2.3 Corollary. *$\mathscr{L}(Q^{\mathrm{H}})$, $\mathscr{L}^{\circledast}$ and $\mathscr{L}^{\mathrm{mII}}$ have the same Löwenheim and the same Hanf numbers. Moreover, they have recursively isomorphic sets of valid sentences.*

Proof. The proof of this result is routine as it follows from standard facts of abstract model theory and from an easy inspection of the above proof (see also Proposition XVII.4.4.2(i)). □

1.2.4 Remark. The gödelized set V^{mII} of valid sentences in $\mathscr{L}^{\mathrm{mII}}$ is not definable in n-th order arithmetic. Indeed, it is not a Σ_n^m subset of the natural numbers, for any $n, m \in \omega$ (see Montague [1965] and also Tharp [1973]). For the Hanf number of $\mathscr{L}^{\mathrm{mII}}$ see Barwise [1972b] and Väänänen [1979b]. The reader should also consult Theorem 2.1.5(i) of the present chapter for more on this notion.

1.3. Further Topics on $\mathscr{L}(Q^{\mathrm{H}})$

In the light of Theorem 1.2.2 the implicit expressive power of $\mathscr{L}(Q^{\mathrm{H}})$ is very strong (see also Theorem 2.1.1 and Proposition 2.1.3). Concerning the explicit expressive power of $\mathscr{L}(Q^{\mathrm{H}})$, we first observe that $\mathscr{L}(Q^{\mathrm{H}}) \geq \mathscr{L}(Q_0)$. Indeed,

$$\begin{aligned} Q_0 x\varphi(x) \quad \text{iff} \quad & \exists t\{\varphi(t) \wedge \exists f, g\, \forall u, v[(u = v \leftrightarrow f(u) = g(v)) \\ & \wedge (\varphi(u) \rightarrow \varphi(f(u)) \wedge f(u) \neq t)]\}. \end{aligned}$$

1.3.1 Proposition. *$\mathscr{L}(Q^{\mathrm{H}})$ is neither (ω, ω)-compact nor axiomatizable, nor does it have the weak Beth property.*

Proof. There is a sentence of $\mathscr{L}(Q^{\mathrm{H}})$ characterizing up to isomorphism the standard model of arithmetic, since Q_0 is EC in $\mathscr{L}(Q^{\mathrm{H}})$. Thus, $\mathscr{L}(Q^{\mathrm{H}})$ cannot be countably compact and, using Gödel's incompleteness theorem, $\mathscr{L}(Q^{\mathrm{H}})$ is not axiomatizable. Failure of the weak Beth property is now a particular case of a result in abstract model theory which holds for every finitely generated logic in which the class $\{\mathfrak{A} \mid \mathfrak{A} \cong \langle \omega, < \rangle\}$ is EC (see, for example, Makowsky–Shelah [1979b, Theorem 6.1], or Theorems XVII.4.1.1 and 4.2.9). □

1.3.2 Theorem. $\mathscr{L}(Q^{\mathrm{H}}) \geq \mathscr{L}(Q_\alpha)$ *iff* $\alpha = 0$.

Proof. We must prove only the ($\Rightarrow$)-direction. To this purpose, it suffices to show, by induction on the complexity of formulas, that for each formula φ in the pure identity language of $\mathscr{L}(Q^{\mathrm{H}})$—that is to say, only the equality ($=$) occurs in φ—there is a formula $\hat{\varphi}$ in the pure identity language of $\mathscr{L}_{\omega\omega}$ equivalent to φ upon restriction to infinite sets (that is, $\kappa \models_{\mathscr{L}(Q^{\mathrm{H}})} \varphi \leftrightarrow \hat{\varphi}$, for each $\kappa \geq \omega$). The only nontrivial step in the proof arises in the case where φ has the form $Q^{\mathrm{H}}\psi$. In this case, one then uses upward and downward Löwenheim–Skolem methods for $\mathscr{L}_{\omega\omega}$ to establish that φ does not distinguish between infinite sets. By contrast, for $\alpha > 0$, Q_α does distinguish between infinite sets. □

1.4. Bibliographical Notes

Henkin's quantifier was introduced in Henkin [1961], while Ehrenfeucht proved that $\mathscr{L}(Q^{\mathrm{H}})$ is neither countably compact, nor axiomatizable (see Henkin [1961]). Theorems 1.1.2 and 1.2.1 are proved in Enderton [1970] and in Walkoe [1970]. The proof of Theorem 1.3.2 given above is due to Lopez-Escobar [1969], who also proved the failure of interpolation. Paulos [1976] proved that both Δ-closure and Beth property fail for $\mathscr{L}(Q^{\mathrm{H}})$. Failure of the weak Beth property is proved in Gostanian–Hrbacek [1976] who used general ideas from Craig [1965]. The reader should also consult Kreisel [1967], Mostowski [1968] and Lindström [1969] for more in this connection. Back-and-forth games for $\mathscr{L}(Q^{\mathrm{H}})$-equivalence are used by Krynicki [1977b] in connection with Theorem 1.3.2. Here the reader should also see Krawczyk–Krynicki [1976] and Weese [1980]. Theorem 1.2.2 is proved in Krynicki [1978] and Krynicki–Lachlan [1979]. In the latter paper, the reader can also find decidability (undecidability) results on $\mathscr{L}(Q^{\mathrm{H}})$. Partially ordered quantifiers are used in Barwise [1976] to find nice first-order axiomatizations for certain classes of structures such as, for example, the class of structures having a nontrivial automorphism f such that $f^2 = $ identity. In Walkoe [1970, 1976] and in Keisler–Walkoe [1973] partially ordered quantifiers are used to prove the following result about *ordinary* model theory: Let Q' and Q'' be first-order prefixes, with $Q' \neq Q''$ and Q' and Q'' having the same length. Then, for some quantifier-free formula φ in $\mathscr{L}_{\omega\omega}$, there is no quantifier-free formula ψ in $\mathscr{L}_{\omega\omega}$ such that $Q'\varphi$ is equivalent to $Q''\psi$. See Harel [1979], Cowles [1981], and Barwise [1979] for further information about Q^{H}.

2. Quantifiers for Comparing Structures

The quantifiers presented in this section express the fact that two structures $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic: In Section 2.1 both $\mathfrak{A}$ and $\mathfrak{B}$ are sets, and in Section 2.2 we add one binary relation; while in Section 2.3 we keep $\mathfrak{A}$ fixed.

2.1. Equicardinality Quantifiers

Recall that Härtig's quantifier I is defined by $I = \{\langle A, R, S\rangle \mid |R| = |S|\}$, so that $Ixy\varphi(x), \psi(y)$ says that $|\{x \mid \varphi(x)\}| = |\{y \mid \psi(y)\}|$. Rescher's quantifier Q^{R} is given by $Q^{\mathrm{R}} = \{\langle A, R, S\rangle \mid |R| < |S|\}$. Chang's quantifier Q^{C} binds only one variable and $Q^{\mathrm{C}}x\varphi(x)$ says that $Ixy\varphi(x), (y = y)$. Clearly, $\mathscr{L}(Q^{\mathrm{C}}) \leq \mathscr{L}(I)$. Also observe that $\mathscr{L}(Q_0) \leq \mathscr{L}(I)$. As a matter of fact, we have

$$(1) \qquad Q_0 x\, \varphi(x) \quad \text{iff} \quad \exists z[\varphi(z) \wedge Ixy\, \varphi(x), \varphi(y) \wedge y \neq z].$$

These points clear, we can now consider

2.1.1 Theorem. $\mathscr{L}(Q_0) < \mathscr{L}(I) < \mathscr{L}(Q^R) < \mathscr{L}(Q^H)$.

Proof. For the proof that $\mathscr{L}(Q_0) \leq \mathscr{L}(I)$ holds we only need give due regard to (1) above. It is trivially true that $\mathscr{L}(I) \leq \mathscr{L}(Q^R)$. Also, $\mathscr{L}(Q^R) \leq \mathscr{L}(Q^H)$ holds, for we have

$$(Ixy\,\varphi(x), \psi(y)) \vee (Q^R xy\,\varphi(x), \psi(y)) \quad \text{iff}$$
$$\exists f, f' \,\forall x, x' \{(x = x' \leftrightarrow f(x) = f'(x')) \wedge [\varphi(x) \rightarrow \psi(f(x))]\}.$$

$\mathscr{L}(Q_0)$ is not equivalent to $\mathscr{L}(I)$, since the former—as a sublogic of $\mathscr{L}_{\infty\omega}$—has the Karp property, while the second logic does not. (*Proof*: The two-cardinal structures $\langle \omega_1, U \rangle$ and $\langle \omega_1, V \rangle$ with $|U| = \omega$ and $|V| = |\omega_1 \sim V| = \omega_1$ are partially isomorphic, but not $\mathscr{L}(Q^C)$-equivalent, and hence not $\mathscr{L}(I)$-equivalent). The fact that $\mathscr{L}(I)$ is not equivalent to $\mathscr{L}(Q^R)$ has been proved by Hauschild [1981] (In this connection, the reader should also see Weese [1981b]). The fact that $\mathscr{L}(Q^R)$ is not equivalent to $\mathscr{L}(Q^H)$ has been proven by Cowles [1981]. □

The following sentence of $\mathscr{L}(I)$ characterizes $\langle \omega, < \rangle$ up to isomorphism:

> $\forall x \neg Iuv(u < x), (v \leq x) \wedge$ "$<$ is a discrete linear order with first element".

Using the above sentence, we immediately obtain

2.1.2 Proposition. *$\mathscr{L}(I)$ and $\mathscr{L}(Q^R)$ are neither (ω, ω)-compact nor axiomatizable nor do they satisfy the weak Beth property.*

Proof. The proof is the same as that given for Proposition 1.3.1. □

As for the implicit expressive power of $\mathscr{L}(I)$ we have

2.1.3 Proposition. *The following are* RPC *in $\mathscr{L}(I)$ and, hence in $\mathscr{L}(Q^R)$ also:*

(i) *the class of well-ordered structures;*
(ii) *the class of well-ordered structures which are isomorphic to some cardinal;*
(iii) *the class of well-founded structures;*
(iv) *the class $\{\langle A, E \rangle | \langle A, E \rangle \cong \langle L(\alpha), \in \rangle$, for some ordinal $\alpha\}$;*
(v) *the class $\{\langle A, E \rangle | \langle A, E \rangle \cong \langle L(\kappa), \in \rangle$, for some cardinal $\kappa\}$.*

Proof. (i), we note that $<$ well-orders its universe of sort s iff there is an additional sort s' and a binary relation Rxx', where $x \in s$ and $x' \in s'$, such that the function $f(x) = |\{x' | Rxx'\}|$ is strictly increasing; that is to say, we have formally that

$$x < y \rightarrow [(Rxx' \rightarrow Ryx') \wedge \neg Iu'v' Rxu', Ryv'].$$

To prove (ii), we add the clause that $\forall z \neg Ixy(x = x), (y < z)$ to the above sentence. The proof of (iii) is the same as for (i). To prove (iv), we use Mostowski's collapsing lemma and standard results on constructible sets to exhibit a finite subtheory of ZF + $V = L$ whose well-founded models are exactly those that are isomorphic to $\langle L(\alpha), \in\rangle$, for some $\alpha \in \mathrm{On}$. Now, recall that well-foundedness is RPC in $\mathscr{L}(I)$, by (iii). To prove (v), we use (iv) and (ii). ☐

2.1.4 Theorem. ($V = L$). *$\Delta\mathscr{L}(I) \equiv \Delta\mathscr{L}(Q^{\mathrm{R}}) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$. $\mathscr{L}(I)$, $\mathscr{L}(Q^{\mathrm{R}})$ and $\mathscr{L}^{\mathrm{mII}}$ have the same Löwenheim number and the same Hanf number. Moreover, they have recursively isomorphic sets of valid sentences.*

Proof. That $\mathscr{L}(I) \leq_{\mathrm{RPC}} \mathscr{L}(Q^{\mathrm{R}})$ is trivially true. The fact that $\mathscr{L}(Q^{\mathrm{R}}) \leq_{\mathrm{RPC}} \mathscr{L}^{\mathrm{mII}}$ is proven by use of pairing functions, as in Claim 1 of Theorem 1.2.2. We must now show that $V = L$ implies that $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(I)$. Let α be a sentence of $\mathscr{L}(I)$ of type τ whose E-reducts are exactly the structures that are isomorphic to $\langle L(\kappa), \in\rangle$, for some cardinal κ, as in Proposition 2.1.3(v) (E is meant as membership). Expand τ by adding a function symbol f, and let sentence β assert that "f is increasing and maps the ordinals one-one onto the infinite cardinals." By 2.1.3(i)(ii), ordinals and cardinals are true (up to isomorphism) ordinals and cardinals, so that f is isomorphic to the aleph function and κ is a fixed point, $\omega_\kappa = \kappa$. Hence, using GCH—a consequence of $V = L$—we have that $\kappa = \beth_\kappa$. Now add two constants c and p and let sentence θ assert that "c is a cardinal and p is the power set of c". In every model of $\alpha \wedge \beta \wedge \theta$, c is indeed isomorphic to a cardinal, and p is isomorphic to the set of constructible subsets of c (use, for example, Theorem 7.4.3(vii) in Chang, Keisler [1977], to the effect that, since $\kappa = \beth_\kappa$, then $L(\kappa) = R(\kappa) \cap L$; recall also that $c < \kappa$). Now, given $\varphi \in \mathscr{L}^{\mathrm{mII}}$ of type τ_φ, we construct $\varphi' \in \mathscr{L}(I)$ as is done in Claim 2 of Theorem 1.2.2 by relativizing each quantified individual variable to $\{x \mid x < c\}$, i.e. to $\{x \mid Exc\}$, and relativizing each set variable to $\{r \mid Erp\}$, and using E instead of $\in$. By $V = L$, p is the power set of c, so that the τ_φ-reducts of models of $\varphi' \wedge \alpha \wedge \beta \wedge \theta$, upon restriction to $\{x \mid Exc\}$ are exactly the models of φ. Whence we have that $\mathscr{L}^{\mathrm{mII}} \leq_{\mathrm{RPC}} \mathscr{L}(I)$. The proof of the theorem is completed by using standard tools. ☐

Remark. Thus, we see that under the assumption that $V = L$, the gödelized set V_I of valid sentences of $\mathscr{L}(I)$ is not a Σ_n^m subset of ω, $\forall n, m \in \omega$ (see Remark 1.2.4). As was remarked by Väänänen [1980b, p. 198], $\Delta\mathscr{L}(I) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$ continues to hold if $V = L$ is weakened to $V = L[0^\#]$, or even to $V = L^\mu$.

2.1.5 Theorem. (i) *If λ is the smallest inaccessible (hyperinaccessible, Mahlo, hyper-Mahlo) cardinal, then the Hanf number of $\mathscr{L}(I)$ is $>\lambda$.*

(ii) *The gödelized set V_I of valid sentences of $\mathscr{L}(I)$ is neither a Σ_2^1, nor a Π_2^1 subset of ω.*

(iii) *The fact that the Löwenheim number of $\mathscr{L}(I)$ is $<2^\omega$ and V_I is a Δ_3^1 subset of ω is consistent, if* ZF *is consistent.*

(iv) *The fact that $\mathscr{L}(I)$ and $\Delta\mathscr{L}(I)$ have different Hanf numbers is consistent, if* ZF *is consistent.*

Proof. (i) Let φ be a sentence of $\mathscr{L}(I)$ of type $\tau = \{E, \ldots\}$ such that the E-reducts of the models of φ are the well-founded models of ZFC + "there are no inaccessible cardinals". The existence of φ then follows from Proposition 2.1.3 together with standard results from axiomatic set theory. Now, $\langle R(\lambda), E\rangle \models \varphi$, where λ is the first inaccessible cardinal and E means membership. Note that $|R(\lambda)| = \lambda$. We claim that for no $\mu > \lambda$, φ has a model of cardinality μ. Otherwise (*absurdum* hypothesis) let $\mathfrak{B} = \langle B, E, \ldots\rangle \models \varphi$ with $|B| = \mu$. By Mostowski's collapsing lemma, we have that $\mathfrak{B} \upharpoonright E$ is (isomorphic to) a transitive model of ZFC. Also, $\lambda \in B$ holds; for otherwise, by the assumed inaccessibility of λ, we would have $|B| < \mu$. For a suitable transitive well-founded (end) extension $\mathfrak{D}$ of $\mathfrak{B}$ we have that $\mathfrak{D} \models$ "λ is inaccessible". Now, "x is not inaccessible" is a Σ_1 predicate. Hence, we cannot have $\mathfrak{B} \models$ "λ is not inaccessible", by a familiar persistence argument. Thus, $\mathfrak{B} \models$ "λ is inaccessible and there are no inaccessibles"—a contradiction. In case λ is hyperinaccessible, *etc*., the proof is the same, since we only need the fact that each of these properties is inherited by transitive submodels.

(ii) Assume that V_I is either Σ^1_2 or Π^1_2 (*absurdum* hypothesis). By Shoenfield's absoluteness lemma, V_I is an element v of, say $L(\omega_1)$. Let ψ be a sentence in $\mathscr{L}(I)$ of type $\tau = \{E, \ldots\}$ such that the E-reducts of the models of ψ are the sets $\langle L(\kappa), E\rangle$ as in Proposition 2.1.3(v). Let χ assert further that an uncountable ordinal is in the universe so that $\kappa > \omega_1$. Now, $x \in V_I$ holds true iff $\psi \wedge \chi \rightarrow x \in v$ holds true. Proceeding as in Tarski's diagonal argument, we now let $y \in W$ mean that y is the Gödel number of a formula $\beta(x)$ having one free variable such that $\beta(y)$ is false. By the above discussion, W is an element w of $L(\omega_1)$ and $x \in W$ holds true iff $\psi \wedge \chi \rightarrow x \in w$ holds true. Let z be the Gödel number of the formula $\theta(x)$ which asserts that "$\psi \wedge \chi \rightarrow x \in w$." Then $z \in W$ iff $z \notin W$. This is, of course, a contradiction.

(iii) This is proven in Väänänen [1980b, Corollary 3.2.3]. The reader should see Example XVII.2.4.3 and Proposition XVII.2.4.7 of this volume.

(iv) is proven in Väänänen [1983]. See also Theorem XVII.4.5.4 of the present volume. □

Let us end this subsection with a brief examination of Q^{C}. On finite structures, Q^{C} may be replaced by $\forall$. On structures of cardinality ω_α, Q^{C} behaves like Q_α: indeed many of the techniques used for the Q_α—notably for Q_1—apply equally well to Q^{C}, as is shown in detail in the textbook by Bell and Slomson [1969, Chapter 13]. These techniques are also extensively discussed in Chapters IV and V of this volume. We will thus limit ourselves to stating, without proof, the following results about Q^{C}.

2.1.6 Theorem. (i) *Let T be a countable set of sentences in $\mathscr{L}(Q^{\mathrm{C}})$ having a denumerable model. Then T has a model of every infinite cardinality.*

(ii) *Assume that all singular cardinals are strong limit. Then $\mathscr{L}(Q^{\mathrm{C}})$ is both axiomatizable and (ω, ω)-compact relative to infinite structures.*

(iii) *Assume* GCH, *then $\mathscr{L}(Q^{\mathrm{C}})$ is compact relative to infinite structures.* □

The logic $\mathscr{L}(Q^{\mathrm{C}})$ is not closed under relativization (and hence, it is not Δ-closed). Indeed, if $\mathscr{L}(Q^{\mathrm{C}})$ allowed relativization, then the relativization of $Q^{\mathrm{C}}x\,\varphi(x)$ to

$\{y \mid \psi(y)\}$ would be equivalent to $Ixy\, \varphi(x) \wedge \psi(x), \psi(y)$, and we could then characterize the standard model of arithmetic in $\mathscr{L}(Q^C)$ as is done in $\mathscr{L}(I)$ by using Section 2.1(1). Thus, we would contradict Theorem 2.1.6(i).

Evidently, $\mathscr{L}(Q^C)$ is not (ω, ω)-compact, for $Q_0 z(z = z)$ can be expressed as $\exists x Q^C z(z \neq x)$. In the above theorem, compactness relative to infinite structures means that for every set T of sentences in $\mathscr{L}(Q^C)$, if each finite $T' \subseteq T$ has an infinite model (that is, a model whose universe is infinite), then T itself has an infinite model.

2.2. *Similarity Quantifier and Its Variants*

The quantifier I says that two *sets* are isomorphic; the *similarity* quantifier S says that two structures with a binary relation are isomorphic; that is,

$$\mathfrak{A} \models Sxyuv\, \varphi(x, y), \psi(u, v) \quad \text{iff} \quad \langle A, \varphi^{\mathfrak{A}} \rangle \cong \langle A, \psi^{\mathfrak{A}} \rangle,$$

where $\varphi^{\mathfrak{A}} = \{\langle a, b \rangle \in A^2 \mid \mathfrak{A} \models \varphi(a, b)\}$. Let α be given by

$$\forall m, n, p[m < n < p \to \neg Sxyx'y'(m < x < y < p), (n < x' < y' < p)].$$

Then a discrete linear ordering with first element is a model of α iff it is isomorphic to $\langle \omega, < \rangle$. By arguing as in Proposition 1.3.1, we see that $\mathscr{L}(S)$ is neither (ω, ω)-compact nor axiomatizable, nor does it have the weak Beth property.

Concerning the implicit expressive power of $\mathscr{L}(S)$, in Väänänen [1980a] it is proven that $\Delta\mathscr{L}(S) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$. The easy direction of this theorem uses pairing functions as in Claim 1 of Theorem 1.2.2. For the other direction, we first show that well-foundedness is RPC-definable in $\mathscr{L}(S)$. As a matter of fact, the quantifier I is clearly RPC in $\mathscr{L}(S)$. But I is also the complement of an RPC-class in $\mathscr{L}(S)$, since $\langle A, U, V \rangle \models \neg IxyUxVy$ iff the disjoint sum B of U and V satisfies $\langle B, U^2 \rangle \not\cong \langle B, V^2 \rangle$. Therefore, I is EC in $\Delta\mathscr{L}(S)$ and, by using Proposition 2.1.3, well-foundedness is RPC in $\mathscr{L}(S)$, as was required. To conclude the proof that $\Delta\mathscr{L}(S) = \Delta\mathscr{L}^{\mathrm{mII}}$, we now try to express genuine power set in $\Delta\mathscr{L}(S)$, and, finally, argue as in Claim 2 of Theorem 1.2.2.

Thus, if we try to express isomorphism as a primitive logical notion, we may well attain the implicit expressive power of $\mathscr{L}^{\mathrm{mII}}$ by means of a single quantifier. Note here the analogy with the case of $\mathscr{L}(Q^H)$ in the framework of partially ordered quantification.

2.2.1 Variants of S. We can consider isomorphism between certain binary relations such as orderings or equivalence relations. Thus, we might define, say, S_{DLO} and S_{EQ} as follows:

$$\mathfrak{A} \models S_{\mathrm{DLO}} xyvw\, \varphi(x, y), \psi(v, w) \quad \text{iff} \quad \langle A, \varphi^{\mathfrak{A}} \rangle \cong \langle A, \psi^{\mathfrak{A}} \rangle,$$

and $\psi^{\mathfrak{A}}$ is a dense linear ordering over its field;

$$\mathfrak{A} \models S_{\mathrm{EQ}}xyvw\ \varphi(x, y), \psi(v, w) \quad \text{iff} \quad \langle A, \varphi^{\mathfrak{A}} \rangle \cong \langle A, \psi^{\mathfrak{A}} \rangle,$$

and $\psi^{\mathfrak{A}}$ is an equivalence relation over its field.

The list of such variants of the quantifier S is potentially infinite. However, we shall limit our attention to S_{DLO} and S_{EQ}. It is not difficult to see that $\Delta\mathscr{L}(S_{\mathrm{EQ}}) \geq \Delta\mathscr{L}(I)$ ($|A| = |B|$ iff the equivalence relation given by equality on A is isomorphic to equality on B), and that $\Delta\mathscr{L}(S_{\mathrm{EQ}}) \leq \Delta\mathscr{L}(I)$ (two equivalence relations $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic iff for every λ, $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of equivalence classes of power λ). Therefore, assuming $V = L$ we can apply Theorem 2.1.4 to the effect that $\Delta\mathscr{L}(S_{\mathrm{EQ}}) = \Delta\mathscr{L}^{\mathrm{mII}}$. Turning to S_{DLO}, we immediately see that $\mathscr{L}(Q_1) \leq \Delta\mathscr{L}(S_{\mathrm{DLO}})$ ($|A| \geq \omega_1$ iff there are two nonisomorphic dense linear orders without endpoints on A). It is also proven in Väänänen [1980a] that $\mathscr{L}(Q_0) \leq \Delta\mathscr{L}(S_{\mathrm{DLO}})$ and that $\mathscr{L}(S_{\mathrm{DLO}}) \leq \Delta\mathscr{L}(S_{\mathrm{EQ}})$ is an independent statement of ZF.

2.3. *The Quantifiers $Q^{I\mathfrak{A}}$ and $Q^{PI\mathfrak{A}}$*

For $\mathfrak{A}$ an arbitrary structure of finite relational type τ, let $Q^{I\mathfrak{A}}$ have as its defining class $I\mathfrak{A} = \{\mathfrak{B} \mid \mathfrak{B} \cong \mathfrak{A}\}$. Clearly, $\mathscr{L}(Q^{I\mathfrak{A}}) \equiv \mathscr{L}_{\omega\omega}$ iff $\mathfrak{A}$ is finite iff $\mathscr{L}(Q^{I\mathfrak{A}})$ is compact. Next, we will consider the denumerable case.

2.3.1 Theorem. *Let $\mathscr{L} = \mathscr{L}(Q^{I\mathfrak{A}})$, with $|A| = \omega$. Then $\mathscr{L}$ does not have the Craig property. Furthermore, $\mathscr{L}$ is (ω, ω)-compact iff there is a first-order sentence α with no finite models whose denumerable models are exactly the models in $I\mathfrak{A}$.*

Proof. The proof is by cases. We will begin with

Case 1. $\exists\alpha \in \mathscr{L}_{\omega\omega}$ whose denumerable models are exactly those in $I\mathfrak{A}$.
Then, let $\psi \in \mathscr{L}$ be defined by $\mathfrak{B} \models \psi$ iff $\mathfrak{B}$ has two sorts s and s' and f maps $\mathfrak{B} \restriction s$ one-one into $\mathfrak{B} \restriction s'$ and $\mathfrak{B} \restriction s' \in I\mathfrak{A}$. Then we see that the class of countable sets is $\mathrm{RPC}_{\mathscr{L}}$. Now, let $\varphi \in \mathscr{L}$ be defined by $\mathfrak{D} \models \varphi$ iff $\mathfrak{D} \notin I\mathfrak{A}$ and $\mathfrak{D} \models \alpha$ and g maps D one-one into $D' \subsetneqq D$. We then see that the class of uncountable sets is $\mathrm{RPC}_{\mathscr{L}}$. Therefore, Q_1 is EC in $\Delta\mathscr{L}$ so that $\Delta\mathscr{L} \geq \Delta\mathscr{L}(Q_1)$. The proof for Case 1 can now be completed as follows:

Subcase 1.1. α may be assumed to have no finite models.
Then $\mathfrak{B} \in I\mathfrak{A}$ iff $\mathfrak{B} \models_{\mathscr{L}(Q_1)} \alpha \wedge \neg Q_1x(x = x)$. Hence, $I\mathfrak{A} \in \mathrm{EC}_{\mathscr{L}(Q_1)}$. Whence $\mathscr{L} \leq \mathscr{L}(Q_1)$. By the above discussion, we have $\Delta\mathscr{L} \equiv \Delta\mathscr{L}(Q_1)$. This shows that $\mathscr{L}$ is (ω, ω)-compact (as is $\mathscr{L}(Q_1)$ and Δ-closure preserves compactness) and that $\mathscr{L}$ does not have the interpolation property ($\Delta\mathscr{L}(Q_1)$ does not, see Hutchinson [1976]).

Subcase 1.2. every α as in Case 1 has some finite model.
Then α need have arbitrarily large finite models; let $\theta \in \mathscr{L}$ be given by $\mathfrak{B} \models \theta$ iff $\mathfrak{B}$ has sorts s, s', s'' and f maps $\mathfrak{B} \restriction s$ one-one into $\mathfrak{B} \restriction s'$, g maps $\mathfrak{B} \restriction s'$ one-one into

$\mathfrak{B} \upharpoonright s''$, $\mathfrak{B} \upharpoonright s'' \in I\mathfrak{A}$, $\mathfrak{B} \upharpoonright s' \notin I\mathfrak{A}$, $\mathfrak{B} \upharpoonright s' \models \alpha$; thus $\mathfrak{B} \upharpoonright s$ can be of every finite (but of no infinite) cardinality and $\neg Q_0$ is $\mathrm{RPC}_{\mathscr{L}}$; trivially Q_0 is $\mathrm{RPC}_{\mathscr{L}}$, so that $\Delta\mathscr{L} \geq \Delta\mathscr{L}(Q_0)$ and $\mathscr{L}$ cannot be (ω, ω)-compact (as $\Delta\mathscr{L}$ is not, and Δ-closure preserves compactness). Actually we can find a recursively enumerable (r.e.) set of $\mathscr{L}$-sentences which is a counterexample to compactness, i.e. $\mathscr{L}$ is not r.e. compact. Then $\mathscr{L}$ does not have the Beth property (hence interpolation fails for $\mathscr{L}$), by a well-known general fact in abstract model theory, to the effect that the Beth property implies r.e. compactness in every finitely generated logic (see, for example, Väänänen [1977b], or Makowsky-Shelah [1979b, Theorem 6.1], or Theorem XVII.4.2.9 of the present volume).

Case 2. $\neg\exists\alpha \in \mathscr{L}_{\omega\omega}$ whose denumerable models are exactly those in $I\mathfrak{A}$.

Subcase 2.1. $\exists\mathfrak{B}$ denumerable such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B} \not\cong \mathfrak{A}$.
Let $\{I_n\}_{n<\omega}: \mathfrak{A} \cong_\omega \mathfrak{B}$, as given by the Fraïssé–Ehrenfeucht characterization of $\equiv$ (see Chapter II.4.2). Rename the sorts and symbols of $\mathfrak{B}$. Let $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{B}, I_0, \omega, <, L, J, f \rangle$, where $L^{\mathfrak{M}}np$ iff $p \in I_n$ (for $p \in I_0$, $n \in \omega$), $J^{\mathfrak{M}}pab$ iff $p(a) = b$ (for $a \in A$, $b \in B$), $f^{\mathfrak{M}}$ maps A one–one onto B. Take a finite subtheory T of $\mathrm{Th}_{\mathscr{L}}\,\mathfrak{M}$ such that for every $\mathfrak{M}' \models T$, $\mathfrak{M}' = \langle \mathfrak{A}', \mathfrak{B}', I_0', D', <', L', J', f' \rangle$, $\langle D', <' \rangle$ is still a discrete linear order with first element, L', J', still codes in $\mathfrak{M}'$ a D'-sequence of sets of partial isomorphisms with the back-and-forth property so that $\mathfrak{A}' \equiv \mathfrak{B}'$, f' maps A' one–one onto B', $\mathfrak{A}' \in I\mathfrak{A}$ and $\mathfrak{B}' \notin I\mathfrak{A}$. For details about T, see, for example, Flum [1975b, proof of Lindström's theorem]. If $\mathscr{L}$ is (ω, ω)-compact (*absurdum* hypothesis) then it would be consistent to assume that $\langle D', <' \rangle$ has an infinitely descending chain. Hence, $\mathfrak{A}' \cong_p \mathfrak{B}'$. Whence, $\mathfrak{A}' \cong \mathfrak{B}'$ by Karp's back-and-forth argument, since f' ensures that $\mathfrak{B}'$ is denumerable also. But, then, the basic isomorphism axiom for $\mathscr{L}$ implies that $\mathfrak{B}' \in I\mathfrak{A}$—a contradiction. We have thus actually proved that $\mathscr{L}$ is not r.e. compact. Hence, by the well-known general results quoted above (see Theorem XVII.4.2.9), $\mathscr{L}$ does not have the Beth (resp., Craig) property.

Subcase 2.2. $\neg\exists\mathfrak{B}$ denumerable such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B} \not\cong \mathfrak{A}$.
For $n = 1, 2, \ldots,$ there are $\mathfrak{B}_n \not\cong \mathfrak{A}$, $|B_n| = \omega$, and $\{I_0, \ldots, I_n\}$ such that $\{I_0, \ldots, I_n\}: \mathfrak{A} \cong_n \mathfrak{B}_n$ (otherwise $\exists\alpha \in \mathscr{L}_{\omega\omega}$ whose denumerable models are exactly those in $I\mathfrak{A}$, by the Fraïssé–Ehrenfeucht characterization of $\equiv$, thus, we contradict our assumptions). So let $\mathfrak{M}_n = \langle \mathfrak{A}, \mathfrak{B}_n, I_0, \omega, <, L, J, f, s \rangle$ as in the above proof of Subcase 2.1, where s is the successor function. Let T_n be a finite theory such that for every $\mathfrak{M}_n' \models T_n$, L' and J' code a finite sequence of sets of partial isomorphisms $\{I_0', \ldots, I_{s\ldots s(0)}'\}: \mathfrak{A}' \cong_n \mathfrak{B}_n'$, with $|B_n'| = |A|$, $\mathfrak{A}' \in I\mathfrak{A}$, $\mathfrak{B}_n' \notin I\mathfrak{A}$. Now, $T = \bigcup T_n$ is inconsistent, by the Fraïssé–Ehrenfeucht characterization of $\equiv$ as well as by our assumptions, and T yields a counterexample to the r.e. compactness of $\mathscr{L}$. Thus, the Beth property also must fail for $\mathscr{L}$, for we can argue as at the end of Case 1. The examination of Subcase 2.2 concludes the proof of our theorem. $\square$

For $\mathfrak{A}$ a structure of finite relational type τ, let $Q^{PI\mathfrak{A}}$ have as its defining class $PI\mathfrak{A} = \{\mathfrak{B} \mid \mathfrak{B} \cong_p \mathfrak{A}\} = \{\mathfrak{B} \mid \mathfrak{B} \equiv_{\mathscr{L}_{\infty\omega}} \mathfrak{A}\}$. Clearly, we have that $\mathscr{L}(Q^{PI\mathfrak{A}}) \equiv \mathscr{L}_{\omega\omega}$ iff $PI\mathfrak{A} \in \mathrm{EC}_{\mathscr{L}_{\omega\omega}}$.

2.3.2 Theorem. *Assume that $PI\mathfrak{A} \notin EC_{\mathscr{L}_{\omega\omega}}$, where $\mathfrak{A}$ need not be denumerable. Then, $\Delta\mathscr{L}(Q^{PI\mathfrak{A}}) \geq \Delta\mathscr{L}(Q_0)$. In particular, $\mathscr{L}(Q^{PI\mathfrak{A}})$ is not (ω, ω)-compact and does not have the Beth property. Moreover $\mathscr{L}(Q^{PI\mathfrak{A}})$ is not axiomatizable.*

Proof. The proof is by cases. We begin with

Case 1. $\exists\mathfrak{B}$ such that $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B} \not\cong_p \mathfrak{A}$.
Let $\{I_n\}_{n<\omega}: \mathfrak{A} \cong_\omega \mathfrak{B}$ and $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{B}, I_0, \omega, <, L, J\rangle$ with L and J coding $\{I_n\}_{n<\omega}$ as in the proof of Subcase 2.1 of Theorem 2.3.1. By a similar argument, we exhibit a finite subtheory T of $\mathrm{Th}_{\mathscr{L}}\mathfrak{M}$ from which a counterexample to r.e. compactness can be obtained. Hence, the Beth property fails also for $\mathscr{L} = \mathscr{L}(Q^{PI\mathfrak{A}})$. A closer examination of T shows that $\langle \omega, <\rangle$ is $\mathrm{RPC}_{\mathscr{L}}$; and, hence, $\mathscr{L}$ is not axiomatizable, by Gödel's incompleteness theorem.

Case 2. $\mathfrak{B} \equiv \mathfrak{A}$ implies $\mathfrak{B} \cong_p \mathfrak{A}$.
Then, for $n = 1, 2, \ldots$, there is a $\mathfrak{B}_n$ such that $\mathfrak{B}_n \cong_n \mathfrak{A}$, and $\mathfrak{B}_n \notin PI\mathfrak{A}$ (otherwise, $PI\mathfrak{A}$ would be EC in $\mathscr{L}_{\omega\omega}$). Now argue as in Subcase 2.2 of Theorem 2.3.1, to obtain a counterexample T to r.e. compactness and hence to the Beth property in $\mathscr{L}(Q^{PI\mathfrak{A}})$. Indeed, T is a recursive set of sentences so that, by a trick method which goes back to Craig and Vaught [1958], one can code T into a single sentence whose $<$-reducts are all isomorphic to $\langle \omega, <\rangle$. Thus, $\langle \omega, <\rangle$ is RPC in $\mathscr{L}(Q^{PI\mathfrak{A}})$, and the proof is concluded by arguing as in Case 1. □

Remarks. Barwise [1974a] proved that $\Delta\mathscr{L}(Q_0) = \mathscr{L}_{\omega^+}$, where $\omega^+ = \omega_1^{\mathrm{CK}}$ is the smallest admissible set to which ω belongs (see also XVII.3.2.2). More generally, for $U \subseteq \omega$, let $\langle \omega, U\rangle^+$ denote the smallest admissible set having ω and U as its elements: then we have

2.3.3 Theorem. $\Delta\mathscr{L}(Q^{PI\langle \omega, <, U\rangle}) \equiv \Delta\mathscr{L}(Q^{I\langle \omega, <, U\rangle}) \equiv \mathscr{L}_{\langle \omega, U\rangle^+}$.

Proof. The reader is referred to Makowsky–Shelah–Stavi [1976, Theorem 4.1]. See also Theorem XVII.3.2.3 of this volume. □

2.4. Bibliographical Notes

The quantifiers Q^{R} and I were introduced respectively by Rescher [1962] and Härtig [1965]. Failure of (ω, ω)-compactness and axiomatizability for $\mathscr{L}(I)$ was proven by Yasuhara [1969] and Issel [1969]. The latter author also proved that ω_ω is the Hanf and the Löwenheim number of the fragment of $\mathscr{L}(I)$ with equality and otherwise only unary relation symbols. Proposition 2.1.3(i) goes back to Lindström [1966a, p. 192]. For Theorem 2.1.4, see, for example, Väänänen [1978] and Pinus [1979b]. Lower bounds for the Hanf number of $\mathscr{L}(I)$ were also discussed by Fuhrken [1972] and Pinus [1978]. Further information on $\mathscr{L}(I)$ can be obtained from Väänänen's papers quoted in Section 2.1 as well as from Väänänen [1978, 1979b]. Named after C. C. Chang, the quantifier Q^{C} is studied in detail in Bell–Slomson [1969]; the fragment containing $=$ but otherwise only unary relations, was studied by Slomson [1968], who proved that ω is both its

Löwenheim and its Hanf number. He also proved the decidability of this fragment—a proof of the decidability of the corresponding fragments of $\mathscr{L}(Q^{\mathrm{H}})$ and $\mathscr{L}(I)$ can be found in Krynicki–Lachlan [1979]. An axiomatization of the fragment of $\mathscr{L}(Q^{\mathrm{C}})$—without equality—was given by Yasuhara [1966a]. The quantifiers S, S_{DLO}, S_{EQ} and their relativized versions are presented in Väänänen [1980a]. The quantifiers $Q^{I\mathfrak{A}}$ and $Q^{PI\mathfrak{A}}$ are studied in Makowsky–Shelah–Stavi [1976]. The reader should also see Makowsky [1973] for more in this connection.

3. Cardinality, Equivalence, Order Quantifiers and All That

In this section, we will consider quantifiers which assert that a structure has a certain property. In Section 3.1 we will study properties of sets and equivalence relations. In Section 3.2, we shall focus attention on linear orderings. Other cases are examined in Section 3.3.

3.1. Cardinality and Equivalence Quantifiers

Let Q have a class of sets as its defining class. By the isomorphism property, Q must express some property of cardinals. As a typical example, consider the quantifier Q_α which asserts that "there are at least ω_α-many elements", where α is an ordinal ≥ 0. The Q_α's are extensively studied in Chapters IV and V. The following result extends to quantifiers of the form $Qx_1 \ldots x_n \varphi_1(x_1), \ldots, \varphi_n(x_n)$.

3.1.1 Theorem. *Assume that each quantifier Q^i occurring in* (i) *through* (iii) *below is a class of sets. Furthermore*:

(i) *Let $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$. If $\mathscr{L}$ is (ω, ω)-compact and Δ-closed, then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*
(ii) *Let $\mathscr{L} = \mathscr{L}(Q^1, \ldots, Q^n)$. If $\mathscr{L}$ obeys interpolation, then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*
(iii) *For $\alpha \geq 1$ a fixed ordinal, let $\mathscr{L} = \mathscr{L}(Q_\alpha, Q^i)_{i \in I}$. Then $\mathscr{L}$ is not Δ-closed.*

For the proof of this result we need the following

3.1.2 Lemma. *For $\kappa, \lambda \geq \omega$, let $\mathfrak{A}^\kappa_\lambda = \langle A, E \rangle$, where E is an equivalence relation on A having λ equivalence classes, each of cardinality κ. Let $\mathscr{L}^0 = \mathscr{L}(Q^j)_{j \in J}$, where each Q^j is a class of sets. Then $\mathfrak{A}^\kappa_\kappa \equiv_{\mathscr{L}^0} \mathfrak{A}^\kappa_\omega$.*

Proof of Lemma 3.1.2. Let $\mathscr{L}_C = \mathscr{L}_{\infty\omega}(Q_\alpha)_{\alpha \in On}$. Then, $\mathfrak{A}^\kappa_\kappa \equiv_{\mathscr{L}_C} \mathfrak{A}^\kappa_\omega$, as was observed by Caicedo [1979, p. 93] with the help of a back-and-forth argument (this refines Keisler's proof that $\mathfrak{A}^{\omega_1}_\omega \equiv_{\mathscr{L}(Q_1)} \mathfrak{A}^{\omega_1}_{\omega_1}$; see II.4.2.8). We also have that $\mathscr{L}_C$-equivalence is finer than $\mathscr{L}^0$-equivalence, as was proven in the same paper by Caicedo ([1979, Lemma 4.2]). Also see Väänänen [1977c].

Proof of Theorem 3.1.1. (i) Assume that $\mathscr{L} \not\equiv \mathscr{L}_{\omega\omega}$. Then, by definition of $\mathscr{L}$, there is a sentence φ in the pure identity language of $\mathscr{L}$ which is not equivalent to any $\mathscr{L}_{\omega\omega}$-sentence. We now consider

Case 1. For some $\lambda > \omega$, λ and ω are separated by φ (say, $\omega \models_{\mathscr{L}} \varphi$ and $\lambda \not\models_{\mathscr{L}} \varphi$). Using a *choice function* from λ into $\mathfrak{A}^{\lambda}_{\lambda}$ (that is, a bijection from λ onto $\mathfrak{A}^{\lambda}_{\lambda}/E$) and a choice function from ω into $\mathfrak{A}^{\lambda}_{\omega}$, we see that $\mathfrak{A}^{\lambda}_{\lambda}$ and $\mathfrak{A}^{\lambda}_{\omega}$ belong to complementary RPC classes in $\mathscr{L}$. So, if we use Δ-closure, $\mathfrak{A}^{\lambda}_{\lambda}$ and $\mathfrak{A}^{\lambda}_{\omega}$ can be separated by some sentence in $\mathscr{L}$—thus contradicting Lemma 3.1.2.

Case 2. For every $\lambda > \omega$, $\omega \models_{\mathscr{L}} \varphi$ iff $\lambda \models_{\mathscr{L}} \varphi$ (say, $\omega \models_{\mathscr{L}} \varphi$).

Subcase 2.1. $\exists n < \omega$ such that φ has no model of cardinality $> n$.
Then, without loss of generality, φ has no finite models, so that $\mathscr{L} \geq \mathscr{L}(Q_0)$, and $\mathscr{L}$ is not even r.e. compact.

Subcase 2.2. Both φ and $\neg\varphi$ have arbitrarily large finite models.
Then, the theory whose sentences are $\neg\varphi$, $\exists^{\geq 1}x(x = x)$, $\exists^{\geq 2}x(x = x), \ldots$ is a counterexample to r.e. compactness.

Subcase 2.3. φ has arbitrarily large finite models, but $\neg\varphi$ does not.
Then φ is first-order, contradicting our assumption.

(ii) By inspection of the proof of (i), we see that r.e. compactness and Δ-closure are actually sufficient to imply that $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$. But, if $\mathscr{L}$ obeys interpolation, then $\mathscr{L}$ has both Beth and Δ-closure. Hence, $\mathscr{L}$ is r.e. compact, since $\mathscr{L}$ is finitely generated by assumption.

(iii) Using choice functions, we see that $\mathfrak{A}^{\omega_\alpha}_{\omega_\alpha}$ and $\mathfrak{A}^{\omega_\alpha}_{\omega}$ belong to complementary RPC classes of $\mathscr{L}(Q_\alpha)$, and hence of $\mathscr{L}$ also. If $\mathscr{L}$ were Δ-closed, then some sentence in $\mathscr{L}$ would separate these two structures, thus contradicting Lemma 3.1.2. This completes the proof of the theorem. $\square$

Let $X = \{\langle A, E\rangle \mid E$ is an equivalence relation on $A\}$. Then Q is an *equivalence quantifier* iff its defining class is a subclass of X.

3.1.3 Theorem. *Let $\mathscr{L}$ be a compact logic with the interpolation and the Feferman–Vaught property* (FVP). *Let Q be an equivalence quantifier which is* EC *in $\mathscr{L}$. Then Q is* EC *in $\mathscr{L}_{\omega\omega}$.*

Proof. We pose a denial, and let K be a class of equivalence relations which is EC in $\mathscr{L}$ but not in $\mathscr{L}_{\omega\omega}$. Then K must separate two elementarily equivalent structures $\mathfrak{A} = \langle A, E\rangle$ and $\mathfrak{A}' = \langle A', E'\rangle$ (say, $\mathfrak{A} \in K$ and $\mathfrak{A}' \notin K$) by a familiar open cover argument using the compactness of $\mathscr{L}$ (for a similar argument see, for example, Theorem III.1.1.5). We now proceed by cases:

Case 1. Each equivalence class of $\mathfrak{A}$ and $\mathfrak{A}'$ has infinitely many elements.
Thus, let N and M be infinite sets such that $|N| = \omega$, $|M| > |A \cup A'|$, $N \equiv_{\mathscr{L}} M$. Such sets N and M clearly exist by the assumed compactness of $\mathscr{L}$. By FVP, we

have that $[\mathfrak{A}, \mathfrak{A}', N] \equiv_{\mathscr{L}} [\mathfrak{A}, \mathfrak{A}', M]$ (as three-sorted structures). By adding two functions f and f', we can expand $[\mathfrak{A}, \mathfrak{A}', M]$ to a model of the sentence φ which asserts that "f and f' are injections of A and A' respectively into the third sort s_3". On the other hand, $[\mathfrak{A}, \mathfrak{A}', N]$ can be expanded to a model of the conjunction ψ of the sentence asserting that "sort s_3 is injected by h and h' into each equivalence class of $\mathfrak{A}$ and $\mathfrak{A}'$, respectively" (where h is, for example, a binary function $h(x, z)$, x in the first sort, $z \in s_3$, and $h(x, \cdot)$ maps s_3 one–one into the equivalence class of x in $\mathfrak{A}$) and of the sentence which asserts that "either g_0 is a bijection of $\mathfrak{A}/E$ onto $\mathfrak{A}'/E'$, or g and g' are injections of s_3 into $\mathfrak{A}/E$ and $\mathfrak{A}'/E'$, respectively". Since $\mathscr{L}$ has compactness and interpolation, then $\mathscr{L}$ satisfies Robinson's consistency, to the effect that $\varphi \wedge \psi$ has a model $[\mathfrak{B}, \mathfrak{B}', P, \ldots]$ which is also a model of $\mathrm{Th}_{\mathscr{L}}[\mathfrak{A}, \mathfrak{A}', N]$ (that is, a model of $\mathrm{Th}_{\mathscr{L}}[\mathfrak{A}, \mathfrak{A}', M]$). In this model, we have that $\mathfrak{B} \cong \mathfrak{B}'$ by the Cantor–Bernstein theorem, and $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}$, $\mathfrak{B}' \equiv_{\mathscr{L}} \mathfrak{A}'$, thus contradicting the isomorphism axiom for $\mathscr{L}$, since K separates $\mathfrak{A}$ and $\mathfrak{A}'$.

Case 2. Each equivalence class of $\mathfrak{A}$ and $\mathfrak{A}'$ has finitely many elements.
Then let $n = 1, 2, \ldots$. Let κ_n, κ_n' be such that in $\mathfrak{A}$ there are κ_n equivalence classes with n elements and in $\mathfrak{A}'$ there are κ_n' such classes. If κ_n is finite, then $\kappa_n = \kappa_n'$ (since $\mathfrak{A} \equiv \mathfrak{A}'$). If κ_n is infinite, then ω can be injected into $\mathfrak{A}/E$ and into $\mathfrak{A}'/E'$. Let $[\mathfrak{A}, \mathfrak{A}', N] \equiv_{\mathscr{L}} [\mathfrak{A}, \mathfrak{A}', M]$ be as above. Then $[\mathfrak{A}, \mathfrak{A}', N]$ can be expanded to a model of the sentence asserting that "f is an injection showing that there are more ($\geq$) than $|N|$ equivalence classes with n elements in $\mathfrak{A}$, and f' does the same for $\mathfrak{A}'$, or else g_0 is a bijection showing that such classes are as many in $\mathfrak{A}$ as in $\mathfrak{A}'$". On the other hand, $[\mathfrak{A}, \mathfrak{A}', M]$ can be expanded to a model of the sentence which asserts that "h and h' show that there are less ($\leq$) than $|M|$ equivalence classes in $\mathfrak{A}$ and $\mathfrak{A}'$ with n elements, or else g is a bijection showing that such classes are as many in $\mathfrak{A}$ as in $\mathfrak{A}'$". Using Robinson's theorem as was done in Case 1, we exhibit a model $[\mathfrak{B}, \mathfrak{B}', P, \ldots]$ of all these sentences together, and of $\mathrm{Th}_{\mathscr{L}}[\mathfrak{A}, \mathfrak{A}', N]$ as well so that $\mathfrak{B} \cong \mathfrak{B}'$ (since $\kappa_n = \kappa_n'$ for all $n \in \omega$), $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ and $\mathfrak{A}' \equiv_{\mathscr{L}} \mathfrak{B}'$, again contradicting $\mathfrak{A} \in K$ and $\mathfrak{A}' \notin K$.

Case 3. Neither Case 1, nor Case 2 occurs.
Then let $\mathfrak{A}_1$ be the substructure of $\mathfrak{A}$ only containing the equivalence classes having infinitely many elements, and let $\mathfrak{A}_2$ be the substructure of $\mathfrak{A}$ containing the equivalence classes with finitely many elements. Let $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ be similarly defined with regard to $\mathfrak{A}'$. Then $\mathfrak{A}_1 \equiv \mathfrak{A}_1'$, and $\mathfrak{A}_2 \equiv \mathfrak{A}_2'$ (by using standard results of first-order model theory, as $\mathfrak{A} \equiv \mathfrak{A}'$); so, by the arguments given for Cases 1 and 2, we see that $\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_1'$ and $\mathfrak{A}_2 \equiv_{\mathscr{L}} \mathfrak{A}_2'$. By FVP, we have that $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv_{\mathscr{L}} [\mathfrak{A}_1', \mathfrak{A}_2']$. Now consider structure $\mathfrak{M} = [\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}, f, g]$, where f and g are the canonical embeddings of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively into $\mathfrak{A}$. Let $\mathfrak{M}'$ be similarly defined, using new symbols for f', g' and $\mathfrak{A}'$. If $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{A}'$ (*absurdum* hypothesis), then $[\mathfrak{A}_1, \mathfrak{A}_2]$ and $[\mathfrak{A}_1', \mathfrak{A}_2']$ have expansions $\mathfrak{M}$ and $\mathfrak{M}'$ with $\mathrm{Th}_{\mathscr{L}} \mathfrak{M} \cup \mathrm{Th}_{\mathscr{L}} \mathfrak{M}'$ inconsistent. Hence, by Robinson's consistency, they are $\mathscr{L}$-inequivalent, thus contradicting the fact that $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv_{\mathscr{L}} [\mathfrak{A}_1', \mathfrak{A}_2']$. Therefore, $\mathfrak{A}$ and $\mathfrak{A}'$ must be $\mathscr{L}$-equivalent. This, in turn, contradicts our initial *absurdum* hypothesis according to which K separates $\mathfrak{A}$ and $\mathfrak{A}'$. $\square$

As actual examples of equivalence quantifiers, consider the Q_α^{E} ($\alpha \geq 0$), which are defined by $Q_\alpha^{\mathrm{E}} = \{\langle A, E\rangle \mid E$ is an equivalence relation on A with at least ω_α distinct equivalence classes$\}$. See Feferman [1975]. We thus have

3.1.4 Theorem. (i) $\mathscr{L}(Q_\alpha) \not\geq \mathscr{L}(Q_\alpha^{\mathrm{E}})$, *whenever* $\alpha > 0$;

(ii) $\Delta\mathscr{L}(Q_\alpha) \equiv \Delta\mathscr{L}(Q_\alpha^{\mathrm{E}})$;

(iii) $\mathscr{L}(Q_\alpha)$ *and* $\mathscr{L}(Q_\alpha^{\mathrm{E}})$ *have recursively isomorphic sets of valid sentences, the same Hanf and same Löwenheim numbers (the latter being equal to* ω_α*), and equal compactness spectrum*;

(iv) *interpolation fails for* $\mathscr{L}(Q_\alpha^{\mathrm{E}})$, *if* ω_α *is regular.*

Proof. (i) Immediate from Lemma 3.1.2.

(ii) Using choice functions, one sees that $\Delta\mathscr{L}(Q_\alpha^{\mathrm{E}}) \leq \Delta\mathscr{L}(Q_\alpha)$; on the other hand, using the equivalence relation given by identity ($=$), one also sees that $\Delta\mathscr{L}(Q_\alpha^{\mathrm{E}}) \geq \Delta\mathscr{L}(Q_\alpha)$.

(iii) Immediate from (ii), together with standard results of abstract model theory; there is no problem for the Hanf number in this case, see Corollary XVII.4.3.4 and Section 4.5 in the same Chapter.

(iv) for ω_α regular, $\Delta\mathscr{L}(Q_\alpha)$ does not obey interpolation (see Hutchinson [1976b]). ☐

Thus each $\mathscr{L}(Q_\alpha^{\mathrm{E}})$ is closely related to $\mathscr{L}(Q_\alpha)$; the latter logics are studied extensively in Chapters IV and V.

3.2. Order Quantifiers

Let $X = \{\langle A, R\rangle \mid R$ is a partial ordering relation$\}$. Then Q is an *order quantifier* iff its defining class is a subclass of X. One notable example of an order quantifier, $Q^{\mathrm{cf}\,\omega} = \{\langle A, <\rangle \mid <$ is a linear ordering of cofinality $\omega\}$, has been discussed in detail in Chapter II, where it is proved that $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ is compact, axiomatizable and has Löwenheim number equal to ω_1. From a general result due to Ebbinghaus [1975b] one can infer that $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ does not have the interpolation property (Counterexample II.7.1.3(c)). It is an open problem whether there exist extensions of $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ generated by a set of quantifiers and satisfying Robinson's consistency theorem. Such extensions (if any) would have many syntactic and algebraic properties in common with first-order logic.

The order quantifier Q^D gives rise to a logic having many properties in common with $\mathscr{L}(Q_1)$, where $Q^D = \{\langle A, <\rangle \mid <$ is a dense linear ordering with a countable dense subset$\}$. As a matter of fact we have:

3.2.1 Theorem. (i) $\mathscr{L}(Q^D)$ *is* (ω, ω)*-compact, axiomatizable and its Löwenheim number is* ω_1;

(ii) $\mathscr{L}(Q^D)$ *does not have the interpolation property.*

Proof. (i) $Q^D xy\, \varphi(x, y)$ iff $\{\langle x, y\rangle \mid \varphi(x, y)\}$ is a dense linear ordering and aa s "s is dense in the ordering $\varphi(x, y)$". Now refer to Section IV.4.

(ii) From Ebbinghaus [1975b]. ☐

By contrast with Q^D, an order quantifier having many properties in common with the quantifier I is $R = \{\langle A, <\rangle \mid <$ has the order type of a regular cardinal$\}$. Note that $\Delta\mathscr{L}(Q^R) \leq \Delta\mathscr{L}(R) \leq \Delta\mathscr{L}^{\mathrm{mII}}$ (for the first inclusion, note that $|A| < |B|$ iff there is a regular cardinal, namely $|A|^+$, and injections of $|A|^+$ into B and of A into an initial segment of $|A|^+$; for the second inclusion, proceed as in Claim 1 of Theorem 1.2.2). Also, by saying that a discrete linear ordering with first element has the order type of a regular cardinal, we can characterize $\langle \omega, <\rangle$ in $\mathscr{L}(R)$. Hence, the latter is not (ω, ω)-compact, not axiomatizable, and does not have the weak Beth property (see Proposition 2.1.2). In addition, Proposition 2.1.3 and Theorem 2.1.5 above can be applied to $\mathscr{L}(R)$ as well.

3.2.2 Theorem. (i) *If $V = L$, then $\Delta\mathscr{L}(I) \equiv \Delta\mathscr{L}(R) \equiv \Delta\mathscr{L}^{\mathrm{mII}}$;*

(ii) *the fact that $\Delta\mathscr{L}(R) \not\equiv \Delta\mathscr{L}(I)$ is consistent, if "*ZF *+ there are uncountably many measurable cardinals" is consistent;*

(iii) *the fact that $\Delta\mathscr{L}(R) \not\equiv \Delta\mathscr{L}^{\mathrm{mII}}$ is consistent, if* ZF *is consistent.*

Proof. The argument for (i) is by the above discussion and by Theorem 2.1.4.

(ii) See Väänänen [1978, 3.1];

(iii) See Väänänen [1980b, Corollary 3.2.5 and the remark following it]. See also Chapter XVII, *passim*. ☐

Our final example of an order quantifier is the *well-order quantifier W*, which is defined by $\mathfrak{A} \models Wxy\,\varphi(x, y)$ iff $\{\langle x, y\rangle \in A^2 \mid \mathfrak{A} \models \varphi(x, y)\}$ well-orders its field. Clearly, we have that $\langle \omega, <\rangle$ can be characterized by a sentence of $\mathscr{L}(W)$, whence $\mathscr{L}(W)$ is not (ω, ω)-compact, not axiomatizable, and does not have the weak Beth property. Theorem 2.1.5(i) can be applied to $\mathscr{L}(W)$ with the same proof.

3.2.3 Theorem. *Let $\mathscr{L} = \mathscr{L}(W)$. Then we have:*

(i) *The gödelized set of valid sentences of $\mathscr{L}$ is the complete Π^1_2 subset of ω;*

(ii) *the Löwenheim number of $\mathscr{L}$ is ω;*

(iii) *$\Delta\mathscr{L} < \Delta\mathscr{L}_{\omega_1\omega_1}$, and $\Delta\mathscr{L} < \Delta\mathscr{L}(I)$;*

(iv) *assuming that $V = L$, the Hanf number of $\mathscr{L}$ equals the Löwenheim number of $\mathscr{L}^{\mathrm{mII}}$;*

(v) *the smallest logic $\mathscr{L}' \geq \mathscr{L}$ having the Beth property is not Δ-closed;*

(vi) *the smallest logic $\mathscr{L}'' \geq \mathscr{L}$ having the weak Beth property is not a sublogic of $\mathscr{L}_{\infty\infty}$.*

Proof. (i) follows from Kotlarski [1978, p. 126]. In this connection, the reader should also see Corollary XVII.4.3.7 of the present volume.

(ii) We extend the usual proof of the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega\omega}$ by witnessing also that $\neg Wxy\varphi(x, y)$ with the help of an infinitely descending chain of constants.

(iii) is immediate from (ii).

(iv) See Väänänen [1979b, p. 316].

(v) See Makowsky–Shelah [1979b, p. 222]. Note that, as a consequence, the Beth property does not imply Craig interpolation.

Finally, for (vi) see Theorem XVII.4.1.3. ☐

Let us conclude this subsection with a note on some general facts about binary quantifiers. Assert that Q is *binary* iff Q has the form $Qx_1y_1 \ldots x_ny_n\,\varphi_1(x_1, y_1), \ldots, \varphi_n(x_n, y_n)$. Krynicki–Lachlan–Väänänen [1984] have proven negative results concerning binary quantifiers along the lines of the negative results about monadic and equivalence quantifiers that were given in Section 3.1 above. For example, binary quantifiers cannot count the dimension of a vector space in much the same way as monadic quantifiers cannot count the number of equivalence classes of an equivalence relation. Furthermore, there exists a ternary quantifier which is not definable by using binary quantifiers only.

3.3. *Other Quantifiers*

In this subsection we briefly deal with other quantifiers occurring in the literature. The reader is referred to Chapter IV for the "almost all" quantifier aa, as well as for the Magidor–Malitz quantifiers. Other classes of quantifiers are considered in Chapter III. Quantifiers arising in connection with infinitary languages are dealt with in Part C. For second-order quantifiers see Chapters XII and XIII. Quantifiers for enriched structures are studied in Chapter XV (but see also Section 4 below).

To introduce our next class of quantifiers we need the following:

3.3.1 Definition. A class K of structures is *inductive* iff it is closed under unions of chains (with respect to the substructure relation $\subseteq$). For λ a cardinal, K is λ-*inductive* iff K is closed under λ-unions, where $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta (\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq, \ldots)$ is a λ-*union* iff for every $B \subseteq A (= \bigcup A_\beta)$ with $|B| < \lambda$, there is $\beta < \alpha$ such that $B \subseteq A_\beta$.

This notion clear, then we have

3.3.2 Theorem. *If K is an arbitrary class of structures of type τ, with K closed under isomorphism, and if λ is an arbitrary cardinal, the following are equivalent*:

(i) *Both K and its complement $\bar{K}$ are λ-inductive*;
(ii) $\forall \mathfrak{A} \in \mathrm{Str}(\tau)\, \exists \mathfrak{A}_0 \subseteq \mathfrak{A}$, *with* $|A_0| < \lambda$ *such that* $\forall \mathfrak{B}$, $\mathfrak{A}_0 \subseteq \mathfrak{B} \subseteq \mathfrak{A}$ *implies* ($\mathfrak{A}_0 \in K$ *iff* $\mathfrak{B} \in K$).

Proof. See Makowsky [1975b, Theorem 2.16]. ☐

Following Makowsky [1975b], we call any class (or quantifier) Q, λ-*securable* iff Q satisfies either of conditions (i) or (ii) in Theorem 3.3.2 above. ω-securable classes are called *continuous* by Tharp [1974]. From the definition it follows that $\exists$ and $\forall$ are 2-securable, Q_α is $\omega_{\alpha+1}$-securable, W is ω_1-securable, Q^{C} is never λ-securable. Moreover, Q^{D} is ω_2-securable if there is no Suslin tree (see Makowsky [1975b]). We also have

3.3.3 Theorem. *Let K be an arbitrary class of type τ, with K closed under isomorphism*:

(i) *K is n-securable, for some $n \in \omega$, iff both K and $\bar{K}$ can be defined by $\forall\exists$-sentences of $\mathscr{L}_{\omega\omega}$*;

(ii) *K is ω-securable iff both K and $\bar{K}$ are inductive;*

(iii) *if K is ω-securable and has type $\tau' = \{U_1, \ldots, U_m\}$, where each U_i is a unary relation, then K is* EC *in $\mathscr{L}_{\omega\omega}$;*

(iv) *for λ a regular cardinal, let $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$, where each Q^i is λ-securable; then, if $\lambda \leq \omega$, the Löwenheim number of $\mathscr{L}$ is ω; if $\lambda > \omega$, then each consistent sentence of $\mathscr{L}$ has a model of cardinality $< \lambda$; in particular, the Löwenheim number of $\mathscr{L}(W)$ is ω;*

(v) *$\mathscr{L}_{\omega_1\omega}$ is the smallest Δ-closed logic containing all the ω-securable quantifiers.*

Proof. (i) See Makowsky [1975b, Corollary 3.11]. Observe here that Tharp [1973] proved that if K is n-securable, then K is EC in $\mathscr{L}_{\omega\omega}$.

(ii) See Makowsky [1975b, Theorem 2.14]; but also see Miller [1979].

(iii) See Tharp [1974, Theorem 5].

(iv) See Tharp [1974, Theorem 7], for the case $\lambda = \omega$; see Makowsky [1975b, Theorem 2.1], however, for the general case. Recall that W is an ω_1-securable quantifier.

(v) See Makowsky [1975b, Corollary 5.6]. □

We now deal with quantifiers which are used to express the fact that "there exist large sets of indiscernibles". Given a structure $\mathfrak{A} \in \mathrm{Str}(\tau)$, let $q^I_{\mathfrak{A}} = \{B \subseteq A \mid B$ contains an infinite set of order indiscernibles in $\mathfrak{A}\}$, and let $q^F_{\mathfrak{A}} = \{B \subseteq A \mid B$ contains arbitrarily large finite sequences of indiscernibles in $\mathfrak{A}\}$. The resulting logics, $\mathscr{L}(Q^I)$ and $\mathscr{L}(Q^F)$ are syntactically the same as, for example, $\mathscr{L}(Q_1)$. Moreover, their semantics is obtained by letting, for instance,

$$\mathfrak{A} \models Q^I x\, \varphi(x) \quad \text{iff} \quad \{x \in A \mid \mathfrak{A} \models \varphi(x)\} \in q^I_{\mathfrak{A}}.$$

Notice the dependence of Q^I on the whole of $\mathfrak{A}$, rather than on its universe only. Steinhorn [1980] has a number of categoricity and quantifier elimination results on Q^I and Q^F (see also [1981]). He also proves that $\mathscr{L}(Q^F)$ does not have the interpolation property.

Thomason [1966] introduced a logic $\mathscr{L}_q$ with free variables for quantifiers. The idea here was to examine those properties which are common to all generalized quantifiers. If Q is any such variable, then $Qx_1 \ldots x_n\, \varphi_1(x_1), \ldots, \varphi_n(x_n)$ is a formula of $\mathscr{L}_q$. If $\psi(\ldots Q)$ is a sentence of $\mathscr{L}_q$, then a model of $\psi(\ldots Q)$ consists of an ordinary structure $\mathfrak{M}$ together with a quantifier (in the sense of Mostowski) $\hat{Q}$ which serves as an interpretation of Q. Sentence $\psi(\ldots Q)$ is valid in $\mathscr{L}_q$ iff $(\mathfrak{M}, \hat{Q})$ satisfies $\psi(\ldots Q)$ for all structures $\mathfrak{M}$ and all interpretations $\hat{Q}$. Yasuhara [1969] wrote down a sentence characterizing the natural numbers in $\mathscr{L}_q$. Therefore, $\mathscr{L}_q$ is neither (ω, ω)-compact, nor axiomatizable. The sets of valid sentences of $\mathscr{L}(I)$ and $\mathscr{L}_q$ are recursively isomorphic.

Now let $\mathscr{L}_Q$ be just as $\mathscr{L}_q$, but with quantifier variables to be interpreted over binary quantifiers. Then, in Väänänen [1980a], it is proved that $\mathscr{L}_Q$ and $\mathscr{L}(Q^{\mathrm{H}})$ have recursively isomorphic sets of valid sentences. Roughly speaking, $\mathscr{L}_q$ is to $\mathscr{L}(I)$ as $\mathscr{L}_Q$ is to $\mathscr{L}(Q^{\mathrm{H}})$.

3.4. Bibliographical Notes

Theorem 3.1.1 is due to Caicedo [1979]. Using the Feferman–Vaught property, Makowsky [1978c] proves a stronger form of Theorems 3.1.3 and 3.1.1 for arbitrary monadic and equivalence quantifiers. The equivalence quantifiers Q^{E}_{α} were first introduced by Feferman [1975], after Keisler's counterexample to Craig's interpolation in $\mathscr{L}(Q_1)$. The quantifier $Q^{\mathrm{cf}\,\omega}$ is studied in Shelah [1975d]. For other compact quantifiers, see Rubin–Shelah [1980], where it is proved that compactness does not imply axiomatizability (if $V = L$). For the quantifier R, see Väänänen [1978, 1979b, 1980b]. For further information about Q^{D} see Makowsky–Shelah–Stavi [1976]. For free quantifier variables and their associated logics, see Thomason–Randolph Johnson Jr. [1969], Yasuhara [1966b], Bell–Slomson [1969], Väänänen [1979d, 1980a], and Anapolitanos–Väänänen [1981].

4. *Quantifiers from Robinson Equivalence Relations*

Although compactness and interpolation are often regarded as desirable properties of logics, in general quantifiers do not take care of such properties. For example, none of the logics described in the preceding sections has the Robinson property. A logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ has compactness and interpolation iff $\mathscr{L}$ has the Robinson property (see Chapter XIX): the latter only depends on $\equiv_{\mathscr{L}}$. Thus, we may naturally ask which equivalence relations $\sim$ with the Robinson property (for short, Robinson equivalence relations) do generate a nice logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$. Recall that $\sim$ is *bounded* iff for every type τ there is κ_τ such that the number of equivalence classes of $\sim$ of type τ is κ_τ. $\sim$ is *preserved under reduct* iff $\mathfrak{A} \sim \mathfrak{B}$ implies $\mathfrak{A}\restriction\tau \sim \mathfrak{B}\restriction\tau$, for each, $\tau \subseteq \tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$. Preservation under renaming is defined analogously. A quantifier Q belongs to *hull*$(\sim)$ iff $\equiv_{\mathscr{L}(Q)}$ is *coarser* than $\sim$ (that is, $\mathfrak{A} \sim \mathfrak{B}$ implies $\mathfrak{A} \equiv_{\mathscr{L}(Q)} \mathfrak{B}$). We say that $\sim$ is *separable by quantifiers* iff whenever $\mathfrak{A}$, $\mathfrak{B}$ are structures of type τ and not-$\mathfrak{A} \sim \mathfrak{B}$ there is $\tau' \subseteq \tau$ and $Q \in \mathrm{hull}(\sim)$ of type τ' such that $\mathfrak{A}\restriction\tau' \in Q$ and $\mathfrak{B}\restriction\tau' \notin Q$ (intuitively, Q separates $\mathfrak{A}$ and $\mathfrak{B}$). We finally let $\mathscr{L}(\sim) = \mathscr{L}\{Q \mid Q \in \mathrm{hull}(\sim)\}$. These ideas clear, we now recall the following results from Chapter XIX:

Theorem. *Let $\sim$ be an arbitrary bounded Robinson equivalence relation on the class of all structures and assume that $\sim$ is preserved under reduct and renaming, is coarser than $\cong$ and finer than $\equiv$. Then, adopting the above notation we have:*

(i) *$\mathscr{L}(\sim)$ is the strongest logic $\mathscr{L}$ such that $\equiv_{\mathscr{L}}$ is coarser than $\sim$;*
(ii) *if, in addition, $\sim$ is separable by quantifiers, then $\mathscr{L}(\sim)$ is the unique (up to equivalence) logic $\mathscr{L}$ such that $\equiv_{\mathscr{L}} = \sim$. Furthermore, $\mathscr{L}(\sim)$ is compact and has the Craig interpolation property.* □

Corollary. *The following hold up to equivalence*:

(i) $\mathscr{L}_{\omega\omega}$ *is the unique logic* $\mathscr{L}$ *such that* $\equiv_{\mathscr{L}} = \equiv$;

(ii) *topological logic* $\mathscr{L}_t$ *is the unique logic* $\mathscr{L}$ *such that* $\equiv_{\mathscr{L}} = \equiv^t$ *holds, where* $\equiv^t$ *is topological* ω*-partial homeomorphism*; *the open and the interior quantifiers and their n-dimensional versions are in* hull($\equiv^t$);

(iii) *the same as the first part of* (ii) *for n-dimensional monotone logic* ($n = 1, 2, 3, \ldots$). □

Note that in two-dimensional monotone logic we have a model-theoretical framework for such notions as uniform continuity and metric completeness (see Robinson [1973, p. 511]). For topological and monotone logic see Chapter XV, and Flum–Ziegler [1980]. The equivalence "Robinson Consistency = Compactness + Craig Interpolation" was first proved in Mundici [1982b] (and was announced in Mundici [1979a, b]) and, independently, in Makowsky–Shelah [1983]. The above theorem, as well as (i) of the corollary were first proven in Mundici [1982a]. Parts (ii) and (iii) of the corollary can be found in Mundici [1982c, II and 198?b].

By the above theorem, any separable Robinson equivalence relation $\sim$ canonically generates a nice set $\{Q^i\}_{i \in I}$ of quantifiers. In order to eliminate redundancy, we may restrict attention to subsets of hull($\sim$) of minimal cardinality but which are still capable of generating $\mathscr{L}(\sim)$. Once $\mathscr{L}(\sim)$ is written out as $\mathscr{L}\{Q \mid Q \in B\}$, for B any such minimal set, the quantifiers in B are enough to give a full account of all the syntactic as well as algebraic properties of $\mathscr{L}(\sim)$.

In the absence of a Kreisel-like program for quantifiers, the above theorem and corollary may also give some hints in the search of (sets of) quantifiers such as B. One might, for instance, investigate whether letting Q range over the elementary classes of $\Delta\mathscr{L}(Q^{\mathrm{cf}\,\omega})$, one can encounter an element of hull($\sim$), for $\sim$ a bounded separable Robinson equivalence relation $\neq \equiv$. As a first step in this direction, one would check whether the compact logic $\mathscr{L}(Q)$ obeys interpolation. The progression from the open and the interior quantifiers, to their multidimensional versions, and from the latter to topological logic $\mathscr{L}_t$ shows that this program is feasible. Incidentally, the rôle played by restricted second-order quantifiers for $\mathscr{L}_t$ shows that the usual first-order quantifiers do not have the sole right of producing good syntaxes (see also Chapters XII and XV in this respect).

In Mundici [1982e], the author tried to obtain Robinson equivalence relations and their associated quantifiers as a byproduct of more fundamental objects, such as (suitably generalized) back-and-forth approximations of isomorphism. Indeed, this can be done for $\equiv$ and $\equiv^t$. In addition, back-and-forth techniques already pervade (abstract) model theory.

Chapter VII
Decidability and Quantifier-Elimination

By A. BAUDISCH, D. SEESE, P. TUSCHIK, and M. WEESE

The decidability of the elementary theory for a given class K of structures reflects a certain low expressive power of the elementary language with respect to that class. Therefore, it is natural to look for stronger logics L such that K has a decidable L-theory. The rigorous establishment of decidability for the L-theory of K often provides results about the L-definable properties and L-equivalence of structures in K. This means, then, that investigations into the decidability of the L-theory of K are closely related to the L-model theory of K.

In this chapter we will investigate the decidability of such logics. We will concentrate on Malitz quantifiers (particularly on cardinality quantifiers) and Härtig quantifiers as well as on stationary logic. The first result in this direction was the decidability of the theory of unary predicates without equality in the logic with the quantifier "there are $\aleph_\alpha$ many". This result was proven in a fundamental paper by Mostowski [1957]. Topological and monadic second-order logics are treated in other chapters of this volume; and, we therefore, will not consider them here. However, we wish to emphasize at this point that results concerning the latter do have important consequences for the material that will be presented in our discussions.

Our chapter is basically organized along the lines sketched below. First, with respect to three main methods of proving decidability, there is a division into three sections which are respectively entitled *Quantifier-Elimination, Interpretations*, and *Dense Systems*. In each of these the general method is introduced and then clarified with respect to several concrete classes of structures. These classes are: the class of modules and abelian groups (Section 1), the class of well-orderings (Section 2), and the classes of linear orderings and boolean algebras (Section 3). At the end of each subsection we refer to some further results without making any claims that the discussions given present a complete picture of the material. However, the reader will find references to most of the corresponding investigations in the bibliography given at the end of the volume.

Much of the material of this chapter is related to our text (see Baudisch–Seese–Tuschik–Weese [1980]), in which the reader can find more detailed proofs as well as some similar investigations on the class of trees.

We wish to express our gratitude to Philipp Rothmaler who contributed so many of his ideas and so much of his time and energy to the creation of this chapter that we can justly say that he is a co-author of this study.

1. Quantifier-Elimination

1.1. The Framework

In general an extended language has a more expressive power than the original language. However, in many cases there are model classes which cannot be further distinguished in the extended language. Such model classes often have interesting properties, and it is this very fact that leads us to the following

Definition. Let L be a sublanguage of a language L' and K a class of L'-structures. We say that *L' is reducible to L with respect to the class K* if for every formula $\varphi(\bar{x})$ of L' there is a formula $\psi(\bar{x})$ of L such that $K \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$. L' is said to be *effectively reducible to L with respect to K* if ψ can be found effectively (depending on φ).

An important special case arises from extensions obtained by adding certain quantifiers. And this case we will examine more closely in the

Definition. Suppose L' arises from L by adding (in the canonical way) an arbitrary quantifier Q to it. If L' is reducible to L with respect to a class K, we say that *K admits the L-elimination of Q*, or *Q is L-eliminable in K*. If K is an L'-elementary class, that is, if $K = \mathrm{Mod}(\mathrm{Th}_{L'}(K))$, then we also say that $\mathrm{Th}_{L'}(K)$ *admits the elimination of Q*, or *Q is eliminable in* $\mathrm{Th}_{L'}(K)$.

The examples below show that important model-theoretic properties are reflected by the notion of reducibility.

Example. Let L' be a first-order language, K an L'-elementary class.

(1) Let L be the set of all open L'-formulas. Clearly, L' is the extension of L by adding the quantifier $\exists$.

   (a) $\mathrm{Th}_{L'}(K)$ is substructure complete iff $\exists$ is eliminable in $\mathrm{Th}_{L'}(K)$.
   (b) If, in addition, L' is the language of fields, then ACF—the class of algebraically closed fields (or $\mathrm{Th}_{L'}(\mathrm{ACF})$ since ACF is L'-elementary)—admits the elimination of the quantifier $\exists$ (see Sacks [1972]).

(2) Let L be the set of all existential L'-formulas. Then $\mathrm{Th}_{L'}(K)$ is model-complete iff L' is reducible to L with respect to K (see Sacks [1972]).

For quantifier-elimination there are two ways to look at the problem. On the one hand, we can regard the existence of a quantifier-elimination as a certain model-theoretic property, this being reflected, for instance, in Example (1a) above. On the other, we might be interested more in the manner of elimination itself. This is especially true when decidability is under consideration. If we take the first position, then we will speak of "eliminability". The case in which L is an elementary language and L' is $L(Q)$, the language obtained from L by adding a certain generalized quantifier Q (or even a set of them), is then of particular interest. In the second subsection, we will consider precisely this situation, admitting the following abuse of language.

Let K be an elementary class axiomatized by the theory T in a first-order language L, and let Q be a certain generalized quantifier. We say *Q is eliminable in K* (or *in T*) if $L(Q)$ is reducible to L with respect to the class $\mathrm{Mod}(T \cup \{Q\bar{x}(\bar{x} = \bar{x})\}$ or, equivalently, if Q is L-eliminable in the class $\{\mathfrak{M} \in K : \mathfrak{M} \models Q\bar{x}(\bar{x} = \bar{x})\}$. Notice then that for eliminability of Q in T it will suffice to eliminate Q in expressions of the form $Q\bar{x}\, \varphi(\bar{x}, \bar{z})$, where φ is first-order.

If we take the second of the positions we have noted, we will speak of "elimination procedures". Observe that by an "elimination procedure for a class K" we do not mean a procedure providing a complete elimination of a given quantifier in K, but rather one that is applicable only up to a certain set of sentences (and, in some cases, formulas also)—the so-called core sentences—which should be easy to survey. Thus, finding an elimination procedure will, in most cases, include finding an appropriate set of core sentences (and definable predicates); and, of course, it will yield eliminability results for those subclasses on which the truth values of the core sentences are constant. In the third subsection we will consider this problem for the class of modules as well as the class of abelian groups. Finally, we emphasize that throughout this section we will be mainly concerned with the Malitz quantifiers Q_α^m ($m < \omega$, α an ordinal), where in the next subsection we will concentrate on the cardinality quantifiers Q_0 $(= Q_0^1)$ and Q_1 $(= Q_1^1)$ and the Ramsey quantifiers Q_0^m ($m < \omega$).

As concerns other generalized quantifiers, we would like to draw the readers' attention to the results of Steinhorn [1980], results which once again fortify our conviction that the method of generalized quantifiers can be an excellent tool for investigations into first-order model theory.

Convention. Throughout this section the length of the sequence $\mathbf{x}$ is assumed to be equal to the arity of the given quantifier and, if not stated otherwise, this to be equal to m.

Recall that a set D is (*weakly*) *homogeneous* for a formula $\varphi(\mathbf{x}, \mathbf{a})$ in a structure $\mathfrak{M}$, $\mathbf{a} \in \mathfrak{M}$, if every m-tuple $\mathbf{d}$ of (distinct) elements of D satisfies $\varphi(\mathbf{x}, \mathbf{a})$ in $\mathfrak{M}$. For an ordinal α, the $\aleph_\alpha$-interpretation Q_α^m of the m-placed Malitz quantifier Q^m is defined for a structure $\mathfrak{M}$ of power not less than $\aleph_\alpha$ by "$\mathfrak{M} \models Q_\alpha^m x_0 \ldots x_{m-1}\, \varphi(x_0, \ldots, x_{m-1})$ iff there is a set of power $\aleph_\alpha$ in $\mathfrak{M}$ which is weakly homogeneous for φ".

Warning. As to the elimination procedure given in the third subsection of this chapter, it is essential to interpret Q^m in the way in which it was there interpreted, with emphasis on "weakly". This attribute does not play any rôle in the investigation of eliminability, since the corresponding two quantifiers (one as given above, and the other having "weakly" omitted) are expressible one by the other. This the reader can easily verify. Accordingly, in the next subsection we will use this quantifier with "weakly" omitted, in the interpretation the omission being for the sake of simplicity.

The reader should consult Chang–Keisler [1973] or Shelah [1978a] for the fundamental concepts of stability theory.

1.2. Eliminability of Generalized Quantifiers

As we have already mentioned, our aim here is to find model-theoretic properties of first-order theories which are equivalent to eliminability of certain generalized quantifiers.

Convention. In this subsection "theory" means "first-order theory having infinite models only", and T will denote such a theory. Moreover, terms such as "definable" or "formula" are used for "first-order definable" or "first-order formula". Two other points are worth mentioning at this juncture.

First, remember the warning given in the first subsection; and, second, we note that although the general concept to be treated here is due to Tuschik, the material was unfortunately, not published in full detail until the work of Baudisch–Seese–Tuschik–Weese [1980]. In this connection, the reader should also see Tuschik [1975, 1977a].

We will begin our exposition with the unary quantifier Q_0 having the interpretation "there are infinitely many", examining first the following

Definition. A formula $\varphi(x, z_1, \ldots, z_n)$ is said to *have a degree* relative to T if there is a natural number k such that, for every model $\mathfrak{M}$ of T and elements $a_1, \ldots, a_n$ of $\mathfrak{M}$, the following holds:

> if $\varphi(x, a_1, \ldots, a_n)$ has finitely many solutions in $\mathfrak{M}$, then it has at most k-many.

T is said to be *graduated* if every formula has a degree relative to T.

The facts stated in the following examples can be derived from the corresponding elimination of (elementary) quantifiers.

Examples. The first-order theories of the following classes of structures are graduated:

(1) The class ACF of algebraically closed fields;
(2) The class RCF of real closed fields;
(3) The class DCF_0 of differentially closed fields of characteristic 0,
(4) The class DLO of dense linear orderings;
(5) The class ABA of atomless boolean algebras;
(6) The class AG_p of infinite elementary abelian p-groups, where p is a prime.

Ryll-Nardzewski's theorem (see Chang–Keisler [1973, Theorem 2.3.13]) yields a wealth of graduated theories:

Proposition. *Every countable $\aleph_0$-categorical theory is graduated.* □

Examples (4), (5), and (6) above are special cases of the assertion in the proposition. The next result shows what graduatedness is related to our general topic.

1.2.1 Theorem. *T is graduated* iff *Q_0 is eliminable in T.*

Clearly, in a graduated theory, Q_0 is eliminable by $\exists^{>n}$, where n ranges over the degrees of all formulas. If Q_0 is eliminable in such a simple manner, then we say that Q_0 is *definable in T*. The theorem just stated thus asserts that if Q_0 is eliminable, then it is definable. This is, *mutatis mutandis*, true for Malitz and other "Malitz-like" quantifiers; and, moreover, it is basic for eliminability investigations. For more on this, the reader should see Baldwin–Kueker [1980]; Baudisch–Seese–Tuschik–Weese [1980]; Rothmaler–Tuschik [1982]; Vinner [1975]. We will prove it here in the following general form, a form which is appropriate for our purposes. The reader may extend it to a more general concept of quantifiers, including that of Steinhorn [1980]. Before proceeding further in this direction, however, we need some additional notation.

For the m-placed Malitz quantifier Q^m, we also introduce finitary interpretations: for a given natural number n, the n-interpretation of Q^m, in terms $Q^m_{(n)}$, is given by "$\mathfrak{M} \models Q^m_{(n)} x_0 \ldots x_{m-1}\, \varphi(x_0, \ldots, x_{m-1})$ iff there is a set of power n in $\mathfrak{M}$ which is homogeneous for $\varphi(x_0, \ldots, x_{m-1})$".

Definition. The quantifier Q^m_α is called *definable in T* if for every (first-order) formula $\varphi(\bar{x}, \bar{z})$, there is a number n_φ such that

$$T \cup \{Q^m_\alpha \bar{x}(\bar{x} = \bar{x})\} \vdash \forall \bar{z}(Q^m_{(n_\varphi)} \bar{x}\, \varphi(\bar{x}, \bar{z}) \to Q^m_\alpha \bar{x}\, \varphi(\bar{x}, \bar{z})).$$

Notice that if φ is first-order, then $Q^m_{(n)} \bar{x}\, \varphi(\bar{x})$ is first order also. Hence, Q^m_α is eliminable in T, if it is definable in T. The definability lemma given below asserts that the converse is true.

The Definability Lemma. *A Malitz quantifier is eliminable in T iff it is definable in T.*

Proof. One direction has been already mentioned. As for the other direction, suppose that $\varphi(\bar{x}, \bar{z})$ and $\psi(\bar{z})$ are (first-order) formulas such that, for $T' = T \cup \{Q^m_\alpha \bar{x}(\bar{x} = \bar{x})\}$, the following holds:

$$(*) \qquad T' \vdash \forall \bar{z}(Q^m_\alpha \bar{x}\, \varphi(\bar{x}, \bar{z}) \leftrightarrow \psi(\bar{z})).$$

We have to show that a number n_φ exists with

$$T' \vdash \forall \bar{z}(Q^m_{(n_\varphi)} \bar{x}\, \varphi(\bar{x}, \bar{z}) \to \psi(\bar{z})).$$

Thus, we assume the contrary. Then, for arbitrarily large numbers n, there are models $\mathfrak{M}_n$ of T' containing sequences $\bar{a}_n$ with $\mathfrak{M}_n \models \neg\psi(\bar{a}_n)$ and sets A_n homogeneous for $\varphi(\bar{x}, \bar{a}_n)$ which have power not less than n. Let $C = \{c_i : i < \aleph_\alpha\}$ be a set and $\bar{a}$ be a sequence of new and distinct constant symbols, and let S denote the union of the following sets of sentences in the corresponding inessential extension of the (first-order) language of T:

(1) $\{c_i \neq c_j : i < j < \aleph_\alpha\}$;

(2) $\{\varphi(\bar{c}, \bar{a}) : \bar{c} \in C^m\}$;

(3) $T \cup \{\neg\psi(\bar{a})\}$.

By assumption, every finite subset of S can be realized in some $\mathfrak{M}_n$. Thus, the compactness theorem (for first-order logic) implies the existence of a model $\mathfrak{M}$ of T consisting of a sequence $\bar{a}$ with $\mathfrak{M} \models \neg\psi(\bar{a})$ and a set A which is homogeneous for $\varphi(\bar{x}, \bar{a})$ and has power $\aleph_\alpha$. But this contradicts the assertion in (*). $\square$

Having proven the definability lemma in the most general form, we now return to unary Malitz quantifiers = usual cardinality quantifier. Theorem 1.2.1 is a special case of that lemma. Together with the proposition above, it implies that Q_0 is eliminable in every countable $\aleph_0$-categorical theory as well as in all the theories of Examples (1) through (6). Let us turn now to the $\aleph_1$-interpretation. We are going to prove a theorem which is due to Tuschik and which links the eliminability of Q_1 with the following well-known property of first-order model theory. First, recall that T has the *Vaught property* if it has a model $\mathfrak{M}$ containing an infinite definable set of power less than $|\mathfrak{M}|$.

We need also Vaught's two-cardinal theorem which asserts that a countable theory having the Vaught property possesses a model of power $\aleph_1$ containing an infinite countable definable set. For a proof of this, consult Chang–Keisler [1973, Theorem 3.2.12] or Sacks [1972, Section 22]. Interestingly enough, a good portion of it yields the next lemma; and, in fact, does so without any restriction on the cardinality of the theory.

Lemma. *A nongraduated theory has the Vaught property.* $\square$

Now we are able to prove the promised theorem.

1.2.2 Theorem. *Let T be countable, then Q_1 is eliminable in T iff T does not have the Vaught property.*

Proof. If T has the Vaught property, then, by the two-cardinal theorem, Q_1 is not definable. Hence, it is not eliminable in T. For the other direction, suppose Q_1 is not eliminable in T. Then, by the definability lemma, there is a formula $\varphi(x, \bar{z})$ and models $\mathfrak{M}_n$ of T containing sequences $\bar{a}_n$ such that $\aleph_1 > |\varphi(\mathfrak{M}_n, \bar{a}_n)| \geq n$, for every number n. If one of these latter sets is infinite, then we are done.

If not, then T is not graduated. Thus, by the above lemma, again T has the Vaught property. □

Vaught's two-cardinal theorem implies no $\aleph_1$-categorical countable theory has the Vaught property. Thus, a corollary follows which was independently obtained by several investigators (see Tuschik [1975], Vinner [1975], or Wolter [1975b]).

Corollary. *Q_1 is eliminable in every countable $\aleph_1$-categorical theory.* □

Together with the above lemma, the preceding theorems on Q_0 and Q_1 yield the next result.

Corollary. *Let T be countable. If Q_1 is eliminable in T, then Q_0 is also.* □

Similarly, two-cardinal considerations show that the eliminability of Q_1 is equivalent to the eliminability of each of the following quantifiers in a given countable theory: Chang's quantifier Q_c (= the unary cardinality quantifier in the equicardinality interpretation) and Härtig's quantifier I. As a further consequence, we remark that Q_1 (and also Q_0) is eliminable in the theories of Examples (1), (2), and (6). For ACF and RCF, this fact was also shown by Vinner [1975].

In the remainder of this subsection, we will present some material that is due to Baldwin–Kueker [1980]. This material concerns the eliminability of Ramsey quantifiers (= Malitz quantifiers in the $\aleph_0$-interpretation) in complete theories. Moreover, we will eventually prove a theorem describing this eliminability within the class of stable theories in terms of the following notion of first-order model theory, a notion that was introduced by Keisler [1967b]. The reader should also see Shelah [1978a] in this connection.

$\varphi(\bar{x}, \bar{z})$ has the *finite cover property* (abbreviated f.c.p.) in T if, for arbitrarily large numbers n, there are models $\mathfrak{M}_n$ containing sequences $\bar{a}_0, \ldots, \bar{a}_{n-1}$ which satisfy

$$\mathfrak{M}_n \models \neg\exists\bar{x} \bigwedge_{j<n} \varphi(\bar{x}, \bar{a}_j) \wedge \bigwedge_{i<n} \exists\bar{x} \bigwedge_{i\neq j<n} \varphi(\bar{x}, \bar{a}_j).$$

T is said to have the *f.c.p.* if some formula has. Note that

$$\psi(x, \bar{v}^\frown u) \leftrightarrow (\varphi(x, \bar{v}) \wedge x \neq u)$$

has the f.c.p. if $\varphi(x, \bar{v})$ is not graduated.

By Keisler [1967b] a countable $\aleph_1$-categorical theory does not have the f.c.p. On the other hand, Shelah proved that every unstable one does have this property (See Shelah [1978a]). The first half of the theorem of Baldwin and Kueker is contained in the next lemma.

Lemma. *If T does not have the f.c.p., then all Ramsey quantifiers are eliminable in T.*

Proof. Assume Q_0^m is not eliminable in T. By the definability lemma, we then have a formula $\varphi(x_0, \ldots, x_{m-1}, \bar{z})$ as well as models $\mathfrak{M}_n$ of T containing sequences $\bar{a}_n$

and finite sets A_n of power not less than n such that A_n is homogeneous for $\varphi(\bar{x}, \bar{a}_n)$ in $\mathfrak{M}_n$ and maximal with respect to that property ($n < \omega$). Let $\psi(x, \bar{z}'^\frown\bar{z})$ be the formula $\varphi(x, z_1, \ldots, z_{m-1}, \bar{z}) \wedge x \neq z_1$, where $\bar{z}' = (z_1, \ldots, z_{m-1})$. We will show that $\psi(x, \bar{z}'^\frown\bar{z})$ has the f.c.p.

To this end, let B_n denote the set of all $(m - 1)$-tuples from A_n. Choose a subset C_n of B_n minimal with respect to the property

$$(1)_n \qquad \mathfrak{M}_n \models \neg\exists x \bigwedge_{\bar{c}\in C_n} \psi(x, c^\frown\bar{a}_n).$$

This is possible, since B_n itself is a finite set having that property, for A_n is maximally homogeneous for $\varphi(\bar{x}, \bar{a}_n)$. Thus, the following holds:

$$(2)_n \qquad \mathfrak{M}_n \models \bigwedge_{\bar{c}'\in C_n} \exists x \bigwedge_{\bar{c}'\neq\bar{c}\in C_n} \psi(x, c^\frown\bar{a}_n).$$

For every subset of B_n consisting of less than n elements, we can choose an element of A_n different from all first components of elements of that subset. Hence, no such subset has the property given in $(1)_n$. Consequently, C_n has at least n elements. This conclusion, together with $(1)_n$ and $(2)_n$, for all n, shows that $\psi(x, \bar{z}'^\frown\bar{z})$ has the f.c.p. in T. ☐

In the other direction of the theorem below we shall utilize Shelah's f.c.p. theorem which asserts that a stable complete theory has the f.c.p. iff there is a formula $\varphi(x, y, \bar{z})$ satisfying the following: For every number n there is a sequence $\bar{c}_n$ of elements in some model $\mathfrak{M}_n$ of T such that $\varphi(x, y, \bar{c}_n)$ defines on the universe of $\mathfrak{M}_n$ an equivalence relation having not less than n, but only finitely many equivalence classes (see Shelah [1978a; Chapter II, Theorem 4.4]).

1.2.3 Theorem. *Let T be stable and complete. Then All Ramsey quantifiers are eliminable in T iff the Ramsey quantifier Q_0^2 is eliminable in T iff T does not have the f.c.p.*

Proof. For the remaining implication, assume T has the f.c.p., then we must show that Q_0^2 is not definable in T. To this end, we choose a formula $\varphi(x, y, \bar{z})$, as well as sequences $\bar{c}_n$ and models $\mathfrak{M}_n$ according to the f.c.p. theorem. Then, clearly we have that

$$\mathfrak{M}_n \models Q_{(n)}^2 x_0 x_1 \neg\varphi(x_0, x_1, \bar{c}_n) \wedge \neg Q_0^2 x_0 x_1 \neg\varphi(x_0, x_1, \bar{c}_n)$$

holds for all n. Whence, the assertion follows. ☐

Using the aforementioned observation of Keisler, we can easily derive the following

Corollary. *All Ramsey quantifiers are eliminable in every countable $\aleph_1$-categorical theory.* ☐

This corollary generalizes the corresponding result for ACF_0 as proven by Cowles [1979a].

Further Results. We will close this section with a few brief remarks sketching some further pertinent results.

(1) Tuschik has provided some further results with regard to the relative strength and effectiveness of eliminability of the unary cardinality quantifiers Q_α. The reader should consult Tuschik [1977a or 1982a]; or Baudisch–Seese–Tuschik–Weese [1980]; or Rothmaler–Tuschik [1982]. Vinner [1975] is also informative.

(2) In Rothmaler [1981 or 1984] it is shown that Q_0 is eliminable in every complete first-order theory of modules. Baudisch [1984] extended this to all Ramsey quantifiers. See the next subsection for more on this.

(3) Further algebraic results can be found in the papers of Cowles, Pinus, and Rothmaler that are cited in the bibliography.

(4) Baudisch [1977b or 1979], and Baldwin–Kueker [1980] prove independently that all Ramsey quantifiers are eliminable in a countable $\aleph_0$-categorical first-order theory, thus showing that the stability assumption made in Theorem 1.2.3 of this section is necessary.

(5) Schmerl–Simpson [1982] provided an effective elimination of all Ramsey quantifiers in Presburger arithmetic. In contrast, however, Kierstead–Remmel [1983] constructed decidable first-order theories admitting elimination of these quantifiers which cannot be made effective.

(6) Baldwin–Kueker [1980] proved the eliminability of the Malitz quantifiers Q_c^m (in the equi-cardinality interpretation) in countable $\aleph_1$- categorical first-order theories. Clearly, this is then true for all other interpretations. This result generalizes the corresponding result for ACF_0 which had been proven by Cowles [1979a].

(7) Rothmaler–Tuschik [1982] generalized the result that is here given as Theorem 1.2.2 to Malitz quantifiers Q_1^m ($m < \omega$) so as to obtain an analog of Theorem 1.2.3 for these. Furthermore, as a corollary, they independently obtained the result mentioned in the preceding remark.

(8) Theorem 1.2.3 asserts, among other things, that in stable theories the eliminability of Q_α^2 implies that of all Q_α^m ($m < \omega$) in the case $\alpha = 0$. Rapp [1982 or 1983] proved that this is also true in the case $\alpha = 1$. Moreover, he showed that in stable theories the eliminability of Q_1^2 implies that of all Malitz quantifiers Q_α^m ($m < \omega$, $\alpha \geq 0$; for $\alpha = 0$ this was already noticed by Rothmaler–Tuschik [1982]).

1.3. Elimination Procedures for Modules and Abelian Groups

The elementary theory of groups is undecidable (see Tarski in Tarski–Mostowski–Robinson [1953]). Furthermore, a good number algebraically interesting classes of groups have an undecidable elementary theory. From Ershov [1974] and Samjatin [1978], it is known that the elementary theory of every non-abelian variety of groups is undecidable. In contrast to this, however, Szmielew [1955] proved the decidability of the elementary theory T_Z of abelian groups. Extending the ordinary language $(+, -, 0)$ of group theory by predicates "$p^n|x$" and defining some core sentences, she gave an effective elimination procedure: every formula is equivalent modulo T_Z to a boolean combination of Szmielew core sentences and atomic formulas. One can extend this elimination procedure to the logics $\mathscr{L}_{\omega\omega}(Q_\alpha)$, $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$, $\mathscr{L}_{\omega\omega}(\mathrm{aa})$, and $\mathscr{L}_{\omega\omega}(I)$, provided the set of core sentences is extended in an appropriate way. For concrete results and references, the reader should see the list below. Moreover, one can find corresponding elimination procedures for arbitrary R-modules. Here we will present just such a procedure for Malitz quantifiers in regular interpretations (Baudisch [1984]).

Convention. Throughout this subsection R is an associative ring with unit 1, $\mathfrak{A}$ is a left R-module, and $\mathbf{a}$ is a sequence from $\mathfrak{A}$. As is usual in first-order model theory of modules, we will consider the first-order language having the following nonlogical symbols: 0, $+$, and, for every $r \in R$, a unary function symbol expressing the left multiplication by r.

For the sake of simplicity we will use L_R to denote the set of all first-order formulas in this language. Then the elementary theory of all (unital) left R-modules can be axiomatized by a set of L_R-sentences. Let $L_R(Q_\alpha^{<\omega})$ and $T_R(Q_\alpha^{<\omega})$ denote the extensions of L_R and T_R respectively to the logic $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$.

A *positive primitive* (abbreviated *p.p.*) formula is a formula of the form $\exists \mathbf{y}\, \psi(\mathbf{x}, \mathbf{y})$, where ψ is a finite conjunction of equations (with coefficients from R). Notice that a p.p. formula $\chi(x_0, \ldots, x_{m-1})$ defines an additive subgroup $\chi(\mathfrak{A}^m)$ in the module $\mathfrak{A}^m$ (and if R is commutative, this is even a submodule), and a p.p. formula $\chi(\mathbf{x}; \mathbf{a})$ defines a coset of the subgroup $\chi(\mathfrak{A}^m; \mathbf{0})$ in $\mathfrak{A}^m$.

Notation. Throughout this discussion we will let $\chi(\mathbf{x}; \mathbf{z})$ be a p.p. formula. Moreover, we will use $\chi^j(x)$ to denote the formula $\chi(0, \ldots, 0, x, 0, \ldots, 0; \mathbf{0})$ obtained from $\chi(\mathbf{x}; \mathbf{0})$ by substituting x for x_j and 0 for the other components of $\mathbf{x}$, $\chi'(x)$ to denote $\bigwedge_{j<m} \chi^j(x)$, and $\chi^d(x; \mathbf{a})$ to denote the formula $\chi(x, \ldots, x; \mathbf{a})$.

Note that by the additivity of p.p. formulas, $\bigwedge_{j<m} \chi^j(x)$ implies $\chi^d(x; \mathbf{0})$. The following implication can be easily derived from additivity of the p.p. formula $\chi(\mathbf{x}; \mathbf{z})$:

$$(1) \qquad T_R \vdash \chi(x_0, \ldots, x_{j-1}, w, x_{j+1}, \ldots, x_{m-1}; \mathbf{z}) \rightarrow \\ [\chi(x_0, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_{m-1}; \mathbf{z}) \leftrightarrow \chi^j(v - w)].$$

It can be easily seen from the first part of the next lemma that, for a p.p. formula, a sufficiently large set is homogeneous if it is weakly homogeneous.

Lemma. *Let $\chi(\mathbf{x}; \mathbf{z})$ be a p.p. formula, then we have*

(i) *A set C of power greater than m which is weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$ is contained in $\chi^d(\mathfrak{A}; \mathbf{a})$ and in $c + \chi'(\mathfrak{A})$, for every $c \in C$; and*
(ii) *Every subset C of $c + \chi'(\mathfrak{A})$, for some $c \in \chi^d(\mathfrak{A}; \mathbf{a})$, is weakly homogeneous for $\chi(x; \mathbf{a})$.*

Proof. To establish (i), we let $\{c_0, \ldots, c_m\}$ be an $(m+1)$-element-subset of C. Since it is weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$, it is not difficult to derive $\mathfrak{A} \models \bigwedge_{j<m} \chi^j(c_i - c_k)$, where $i, k \leq m$. Hence, all elements of C lie in the same coset of $\chi'(\mathfrak{A})$. Using (1), $C \subseteq \chi^d(\mathfrak{A}; \mathbf{a})$ follows. The proof of part (ii) is an immediate consequence of (1). $\square$

Key Lemma. *Let C be an infinite subset of $\mathfrak{A}$ of regular cardinality such that for every $j < m$, there are less than $|C|$ elements of C in every coset of $\chi^j(\mathfrak{A})$. Then C contains a subset of the same cardinality which is weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$.*

Proof. We first note that by definition a set of power less than m is weakly homogeneous for arbitrary m-placed formulas; this fact provides the initial step of the following induction

It suffices to show that to every subset E of C of power less than $|C|$ which is weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$ one can add some $c \in C - E$ and still not disturb the weak homogeneity for $\neg\chi(\mathbf{x}; \mathbf{a})$. To do so, however, we must prove that the set of elements in C which one cannot add to E has power less than $|C|$. To this purpose, then, let $c \in C - E$ such that $E \cup \{c\}$ is not weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$. Then there are $j < m$ and distinct elements $e_0, \ldots, e_{j-1}, e_{j+1}, \ldots, e_{m-1}$ in E such that

$$(*) \qquad \mathfrak{A} \models \chi(e_0, \ldots, e_{j-1}, c, e_{j+1}, \ldots, e_{m-1}; \mathbf{a}).$$

By (1) above, all c's satisfying $(*)$ lie in the same coset of $\chi^j(\mathfrak{A})$. Hence, by hypothesis, there are less than $|C|$ such elements.

Since C is infinite and $|E| < |C|$, there are less than $|C|$ $(m-1)$-tuples in E. Consequently, the whole set of elements that one cannot add to E must have power less than $|C|$ also. $\square$

This lemma enables us to prove a strong "Ramsey-like" property for p.p. formulas.

Lemma. *Every infinite subset of $\mathfrak{A}$ of regular cardinality contains a subset of the same cardinality which is weakly homogeneous either for $\chi(\mathbf{x}; \mathbf{a})$ or for $\neg\chi(\mathbf{x}; \mathbf{a})$.*

Proof. Using induction on m, we can clearly assume the assertion is true for $m - 1 \geq 1$. Let C be an infinite set in $\mathfrak{A}$ not containing a subset of cardinality $|C|$ which is weakly homogeneous for $\neg\chi(\mathbf{x}; \mathbf{a})$. By the Key Lemma there are some $j < m$ (for the sake of simplicity, say $j = 0$), some $c \in C$, and some subset E of power $|C|$ in $\chi^0(\mathfrak{A})$ with $c + E \subseteq C$. By the induction hypothesis, E contains a subset D of power $|E| = |C|$ which is weakly homogeneous for $\chi(c, c + x_1, \ldots, c + x_{m-1}; \mathbf{a})$ or for its negation. Since $D \subseteq \chi^0(\mathfrak{A})$, by (1) above, $c + D$ is then weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ or $\neg\chi(\mathbf{x}; \mathbf{a})$, respectively. $\square$

Corollary. *Let $\varphi_i(\mathbf{x}; \mathbf{z})$ be a conjunction of p.p. and negated p.p. formulas $(i < n)$. Then every infinite set of regular cardinality weakly homogeneous for $\bigvee_{i<n} \varphi_i(\mathbf{x}; \mathbf{a})$ in $\mathfrak{A}$ contains a subset of the same cardinality which is weakly homogeneous for some $\varphi_{i_0}(\mathbf{x}; \mathbf{a})$, $i_0 < n$.*

Proof. Let C be infinite and weakly homogeneous for $\bigvee_{i<n} \varphi_i(\mathbf{x};\mathbf{a})$. Using induction on n, we assume that there is no subset of C of power $|C|$ which is weakly homogeneous for $\bigvee_{0<i<n} \varphi_i(\mathbf{x};\mathbf{a})$ in $\mathfrak{A}$. Let $\varphi_0(\mathbf{x};\mathbf{a})$ be $\bigwedge_{i<k} \chi_i(x;a)$, where the χ_i are p.p. or negated p.p. formulas. Step by step, we will construct a subset of C of power $|C|$ which is weakly homogeneous for $\varphi_0(\mathbf{x};\mathbf{a})$ in $\mathfrak{A}$. For this, assume that $C' \subseteq C$ is weakly homogeneous for $\bigwedge_{i<j} \chi_i(\mathbf{x};\mathbf{a})$ and $|C'| = |C|$, where $j \leq k$ (if $j = 0$, let $C = C'$). By the preceding lemma, it suffices to show that C' contains no subset of power $|C|$ which is weakly homogeneous for $\neg\chi_j(\mathbf{x};\mathbf{a})$. But this is clear, since every subset D of C that is weakly homogeneous for $\neg\chi_j(\mathbf{x};\mathbf{a})$ would be weakly homogeneous for $\bigvee_{0<i<n} \varphi_i(\mathbf{x};\mathbf{a})$, thus contradicting the assumption. ∎

Using an infinitary version of B. H. Neumann's lemma, we obtain the next lemma. (See Baudisch [1984]).

Lemma. *Let χ, η_i be p.p. formulas ($i < n$). Then*

$$T_R(Q_\alpha^2) \vdash Q_\alpha^2 x_0 x_1\left(\chi(x_0) \wedge \bigwedge_{i<n} \neg\eta_i(x_0 - x_1)\right)$$
$$\leftrightarrow \bigwedge_{i<n} Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta_i(x_0 - x_1)).$$

Before we prove the main theorem of this subsection, we will introduce some more notation and state a theorem on the existence of an elementary elimination procedure which is due to Baur [1976] and Monk [1975] and which is basic for the the first-order model theory of modules.

If χ and η are p.p. formulas with $T_R \vdash \eta(x) \to \chi(x)$ then let $(\chi/\eta)(\mathfrak{A})$ denote the cardinality of the factor group $\chi(\mathfrak{A})/\eta(\mathfrak{A})$. Clearly there are elementary $\exists\forall$-sentences expressing $(\chi/\eta)(\mathfrak{A}) \geq k$ for every natural number k. Call these *elementary core sentences.* Now the theorem of Baur and Monk states that every formula of L_R is equivalent modulo T_R to a boolean combination of elementary core sentences and p.p. formulas.

Our goal is to prove an analogue to this theorem for the language $L_R(Q_\alpha^{<\omega})$. First note that in this language we can express $(\chi/\eta)(\mathfrak{A}) \geq \aleph_\alpha$ by $Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta(x_0 - x_1))$. Those sentences, together with the elementary core sentences, will be called *Q_α^2-core sentences.*

Theorem. *Every formula of $L_R(Q_\alpha^{<\omega})$ is equivalent modulo $T_R(Q_\alpha^{<\omega})$ to a boolean combination of Q_α^2-core sentences and p.p. formulas. This* boolean combination *can be effectively found relative to the elementary procedure provided by the Theorem of Baur and Monk.*

Proof. We show the theorem for regular ω_α only. For a complete proof, see Baudisch [1984]. By the theorem of Baur and Monk and induction on the complexity of formulas, it suffices to consider the case $Q_\alpha^m \mathbf{x}\, \varphi(\mathbf{x};\mathbf{z})$, where $\varphi(\mathbf{x};\mathbf{z})$ is a boolean combination of p.p. formulas. The above corollary thus reduces this to the case in which φ is a conjunction of p.p. and negated p.p. formulas. Since a conjunction of p.p. formulas is equivalent to a p.p. formula, we can further suppose that $\varphi(\mathbf{x};\mathbf{z})$ is of the form $\chi(\mathbf{x};\mathbf{z}) \wedge \bigwedge_{i<k} \neg\eta_i(\mathbf{x};\mathbf{z})$, where χ and η_i are p.p. $(i<k)$. We will now construct the desired boolean combination in the following development.

Let H be the set of all partitions $\{I, J\}$ of $\{(i,j): i<k, j<m\}$. For each $\{I, J\} \in H$ define $F(I) = \{i<k: \text{for all } j<m\ (i,j) \in I\}$. Now let $\psi(\mathbf{z})$ be the disjunction of the following formulas, where $\{I, J\}$ runs over all the partitions in H:

$$\exists y\left[\chi^d(y;\mathbf{z}) \wedge \bigwedge_{i\in F(I)} \neg\eta_i^d(y;\mathbf{z})\right]$$
$$\wedge\ Q_\alpha^2 x_0 x_1\left[\chi'(x_0) \wedge \bigwedge_{(i,j)\in I} \eta_i^j(x_0) \wedge \bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1)\right].$$

Using the preceding lemma and the elementary elimination procedure, it is not difficult to show that ψ is indeed equivalent to a boolean combination of p.p. formulas and Q_α^2-core sentences. Thus, it suffices to verify that

$$T_R(Q_\alpha^{<\omega}) \vdash \forall\mathbf{z}(Q_\alpha^m\mathbf{x}\ \varphi(\mathbf{x};\mathbf{z}) \leftrightarrow \psi(\mathbf{z})).$$

To prove this in the direction from left to right, we let C be a set of power $\aleph_\alpha$ which is weakly homogeneous for $\varphi(\mathbf{x};\mathbf{a})$ in $\mathfrak{A}$. By part (i) of the first lemma, C is weakly homogeneous for $\chi'(x_0 - x_1)$. Thus it is trivially so for

$$\bigvee_{\{I,J\}\in H}\left(\chi'(x_0 - x_1) \wedge \bigwedge_{(i,j)\in I} \eta_i^j(x_0 - x_1) \wedge \bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1)\right).$$

The corollary above yields some $\{I, J\} \in H$ and some set $C' \subseteq C$ of power $\aleph_\alpha$ which is weakly homogeneous for

$$\chi'(x_0 - x_1) \wedge \bigwedge_{(i,j)\in I} \eta_i^j(x_0 - x_1) \wedge \bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1).$$

Let $c \in C'$ and E a set with $c + E = C'$. By (i) of the first lemma, $\mathfrak{A} \models \chi^d(c;\mathbf{a})$. Further,

(2) $\qquad E$ is weakly homogeneous for $\bigwedge_{(i,j)\in J} \neg\eta_i^j(x_0 - x_1)$

and

(3) $\qquad E \subseteq \chi'(\mathfrak{A}) \cap \bigcap_{(i,j)\in I} \eta_i^j(\mathfrak{A}).$

Notice that (3) implies

(4) $\qquad E \subseteq \eta_i'(\mathfrak{A})\quad$ for all $\quad i \in F(I),$

since $\vdash \bigwedge_{j<n} \eta_i^j(x) \leftrightarrow \eta_i'(x)$. Then $\mathfrak{A} \models \bigwedge_{i \in F(I)} \neg \eta_i^d(c; \mathbf{a})$; for, otherwise $\mathfrak{A} \models \eta_i^d(c; \mathbf{a})$ together with (4) and (ii) of the first lemma would imply that $c + E$ were weakly homogeneous for $\eta_i(\mathbf{x}; \mathbf{a})$. Recalling that $c + E \subseteq C$, we thus have a contradiction.

To establish the other direction of the above equivalence, we first choose a partition $\{I, J\} \in H$, a set E of power $\aleph_\alpha$ satisfying (2) and (3) (and hence, must satisfy (4) also), and an element $c \in \mathfrak{A}$ with

$$\mathfrak{A} \models \chi^d(c; \mathbf{a}) \wedge \bigwedge_{i \in F(I)} \neg \eta_i^d(c; \mathbf{a}).$$

We will eventually show that $c + E$ contains a subset of power $\aleph_\alpha$ which is weakly homogeneous for $\varphi(\mathbf{x}; \mathbf{a})$. By (ii) of the first lemma, $c + E$ is weakly homogeneous for $\chi(\mathbf{x}; \mathbf{a})$ as well as for $\bigwedge_{i \in F(I)} \neg \eta_i(\mathbf{x}; \mathbf{a})$; since, otherwise, by additivity, (4) would imply $\mathfrak{A} \models \eta_i^d(c; \mathbf{a})$, thus contradicting the choice of c.

It thus remains to prove the following

Claim. *For every subset E' of E of the same power and for every $i < n$ with $i \notin F(I)$, $c + E'$ contains a subset of the same power which is weakly homogeneous for $\neg \eta_i(x; \mathbf{a})$.*

To establish this claim, we fix some $i \notin F(I)$ and, without loss of generality, assume that $(i, m - 1) \in J$. That done, we first consider the case $\{(i, j) : j < m\} \subseteq J$. Then, by (2), all the elements of $c + E'$ lie in different cosets of $\eta_i^j(\mathfrak{A})$ for all $j < m$. Thus, the hypothesis of the Key Lemma is trivially satisfied. Thereby establishing the claim for this case.

Turning now to the general case, we let the variables be ordered in such a way that "$(i, j) \in I$ iff $j < k$" for some $k < m$. We then apply the same argument to the formula $\eta_i(c, \ldots, c, x_k, \ldots, x_{m-1}; \mathbf{a})$ in order to obtain a subset $c + E'' \subseteq c + E'$ which has the same power and which is weakly homogeneous for its negation. Since $E'' \subseteq \bigcap_{j<k} \eta_i^j(\mathfrak{A})$ by (3), it is easy to see that $c + E''$ is weakly homogeneous even for $\neg \eta_i(\mathbf{x}; \mathbf{a})$; whence, the claim is proven. □

Corollary. *For modules, $\mathscr{L}_{\omega\omega}(Q_\alpha^2)$ has the same expressive power as $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$.* □

Corollary. *All Ramsey quantifiers Q_0^m ($m < \omega$) are eliminable in every complete (first-order) extension of T_R.* □

By Baur [1975], every complete first-order theory of modules is stable. Hence, the preceding corollary, together with Theorem 1.2.3, has as a consequence the following

Corollary. *No complete first-order theory of modules has the f.c.p.* □

Finally, we specify the theorem to the case of abelian groups. We begin by making a general remark.

Assume Σ_1 to be a set of p.p. formulas that is closed under substitution of free variables by 0 and Σ_2 to be a set of elementary core sentences such that the elementary elimination procedure only needs formulas from Σ_1 and Σ_2. Then, the theorem holds true for boolean combinations of formulas from $\Sigma_1 \cup \Sigma_2$ and Q_α^2-core sentences of the form $Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta(x_0 - x_1))$, where χ is a conjunction of formulas from Σ_1 and η is in Σ_1.

Now let R be the ring Z of all integers, Σ_1 the set of all atomic L_Z-formulas and all formulas $\exists y(p^n y = \sum_{i<k} r_i x_i)$, where p is a prime, n a natural number, and r_i an integer. Furthermore, let Σ_2 be the set of all Szmielew core sentences; that is, Σ_2 is the set of all sentences $(\chi/\eta) \geq k$, where

(#) either $\chi(x)$ is $px = 0 \wedge p^{n-1}|x$ and $\eta(x)$ is $x = 0$;
or $\chi(x)$ is $p^{n-1}|x$ and $\eta(x)$ is $p^n|x$;
or $\chi(x)$ is $px = 0 \wedge p^{n-1}|x$ and $\eta(x)$ is $px = 0 \wedge p^n|x$;
or $\chi(x)$ is $rx = 0$ and $\eta(x)$ is $x = 0$,

for some prime p and natural numbers n and r, with $1 \leq n$.

Call all sentences of Σ_2 and all sentences $Q_\alpha^2 x_0 x_1(\chi(x_0) \wedge \neg\eta(x_0 - x_1))$, for (χ, η) from (#), *Q_α^2-Szmielew-core-sentences.* The new sentences express that the corresponding Szmielew invariants are of power at least $\aleph_\alpha$. By Szmielew [1955], the elementary elimination procedure for Z-modules (= Abelian groups) only needs formulas from Σ_1 and Σ_2. The above theorem can be sharpened in this context.

Theorem. *For every formula of $L_Z(Q_\alpha^{<\omega})$, we can effectively find a boolean combination of formulas from Σ_1 and Q_α^2-Szmielew-core-sentences to which it is equivalent modulo $T_Z(Q_\alpha^{<\omega})$.* □

Corollary. *$T_Z(Q_\alpha^{<\omega})$ is decidable.* □

We now collect corresponding results into the following table.

Table of Elimination Procedures and Decidability

	Abelian Groups	Modules
$\mathscr{L}_{\omega\omega}$	Szmielew [1955]	Baur [1976], Monk [1975]
$\mathscr{L}_{\omega\omega}(Q_\alpha)$	Baudisch [1976]	Rothmaler [1981 or 1984]
$\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$	Baudisch [1983]	Baudisch [1984]
$\mathscr{L}_{\omega\omega}(\text{aa})$	Eklof–Mekler [1979] Baudisch [1981a]	Eklof–Mekler [1979]
$\mathscr{L}_{\omega\omega}(I)$	Baudisch [1977b or c]	Similar to Baudisch [1977c]
	Decidability Problem is open	

That $T_Z(I)$ is decidable iff the I-theory of all finite abelian groups is decidable follows from Baudisch [1977c]. Moreover, in Baudisch [1980] the I-theory of abelian p-groups is shown to be decidable. In the same vein, Schmitt [1982] has shown the decidability of the $\mathscr{L}(Q_\alpha)$-theory of ordered abelian groups for $\alpha = 0$ and $\alpha = 1$. Furthermore, in the language, considered, he allows quantification (with $\exists$ and Q_α) over convex subgroups generalizing Gurevich [1977a]. By adding suitable definable predicates, an elimination procedure for first-order quantifiers is given. In order to decide the remaining sentences, the order structure of the convex subgroups is considered in appropriate elementary languages.

In the logics that we have mentioned above, elimination procedures are also applied to other classes of structures. Thus, for example, Cowles has results for certain fields (see Cowles [1977, 1979a, b]) and Wolter for Pressburger arithmetic and for well-orderings (see Wolter [1975a, b]).

In the case of the Henkin-quantifier (the reader is referred to Section VI.2.13 of this volume), Krynicki–Lachlan [1979] used this method to prove the decidability of the corresponding theory of finitely many unary predictes with equality. For more on boolean algebras, the reader should also see the results of Molzahn [1981b] that are cited at the end of the third section of this chapter. Finally, some material on the elimination of quantifiers in stationary logic and its applications are given in the next subsection.

1.4. Elimination of Quantifiers for Stationary Logic

The reader should consult Chapter IV for the basic notions concerning $L(\mathrm{aa})$. Throughout the present subsection, L will be taken as a countable elementary language and T as an $L(\mathrm{aa})$-theory. Since generalization over second-order variables is not allowed in $L(\mathrm{aa})$, the appropriate notion of eliminability of quantifiers is the one that is defined below (see Eklof and Mekler [1979] where it is called *strong elimination of quantifiers*).

Definition. T is said to *admit elimination of quantifiers* if, for every formula $\varphi(\bar{s}, \bar{x})$, there is a quantifier-free formula $\psi(\bar{s}, \bar{x})$ such that $T \vdash \mathrm{aa}\, \bar{s}\, \forall \bar{x}(\varphi(\bar{s}, \bar{x}) \leftrightarrow \psi(\bar{s}, \bar{x}))$.

By generalizing ideas of Eklof–Mekler [1979], Mekler [1984] found the following criterion for eliminability of quantifiers in $L(\mathrm{aa})$-theories. This criterion is an analogue of that for the elementary case and the notation "$\equiv^0$" is used to denote equivalence with respect to quantifier-free formulas.

Theorem. *T admits elimination of quantifiers iff whenever $\mathfrak{A}$, $\mathfrak{B} \models T$ and $|\mathfrak{A}|$, $|\mathfrak{B}| \le \aleph_1$, there are cubs C and D for $\mathfrak{A}$ and $\mathfrak{B}$ such that for all $\bar{A} \in C$, and $\bar{B} \in D$ and $\bar{a} \in \mathfrak{A}$ and $\bar{b} \in \mathfrak{B}$, if $\langle \mathfrak{A}, \bar{A}, \bar{a} \rangle \equiv^0 \langle \mathfrak{B}, \bar{B}, \bar{b} \rangle$ holds, then $\langle \mathfrak{A}, \bar{A}, \bar{a} \rangle \equiv_{\mathrm{aa}} \langle \mathfrak{B}, \bar{B}, \bar{b} \rangle$.*

Proof. We will present the proof for the nontrivial direction. Assume, then, that $\varphi(\bar{s}, \bar{x})$ is not equivalent to a quantifier-free formula. Let $\{\psi_n(\bar{s}, \bar{x}) : n < \omega\}$ be an enumeration of all the corresponding quantifier-free formulas. For $t \in {}^k2$, define $\psi^t = \bigwedge_{i<k} \psi_i(\bar{s}, \bar{x})^{t(i)}$, where $\psi_i^0 = \psi_i$ and $\psi_i^1 = \neg \psi_i$.

Let $\mathfrak{T}$ be the tree of all $t \in {}^{<\omega}2$ such that for all $t' < t$

$$(*) \qquad \begin{array}{ll} \text{neither} & \text{(i) } T \vdash \text{aa } \bar{s}\, \forall \bar{x}(\psi^{t'}(\bar{s}, \bar{x}) \to \varphi(\bar{s}, \bar{x})) \\ \text{nor} & \text{(ii) } T \vdash \text{aa } \bar{s}\, \forall \bar{x}(\psi^{t'}(\bar{s}, \bar{x}) \to \neg\varphi(\bar{s}, \bar{x})). \end{array}$$

Then $\mathfrak{T}$ must be infinite, because otherwise there would be some $k < \omega$ such that for all $t' \in {}^k2$ either (i) or (ii). This would imply $T \vdash \text{aa } \bar{s}\, \forall \bar{x}(\varphi(\bar{s}, \bar{x}) \leftrightarrow \bigvee_{t' \in I} \psi^{t'}(\bar{s}, \bar{x}))$, where I is the set of all $t' \in {}^k2$ with property (i). By König's lemma, there is an infinite branch $\eta \in {}^{\omega}2$ of $\mathfrak{T}$. So, by $(*)$ and the construction of $\mathfrak{T}$, we have

$$(**) \qquad \begin{array}{l} T \cup \{\text{stat } \bar{s}\, \exists \bar{x}(\psi^{\eta \restriction k}(\bar{s}, \bar{x}) \wedge \neg\varphi(\bar{s}, \bar{x}))\} \quad \text{and} \\ T \cup \{\text{stat } \bar{s}\, \exists \bar{x}(\psi^{\eta \restriction k}(\bar{s}, \bar{x}) \wedge \varphi(\bar{s}, \bar{x}))\} \end{array}$$

are consistent for every $k < \omega$. Assume now that $\bar{s} = (s_1, \ldots, s_n)$ and $\bar{x} = (x_1, \ldots, x_m)$. We will introduce new predicates $U_i(s_1, \ldots, s_i)$ and functions $f_j(\bar{s})$, where $0 < i \leq n$, $0 < j \leq m$. We also define T' to be the extension of T by the following axioms:

$$\text{stat } s_1 U_1(s_1), \text{ aa } s_1 \text{ stat } s_2 U_2(s_1, s_2), \ldots;$$

$$\text{aa } s_1 \text{aa } s_2 \ldots aa\, s_{n-1} \text{ stat } s_n U_n(s_1, \ldots, s_n); \text{ and}$$

$$\begin{aligned} &\text{aa } s_1 \cdots s_n(U_1(s_1) \wedge U_2(s_1, s_2) \wedge \cdots \wedge U_n(s_1, \ldots, s_n) \\ &\quad \to \psi^{\eta \restriction k}(\bar{s}, f_1(\bar{s}), \ldots, f_m(\bar{s})) \quad \text{for every} \quad k < \omega. \end{aligned}$$

Using compactness, we see that $(**)$ implies the consistency of

$$\begin{aligned} T_0 = T' \cup \{&\text{aa } s_1 \ldots s_n(U_1(s_1) \wedge \cdots \wedge U_n(s_1, \ldots, s_n) \\ &\to \neg\varphi(\bar{s}, f_1(\bar{s}), \ldots, f_m(\bar{s}))\}; \end{aligned}$$

and

$$\begin{aligned} T_1 = T' \cup \{&\text{aa } s_1 \ldots s_n(U_1(s_1) \wedge \cdots \wedge U_n(s_1, \ldots, s_n) \\ &\to \varphi(\bar{s}, f_1(\bar{s}), \ldots, f_m(\bar{s}))\}. \end{aligned}$$

Now let $\mathfrak{A}$ and $\mathfrak{B}$ be the reducts to L of models of T_0 and T_1, and let C and D be the cubs in $\mathfrak{A}$ and $\mathfrak{B}$, given by the criterion. By the axioms of T', there are chains $A_1 \subseteq \cdots \subseteq A_n$ of elements of C and chains $B_1 \subseteq \cdots \subseteq B_n$ of elements of D all of which fulfill $U_1(s_1) \wedge \cdots \wedge U_n(s_1, \ldots, s_n)$. Furthermore, $\bar{A}$ and $\bar{B}$ can be chosen so that

$$\langle \mathfrak{A}, \bar{A}, f_1(\bar{A}), \ldots, f_m(\bar{A}) \rangle \equiv^0 \langle \mathfrak{B}, \bar{B}, f_1(\bar{B}), \ldots, f_m(\bar{B}) \rangle;$$

and

$$\mathfrak{A} \models \neg\varphi(\bar{A}, f_1(\bar{A}), \ldots, f_m(\bar{A})) \quad \text{and} \quad \mathfrak{B} \models \varphi(\bar{B}, f_1(\bar{B}), \ldots, f_m(\bar{B})).$$

However, this contradicts the condition of the criterion. $\square$

Applying this notion to modules, Eklof–Mekler [1979] found an elimination procedure for the $L(\mathrm{aa})$-theory of all R-modules. They used $L(\mathrm{aa})$-core-sentences of the form

$$\mathrm{aa}\, s\, \forall x(\chi(x) \to \exists y(y \in s \wedge \chi(x - y) \wedge \eta(x - y))),$$

where χ and η are p.p. formulas. It is easy to see that such a core sentence is equivalent to $(\chi/\eta)(\mathfrak{A}) \leq \aleph_0$—which is the negation of a Q_1^2-core sentence—so that $L(\mathrm{aa})$-equivalence and Q_1^2-equivalence coincide. To verify that the criterion does indeed hold on modules, Eklof and Mekler used the work of Fisher [1977] on injective elements in abelian classes, which continues the work of Eklof–Fisher [1972] on the description of saturated abelian groups to give a model-theoretic proof of the results of Szmielew [1955].

Specifying this development to abelian groups, Eklof and Mekler proved decidability of the $L(\mathrm{aa})$-theory. Similar results on abelian groups were independently obtained by Baudisch [1981a]. Along these same lines, we note that further applications of this method to fields and orderings can be found in Eklof–Mekler [1979]. There decidability is shown for the $L(\mathrm{aa})$-theories of complex, real, and p-adic numbers.

2. *Interpretations*

The method of syntactic interpretation was used by Tarski–Mostowski–Robinson [1953] to deduce decidability or undecidability of theories from other theories (see also Rabin [1965]).

The actual method has many applications to decidability problems, and we will give a short description of it here. Let K and K' be model classes in languages L and L' respectively, where L and L' are not necessarily elementary. Then, we say that an *interpretation* I assigns to every relational symbol R of L a formula ψ_R of L', and the formula $x = x$ corresponds to a formula $\varphi(x)$ of L'. The interpretation of the basic symbols of the language L is inductively extended to all formulas χ of L. The interpreted formula is then denoted by χ^I and is built according to the following rules, where, for the sake of notational simplicity, we let L be an elementary language with only one binary relation symbol R.

(i) $(x = y)^I := (x = y)$;
(ii) $(R(x, y))^I := \psi_R(x, y)$;
(iii) $(\neg\chi)^I := \neg(\chi^I)$;
(iv) $(\chi_1 \vee \chi_2)^I := \chi_1^I \vee \chi_2^I$; and
(v) $(\exists x\chi)^I := \exists x(\varphi(x) \wedge \chi^I)$.

If $\mathfrak{B} = (B, \ldots)$ is a model of K', then we obtain—with the help of I—a model $\mathfrak{B}^I$ for L. The domain of $\mathfrak{B}^I$ is simply the set of all elements of the domain of $\mathfrak{B}$ satisfying the formula $\varphi(x)$. The symbol R of L is interpreted by the relation

$$\{(a, b) \in B^2/\mathfrak{B} \models \varphi(a) \wedge \varphi(b) \wedge \psi_R(a, b)\}.$$

The next lemma is easily proven by induction on the complexity of formulas.

Lemma (Rabin [1965]). *For each formula χ of L and for each structure $\mathfrak{B}$ of K':*

$$\mathfrak{B} \models \chi^I \text{ iff } \mathfrak{B}^I \models \chi. \quad \square$$

The theory $\mathrm{Th}_L(K)$ is said to be *interpretable in* $\mathrm{Th}_{L'}(K')$ if

(i) for every structure $\mathfrak{A} \in K$ there is a $\mathfrak{B} \in K'$ so that $\mathfrak{B}^I$ and $\mathfrak{A}$ are isomorphic;
(ii) for every structure $\mathfrak{B} \in K'$, the structure $\mathfrak{B}^I$ is isomorph to a structure of K.

The main property of interpretations with respect to decidability is expressed in the following result.

2.1 Theorem (Rabin [1965]). *Let K and K′ be model classes and let L and L′, respectively, be suitable languages, where L is assumed to be elementary. If* $\mathrm{Th}_L(K)$ *is interpretable in* $\mathrm{Th}_{L'}(K')$, *then the decidability of* $\mathrm{Th}_{L'}(K')$ *implies the decidability of* $\mathrm{Th}_L(K)$. $\square$

The proof is a straightforward application of the preceding lemma. There are obvious generalizations of the notion of interpretability, and a result similar to Theorem 2.1 can be proven for them. Thus, for example, we may admit

(a) any finite signature;
(b) the identity can be handled as a non-logical symbol; that is to say, it is interpreted by a congruence relation;
(c) n-tuples of elements from the domain of $\mathfrak{B}$ can be used as individuals of $\mathfrak{B}^I$; and
(d) both languages can be non-elementary.

In the next subsection, we will give some examples that show how to apply interpretability in investigations on decidability. In particular, we will embed these examples in an investigation of well-orderings.

Well-orderings. The elementary theory of the class WO of all well-orderings was proven to be decidable by Mostowski–Tarski [1949]. The proof uses the method of elimination of quantifiers and was published in Doner–Mostowski–Tarski [1978].

One of the simplest methods used to prove the undecidability of a theory is that of trying to show that a theory which is known to be undecidable is interpretable in it. This holds also for extended logics.

The following example shows that the expressive power of the logic with the equicardinality quantifier I is great enough to make the theory of well-orderings undecidable.

Let $K = \{\mathfrak{N}\}$, where $\mathfrak{N}$ is the structure of natural numbers with addition and multiplication, and let $K' = \{\mathfrak{M}\}$, where $\mathfrak{M} = (M, <)$ is a linear ordering of order type ω^2. Furthermore, L and L', respectively, will denote the corresponding elementary languages. $L'(I)$ arises from L' by adding the equicardinality quantifier

I. We shall show that $\mathrm{Th}_L(K)$ is interpretable in $\mathrm{Th}_{L'(I)}(K')$. In fact, the interpretation J is defined as follows:

(i) $$(x = y)^J :\equiv \exists y(y < x \wedge \neg \exists y(y < x \wedge \forall z(z \leq y \vee x \leq z))) = \varphi(x);$$

(ii) $$(x + y = z)^J :\equiv Iu(\varphi(u) \wedge u \leq x, \varphi(u) \wedge y \leq u \wedge u \leq z); \text{ and}$$

(iii) $$(x \cdot y = z)^J :\equiv Iu(\varphi(u) \wedge u \leq z, \exists v \, \exists w(\varphi(v) \wedge v < y \wedge v \leq u \wedge u < w \wedge Iw'(\varphi(w') \wedge w' < x, v \leq w' \wedge w' < w))).$$

Here φ defines the elements of $\mathfrak{N}$ as limit-elements of $\mathfrak{M}$ (see Fig. 1). The formula

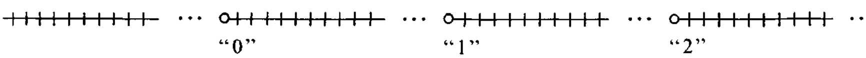

Fig. 1

on the right side of (ii) defines the addition by using the fact that between b and $a + b$ there must be just a elements. See Fig. 2, where the addition "2" + "4" = "6" is presented.

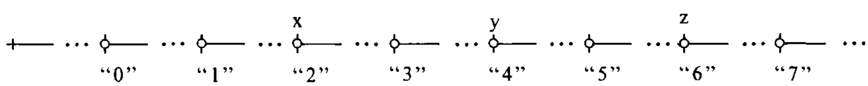

Fig. 2

To illustrate the meaning of the formula on the right side of (iii), we present the example "2" · "3" = "6" in Fig. 3. Here $x = 2$, $y = 3$, and $z = 6$.

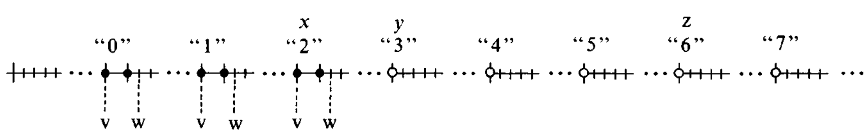

Fig. 3

The solid circles are precisely points satisfying the formula:

$$\exists v \, \exists w(\varphi(v) \wedge v < 3 \wedge Iw'(\varphi(w') \wedge w' < 2, v \leq w' \wedge w' < w) \wedge v \leq u \wedge u < w).$$

But there are just $x \cdot y = 2 \cdot 3$ points u, satisfying this formula. $\mathrm{Th}_L(K)$ is interpretable in $\mathrm{Th}_{L'(I)}(K')$ iff $(M, <)^J \cong \mathfrak{N}$. But this can be easily verified if we regard the meaning of the formulas $(x = x)^J$, $(x + y = z)^J$, and $(x \cdot y = z)^J$. Hence, we obtain the undecidability of $\mathrm{Th}_{L'(I)}$ by Theorem 2.1, since it is well-known that $\mathfrak{N}$ has an undecidable elementary theory.

From the above example, we infer the following result from Weese [1977c].

Lemma. $\mathrm{Th}_{L'(I)}(\mathrm{WO})$ *is undecidable.*

Proof. An easy way to prove the lemma is to use the fact that $\mathfrak{N}$ is strongly undecidable (see Shoenfield [1967, Theorem 2 and Theorem 3 on pages 134 and 135, respectively]). We go another way in demonstrating how to extend the notion of interpretability to languages containing the quantifier I.

We first show that $\mathrm{Th}_{L'}(\mathfrak{M})$ is interpretable in $\mathrm{Th}_{L'}(\mathrm{WO})$, where, as above, $\mathfrak{M}$ is a well-ordering of order-type ω^2. Let $\varphi_0(x)$ be a formula in L' expressing the notion

"x is the least limit-point which is a limit of limit-points".

Assume that $\mathfrak{M}'$ is a well-ordering of order-type greater than ω^2 and let a be an element of $\mathfrak{M}'$ with $\mathfrak{M}' \models \varphi_0(a)$. Then obviously

$$\mathfrak{M}' \restriction \{b/b \in |\mathfrak{M}'| \text{ and } b < a\}$$

has order-type ω^2. Hence, we get the desired interpretation, an interpretation defining $\varphi(x)$ to be $\exists y(\varphi_0(y) \wedge x < y)$ and defining $\psi_<(x, y)$ to be $x < y$.

Now we can extend this interpretation to an interpretation of $\mathrm{Th}_{L'(I)}(\mathfrak{M})$ in $\mathrm{Th}_{L'(I)}(\mathrm{WO})$. To this, we add rule (vi) as given below to rules (i) through (v) in the definition of χ^I:

(vi) $(Ix(\chi_1, \chi_2))^I := Ix(\varphi(x) \wedge \chi_1^I, \varphi(x) \wedge \chi_2^I)$.

It is easy to prove Theorem 2.1 as well as the lemma preceding it for this notion of interpretation. Hence, we obtain the undecidability of $\mathrm{Th}_{L'(I)}(\mathrm{WO})$ by the above example. □

The strongest result for the decidability of classes of well-orderings in extended logics are the results for monadic second-order theories (see Chapter XIII, Section 4.2 of this volume). They imply many other results using the method of interpretability. The following result was proved first by Slomson (see Slomson [1976]) using the method of dense systems and a game-theoretical examination of the structure of well-orderings.

2.2 Theorem. $\mathrm{Th}_{Q_0^{<\omega}}(\mathrm{WO})$ *and* $\mathrm{Th}_{Q_1^{<\omega}}(\mathrm{WO})$ *are decidable.*

Proof. Shelah [1975e] proved the decidability of the monadic theory of the class of all well-orderings $<\omega_2$, which is briefly denoted by $\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2\})$. (The reader should also see Chapter XIV, Section 4.2 or this volume for more on this). We will use this result to prove Theorem 2.2 by interpretability. Obviously the following relations are expressible by formulas in the monadic language for ordinals:

"$Y \subseteq X$" is expressible by $\forall y(y \in Y \to y \in X)$;

"$X \neq \varnothing$" is expressible by $\exists x\, x \in X$;

"x has a successor in Y" is expressible by $\exists y(y \in Y \wedge x < y)$;

"x is not the first element of Y" is expressible by $\exists y(y < x \wedge y \in Y)$;

"x is a limit in Z" is expressible by

$$\forall z(z \in Z \wedge z < x \to \exists y(y \in Z \wedge z < y < x));$$

and

"Z is confinal in Y" is expressible by $\forall y(y \in Y \to \exists z(y \leq z \wedge z \in Z))$.

Then define $\chi_0(X)$ and $\chi_1(X)$ as follows:

$$\chi_0(X) := \exists Y(\text{“}Y \subseteq X\text{”} \wedge \text{“}Y \neq \varnothing\text{”} \wedge \forall x(x \in Y \to \text{“}x \text{ has a successor in } Y\text{”}));$$

$$\chi_1(X) := \exists Y(\text{“}Y \subseteq X\text{”} \wedge \chi_0(Y) \wedge \forall Z(\text{“}Z \subseteq Y\text{”} \wedge \text{“}Z \text{ is cofinal in } Y\text{”} \to \exists x(x \in Z \wedge \text{“}x \text{ is limit in } Z\text{”}))).$$

For each well-ordering $\mathfrak{M}$ and each subset B of the domain of $\mathfrak{M}$, the following holds:

(*) $\quad \mathfrak{M} \models \chi_i(B)$ iff $|B| \geq \aleph_i$ for each $i = 0, 1$.

The downward Löwenheim–Skolem Theorem for $\mathscr{L}_{\omega\omega}(Q_i^{<\omega})$ gives

$$\mathrm{Th}_{Q_i^{<\omega}}(\mathrm{WO}) = \mathrm{Th}_{Q_i^{<\omega}}(\{(\alpha, <)/\alpha < \omega_2\}) \quad \text{for} \quad i = 0, 1.$$

We will show that $\mathrm{Th}_{Q_i^{<\omega}}(\{(\alpha, <)/\alpha < \omega_2\})$ is interpretable in

$$\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2\}).$$

To this end, let $\varphi(x)$ be the formula $x = x$ and define $\psi_<(x, y)$ to be $x < y$. We extend the rules (i) through (v) by one of the following sets of rules

(vi)$_0$ $(Q_0^n \mathbf{x}\chi)^I := \exists X(\chi_0(X) \wedge \forall \mathbf{x}\; \mathbf{x} \in X \to \chi)$ for each $0 < n \in \omega$;

(vi)$_1$ $(Q_1^n \mathbf{x}\chi)^I := \exists X(\chi_1(X) \wedge \forall \mathbf{x}\; \mathbf{x} \in X \to \chi)$ for each $0 < n < \omega$.

Condition $(*)$ guarantees that $(Q_0^n x\chi)^I$ and $(Q_1^n x\chi)^I$ get the correct interpretation by $(\mathrm{vi})_0$ and $(\mathrm{vi})_1$. We will leave it to the reader to verify that Theorem 2.1 also holds for this notion of interpretability, which proves Theorem 2.2. $\square$

At first sight stationary logic is a strengthening of $\mathscr{L}_{\omega\omega}(Q_1)$ which stands closer to monadic second-order logic than does $\mathscr{L}_{\omega\omega}(Q_1^{<\omega})$. Although the theory of well-orderings in stationary logic is decidable, there are models of set theory in which monadic second-order theory is undecidable (see, for example, Chapter XIII of this volume).

The former was proven by Mekler [1984] who used elimination of quantifiers and by Seese [1981b] who employed dense systems. Hence, it would be interesting to know whether or not this result might be inferred by interpretability from the decidability of $\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2\})$. For each natural number n, this is indeed the case for $\mathrm{Th}_{\mathrm{aa}}(\{(\alpha, <)/\alpha < \omega_1 \cdot n)$.

Exercise. Show that $\mathrm{Th}_{\mathrm{aa}}((\omega_1 \cdot n, <))$ is interpretable in

$$\mathrm{Th}_{II}(\{(\alpha, <)/\alpha < \omega_2) \quad \text{for each} \quad n \in \omega.$$

Hint. Extend the interpretability result Theorem 2.1 to $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ and then use the definability of $\omega_1 \cdot n$ in the monadic second-order logic and the fact that the initial-intervals of ω_1 build a canonical closed and unbounded set-system for $(\omega_1, <)$.

Aside from what has already been pointed out, this interpretability result gives the decidability of the theory of $(\omega_1 \cdot n, <)$, for all $n \in \omega$, in the language $\mathscr{L}_{II}(\mathrm{aa})$. It is not possible to extend this interpretability to $(\omega_1 \cdot \omega, <)$, as the following example will show. Moreover, the theorem given below shows that even an extension of $\mathrm{Th}_{\mathrm{aa}}((\omega_1 \cdot \omega, <))$ by unary predicates yields an undecidable theory.

2.3 Theorem. *Let* WOP *denote the following class of structures*

$$\{(\alpha, <, P)/(\alpha, <) \in \mathrm{WO} \text{ and } P \subseteq \alpha\}.$$

Then $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WOP})$ *is undecidable.*

Proof. The proof falls into three steps. First, we prove that the elementary theory of countable, symmetric, and reflexive graphs, a theory that is known to be undecidable (see, for example, Rabin [1965]), is interpretable in

$$\mathrm{Th}_{\mathrm{aa}}\left(\left\{\dot{\bigcup_{i<\gamma}} (\omega_1, <, P_i) \colon \gamma \leq \omega, P_i \subseteq \omega_1 \ (i < \gamma)\right\}\right).$$

Moreover, we should here remark that $\dot{\bigcup}$ is the disjoint union and not the sum of orders. The basic idea used here is the following.

Let $\mathfrak{G} = (G, R)$ be a countable, symmetric, and reflexive graph. We assume that $G = \gamma$, for some $\gamma \leq \omega$, and show that $\mathfrak{G}$ can be defined in a uniform way in a structure of the above class. By a theorem of Ulam [1930], there is a set

$$\{S_{\alpha\beta}: \alpha \leq \beta < \omega_1\}$$

of pairwise disjoint stationary subsets of ω_1. For $i < \gamma$, let

$$A_i := \bigcup \{S_{ij}: (i, j) \in R \text{ and } i < j\}.$$

Then each A_i is stationary on ω_1 and

$$(\#) \qquad A_i \cap A_j \text{ is stationary} \quad \text{iff} \quad (i, j) \in R.$$

This is used to define $\mathfrak{G}$ in $\dot{\bigcup}_{i<\gamma} (\omega_1, <, A_i)$ by the following formulas:

$$\varphi_0(x) := \neg \exists y (y < x),$$

and

$$\begin{aligned}\varphi_1(x, y) := \varphi_0(x) \wedge \varphi_0(y) \wedge (\text{stat } s)\, \exists z\, \exists u (&\text{“}\sup(s \cap \{v: v < z\}) = z\text{”} \\ &\wedge \text{“}\sup(s \cap \{v: v < u\}) = u\text{”} \\ &\wedge x < z \wedge y < u \wedge P(z) \wedge P(u)).\end{aligned}$$

Here "$\sup(s \cap \{v: v < z\}) = z$" and "$\sup(s \cap \{v: v < u\}) = u$" are abbreviations of the corresponding formulas. $\varphi_1(x, y)$ expresses just the left side of $(\#)$, while $\varphi_0(x)$ defines the domain of $\mathfrak{G}$. Hence, by using as $\varphi(x)$, $\varphi_R(x, y)$ the formulas $\varphi_0(x)$, $\varphi_1(x, y)$, respectively, we get the desired interpretation. Moreover, we must have to add the rules (vi) and (vii) to the rules (i) through (v):

(vi) $(s(x))^I := s(x) \wedge \varphi(x)$; and

(vii) $(\text{aa } s\chi)^I := \text{aa } s\chi$.

Theorem 2.1 also holds for this notion of interpretability. Thus,

$$\mathrm{Th}_{\mathrm{aa}}\left(\left\{\dot{\bigcup_{i<\gamma}} (\omega_1, <, P_i): \gamma \leq \omega, P_i \subseteq \omega_1 \ (i < \gamma)\right\}\right)$$

is undecidable.

The second step in our argument is to interpret this latter theory in

$$\mathrm{Th}_{\mathrm{aa}}(\{(\omega_1 \cdot \omega, <, P): P \subseteq \omega_1 \cdot \omega\}).$$

This can be easily done using the notion of interpretability given above for stationary logic and we leave the details of it to the reader. Finally, we notice that this theory is interpretable in $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WOP})$. We give only a hint for this third interpretation: For each well-ordering $(\alpha, <)$ with $\alpha > \omega_1 \cdot \omega$ the point $\omega_1 \cdot \omega$ is uniformly definable that is, it is definable independently of α in $(\alpha, <)$ by a formula from $\mathscr{L}_{\omega\omega}(Q_1)$. This completes the proof. □

We will conclude this subsection with a few additional facts and some historical notes.

Further Results. In addition to Theorem 2.2, Slomson [1976] proved that $\mathrm{Th}_{Q_\alpha^{<\omega}}(\mathrm{WO})$ is decidable for all ordinals α. He used the method of dense systems and a game-theoretical examination of the structure of well-orderings. Moreover, he proved that for all α, $\beta > 0$ the theory $\mathrm{Th}_{Q_\alpha^{<\omega}}(\mathrm{WO})$ equals the theory $\mathrm{Th}_{Q_\beta^{<\omega}}(\mathrm{WO})$, while $\mathrm{Th}_{Q_0^{<\omega}}(\mathrm{WO})$ differs from these theories. These results had already been proven by Vinner [1972] for $\mathscr{L}_{\omega\omega}(Q_\alpha)$ rather than for $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$. The reader should also consult Lipner [1970] and Slomson [1972] for more on this. A further generalization of Theorem 2.2 was proven in Tuschik [1982b]. In particular, let Δ be a set of ordinals, and let $L(\Delta)$, $L(\Delta)^{<\omega}$, respectively, be the language L with the additional quantifiers Q_α (for $\alpha \in \Delta$) and Q_α^n, for $\alpha \in \Delta$ and $n \geq 1$, respectively. Assuming GCH, Tuschik [1982b] proceed to prove that $\mathrm{Th}_{L(\Delta)}(\mathrm{WO})$ is decidable for each finite set Δ of non-limit ordinals. In this connection we note that Wolter [1975b] proved this for $\Delta = \{0, \alpha\}$. Moreover, assuming GCH, Tuschik [1982b] proved that, for any finite Δ, $L(\Delta)^{<\omega}$ is reducible to $L(\Delta)$ with respect to the class WO and that $\mathrm{Th}_{L(\Delta)^{<\omega}}(\mathrm{WO})$ is decidable. By performing ordered sums of finitely determinate linear orderings, the proof of the decidability of $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WO})$ given in Seese [1981b] used the method of dense systems and an investigation of the preservation of $\equiv_n (L(\mathrm{aa}))$. Interestingly enough, the proof yields that all well-orderings are finitely determinate, a fact which was also proven by Mekler [1984]. Moreover, Mekler [1984] proved that a simple extension of $\mathrm{Th}_{\mathrm{aa}}(\mathrm{WO})$ by unary predicates and defining axioms for it admits elimination of second-order quantifiers.

Some further results on this can be found in Caicedo [1978], Kaufmann [1978a, b], and Mekler [1984], as well as in Seese–Weese [1982]. The reader should also see Chapter XIII of this volume for material on these notions. Finally, we note that the results cited at the end of the next section also provide some material on boolean algebras.

3. Dense Systems

The method of dense systems was used by Ershov [1964b] and by Läuchli–Leonard [1966] to obtain the decidability of the theories of boolean algebras and linear orderings, respectively. The method used in these studies can be formulated in a general form, a form which is applicable both to axiomatizable and non-axiomatizable logics. In order to develop this form, we first let K be a class of models and L any logic, and then make the following

Definition. A countable subset $M \subseteq K$ of models is:

(i) *dense for K (with respect to L)* if any sentence of L which is satisfiable in K already has a model in M; and
(ii) *is uniformly recursive with respect to L* if the relation "$A \models \varphi$" is recursive, where A varies over models of M and φ over sentences of L (we assume a fixed Gödel-numbering).

3.1 Theorem. *Suppose K and L are as above and M is dense for K and uniformly recursive with respect to L. Then the theory $\mathrm{Th}_L(K)$ is decidable if either:*

(i) *L and K are (recursively) axiomatizable; or*
(ii) *there is a recursive function f so that for each sentence φ of L having a model in K, there is a model $A \in M$, $A \models \varphi$ with $\ulcorner A \urcorner < f(\ulcorner \varphi \urcorner)$. Here, the notation $\ulcorner A \urcorner$ and $\ulcorner \varphi \urcorner$ denote the corresponding Gödel-numbers.*

Generally, to obtain a Gödel-numbering, the set M is generated from simple structures by some operations such as sums, products *etc.* A logic L which preserves L-elementary equivalence for these operations is especially convenient to obtaining decidability. If it preserves L-elementary equivalence for the direct product it has the *product property*. For instance, the elementary logic has the product property as well as the logic with the additional quantifier Q_α. But, on the other hand, the logic with Malitz quantifiers Q_α^n $(n > 1)$ and stationary logic do not possess this property. However, stationary logic does have the product property if only finitely determinate structures are considered.

In the following two subsections, the sets M are constructed for the classes of linear orderings and boolean algebras, respectively. This will clarify the abstract notions that are given above. Furthermore, we obtain some insight into the expressive power of cardinality quantifiers for these two classes.

3.1. Linear Orderings

Let us consider the class of linear orderings LO in the logic $L(Q_1)$ which has the cardinality quantifier Q_1. The decidability of the elementary theory of LO was established in Ehrenfeucht [1959b].

3.1.1 Theorem. *The theory* Th_{Q_1} (LO) *is decidable.*

This result was shown by Tuschik [1977b] and by Herre–Wolter [1977]. The proof closely follows the line given by Läuchli–Leonard [1966] for the elementary case, but with some important exception: the Ramsey theorem cannot be used, since it is no longer valid in the uncountable case. However, Shelah's theorem for additive colourings [1975e] is a useful substitute.

As to Theorem 3.1, it follows that it is enough to have a dense set M which is uniformly recursive. The models in M are called *term-models*. In order to define M, we need some special dense linear orderings $\sigma^{m,n}$ which we will briefly describe as follows: The $\sigma^{m,n}$ are uncountable dense linear orderings with finitely many predicates $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ which form a partition of the underlying set of $\sigma^{m,n}$, so that $X_1, \ldots, X_m$ are countable dense subsets and $Y_1, \ldots, Y_n$ are ω_1-dense subsets, where Y_i is said to be ω_1-dense if between any two elements there are uncountably many elements of Y_i. Suppose that $F = (A_1, \ldots, A_m)$ and $G = (B_1, \ldots, B_n)$ are two finite sequences of linear orderings, then $\sigma(F, G)$ results from $\sigma^{m,n}$ by replacing each point from X_i or Y_j by a copy of A_i or B_j, respectively. $\sigma(F, G)$ is thus called the *shuffle* of (F, G).

Now, the set M is the smallest set containing $\mathbf{1}$ (the unique one-element order), so that we have the following:

(i) If $A, B \in M$ then $A + B \in M$;
(ii) If $A \in M$, *so are* $A \cdot \omega$, $A \cdot \omega^*$, $A \cdot \omega_1$, and $A \cdot \omega_1^*$; and
(iii) If F and G are finite sequences of models from M, then $\sigma(F, G)$ belongs to M also.

The operations above are defined as usual for linear orderings. To show that M is dense, it is convenient to use n,1-isomorphisms, these having been introduced in Chapter II, Section 4.2. In the original papers, the game-theoretic equivalent of $\simeq_{n,1}$ was used (see Lipner [1970], Brown [1972]). To mark this difference we will denote $\simeq_{n,1}$ by $\overset{n}{\sim}$ in the following. The proof of the following lemma is omitted.

Lemma. *The operations which generate M preserve* $\overset{n}{\sim}$. □

A linear ordering A is called *n-term-like* iff there is a term-model B such that $A \overset{n}{\sim} B$. The crucial point consists in proving the following fact:

Lemma. *Suppose every bounded convex subset of A is n-term-like, then A itself is n-term-like.*

Proof. We may suppose that A has a least element (otherwise, we can partition $A = B + C + D$, where B and D have a greatest or least element, respectively, and C is bounded and convex). By the Löwenheim–Skolem theorem for $L(Q_1)$, we can assume A has cardinality $\aleph_1$.

However, A then possesses an increasing cofinal κ-sequence, where κ is 1, ω, or ω_1. In the first case, the stated property follows immediately, since then A is bounded. To establish the other two cases, we remark that the equivalence relation $\overset{n}{\sim}$ has only finitely many equivalence classes. Hence, $\overset{n}{\sim}$ induces a colouring by assigning to the pair $\langle a, b\rangle$ the equivalence class of the interval $(a, b]$ (as an ordered set) with respect to $\overset{n}{\sim}$. This colouring is additive since $+$ preserves $\overset{n}{\sim}$ as was stated in the lemma above. Now, we can apply Shelah's theorem on additive colourings to choose homogeneous subsets. Hence, there is a subset $X \subseteq A$ of order-type ω_1 (or ω in the second case, respectively) so that for any element $a < b$ and $c < d$ of X we have

$$(a, b] \overset{n}{\sim} (c, d].$$

We refer to the original papers concerning the relation

$$A \overset{n}{\sim} A_0 + A_1 \cdot (\omega^* + \omega) \cdot \omega_1,$$

where A_0 and A_1 are bounded segments of A (namely, $A_0 = A^{\leq x_0}$ and $A_1 = (x_0, x_1]$ and x_0 and x_1 are the first two elements of X).

By the hypothesis, A_0 and A_1 are n-term-like, say $A_0 \overset{n}{\sim} B_0$ and $A_1 \overset{n}{\sim} B_1$ with $B_0, B_1 \in M$. Thus, we also have

$$A \overset{n}{\sim} B_0 + B_1 \cdot \omega + B_1 \cdot (\omega^* + \omega) \cdot \omega_1.$$

However, the right side itself is a term-model. Thus, A is n-term-like, and the lemma is proven. □

From the following lemma we can easily conclude that M is dense in LO.

Lemma. *Every linear ordering A is n-term-like.*

Proof. By the Löwenheim–Skolem theorem for $L(Q_1)$, we may again suppose that A is of cardinality $\leq \aleph_1$. We define an equivalence relation $\approx$ on A as follows:

$x \approx y$ iff every segment of the closed interval $[x, y]$ is n-term-like. Clearly, $\approx$ is convex. Furthermore, by the preceding lemma, every equivalence class itself is n-term-like.

Claim. *There is only one equivalence class.*

Assume there are two different equivalence classes $C < D$ in $A/\approx$. If D is a successor of C, then we can prove that the elements of C and D are equivalent, since M is closed under addition. However, this would contradict the assumption about $C < D$. But otherwise $A/\approx$ has to be dense. The elements of $A/\approx$ are themselves linear orderings, and, as we have already proven, they are n-term-like. Thus, there are term-models $A_1, \ldots, A_k$ so that every $C \in A/\approx$ is equivalent to some A_i, $1 \leq i \leq k$. For $C < D \in A/\approx$, we let $F(C, D) \subseteq \{A_1, \ldots, A_k\}$ be the subset of those term-models A_i which are $\overset{n}{\sim}$-equivalent to some E between C and D. Similarly, let $G(C, D) \subseteq \{A_1, \ldots, A_k\}$ be the subset of all term-models A_i so that there are uncountably many E between C and D, with $E \overset{n}{\sim} A_i$. Now, choose $C < D$ in $A/\approx$ with $F(C, D)$ and $G(C, D)$ minimal. Clearly, this implies that, for $C < E < F < D$, $F(C, D) = F(E, F)$ and $G(C, D) = G(E, F)$. Then, it is not difficult to prove that

$$\bigcup_{N \in (E, F)} N \overset{n}{\sim} \sigma(F(C, D), G(C, D)),$$

for any E and F, with $C < E < F < D$. Before continuing our argument, we should remark that (E, F) is the open interval in $A/\approx$ with endpoints E and F. Returning to our line of argument we note that by definition, $\sigma(F(C, D), G(C, D))$ is again a term-model. Thus, we may conclude that the elements of C and D are $\approx$-equivalent.

But this would be a contradiction to $C < D$. Thus, the claim holds and the lemma is proven. □

Corollary. If some sentence φ of $L(Q_1)$ *has an ordered set as a model, then it also has a term-model as a model.*

Proof. Let A be a model of φ. By the Löwenheim–Skolem theorem, we may assume that A has cardinality $\leq\aleph_1$. Suppose the quantifier rank of φ is n. Then, by the preceding lemma, there is a term-model B, so that $A \overset{n}{\sim} B$. However, using claim (∗) from the proof of Corollary 4.2.4 in Chapter II, this implies $B \models \varphi$. □

From the definition of the set M, we know that its members have a very determined structure. This idea is used to prove that M is uniformly recursive with respect to $L(Q_1)$.

Lemma. *M is uniformly recursive with respect to $L(Q_1)$.* □

The proof of the above lemma can be accomplished by induction on the complexity of the term-models and the sentences.

Now, using Theorem 3.1 we obtain the decidability of $\mathrm{Th}_{Q_1}(\mathrm{LO})$, and hence Theorem 3.1.1 is proven.

We have illustrated the main idea in order to prove the decidability of the theory of linear orderings in a language with the quantifier Q_1. Now, let us mention some further results about the class of linear orderings for logics with other generalized quantifiers.

(1) First of all, we refer the reader to Chapter XIII where second-order quantifiers are considered.

(2) (GCH) $\mathrm{Th}_{Q_\alpha}(\mathrm{LO})$ is decidable for every ordinal α. The case $\alpha = 0$ follows from Läucnli [1968]. Since, for regular $\aleph_\alpha$, this theory is the same as $\mathrm{Th}_{Q_1}(\mathrm{LO})$, it is clear that its decidability follows from Theorem 3.1.1, Herre–Wolter [1979b] provides a proof of it for singular $\aleph_\alpha$.

(3) Let Δ be a finite set of ordinals such that for all $\alpha \in \Delta$, $\aleph_\alpha$ is regular. L_Δ denotes the language of linear orderings with the additional generalized quantifiers Q_α, $\alpha \in \Delta$. Under some conditions that are weaker than GCH, Tuschik [1980] proved the decidability of $\mathrm{Th}_{L_\Delta}(\mathrm{LO})$.

(4) Let $L_\Delta^{<\omega}$ be the language L with the additional Malitz quantifiers Q_α^m, for all $\alpha \in \Delta$. If we only add the binary Malitz quantifier, the extended language will then be denoted by L_Δ^2. Suppose that, for all $\alpha \in \Delta$, $\aleph_\alpha$ is regular, then Tuschik [1982b] has shown that $L_\Delta^{<\omega}$ is reducible to L_Δ^2 for the class of linear orderings. Furthermore, $\mathrm{Th}_{L_\Delta^{<\omega}}(\mathrm{LO})$ is decidable. For the limit cardinal number $\aleph_\omega$, it is also shown that $L_{\{\omega\}}^{<\omega}$ is reducible to $L_{\{\omega\}}^2$ for linear orders.

(5) In contrast to the results mentioned above, the theories $\mathrm{Th}_I(\mathrm{LO})$ and $\mathrm{Th}_{\mathrm{aa}}(\mathrm{LO})$ are undecidable. The undecidability of $\mathrm{Th}_I(\mathrm{LO})$ follows immediately from that of $\mathrm{Th}_I(\mathrm{WO})$ (see Section 2), while that of $\mathrm{Th}_{\mathrm{aa}}(\mathrm{LO})$ is proven in Seese–Tuschik–Weese [1982].

3.2. Boolean Algebras

The decidability of the elementary theory of boolean algebras Th(BA) was proved by Tarski [1949]. Some years later, Ershov [1964b] showed that the theory of boolean algebras with a distinguished prime ideal is also decidable. Here we will consider the class of boolean algebras in the logic with the additional cardinality quantifier Q_α, for arbitrary ordinals α.

First, we will compare the various cardinality quantifiers with each other. Therefore, throughout this subsection we will work in a fixed model of set theory, where δ is that ordinal which satisfies $\aleph_\delta = \beth_\omega$. Weese [1976b] showed the following

Theorem. *For every ordinal* $\alpha > 0$, *we have*

(i) $\mathrm{Th}_{Q_\alpha}(\mathrm{BA}) = \mathrm{Th}_{Q_1}(\mathrm{BA})$ *iff there is some* $\beta < \alpha$, *with* $2^{\aleph_\beta} \geq \aleph_\alpha$;
(ii) $\mathrm{Th}_{Q_\alpha}(\mathrm{BA}) = \mathrm{Th}_{Q_\delta}(\mathrm{BA})$ *iff* $2^{\aleph_\beta} < \aleph_\alpha$, *for every* $\beta < \alpha$. □

Remark. In fact, $L(Q_\alpha)$ and $L(Q_\beta)$ represent one and the same language $L(Q)$. The ordinal subscript only serves to mark the different interpretation. If we make comparisons such as the above, we can consider the theories Th_{Q_α} as subsets of $L(Q)$.

From the theorem, we see that there are at most three different theories of boolean algebras in logics with cardinality quantifiers, namely $\mathrm{Th}_{Q_0}(\mathrm{BA})$, $\mathrm{Th}_{Q_1}(\mathrm{BA})$, and $\mathrm{Th}_{Q_\delta}(\mathrm{BA})$. The connection between these theories is illustrated in the next proposition.

Proposition. $\mathrm{Th}_{Q_1}(\mathrm{BA}) \subsetneqq \mathrm{Th}_{Q_\delta}(\mathrm{BA}) \subsetneqq \mathrm{Th}_{Q_0}(\mathrm{BA})$.

Proof. We will only prove that the inclusions are proper. Let At(x), at(x), and atl(x) be formulas of the elementary language of boolean algebras which express the properties "x is an atom", "x is atomic", and "x is atomless", respectively. Set

$$\varphi := \forall x(\mathrm{atl}(x) \rightarrow Qy(y \leq x)),$$

and

$$\psi := \forall x(\mathrm{at}(x) \wedge Qy(y \leq x) \rightarrow Qy(\mathrm{At}(y) \wedge y \leq x)).$$

Then it is immediately seen that

$$\varphi \in \mathrm{Th}_{Q_0}(\mathrm{BA})\backslash\mathrm{Th}_{Q_\delta}(\mathrm{BA}) \quad \text{and} \quad \psi \in \mathrm{Th}_{Q_\delta}(\mathrm{BA})\backslash\mathrm{Th}_{Q_1}(\mathrm{BA}). \quad \square$$

Now, to prove the decidability of these theories, we want to establish dense sets M_0, M_1, and M_σ. For the sake of simplicity, we will restrict ourselves to the construction of M_0 in the following discussion. The constructions of M_1 and M_δ would require some further operations, so that we will omit them entirely and refer the reader to the literature. Before we can define the set M_0, we must introduce two operations for boolean algebras. Let η be the set of rational numbers. Then $\bigoplus_\eta B$ and $\prod^\eta B$ are subalgebras of the Cartesian product $\prod_{i\in\eta} B_i$, where $B_i = B$ for all $i \in \eta$. $\bigoplus_\eta B$ is the subalgebra generated by the elements $\{a_i : i \in \eta\}$, where $\{i \in \eta : a_i \neq 0\}$ is finite. This kind of product is also called a *direct sum.*

Let $I(\eta)$ be the boolean subalgebra of the power set of η, which is generated by the intervals. Then $\prod^\eta B$ is generated by the elements $\{a_i : i \in \eta\}$ with the properties that $\{i \in \eta : a_i \neq 0\}$ belongs to $I(\eta)$, and $\{i \in \eta : a_i \neq 0 \text{ and } a_i \neq 1\}$ is finite.

We are now ready to define M_0. Let **2** be the unique boolean algebra with only two elements and let P be any fixed countable atomless boolean algebra. Then, M_0 is the smallest set containing **2** and P such that the following hold:

(i) if A and B belong to M_0, then so does their direct product $A \times B$;
(ii) if $B \in M_0$, then $\bigoplus_\eta B$ and $\prod^\eta B$ also belong to M_0.

The algebras in M_0 are called *term-models.* To show that M_0 is dense it is convenient to use n, 0-isomorphisms, these latter having been introduced in Chapter II, Section 4.2. In the original paper the game-theoretic equivalent of $\cong_{n,0}$ was used (see Lipner [1970] and Brown [1972]). We observe that it has an especially simple form for boolean algebras. To mark this difference, we denote $\cong_{n,0}$ by $\overset{n}{\sim}$ in the following discussion. The proof of the lemma given below is omitted.

Lemma. *The operations which generate M_0 preserve $\overset{n}{\sim}$ also.* $\square$

A boolean algebra A is *n-term-like* iff there is a term-model B so that $A \overset{n}{\sim} B$. If a is an element of the boolean algebra B, then the ideal generated by a is denoted by $(a)_B = \{b \in B : b \leq a\}$. If no confusion can arise, we omit the subscript B altogether. By interpreting the constant 1 by the element a, we see that the structure $(a)_B$ becomes a boolean algebra. For each boolean algebra B, we can thus define the subset $D_n(B)$ of n-term-like elements as

$$D_n(B) = \{a \in B : \text{for every non-zero } b \in (a), (b) \text{ is } n\text{-term-like}\}.$$

Lemma. *$D_n(B)$ is an ideal.*

Proof. Clearly, if $a \in D_n(B)$ and $b \leq a$, then $b \in D_n(B)$. Let be $a, b \in D_n(B)$. If $a \leq b$ or if $b \leq a$, then obviously $a \cup b \in D_n(B)$. Otherwise, $a \cup b = a \cup (b \backslash a)$ and $a \neq 0$ and $b \backslash a \neq 0$. Since a and $(b\backslash a)$ are n-term-like, there are term-models A_1 and A_2 such that $(a) \overset{n}{\sim} A_1$ and $(b\backslash a) \overset{n}{\sim} A_2$. However, since a and $(b\backslash a)$ are disjoint, we get that $a \cup (b\backslash a) \overset{n}{\sim} A_1 \times A_2$. By definition, $A_1 \times A_2$ is again a term-model. Hence, $a \cup b$ is n-term-like. If $c \leq a \cup b$, then we can repeat the proof for $a \cap c$ and $(b\backslash a) \cap c$. Hence, the element $a \cup b$ belongs also to $D_n(B)$. $\square$

From the next lemma we can easily conclude that M_0 is dense in BA.

Lemma. *Every boolean algebra is n-term-like.*

Proof. By the preceding lemma, we know that $D_n(B)$ is an ideal for every boolean algebra B. We will show that $D_n(B)$ is not proper. Then $B = (1)$ is n-term-like and the lemma is proved. Assume that $1 \notin D_n(B)$. Since $\overset{n}{\sim}$ has only finitely many equivalence classes, there are $A_1, \ldots, A_k \in M_0$ such that any n-term-like boolean algebra is $\overset{n}{\sim}$-equivalent to some A_i, $1 \leq i \leq k$. For each $b \in B \setminus D_n(B)$, let

$$T_n(b) = \{i\colon \text{there is some } c \in D_n(B) \text{ with } c \leq b \text{ such that } (c) \overset{n}{\sim} A_i\}.$$

Let $a \in B \setminus D_n(B)$ be minimal. That is, for every $b \in B \setminus D_n(B) \cap (a)$ $T_n(b) \supseteq T_n(a)$. Clearly, we may assume that either $a/D_n(B)$ is an atom or atomless. We will show that in either cases a is n-term-like.

Case 1. $a/D_n(B)$ is an atom.
If $D_n(B)$ restricted to (a) is the zero-ideal, then a is an atom in B also; thus $(a) \overset{n}{\sim} \mathbf{2}$ and a is n-term-like. Otherwise, $D_n(B)$ is not the zero-ideal and we can prove that

$$(a) \overset{n}{\sim} \bigoplus_{\eta} A, \quad \text{where} \quad A = \prod_{i \in T_n(a)} A_i.$$

Since M_0 is closed under direct product, the algebra A belongs to M_0. Furthermore, M_0 is also closed under the direct sum of an algebra. Hence, $\bigoplus_\eta A$ is a term-model and (a) is n-term-like.

Case 2. $a/D_n(B)$ is atomless.
If $D_n(B)$ restricted to (a) is the zero-ideal, then a is atomless in B also. Thus, $(a) \overset{n}{\sim} P$ and a is n-term-like. Otherwise, $D_n(B)$ is not the zero-ideal, and we can prove that

$$(a) \overset{n}{\sim} \prod\nolimits^{\eta} A, \quad \text{where} \quad A = \prod_{i \in T_n(a)} A_i.$$

As in the first case, $\prod^\eta A$ is a term-model, and hence (a) is n-term-like.

If $b \leq a$, then either $b \in B \setminus D_n(B)$ or $b \in D_n(B)$. In both cases b is n-term-like (in the first case, the proof is the same as for the element a above). However, a must then be an element of $D_n(B)$, which is a contradiction. Hence $D_n(B) = B$. □

Corollary. M_0 *is dense for* BA *with respect to* $L(Q_0)$. □

The proof is similar to the corresponding proof of the corollary of Theorem 3.1.1.

An easy construction of the term-models is used to prove the following

Lemma. M_0 *is uniformly recursive with respect to* $L(Q_0)$.

Proof. The proof is by induction on the complexity of the term-models and the sentences. □

As a conclusion we obtain the following theorem, a result that was proved by Pinus [1976] and by Weese [1977a].

Theorem. *The theory* $\mathrm{Th}_{Q_0}(\mathrm{BA})$ *is decidable.* □

In a similar way (by using rather complicated term-models), we can prove the decidability of the theories $\mathrm{Th}_{Q_1}(\mathrm{BA})$ and $\mathrm{Th}_{Q_\delta}(\mathrm{BA})$. In connection with the first theorem of this subsection, we may conclude the following result due to Weese [1976b].

Theorem. *For every ordinal number* α, *the theory* $\mathrm{Th}_{Q_\alpha}(\mathrm{BA})$ *is decidable.* □

Now, we want to compare the expressive power of $L(Q_0)$ with those of the elementary language L and weak second-order logic L_{ws}. Let F be the boolean subalgebra of the power set of ω generated by the finite sets. Then $F \equiv F \times F\,(L)$; however, in $L(Q_0)$, they can be distinguished by the sentence φ, where

$$\varphi := \exists x\, \exists y(x \cap y = 0 \wedge Q_0 z(z \leq x) \wedge Q_0 z(z \leq y)).$$

Hence, $L(Q_0)$ is really more expressive. On the other hand, we have, for any boolean algebras A and B,

$$A \equiv B\,(L_{\mathrm{ws}}) \quad \text{iff} \quad A \equiv B\,(L(Q_0)).$$

Thus, L_{ws} and $L(Q_0)$ are of the same expressive power. However, while $\mathrm{Th}_{Q_0}(\mathrm{BA})$ is decidable, $\mathrm{Th}_{\mathrm{ws}}(\mathrm{BA})$ is not, as was proved by Paljutin [1971].

In the following discussion, we will mention further decidability results for the class of boolean algebras.

(1) First of all, we refer to the results of Rabin [1969, 1977], who proved the decidability of the theory $\mathrm{Th}_{LI}(P)$, where P is a countable atomless boolean algebra and LI is a second-order language appropriate for boolean algebras whose set variables range over ideals. Rabin interpreted this theory in $S2S$, the monadic theory of two successor functions. Using the fact that for each countable boolean algebra A there is an ideal I on P so that $A \cong P/I$, he concluded that the theory of all countable boolean algebras in the logic LI is also decidable. As a corollary, he obtained the decidability of the elementary theory of boolean algebras with a sequence of distinguished ideals, an accomplishment generalizing the result of Ershov that was mentioned at the beginning of the subsection.

(2) In this discussion, CH is assumed. Using a result of Sierpinski on the existence of special families of linear orderings, Rubin [1982] established the undecidability of $\mathrm{Th}_{Q_1^2}(\mathrm{BA})$, the theory of boolean algebras in the logic with the binary Malitz quantifier in the $\aleph_1$-interpretation.

(3) In contrast to the preceding fact, Molzan [1981b] proved the decidability of $\mathrm{Th}_{Q_0^2}(\mathrm{BA})$ by a quantifier elimination procedure.

(4) The undecidability of the theory $\mathrm{Th}_I(\mathrm{BA})$ in the logic with the Härtig quantifier I was proved by Weese [1976c] by means of interpretation.

(5) Interpretability also yields the undecidability of the theory $\mathrm{Th}_{\mathrm{aa}}(\mathrm{BA})$ of boolean algebras in the stationary logic. This fact was proven by Seese–Tuschik–Weese [1982].

Open Problems

(1) Find appropriate "first-order" conditions equivalent to the eliminability of all Ramsey quantifiers Q_0^m or to the eliminability of all Malitz quantifiers Q_1^m ($m < \omega$) in unstable (countable) complete first-order theories. For stable theories this is known (see Theorem 1.2.3 and Remark 7 at the end of Section 1.2).

(2) Investigate the relative strength of eliminability of Q_α^m for various ordinals α and fixed $m < \omega$. For stable theories, this is known in the case $m = 1$ (see Remark 1 at the end of Section 1.2). In the case $m > 1$, only some partial information is presently available (see Remark 8 at the end of Section 1.2).

(3) Investigate the relative strength of eliminability of Q_α^m for various numbers m (and fixed ordinals α). For stable theories, this is known in case $\alpha = 0$ and $\alpha = 1$ (see Theorem 1.2.3 and Remark 8 at the end of Section 1.2, respectively).

(4) Is $T_Z(I)$, the theory of abelian groups in the logic with the Härtig quantifier, decidable?

(5) Is the theory of well-founded trees in the logic with Q_1 decidable?

(6) Is it consistent with ZFC that $\mathrm{Th}_{Q_1^2}(\mathrm{BA})$ is decidable? Under CH it is not (see Remark 2 at the end of Section 3.2).

Part C

Infinitary Languages

This part of the book is devoted to languages with infinitely long formulas and their applications. Again the structures are of the sort studied in first-order model theory. Languages with richer structures and infinitely long formulas are studied in Part E. The study of infinitely long formulas is more developed than some of the other parts of extended model theory. In particular, there are several books treating various aspects of the subject, notably Keisler [1971a] and Dickmann [1975]. This part of the present book was planned with the existence of these references in mind, containing chapters that give an introduction to the subject leading into these books as well as chapters that discuss more recent advances.

Chapter VIII presents a wealth of material on $\mathscr{L}_{\omega_1\omega}$ and some of its sublogics. Starting with the original motivations for studying languages with infinitely long formulas, the chapter provides both a basic introduction and an explanation of many of the developments that have taken place since Keisler's [1971a] publication. In addition, it discusses extensions of $\mathscr{L}_{\omega_1\omega}$ by new propositional connectives. The importance of these extensions is not for their intrinsic interest so much, as for the fact that they seem to have all the nice properties of $\mathscr{L}_{\omega_1\omega}$, and so make it difficult to find a characterization of $\mathscr{L}_{\omega_1\omega}$ by its model-theoretic properties.

Chapter IX presents an introduction to the stronger logics $\mathscr{L}_{\kappa\lambda}$, one that leads into Dickmann's book [1975] on this topic but also goes beyond it with the presentation of some more recent results. Special emphasis is given to partial isomorphisms and their applications, and to Hanf number computations.

One of the more recent developments in infinitary logic is that dealing with game quantification which has grown out of the work of Svenonius [1965], Moschovakis [1972] and Vaught [1973b]. The logic $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\infty\omega}$ allow only finite strings of quantifiers at any stage in the transfinite process of building formulas. $\mathscr{L}_{\omega_1\omega_1}$ and $\mathscr{L}_{\infty\omega_1}$ permit infinitely long strings of the forms

$$\forall x_1 \forall x_2 \ldots \phi(x_1, x_2, \ldots)$$

and

$$\exists x_1 \exists x_2 \ldots \phi(x_1, x_2, \ldots).$$

The logics $\mathscr{L}_{\infty G}$ and $\mathscr{L}_{\infty V}$ studied in this chapter are stronger than $\mathscr{L}_{\infty\omega}$ but are not comparable with $\mathscr{L}_{\infty\omega_1}$. They contain more powerful forms of infinite quantification, by allowing infinite strings with alternations.

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \phi(x_1, y_1, x_2, y_2, \ldots).$$

However, they are more restrictive in terms of the form of the matrix ϕ that can follow the quantifiers. As the name "game quantification" suggests, a basic motivation comes from game theory. We imagine a two-person game of perfect information played by "$\forall$" and "$\exists$". They are allowed to play in turns. The formula is true in some structure if "$\exists$" has a winning strategy. The restriction on the matrix ϕ represents a restriction on the complexity of the games they are allowed to play. Basically, the games should be "open" or "closed", so that one of the players has a winning strategy. As a consequence, one has

$$\neg(\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \phi)$$

logically equivalent to

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \ldots \neg\phi$$

and equivalence which would fail without some such restriction. It is also exactly these open and closed games that arise in the analysis of inductive definitions, as Moschovakis showed. Vaught showed how these game formulas can be approximated by formulas of $\mathscr{L}_{\omega_1\omega}$, leading to interesting proofs of results about the latter logic. Svenonius' theorem relates the logics to the study of second-order logic on countable structures. All of these results are covered in Chapter X, as well as some of the connections with generalized recursion theory and descriptive set theory.

Chapter XI, the final one in Part C, presents several applications of infinitary logics to algebra. The chapter is organized by algebraic subject matter. The first two sections, on universal locally finite groups and on subdirectly irreducible algebras, respectively, contain "pure" applications, applications of infinitary logic to prove results that can be stated in standard algebraic terms without reference to concepts from logic. The remaining sections, on Lefschetz's principle, abelian groups, almost-free algebras, and concrete constructions, present the conceptually more interesting kind of application where concepts from logic are brought in to enrich some domain.

Chapter VIII

$\mathscr{L}_{\omega_1\omega}$ and Admissible Fragments

by M. Nadel

Of the many strengthenings of first-order logic that the reader will encounter in the course of this book, $\mathscr{L}_{\infty\omega}$ and its admissible fragments have attracted the most attention by a wide margin. Unlike many of the others, these logics are often studied by investigators who are not otherwise involved with questions of abstract model theory. A large body of "hard" model theory has already been developed, and it continues to grow. Such a wealth of material, when coupled with stringent space limitations, creates obvious difficulties for any researcher aiming to present an exposition of this fascinating and ever-growing theory. We have attempted to contend with these difficulties in as reasonable a way as possible while all the time fully recognizing that even the catalog of results that we do present here is indeed far from complete. In fact, entire areas are omitted. We have tried to compensate for this, at least to some extent, through an appendix. Moreover, of the topics we do cover, we try to mention at least the most basic results and then direct the reader to other sources for further information.

In keeping with the procedure sketched in the preceding paragraph, we have tried to strike a reasonable balance between "hard" and "soft" material, but have steered clear of results in the direction of stability theory. Sections 3 and 6 are concerned mainly with "softer" considerations, while Sections 4 and 7 deal mainly with those "harder" aspects that are particularly characteristic of infinitary logic. The distinction here is not absolute, of course, nor is it strictly observed. Sections 1 and 5 provide the necessary background material while Section 2 is concerned with elementary equivalence. Section 8 deals with propositional extensions, and is, perhaps, the "icing on the cake"—a part which some may like best, but which others may prefer to avoid. In any event, the methods used in that section make it a worthwhile discussion even for the reader whose interest in abstract logic is quite limited.

Again, we would like to emphasize that within the limitations imposed by strict space requirements and an already large (and rapidly growing) body of theory, it is hardly possible to completely eliminate one's own prejudices and preferences either with respect to the topics to be treated or to the treatment they are to receive. Fully aware of this, we have nevertheless tried to present a reasonably orthodox treatment of the subject. We hope we have succeeded.

Part I. Compactness Lost

1. Introduction to Infinitary Logics

1.1. Why We Need Infinitary Logic

In the practice of model theory, and in more general mathematics as well, it often becomes necessary to consider structures satisfying certain collections of sentences rather than just single sentences. This consideration leads to the familiar notion of a theory in a logic. For example, in ordinary finitary logic, $\mathscr{L}_{\omega\omega}$, if φ_n is a sentence which expresses that there are at least n elements, then the theory $\{\varphi_n: n \in \omega\}$ would express that there are infinitely many elements. Similarly, in the theory of groups, if ψ_n is the sentence $\forall x[x^n \neq 1]$, then $\{\psi_n: n \in \omega\}$ expresses that a group is torsion free.

Suppose we want to express the idea that a set is finite, or that a group is torsion. A simple compactness argument would immediately reveal that neither of these notions can be expressed by a theory in $\mathscr{L}_{\omega\omega}$. What we need to express in each case is that a certain theory is not satisfied, that is, that at least one of the sentences is false. While theories are able to simulate infinite conjunctions, there is no apparent way to simulate infinite disjunctions—which is just what is needed in this case.

A similar phenomenon occurs with respect to the description of the elements in a structure. In order to specify that there is some element satisfying a certain set of formulas—for instance, $x \neq \mathbf{0}$, $x \neq \mathbf{1}$, $x \neq \mathbf{2}$, and so on—we might simply introduce a new constant symbol, say c, and then consider the theory in the language augmented by c, containing $c \neq \mathbf{0}$. $\mathbf{c} \neq \mathbf{1}$, $\mathbf{c} \neq 2, \ldots$. Suppose, however, that we want to consider structures, say models of set theory, in which the set of natural numbers is standard. Here we must introduce the notion of a *type*; that is, a consistent set of formulas in some fixed finite set of variables. We say that a model $\mathfrak{M}$ *realizes* the type $\Phi(x) = \{\varphi_k(x): k \in \omega\}$ if there is some $m \in M$, such that for each $k \in \omega$, $\mathfrak{M} \models \varphi_k[m]$, or simply, $\mathfrak{M} \models \Phi(m)$. Otherwise, we say that $\mathfrak{M}$ *omits* Φ. In the example above, we want our structures to omit the type $\{x \in \omega, x \neq \mathbf{0}, x \neq \mathbf{1}, \ldots\}$. Of course, this is the same as requiring that each element satisfy at least one of the formulas $x \notin \omega$, $x = \mathbf{0}$, $x = 1, \ldots$. The original results on *omitting types* are due to Henkin [1954, 1957], Orey [1956], and Morley [1965].

The logics we will consider allow us to replace some or all types in the logic by formulas of the logic. Thus, the notion of omitting a type may be equivalent to satisfying a certain sentence. In fact, these logics may be viewed as being formed by closing under "omitting types" as well as the other standard logical operations. Somewhat earlier, model theorists considered *ω-logic* (See Keisler [1966]) in which there is a fixed unary relation symbol, say $\mathfrak{N}$, whose realization in all ω-models is taken to be the same, *viz.*, the set of standard natural numbers. However, as research developed, attention has moved from ω-logic to the more flexible setting which we will discuss in the remainder of this chapter.

1.2. Definition of the Infinitary Logics

We now formally define the formulas of the logic $\mathscr{L}_{\infty\omega}$ as the smallest class closed under the usual connectives and quantifiers of finitary logic and, in addition, under the conjunction of arbitrary sets of formulas. Thus, if Φ is a set of formulas of $\mathscr{L}_{\infty\omega}$, so is $\bigwedge \Phi$. The semantics for $\bigwedge \Phi$ is the obvious one, and the disjunction $\bigvee \Phi$ may be defined using de Morgan's law as $\neg \bigwedge \{\neg \varphi: \varphi \in \Phi\}$. We assume that the reader can supply correct definitions for such standard concepts as subformula, free variable, sentence, *etc.* In cases of doubt, the reader should consult Keisler [1971a] or Barwise [1975].

Formulas, as we have so far defined them, may have infinitely many free variables. However, from now on we will restrict our discussions to those formulas with only finitely many free variables. It should be noted that a subformula of such a formula—and specifically of a sentence—will again have only finitely many free variables.

For any infinite regular cardinal κ we define the sublogic $\mathscr{L}_{\kappa\omega}$ of $\mathscr{L}_{\infty\omega}$ by restricting the conjunctions to be of sets of cardinality less than κ. For κ singular, the definition is a bit different. This is so in order to prevent the conjunction of conjunctions from simulating a conjunction of cardinality κ, and we omit it here. Of special interest is $\mathscr{L}_{\omega_1\omega}$, in which only countable conjunctions and disjunctions occur. $\mathscr{L}_{\omega\omega}$ is simply the familiar finitary logic. For the sake of later comparison, we also introduce the stronger logic $\mathscr{L}_{\infty\infty}$, which, in addition to arbitrary conjunctions and disjunctions, allows either existential or universal quantification over an arbitrary set of variables; that is, if φ is a formula of $\mathscr{L}_{\infty\infty}$ and X is a set of variables, then $\exists X\varphi$ is a formula of $\mathscr{L}_{\infty\infty}$. Again, we leave the standard definitions to the reader. $\mathscr{L}_{\infty\lambda}$ is the sublogic of $\mathscr{L}_{\infty\infty}$ in which the quantifiers are over sets of variables of cardinality less than λ. By analogy to the situation for $\mathscr{L}_{\infty\omega}$, one only considers those formulas of $\mathscr{L}_{\infty\lambda}$ having fewer than λ free variables. The reader should consult Chapter IX for further details.

The structures for these logics are simply the structures of ordinary model theory, and we assume that the notions of satisfaction are self-explanatory. Structures will generally be denoted by $\mathfrak{M}$ or $\mathfrak{N}$ with their universes denoted by M and N, respectively. We save the letters A and B for other purposes. As is the custom in this book, when we wish to call attention to a particular vocabulary τ, we write $\mathscr{L}_{\infty\omega}(\tau)$ instead of $\mathscr{L}_{\infty\omega}$, *etc.*

1.3. Expressive Power

We next offer a few examples of the expressive power of the various logics that have been introduced. Some of these are quite simple; others take considerable ingenuity. It is easy to write a sentence of $\mathscr{L}_{\omega_1\omega}$ in the language with just equality that says that a structure is finite. Similarly, we can write a sentence of $\mathscr{L}_{\omega_1\omega}$ that says a group is torsion or finitely generated, or that a structure with distinguished unary predicate and constant symbols for the natural numbers is an ω-model. In fact, given any countable type in $\mathscr{L}_{\omega\omega}$ or $\mathscr{L}_{\omega_1\omega}$, it is easy to write a sentence in $\mathscr{L}_{\omega_1\omega}$ expressing that the type is omitted.

That an abelian group is $\aleph_1$-free, i.e. every countable subgroup is free, can be expressed by a sentence of $\mathscr{L}_{\omega_1\omega}$ (see Barwise [1973b]). On the other hand, whether or not there is a sentence of $\mathscr{L}_{\infty\infty}$ defining the class of free abelian groups depends upon the particular universe of set theory. See Chapter XI for more details. The Ulm invariants for a countable abelian torsion group can be "written" in $\mathscr{L}_{\omega_1\omega}$ (see Barwise [1973b]). One can do the same for uncountable groups, obtaining sentences of $\mathscr{L}_{\infty\omega}$ which, rather than characterize the group up to isomorphism, characterize its $\mathscr{L}_{\infty\omega}$ elementary class.

Turning now to the vocabulary of linear orderings, it is easy to characterize the well-orderings (at least when the axiom of choice is assumed) by a sentence of $\mathscr{L}_{\omega_1\omega_1}$. However, it can be shown (see Lopez-Escobar [1966a]) that no sentence of $\mathscr{L}_{\infty\omega}$ characterizes the well-orderings. In fact, this class is not even PC. As an exercise, the reader should show that for each ordinal α there is a sentence φ of $\mathscr{L}_{\infty\omega}$ characterizing it up to isomorphism. This can be accomplished by induction on α. While it is true (see Nadel [1974b]) that for any scattered linear order—that is, any linear order without a dense subordering—there is a sentence of $\mathscr{L}_{\infty\omega}$ characterizing it up to isomorphism, there is nevertheless no sentence in $\mathscr{L}_{\infty\omega}$ that characterizes the scattered linear orderings, though obviously there is one in $\mathscr{L}_{\omega_1\omega_1}$.

Finally, we mention that for each countable structure (and we will always assume the underlying vocabulary is countable as well) there is a sentence of $\mathscr{L}_{\omega_1\omega}$ which characterizes it, up to isomorphism, among countable structures. This very early and very fundamental result is due to Scott [1965] and will be considered in Section 4. We point out here that more generally, in the context of any logic $\mathscr{L}$, we may speak of a *Scott sentence* φ of a structure $\mathfrak{M}$ as a sentence of $\mathscr{L}$ which characterizes M up to elementary equivalence in $\mathscr{L}$. The reader should consult Chapter IX for a more complete discussion of the examples.

1.4. Reduction to Omitting Types

In this section we will give a paraphrase of a result which once again emphasizes the connection between $\mathscr{L}_{\omega_1\omega}$ and omitting types in $\mathscr{L}_{\omega\omega}$. See Chapter XI of this volume for details.

Let $\mathscr{L}_B(\tau)$ be a countable fragment of $\mathscr{L}_{\omega_1\omega}(\tau)$ (in a sense to be made precise later). Then, by adding countably many new symbols, τ can be expanded to a larger vocabulary τ' in which there is a set of types such that each τ-structure has a unique expansion to a τ'-structure omitting these types; and, on these τ'-structures, each formula of $\mathscr{L}_B(\tau)$ is equivalent to a formula of $\mathscr{L}_{\omega\omega}(\tau')$, and *vice versa.*

Remark. A similar result holds for arbitrary $\mathscr{L}_{\kappa\omega}$ and is discussed in Section 1.3 of Chapter IX.

1.5. $\mathscr{L}_{\omega_1\omega}$ of an Abstract Logic

Let $\mathscr{L}^*$ be some abstract logic. Beginning with $\mathscr{L}^*$, can be form an infinitary version of $\mathscr{L}^*$? For the sake of this discussion, let us consider a version which we will call $\mathscr{L}^*_{\omega_1\omega}$ and which allows closure under countable conjunctions and disjunctions, rather than the full $\mathscr{L}_{\infty\omega}$ analogue. A naive approach would be to close $\mathscr{L}^*$ under countable conjunctions and disjunctions, negation, and existential and universal quantifiers as well. However, this is really not what is wanted here. Suppose $\mathscr{L}^*$ is $\mathscr{L}(Q_1)$. Then in $\mathscr{L}^*_{\omega_1\omega}$ we would like to be able to have sentences of the form $Q_1x\varphi$, where φ is already a formula of $\mathscr{L}^*_{\omega_1\omega}$. In this situation, it is clear how to proceed. In addition to the above closure conditions, we also close $\mathscr{L}^*_{\omega_1\omega}$ under the "closure operations" of $\mathscr{L}^*$. The problem arises in the general context in which $\mathscr{L}^*$ may not be given in terms of "closure operations".

While the method we will use here and later in Section 6.6 is based on Barwise [1981], there are some difficulties involved in the treatment given there. First of all, the definition for $\mathscr{L}^*_{\omega_1\omega}$ used in that work does not seem to be adequate for the intended purposes; accordingly, we modify it slightly. Even more importantly, the discussion given there purports to include the case of logics involving second-, as well as, first-order variables, e.g. $L(\mathrm{aa})$. As a matter of fact, however, the argument there does not really include this case. We will limit our attention to the first-order case, with the case of $L(\mathrm{aa})$ being considered only briefly in Chapter IV.

In addition to requiring that $\mathscr{L}^*_{\omega_1\omega}$ include $\mathscr{L}^*$ and be closed under countable conjunction and disjunction in the obvious way, we impose a further condition in order to simulate "closing under $\mathscr{L}^*$ itself". This condition is as follows:

(*) If $\varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k)$ is an $\mathscr{L}^*$ sentence, and $\psi_i(c_{i_1}, \ldots, c_{i_{n_i}})$, are $\mathscr{L}^*_{\omega_1\omega}$ sentences, where $\mathfrak{R}_i$ is an n_i-ary relation symbol which does not occur in ψ_i, and $c_{i_1}, \ldots, c_{i_{n_i}}$ do not occur in φ, for $i = 1, \ldots, k$, then $\varphi(\psi_1/\mathfrak{R}_1, \ldots, \psi_k/\mathfrak{R}_k)$ is an $\mathscr{L}^*_{\omega_1\omega}$-sentence in which neither R_i nor $c_i, \ldots, c_{i_{n_i}}$ occur, for $i = 1, \ldots, k$.

The corresponding semantical clause is given by

(*) $\mathfrak{M} \models \varphi(\psi_1/\mathfrak{R}_1, \ldots, \psi_k/\mathfrak{R}_k)$ iff $(\mathfrak{M}, R_1, \ldots, R_k) \models \varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k)$, where $R_i = \{(a_{i_1}, \ldots, a_{i_{n_i}}) \colon (\mathfrak{M}, a_{i_1}, \ldots, a_{i_{n_i}}) \models \psi_i\}$, for $i = 1, \ldots, k$.

Using the above definition we have now formally introduced $\mathscr{L}^*_{\omega_1\omega}$. However, yet another point remains to be considered. Suppose $\mathscr{L}^*$ itself were not closed under the analogue of (*). Barwise [1981] refers to the closure condition as the *substitution axiom.* Then, even without adding any infinite conjunctions or disjunctions, new sentences may be added because of (*) and this may ruin certain properties of $\mathscr{L}^*$, e.g. compactness. Thus, we will only consider $\mathscr{L}^*_{\omega_1\omega}$ for $\mathscr{L}^*$ satisfying the substitution axiom.

It is now easy to see that $\mathscr{L}^*_{\omega_1\omega}$ is closed, for example, under the conjunction of two sentences (for future use it is important to distinguish finite from infinite conjunctions and disjunctions), viz. the correct semantics for θ & ψ will apply to $\mathfrak{N}_1$ & $\mathfrak{N}_2(\theta/\mathfrak{N}_1, \psi/\mathfrak{N}_2)$. Since in (*) φ is required to be an $\mathscr{L}^*$-sentence, rather than an $\mathscr{L}^*_{\omega_1\omega}$-sentence, it is not clear, *a priori*, that $\mathscr{L}^*_{\omega_1\omega}$ will satisfy the substitution axiom. However, a simple argument by induction on the formation of φ shows that $\mathscr{L}^*_{\omega_1\omega}$ does.

Now, having obtained the definition of $\mathscr{L}^*_{\omega_1\omega}$ in working order, an entire new aspect of abstract model theory presents itself. Suppose P_1 and P_2 are properties of logics. We can then hope to prove theorems of the following form:

"Suppose that $\mathscr{L}^*$ satisfies P_1, then $\mathscr{L}^*_{\omega_1\omega}$ satisfies P_2."

We will mention some impressive results of this type in Section 6.6. In the meantime, let us note that the result we mentioned in Section 1.4 holds in the general context of $\mathscr{L}^*_{\omega_1\omega}$. It would be a worthwhile exercise for the reader to fill in the extra step in the proof for (*) and note where the substitution axiom is needed.

2. *Elementary Equivalence*

One reason that $\mathscr{L}_{\infty\omega}$ is such a fruitful logic is that its elementary equivalence relation $\equiv_{\infty\omega}$ (we write this instead of $\equiv_{\mathscr{L}_{\infty\omega}}$) is very natural. Below we will give two useful characterizations of $\equiv_{\infty\omega}$. Lest the inexperienced reader jump to unfounded conclusions, we point out that there are logics other than $\mathscr{L}_{\infty\omega}$ with the same elementary equivalence relation (for example, see Keisler [1968a]).

2.1. The Back-and-Forth Property

A function f from a structure $\mathfrak{M}$ to a structure $\mathfrak{N}$ (for the same vocabulary) is said to be a *partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ if f extends to an isomorphism of the substructure of $\mathfrak{M}$ generated by dom f onto the substructure of $\mathfrak{N}$ generated by range f.

Let κ be a cardinal. A set F of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$ is said to be a *κ-back and forth set* if for any $f \in F$:

(i) $\forall X \subseteq M[|X| < \kappa \to \exists g \in F[f \subseteq g \mathbin{\&} X \subseteq \operatorname{dom} g]]$;
(ii) $\forall Y \subseteq N[|Y| < \kappa \to \exists h \in F[f \subseteq h \mathbin{\&} Y \subseteq \operatorname{ra} h]]$.

If such a set F exists, then we say that $\mathfrak{M}$ and $\mathfrak{N}$ have the *κ-back and forth property* or are *κ-partially isomorphic*, and write $\mathscr{L} \cong_{p,\kappa} \mathfrak{N}$.

It is easy to see that if we take $\kappa = \omega$, we get the same condition as by taking $\kappa = n$, for $2 \le n < \omega$. In this case we will simply omit κ from the notation. This property was first studied by Karp [1965], and for that reason a logic is said to have the *Karp property* if whenever $\mathfrak{M} \cong_p \mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent in that logic. The uninitiated reader should become more familiar with these notions by convincing himself that if $\mathfrak{M}$ and $\mathfrak{N}$ are dense linear orderings without endpoints, then $\mathfrak{M} \cong_p \mathfrak{N}$. But if $\mathfrak{M}$ and $\mathfrak{N}$ are algebraically closed fields of transcendence rank distinct natural numbers, then $\mathfrak{M} \not\cong_p \mathfrak{N}$.

The first characterization of $\equiv_{\infty\omega}$, given below in Karp's theorem, is proved by a straightforward induction on the formation of formulas [see Chapter IX for a detailed discussion]. It should be mentioned that an earlier characterization of $\equiv_{\omega\omega}$ in a similar way has been given by Ehrenfeuct [1961] and Fraïssé [1954b]. The reader should consult Section IX.4 for a more detailed historical survey.

2.1.1 Theorem (Karp's Theorem). $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ *iff* $\mathfrak{M} \cong_p \mathfrak{N}$.

If $\mathfrak{M}$ and $\mathfrak{N}$ are countable, then, in the process of going back-and-forth between them, we can use all the elements of each and obtain the following weak form of Scott's theorem.

2.1.2 Corollary. *If* $|M| = |N| = \aleph_0$ *and* $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, *then* $\mathfrak{M} \cong \mathfrak{N}$.

2.1.3 Remarks. (1) The analogue of Karp's theorem for arbitrary infinite κ holds. However, the analogue of the corollary given in Corollary 2.1.2 does not—except for the case $\mathrm{cf}(k) = \omega$, a result which is due to Chang [1968c]. Quite early in the development of this area, Morley gave an example of two structures $\mathfrak{M}$ and $\mathfrak{N}$ of cardinality $\aleph_1$ such that $\mathfrak{M} \equiv_{\infty\omega_1} \mathfrak{N}$, but $\mathfrak{M} \not\cong \mathfrak{N}$. The reader may consult Nadel–Stavi [1978] for a fuller description of such examples. However, we note that contrary to the assertion there, the question of finding non-isomorphic structures $\mathfrak{M}$ and $\mathfrak{N}$ of power λ, for λ-singular, $\mathrm{cf}(\lambda) > \omega$, $\lambda^\omega = \lambda$, such that $\mathfrak{M} \equiv_{\infty\lambda} \mathfrak{N}$ has only recently been solved by S. Shelah. Given a structure $\mathfrak{M}$ of cardinality λ, let $n(M)$ be the number of non-isomorphic models $\mathfrak{N}$, such that $|N| = \lambda$ and $\mathfrak{M} \equiv_{\infty\lambda} \mathfrak{N}$. Under the assumption that $V = L$, Shelah [1981b] has shown that if λ is regular and not weakly compact, than $n(\mathfrak{M}) = 1$ or 2^λ. However, if λ is weakly compact, then $n(\mathfrak{M})$ can be any cardinal $\mu \leq \lambda$, as shown in Shelah [1982b].

(2) There are results analogous to Theorem 2.2.1, as well as for certain other results to follow, for the properties of a structure being embeddable in or a homomorphic image of another structure. These results can be found in Chang [1968c], or Nadel [1974b], or Chapter IX, and we will not discuss them further here.

2.2. *Potential Isomorphism*

The notion of partial isomorphism is of an algebraic nature. The characterization of $\equiv_{\infty\omega}$ we present in this section is metamathematical and involves the set-theoretic notions of forcing or boolean-valued models (see Jech [1978]). It is due independently to Barwise [1973b] and Nadel [1974b].

We say that structures $\mathfrak{M}$ and $\mathfrak{N}$ are *potentially isomorphic* iff they are isomorphic in some boolean extension of the universe, that is, iff for some complete boolean-algebra B, $[\check{\mathfrak{M}} \cong \check{\mathfrak{N}}]^{\mathbb{B}} = 1$. It is quite easy to show the equivalence given in

2.2.1 Theorem. *$\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ iff $\mathfrak{M}$ and $\mathfrak{N}$ are potentially isomorphic.*

To prove the equivalence one must first observe that $\equiv_{\infty\omega}$ is absolute. To see that $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$ is preserved in a boolean extension, we use Karp's theorem (2.1.1). To see that $\mathfrak{M} \not\equiv_{\infty\omega} \mathfrak{N}$ is preserved in a boolean extension, we merely use the absoluteness of satisfaction for sentences of $\mathscr{L}_{\infty\omega}$. Now, if $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, to make $\mathfrak{M}$ and $\mathfrak{N}$ isomorphic, go to a boolean extension in which both $\mathfrak{M}$ and $\mathfrak{N}$ are countable and then use Corollary 2.1.2.

We have found the notion of potential isomorphism to be a very useful conceptual tool. As simple examples, note that it is now obvious that well-ordered structures of distinct order types are not $\equiv_{\infty\omega}$, while any two algebraically closed fields of infinite transcendence rank are $\equiv_{\infty\omega}$.

2.2.2 Remarks. It is natural to wonder if there are notions of potential isomorphism corresponding to $\equiv_{\infty\lambda}$ for $\lambda > \omega$. This question is investigated in some detail in Nadel–Stavi [1978] where it is shown that, for λ a successor cardinal, there is no such notion in quite a general sense. It is also suggested that one could begin with some very natural notion of potential isomorphism and then use it to fashion a logic with a corresponding notion of elementary equivalence. This idea was the motivation behind the paper by Nadel [1980a]. The investigation begun there was developed much further by D. Mundici and is described in Chapter V.

3. General Model-Theoretic Properties

In this section we will consider the most fundamental results in the model theory of $\mathscr{L}_{\infty\omega}$, or, more accurately, in $\mathscr{L}_{\omega_1\omega}$, since as we shall see, countability will make a very big difference. In fact, we will need to consider countable pieces of $\mathscr{L}_{\omega_1\omega}$. To this end, we now define our first—and quite weak—version of a "nice" piece of $\mathscr{L}_{\infty\omega}$. Later in Section 5, we will give a much stronger version.

3.1. The Model Existence Theorem

3.1.1 Definition. A *fragment* of $\mathscr{L}_{\infty\omega}(\tau)$ is a set $L_B(\tau)$ of formulas and variables of $\mathscr{L}_{\infty\omega}(\tau)$ such that:

(i) $\mathscr{L}_{\omega\omega}(\tau) \subseteq L_B(\tau)$;
(ii) if $\varphi \in L_B(\tau)$, then every subformula and variable of φ is in $L_B(\tau)$;
(iii) if $\varphi(v) \in L_B(\tau)$ and σ is a term of τ all of whose variables lie in $L_B(\tau)$, then $\varphi(\sigma/v) \in L_B(\tau)$; and

(iv) if φ, ψ and $v \in L_B(\tau)$, so are $\neg\varphi$, $\exists v\varphi$, $\forall v\varphi$, $\varphi \,\&\, \psi$, $\varphi \vee \psi$ and $\sim\varphi$, where $\sim\varphi$ is defined inductively as follows: $\sim\theta$ is $\neg\theta$ if θ is atomic, $\sim(\neg\theta)$ is θ, $\sim(\bigwedge\Theta)$ is $\bigvee\{\sim\theta : \theta \in \Theta\}$, $\sim(\bigvee\Theta)$ is $\bigwedge\{\sim\theta : \theta \in \Theta\}$, $\sim(\exists v\varphi)$ is $\forall v \neg\varphi$, $\sim(\forall v\varphi)$ is $\exists v \neg\varphi$.

Closure under $\sim$ is merely to guarantee that $L_B(\tau)$ is closed under taking equivalent formulas of a certain simple type. (A convention on terminology will be helpful here: We will use L_B rather than $L_B(\tau)$ to represent a fragment when the vocabulary τ does not come into play. In particular, we will speak of $L_{\omega_1\omega}$ and $L_{\omega\omega}$ as fragments, where the former corresponds to an arbitrary $L_{\omega_1\omega}(\tau)$, etc. Moreover, we may speak of L_B rather than τ, having certain symbols).

The following definition and the subsequent theorem due to Makkai [1969b] is the principal tool for building models. The precise formulation given here is from Barwise [1975].

3.1.2 Definition. Suppose that the fragment L_B contains a set of constant symbols $C = \{c_n : n \in \omega\}$. A *consistency property* for L_B is a set S such that each $s \in S$ is a set of sentences of L_B and such that the following hold for each $s \in S$:

(C0) $0 \in S$; if $s \subseteq s' \in S$, then $s \cup \{\varphi\} \in S$, for each $\varphi \in s'$;
(C1) If $\varphi \in s$, then $\neg\varphi \notin s$;
(C2) If $\neg\varphi \in s$, then $s \cup \{\sim\varphi\} \in S$;
(C3) If $\bigwedge \Phi \in s$, then for all $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$;
(C4) If $(\forall v\varphi(v)) \in s$, then for every $c \in C$, $s \cup \{\varphi(c)\} \in C$;
(C5) If $\bigvee \Phi \in s$, then for some $\varphi \in \Phi$, $s \cup \{\varphi\} \in S$;
(C6) If $(\exists v\varphi(v)) \in s$, then for some $c \in C$, $s \cup \{\varphi(c)\} \in S$;
(C7) Let t be any term of the form $F(c_{i_1}, \dots, c_{i_n})$,

F an n-any function symbol of L_B, and $c_{i_1}, \dots, c_{i_n}, c, d, \in C$

(i) If $(c = d) \in s$, then $s \cup \{d = c\} \in S$;
(ii) If $\{\varphi(t), (c = t)\} \in s$ then $s \cup \{\varphi(c)\} \in S$;
(iii) For some $e \in C$, $s \cup \{e = t\} \in S$.

Condition (C0) is not essential at this stage, although it does come into play later when we are trying to obtain more refined results. The remaining conditions are just what is needed to build a canonical model in ω stages using the Henkin construction, where a canonical model is simply one in which each element interprets a constant. The point here is that unlike the case of $L_{\omega\omega}$ where compactness holds, one must actually have constructed the entire model after ω stages. It is usually not possible to iterate a construction beyond a limit stage.

3.1.3 Model Existence Theorem. (i) *Let L_B be a countable fragment, and let S be a consistency property for L_B. For each $s \in S$, there is a canonical model $\mathfrak{M} \models \bigwedge s$.*

(ii) **(Extended Version).** *If in addition, T is a set of sentences of L_B such that, for each $s \in S$ and $\varphi \in T$, $s \cup \{\varphi\} \in S$, then, for each $s \in S$, there is a canonical model of $T \cup \{s\}$.* □

We mention at this point, that the model existence theorem does not hold in the absence of the assumption of countability (allowing, of course, an uncountable set of constants in C). We will point out an example of this later.

3.2. *Provability and Completeness*

The first completeness result for $L_{\omega_1\omega}$ was given by Karp [1964]. To the usual Hilbert style proof system for $L_{\omega\omega}$ one adds for each sentence $\bigwedge \Phi$ and $\varphi \in \Phi$, the axiom

$$(\bigwedge \Phi) \rightarrow \varphi$$

and the rule of inference

$$\text{From } \psi \rightarrow \varphi, \quad \text{for all} \quad \varphi \in \Phi, \quad \text{infer } \psi \rightarrow \bigwedge \Phi.$$

Since an application of this rule involves infinitely many premises, proofs may be infinite in length. We now consider an extended form of completeness that is appropriate for countable fragments, and in Section 6 we will consider a more subtle version. We fix a fragment L_B and require that all formulas involved in proofs be in L_B as well as that the proofs be of countable length. We use the standard provability symbol $\vdash_{L_B}$ in the usual way to refer to this system.

3.2.1 Completeness Theorem. *Let L_B be a countable fragment of $\mathscr{L}_{\omega_1\omega}$. Then for any sentence φ of L_B and set of sentences T of L_B, $T \models \varphi$ iff $T \vdash_{L_B} \varphi$.* □

Karp's original proof was boolean-algebraic. Alternatively, we can add to the vocabulary a countable set C of new constant symbols and show that the set $S = \{s\colon s$ is a finite set of sentences of L_B each containing only finitely many constants from C and not $T \vdash_{L_B} \neg \bigwedge s\}$ is a consistency property, and then appeal to the extended version of the model existence theorem.

3.2.2 Remarks. As a result of the completeness theorem, we see that the validity of a sentence of $\mathscr{L}_{\omega_1\omega}$ is absolute (for models of ZFC). On the other hand, it is easy to give examples showing that validity for sentences in $\mathscr{L}_{\infty\omega}$ is not generally absolute and thus no similar absolute notion of provability could give a completeness theorem. For uncountable fragments, being provable in the obvious generalization of the above sense is equivalent to validity in boolean-valued extensions of the universe rather than validity in V itself. That is, φ is provable iff "$\models \varphi$" has value 1 in every boolean-valued extension of V. It is easy to see that provable sentences are boolean valid. To see the other direction, one needs the absoluteness of provability which shall be obtained in Section 6.

Alternatively, (see Mansfield [1972]), there is a completeness theorem for $\mathscr{L}_{\infty\omega}$ where the models themselves (rather than the set-theoretical universe) are taken to be boolean-valued. Thus, provability as above is equivalent to boolean-validity in this second sense also.

3.3. Interpolation

The interpolation theorem for $\mathscr{L}_{\omega_1\omega}$ was first proved by Lopez-Escobar [1965b]. Since the idea involved in his proof can be used in other settings, we shall say a few words about it. The first step—which is the more difficult one—is to find a cut-free Gentzen system which is complete for $\mathscr{L}_{\omega_1\omega}$. This can be done either purely semantically as in Lopez-Escobar [1965b], where completeness is simply proven directly for the cut-free system or, more proof-theoretically, as in Feferman [1968a] where completeness is shown for the system with cut (another name for *modus ponens*), and then "cut elimination" is proven by examining proofs. This second method provides certain ordinal bounds as well.

The idea of the proof is to find the interpolant by induction on the derivation of the implication. For example, suppose the final step in a derivation uses the so-called ($\supset \bigwedge$-rule):

$$\frac{\varphi \supset \psi_i}{\varphi \supset \bigwedge\{\psi_i : i \in \omega\}} \quad \text{for all } i \in \omega.$$

Suppose, by induction, that for each $i \in \omega$ there is some interpolant θ_i such that $\varphi \supset \theta_i$ and $\theta_i \supset \psi_i$ are each derivable. Then, using the ($\supset \bigwedge$-rule) we may obtain $\varphi \supset \bigwedge\{\theta_i : i \in \omega\}$. By using the matching ($\bigwedge \supset$-rule), we may obtain, for each $i \in \omega$, $\bigwedge\{\theta_i : i \in \omega\} \supset \psi_i$. Using the ($\supset \bigwedge$-rule) again we obtain $\bigwedge\{\theta_i : i \in \omega\} \supset \bigwedge\{\psi_i : i \in \omega\}$. It is now easy to check that $\bigwedge\{\theta_i : i \in \omega\}$ is an interpolant. The problem with the cut-rule is that this sort of induction step simply does not work, and that is why cut must be eliminated.

An alternate proof for a countable fragment L_B using the model existence theorem is given in Keisler [1971a]. We describe it very briefly. Suppose $\models \varphi \to \psi$. First, we add an infinite set of new constant symbols $C = \{c_1, c_2, \ldots\}$ to the alphabet. We define S_φ to be the set of all sentences φ' of L_B such that every symbol of the original alphabet that occurs in φ' also occurs in φ; and, in addition, finitely many of the c_n's may occur. S_ψ is defined analogously. We let S be the set of all finite sets of sentences which can be written as $s_1 \cup s_2$, where $s_1 \subseteq S_\varphi$, $s_2 \subseteq S_\psi$; and, if $\theta_1, \theta_2 \in S_\varphi \cap S_\psi$ and $\models \bigwedge s_1 \to \theta_1$, $\models \bigwedge s_2 \to \theta_2$, then $\theta_1 \,\&\, \theta_2$ is consistent. We then show that S is a consistency property and apply the model existence theorem. Since $\models \varphi \to \psi$, we have that $\{\varphi, \neg\psi\} \notin S$. But this means there must be $\theta_1, \theta_2 \in S_\varphi \cap S_\psi$ such that $\models \varphi \to \theta_1$, $\models \neg\psi \to \theta_2$ and $\theta_1 \,\&\, \theta_2$ is inconsistent. Thus, $\models \theta_1 \to \neg\theta_2$. Now, since $\models \neg\theta_2 \to \psi$, we have $\models \theta_1 \to \psi$. Now, by quantifying out the new constants in θ_1 we get the desired interpolant.

There are other more refined interpolation results of Lopez-Escobar [1965b] and Malitz [1969]. A good reference is Keisler [1971a].

The automatic consequences of interpolation, such as the Beth property, naturally hold. Robinson joint consistency fails, but a weaker version of it, a version in which the joint theory T is complete for $L_{\omega_1\omega}$ rather than just for L_B, does hold.

3.3.1 Remarks. The reader should consult Chapter IX for a full discussion of interpolation and definability results for infinitary logics. In particular, it is worth emphasizing in this context that interpolation fails for $\mathscr{L}_{\infty\omega}$.

3.3.2 Remarks. One of the main uses for interpolation results is in obtaining preservation theorems. As in the case of $\mathscr{L}_{\omega\omega}$, the more refined interpolation theorems alluded to above give rise to preservation theorems. For example, Malitz's interpolation theorem shows that a sentence φ of $L_{\omega_1\omega}$ is preserved under submodels relative to some other sentence ψ of $L_{\omega_1\omega}$ (that is, if $\mathfrak{M}, \mathfrak{N} \models \psi$, $\mathfrak{M} \subseteq \mathfrak{N}$ and $\mathfrak{N} \models \varphi$, then $\mathfrak{M} \models \varphi$) iff there is some universal sentence θ such that $\psi \models \varphi \leftrightarrow \theta$. By a universal sentence we mean a sentence which is formed from atomic and negated atomic formulas using only $\bigwedge$, $\bigvee$ and $\forall$. For a fuller discussion of preservation results the reader should consult Chapters 6 and 7 of Keisler [1971a].

3.4. Kueker's Filter

The reader will have noticed by now that many fundamental facts about $\mathscr{L}_{\omega_1\omega}$ fail to extend to $\mathscr{L}_{\infty\omega}$. Some outstanding examples of this are the corollary to Karp's theorem; completeness, and interpolation. D. Kueker [1972, 1977, 1978] (see also Barwise [1974b]) found a way of reformulating these and other results so that they do extend to $\mathscr{L}_{\infty\omega}$. Kueker's reformulation involves countable approximations to structures and formulas as well as a notion of "almost everywhere" corresponding to the closed unbounded filter on $\mathscr{P}_{<\omega_1}(X)$. A description of this very interesting approach can be found in Chapter XVII.

3.5. Omitting Types

Given a fragment L_B, we speak of types over L_B just as we do for $L_{\omega\omega}$, that is, sets of formulas in L_B in some fixed finite set of free variables. Then, using the model existence theorem (see Keisler [1971a] for details), we see that an omitting types theorem can be proved in much the same way as the original Henkin–Orey result for $L_{\omega\omega}$. Since the infinite disjunction is now officially available, it is customary to use it in the statement.

3.5.1 Theorem (Omitting Types Theorem). *Let L_B be a countable fragment of $L_{\omega_1\omega}$ and let T be a set of sentences of L_B which has a model. For each $n \in \omega$, let Φ_n be a set of formulas of L_B in the free variables $v_1, \ldots, v_{k_n}$. Assume that for each $n \in \omega$ and formula $\psi(v_1, \ldots, v_{k_n})$ of L_B, if $T \cup \{\exists v_1 \ldots v_{k_n}\psi\}$ has a model, so does $T \cup \{\exists v_1 \ldots v_{k_n}(\psi \mathbin{\&} \varphi)\}$, for some $\varphi \in \Phi_n$. Then there is a model of*

$$T \cup \left\{\bigwedge_{n\in\omega} \forall v_1 \ldots v_{k_n} \bigvee_{\varphi\in\Phi_n} \varphi\right\}. \quad \square$$

The omitting types theorem is, of course, closely related to the ω-completeness theorem. The latter—especially the ω-rule, *viz.*, from $\varphi(n)$, for each $n \in \omega$, infer $\forall x(N(x) \rightarrow \varphi(x))$—is an important precursor of the study of infinitary logic in its present form.

3.5.2 Remarks. Shelah [1978a] has shown that a stronger version of omitting types is true. In that version there are fewer than continuum many Φ's over the fixed countable fragment L_B. The proof of this may be gleaned from the proof of Lemma 8.2.2 and, hence, we will omit it here.

It should be mentioned that because of the omitting types theorem, we are able to obtain the equivalence of prime models with countable atomic models, just as can be done for $\mathscr{L}_{\omega\omega}$. We shall have more to say about omitting types in Section 6.6.

3.6. Löwenheim–Skolem Results

Since the model existence theorem produces a countable model, we have, in effect, already shown that $\mathscr{L}_{\omega_1\omega}$ has Löwenheim number $\aleph_0$. That is to say, if a sentence of $\mathscr{L}_{\omega_1\omega}$ has a model, it has a countable model. The upward Löwenheim–Skolem result is more complicated. Unlike $\mathscr{L}_{\omega\omega}$, the Hanf number of $\mathscr{L}_{\omega_1\omega}$ is not $\aleph_0$. Examples showing this are easy to find. The proof for $\mathscr{L}_{\omega\omega}$ is simple enough using compactness, but that is not available. It is not surprising that the results for $\mathscr{L}_{\omega_1\omega}$ resemble rather the Hanf number results for omitting types over $\mathscr{L}_{\omega\omega}$, results which were proven slightly earlier by Morley [1965b]. The next result first appeared in Lopez-Escobar [1966a] who credits it to Helling.

3.6.1 Theorem (Upward Löwenheim–Skolem Theorem). *The Hanf number of $\mathscr{L}_{\omega_1\omega}$ is $\beth_{\omega_1}$. This means,*

(i) *if φ is a sentence of $\mathscr{L}_{\omega_1\omega}$ with models of all cardinalities $\beth_\alpha$, $\alpha < \omega_1$, then φ has models of all infinite cardinalities;*
(ii) *for each $\kappa < \beth_{\omega_1}$, there is a sentence φ with a model of cardinality at least κ with no model of cardinality $\beth_{\omega_1}$.* $\square$

3.6.2 Remarks. There is also an upward Löwenheim–Skolem theorem for arbitrary $\mathscr{L}_{\kappa\omega}$ given in Lopez-Escobar [1966a]. This result is discussed in Chapter IX.

Part (i) of Theorem 3.6.1, the difficult part of the result, is proven by using the hypothesis, together with a combinatorial property known as the Erdös–Rado theorem (Erdös and Rado [1956]), to produce a model generated by indiscernibles. The reader should consult Kunen [1977] for a nice treatment of the Erdös–Rado result.

To obtain (ii) for each $\alpha < \omega_1$, Morley gave a sentence φ_α that had models in all cardinalities up to $\beth_\alpha$. In essence, φ_α says that the model is a subset of V_α, the set of all sets of rank $\leq \alpha$. Morley also shows how to get φ_α for $\aleph_\alpha$ instead of $\beth_\alpha$. To do this, one "says" of a linear ordering that it is $\aleph_\alpha$-like.

3.6.3 Remarks. We can ask a similar question about sentences that are complete for $\mathscr{L}_{\omega_1\omega}$. Trivially, the Hanf number is at most $\beth_{\omega_1}$. Malitz [1968] using GCH showed that it is $\beth_{\omega_1}$ and found a sentence φ_α for each $\beth_\alpha$ as above. Later, Baumgartner [1974] was able to accomplish this without the GCH. Shelah [1974a], in a related result, showed that the Hanf number for omitting complete types over $\mathscr{L}_{\omega\omega}$ is $\beth_{\omega_1}$ and obtained a complete type for each $\beth_\alpha$. Can a complete sentence of $\mathscr{L}_{\omega_1\omega}$ be obtained for $\aleph_\alpha$? At this time the only result in this direction is due to Knight [1977] who has found a complete sentence for $\aleph_1$.

3.6.4 Remarks. There is an attractive result of Landraitis [1980] on linear orderings that is worth mentioning at this point, and this we do in

3.6.5 Theorem. *Let $\mathfrak{M}$ be a denumerable linear ordering and let φ be a Scott sentence of $\mathfrak{M}$ in $\mathscr{L}_{\omega_1\omega}$. The spectrum of φ, $S(\varphi) = \{\kappa : \kappa = |\mathfrak{N}|$ for some $\mathfrak{N} \models \varphi\}$ is either*

(i) $\aleph_0$ *iff each (isomorphism) orbit of $\mathfrak{M}$ is scattered;*
(ii) *all infinite cardinals iff $\mathfrak{M}$ has a self-additive interval or*
(iii) $\{\kappa : \aleph_0 \leq \kappa \leq 2^{\aleph_0}\}$, *otherwise;*

and each case occurs. ☐

4. "Harder" Model Theory

4.1. Scott Sentences

Certainly the most striking of the early results in infinitary logic was Scott's theorem which is stated without proof in Scott [1965].

4.1.1 Theorem (Scott's Theorem). *For each countable structure $\mathfrak{M}$ for a countable vocabulary τ there is a sentence φ of $\mathscr{L}_{\omega_1\omega}(\tau)$ such that for any countable τ-structure $\mathfrak{N}$, $\mathfrak{M} \cong \mathfrak{N}$ iff $\mathfrak{N} \models \varphi$.*

We will now proceed to sketch a proof of Scott's theorem. We will assume that the reader can supply the obvious inductive definition of the quantifier rank of a formula of $\mathscr{L}_{\infty\omega}$. We write $\mathfrak{M} \equiv_\alpha \mathfrak{N}$ to mean that $\mathfrak{M}$ and $\mathfrak{N}$ agree on all sentences of $\mathscr{L}_{\infty\omega}$ of quantifier rank at most α. Karp [1965] gave an algebraic characterization of $\equiv_\alpha$.

4.1.2 Lemma. *For any structures $\mathfrak{M}$ and $\mathfrak{N}$ for the same vocabulary, and any ordinal α the following are equivalent:*

(i) $\mathfrak{M} \equiv_\alpha \mathfrak{N}$.
(ii) *There is a sequence $I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\alpha$ of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$ such that if $\beta + 1 \leq \alpha$ and $f \in I_{\beta+1}$, then for each $m \in M$* (resp. $n \in N$), *there is some $g \in I_\beta$, $g \supseteq f$ with $m \in$* dom g (resp. $n \in$ ra g).

The proof of Lemma 4.1.2 is by induction on α and is very similar to that of Karp's theorem (2.1.1).

Now, for each structure $\mathfrak{M}$, $m_1, \ldots, m_k \in M$, and ordinal α, we define a formula $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k)$ of $\mathscr{L}_{\infty\omega}$ by induction on α.

4.1.3 Definition. (i) For $\alpha = 0$, $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) = \bigwedge \{\theta(x_1, \ldots, x_k)\colon \theta$ is atomic or the negation of an atomic formula and $\mathfrak{M} \models \theta(m_1, \ldots, m_k)\}$.

(ii) For $\alpha = \beta + 1$,

$$\begin{aligned}\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) = {} & \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) \\ & \mathbin{\&} \forall x_{k+1} \bigvee_{m \in M} \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k, m}(x_1, \ldots, x_k, x_{k+1}) \\ & \mathbin{\&} \bigwedge_{m \in M} \exists x_{k+1} \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k, m}(x_1, \ldots, x_k).\end{aligned}$$

(iii) For α a limit,

$$\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k) = \bigwedge_{\beta < \alpha} \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k).$$

It is obvious from inspection that $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}$ has quantifier rank α, and that $\mathfrak{M} \models \sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(m_1, \ldots, m_k)$. More importantly, this formula is complete for formulas of quantifier rank of at most α.

4.1.4 Lemma. *For any structures $\mathfrak{M}$, $\mathfrak{N}$, for the same vocabulary, elements $m_1, \ldots, m_k \in M$, $n_1, \ldots, n_k \in N$ and ordinal α, the following are equivalent:*

(i) $(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\alpha} (\mathfrak{N}, n_1, \ldots, n_k)$;

(ii) $\mathfrak{N} \models \sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}(n_1, \ldots, n_k)$;

(iii) $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k} = \sigma^{\alpha}_{\mathfrak{N}, n_1, \ldots, n_k}$.

The only non-trivial step in the proof is that of showing that (ii) implies (i). This fact follows from Lemma 4.1.2 if we define for each $\beta \leq \alpha$,

$$\begin{aligned}I_\beta = \{f\colon & \operatorname{dom} f = \{m_1, \ldots, m_i\}, f(m_j) = n_j, \text{ for } j \leq i \text{ and} \\ & \mathfrak{N} \models \sigma^{\beta}_{\mathfrak{M}, m_1, \ldots, m_i}(n_1, \ldots, n_i)\}.\end{aligned}$$

Two observations are now needed to find the sentence which will characterize a structure up to $\equiv_{\infty\omega}$. First, if it happens that $I_\alpha = I_{\alpha+1}$ for some α, then I_α is easily seen to be a back-and-forth set. Second, for any $\mathfrak{M}$, there is an ordinal α, such that for any $k \in \omega$, $m_1, \ldots, m_k, m'_1, \ldots, m'_k \in M$,

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\alpha} (\mathfrak{M}, m'_1, \ldots, m'_k)$$

implies

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\infty\omega} (\mathfrak{M}, m'_1, \ldots, m'_k).$$

The least ordinal for which this happens is called the *Scott height* of $\mathfrak{M}$ and is denoted SH($\mathfrak{M}$). Using the first observation, we see that the Scott height of $\mathfrak{M}$ is the first ordinal α such that, for all $k \in \omega$, $m_1, \ldots, m_k, m'_1, \ldots, m'_k \in M$,

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_\alpha (\mathfrak{M}, m'_1, \ldots, m'_k)$$

implies

$$(\mathfrak{M}, m_1, \ldots, m_k) \equiv_{\alpha+1} (\mathfrak{M}, m'_1, \ldots, m'_k).$$

Thus, it is easy to see that the Scott height of $\mathfrak{M}$ is below $|\mathfrak{M}|^+$. In Section 7 we will obtain a better bound.

4.1.5 Definition. We now define the sentence $\sigma(\mathfrak{M})$ to be

$$\sigma^\alpha_{\mathfrak{M}} \mathbin{\&} \bigwedge_{\substack{k \in \omega \\ m_1, \ldots, m_k \in M}} \forall x_1 \ldots x_k[\sigma^\alpha_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k)$$
$$\leftrightarrow \sigma^{\alpha+1}_{\mathfrak{M}, m_1, \ldots, m_k}(x_1, \ldots, x_k)]$$

where $\alpha = \mathrm{SH}(\mathfrak{M})$.

This sentence appears first in Chang [1968c] and is called the *canonical Scott sentence* of $\mathfrak{M}$ in view of the next theorem.

4.1.6 Theorem. *For any structures $\mathfrak{M}$ and $\mathfrak{N}$ for the same vocabulary, the following are equivalent*:

(i) $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$;
(ii) $\mathfrak{N} \models \sigma(\mathfrak{M})$;
(iii) $\sigma(\mathfrak{M}) = \sigma(\mathfrak{N})$. $\square$

The non-trivial implication from (ii) to (i) is established much as in Lemma 4.1.4.

We see from Theorem 4.1.6 that $\sigma(\mathfrak{M})$ characterizes $\mathfrak{M}$ up to $\equiv_{\infty\omega}$ and depends only on the $\mathscr{L}_{\infty\omega}$-theory of $\mathfrak{M}$. If $\mathfrak{M}$ is countable, then, by Corollary 2.1.2, $\sigma(\mathfrak{M})$ is the sentence required in Scott's theorem (4.1.1). The quantifier rank of $\sigma(\mathfrak{M})$ is $\mathrm{SH}(\mathfrak{M}) + \omega$ and there are often Scott sentences for $\mathfrak{M}$ of lower quantifier rank. However, it will be observed in Section 7 that at least for countable $\mathfrak{M}$, $\sigma(\mathfrak{M})$ cannot have quantifier rank too much above any other Scott sentence for $\mathfrak{M}$.

4.2. Automorphisms and Local Definability in Countable Models

It was observed by Scott [1965] and follows quite readily from the preceding discussion that a countable model $\mathfrak{M}$ is *rigid* (that is to say, has no non-trivial automorphisms) iff each element of $\mathfrak{M}$ is definable in $\mathfrak{M}$ by a formula of $\mathscr{L}_{\omega_1\omega}$.

A similar result holds for countable models having fewer than continuum many automorphisms. This result has been shown by Kueker [1968].

4.2.1 Theorem. *Let $\mathfrak{M}$ be a countable structure. The following are equivalent:*

(i) *$\mathfrak{M}$ has countably many automorphisms.*
(ii) *$\mathfrak{M}$ has fewer than continuum many automorphisms.*
(iii) *There is some tuple of elements $n_1, \ldots, n_j \in M$ such that $(\mathfrak{M}, n_1, \ldots, n_j)$ is rigid.*
(iv) *There is some tuple of elements $n_1, \ldots, n_j \in M$ such that for each $m \in M$ there is a formula $\varphi(x_1, \ldots, x_j, y)$ of $\mathscr{L}_{\omega_1\omega}$ such that*

$$M \models \exists!\, y\varphi(n_1, \ldots, n_j, y) \;\&\; \varphi(n_1, \ldots, n_j, m),$$

that is, m is definable from $n_1, \ldots, n_j$ in $\mathfrak{M}$ by a formula of $\mathscr{L}_{\omega_1\omega}$.

The main step in the proof comes in showing that (ii) implies (iii). This can be accomplished by using the negation of (iii) to construct a full binary tree all of whose branches give rise to distinct automorphisms of $\mathfrak{M}$. It should be observed that the equivalence of (i) and (ii) can be obtained via general descriptive set-theoretic considerations, since the set of automorphisms of $\mathfrak{M}$ forms a Σ_1^1 set. In Section 7 we will also get a better bound on the defining formulas in (iv).

It follows easily from Theorem 4.2.1 that if $\mathfrak{M}$ is countable, $\mathfrak{N}$ uncountable and $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, then $\mathfrak{M}$ will have $2^{\aleph_0}$ automorphisms.

Another result that was already noted in Scott [1965] is that if $\mathfrak{M}$ is countable and R is a relation on $\mathfrak{M}$, then R is definable by a formula of $\mathscr{L}_{\omega_1\omega}$ iff every automorphism of $\mathfrak{M}$ is an automorphism of $(\mathfrak{M}, R)$. This is a local version of Beth definability and follows from Beth definability for $\mathscr{L}_{\omega_1\omega}$ together with Scott's theorem, if one assumes the vocabulary is countable. However, there is an even more elementary proof. For the non-trivial direction, if each automorphism of $\mathfrak{M}$ is an automorphism of $(\mathfrak{M}, R)$, then for each $\overline{m} = (m_1, \ldots, m_k) \in R$ and $\overline{m}' = (m_1', \ldots, m_k') \notin R$ there is some $\varphi_{\overline{m},\overline{m}}$ in $\mathscr{L}_{\omega_1\omega}$ such that $\mathfrak{M} \models \varphi_{\overline{m},\overline{m}'}(m_1, \ldots, m_k)$ but $\mathfrak{M} \models \neg\varphi_{\overline{m},\overline{m}'}(m_1', \ldots, m_k')$. Now R is definable by $\bigvee_{\overline{m}\in R} \bigwedge_{\overline{m}'\notin R} \varphi_{\overline{m},\overline{m}'}$.

By analogy to the case for rigid models that was considered above, Kueker [1968] and Reyes [1967] have shown the result given in

4.2.2 Theorem. *Let $\mathfrak{M}$ be a countable structure. Let R be a k-ary relation on M and define $S = \{Q\colon (\mathfrak{M}, R) \cong (\mathfrak{M}, Q)\}$. The following are equivalent:*

(i) $|S| = \aleph_0$.
(ii) $|S| < 2^{\aleph_0}$.
(iii) *There is some formula $\varphi(x_1, \ldots, x_j, y_1 \ldots y_k)$ in $\mathscr{L}_{\omega_1\omega}$ and $n_1, \ldots, n_j \in M$ such that*

$$R = \{(m_1, \ldots, m_k)\colon \mathfrak{M} \models \varphi(n_1, \ldots, n_j, m_1, \ldots, m_k)\}. \quad \square$$

In Section 7 we will give a better bound for this result also.

Part II. Compactness Regained

5. *Admissibility*

In passing from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}_{\infty\omega}$ a very substantial gain in expressive power is achieved. As is to be expected, however, there is a considerable price to pay. Many of the very useful properties of $\mathscr{L}_{\omega\omega}$—most notably compactness—are no longer enjoyed by $\mathscr{L}_{\infty\omega}$. If we restrict our attention to $\mathscr{L}_{\omega_1\omega}$, then some of these properties are salvaged. For example, interpolation, and a reasonable form of completeness can be thus regained. Compactness, however, clearly still fails. To obtain an omitting types result, we considered countable fragments L_B of $\mathscr{L}_{\omega_1\omega}$. Though completeness looks even better in this framework, interpolation, for example, fails. Thus, while on the one hand we want to deal with parts of $\mathscr{L}_{\omega_1\omega}$ small enough to be manageable, on the other hand, we would nevertheless like them to be large enough to be closed, for example, under finding interpolants. For this latter consideration, it would be preferable if the pieces that we deal with were given in some absolute way, since then, using them to give bounds would be more meaningful from "the first-order" point of view. $L_{\omega_1\omega}$ itself, as a fragment of $\mathscr{L}_{\infty\omega}$, is given by cardinality conditions, and so is certainly not "first-order".

In order to introduce the notion that has proven fruitful in this respect, we will assume, first of all—without doing any of this explicitly—that the syntax and semantics of $\mathscr{L}_{\infty\omega}$ are given within set theory. That is, we assume that sentences are sets, structures are sets, satisfaction is a ternary relation between structures, formulas, and functions from variables, *etc.* For any transitive set B we will thus be able to define $L_B = L_{\infty\omega} \cap B$; that is, the formulas of L_B are those formulas of $L_{\infty\omega}$ in B. Mild assumptions on B will guarantee that L_B is a fragment in the sense we have been using. Somewhat stronger conditions will give us a great deal of closure, and, when combined with countability, will even give a form of compactness.

5.1. KP *and Admissible Sets*

An *admissible set* is a transitive set A, such that $\langle A, \in\rangle$ is a model of a certain theory KP, the initials standing for Kripke and Platek. Kripke [1964a, b] and Platek [1966] were engaged in trying to generalize recursion theory to the ordinals. They were following the earlier work of Takeuti [1960], [1965] and Tugué [1964] who were studying recursion on the set of all ordinals, and Kreisel-Sacks [1965] whose metarecursion theory, in turn, followed from earlier work of Kleene [1955b] on recursive ordinals and hyperarithmetic sets. For a more complete history, the reader should consult the introduction to Barwise [1975].

In order to present the theory KP, we must first recall the Lévy hierarchy of formulas of a language containing the binary relation symbol $\in$ and perhaps other symbols as defined in Lévy [1965]. The collection of Δ_0-*formulas* is the smallest collection of formulas containing the atomic formulas, closed under the boolean connectives of $\neg$, & and $\vee$, and under bounded quantification. (That is, if φ is a

Δ_0-formula and u and v are variables, $\exists u \in v\varphi$ and $\forall u \in v\varphi$ are Δ_0-formulas, where $\exists u \in v\varphi$ stands for $\exists u[u \in v \,\&\, \varphi]$, *etc.*). The *$\Sigma_1$-formulas* are formulas of the form $\exists v\varphi$, where φ is a Δ_0-formula. The collection of *Σ-formulas* is obtained from the Δ_0-formulas by closing under &, $\vee$, bounded quantifiers, and existential quantifiers. A relation on a structure is said to be Σ-definable, or simply Σ, if it can be defined by a Σ-formula. A relation is $\boldsymbol{\Sigma}$, if it can be defined by a Σ-formula using parameters. A relation is $\boldsymbol{\Pi}$ if its complement is $\boldsymbol{\Sigma}$, and is $\boldsymbol{\Delta}$ if it is both $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$. All other similar definitions should follow easily from this sample.

The reason that the above classes of formulas are important is related to the notion of an *end extension.* A structure $\langle B, F, \ldots\rangle$ is an end extension of a structure $\langle A, E, \ldots\rangle$, where E and F are binary, if $\langle A, E, \ldots\rangle$ is a submodel of $\langle B, F, \ldots\rangle$ and whenever $a \in A$ and $(c, a) \in F$, then $c \in A$. In words, elements of A do not get any new F-members in B. It is then quite easy to show inductively that Σ-formulas are preserved in going to end extensions. We call such formulas *persistent.* If we insist that all the structures involved be models of some theory T we arrive at the notion of *persistent relative to T.* A formula is *absolute relative to T* if it holds in a model of T iff it holds in any end extension which is a model of T. Clearly φ is absolute relative to T iff both φ and $\neg\varphi$ are persistent relative to T. There is a converse to the simple observation that Σ-formulas are persistent. Feferman and Kreisel [1966] (see Feferman [1968b]) have shown that if φ is persistent relative to T, then there is a Σ-formula ψ such that $T \vdash \varphi \leftrightarrow \psi$. Hence, if φ is absolute relative to T, then in T φ is provably equivalent to both a Σ- and a Π-formula.

We can now give a set of axioms for KP. First, there are the axioms of extensionality, pairing and union, and the foundation scheme for arbitrary formulas (since the set existence axioms are weak). In addition, we have the following two schemes:

Δ_0-Separation: $\exists v\,\forall x(x \in v \leftrightarrow x \in u \,\&\, \varphi(x))$, for each Δ_0-formula φ in which v does not occur free.

Δ_0-Collection: $\forall x \in u\, \exists y\varphi(x, y) \rightarrow \exists v\, \forall x \in u\, \exists y \in v\varphi(x, y)$, for each Δ_0-formula φ in which v does not occur free.

Now, a structure $\mathfrak{A} = \langle A, \in, \ldots\rangle$ is *admissible* if $\langle A, \in\rangle$ is transitive and $\langle A, \in, \ldots\rangle \models$ KP. A transitive set A is admissible if $\langle A, \in\rangle$ is an admissible structure. It is often of interest to consider structures $\langle A, \in, \mathscr{P}\rangle$, where $\mathscr{P}$ is the power set operation, and A is closed under power set. Even if $\langle A, \in\rangle$ is admissible and A is closed under power set, $\langle A, \in, \mathscr{P}\rangle$ need not be admissible. As a notational convention, we write L_A to denote such an admissible fragment even when considering an admissible structure $\mathfrak{A} = \langle A, \in, \ldots\rangle$.

For later use, we mention two classes of sets given by conditions weaker than admissibility. Transitive sets $\langle B, \in\rangle$ satisfying all axioms of KP–except perhaps that of Δ_0-collection—are called *rudimentary.* The *primitive recursive set functions* of Jensen–Karp [1971] contain certain innocuous functions, such as the zero function, the pairing function and the union function, and are closed under composition and recursion. Transitive sets closed under these functions are called *primitive recursively closed sets* and are easily seen to be rudimentary, though they are not necessarily admissible.

5.2. Some Admissible Sets

It will be useful to have some examples of admissible sets. The first example is from the set-theoretic point of view. For x a set, let $\mathrm{TC}(x)$ denote the transitive closure of x; that is, it is the smallest transitive set with x as a subset. For κ an infinite cardinal let $H(k) = \{x : |\mathrm{TC}(x)| < \kappa\}$, the set of all sets of hereditary cardinality less than κ. If κ is regular, then $H(\kappa)$ is easily seen to be admissible. If $\kappa = \aleph_0$ all axioms of ZF *except infinity* hold, while if $\kappa > \aleph_0$, all axioms *except perhaps power set* hold. (Note that $H(\beth_\omega)$ is closed under power set, but $(H(\beth_\omega))$, $\in, \mathscr{P})$ is not admissible.) $H(\aleph_0)$ and $H(\aleph_1)$ are usually denoted by HF and HC, respectively. Assuming that the underlying language is coded appropriately, then $L_{H(\kappa)}$ is simply $L_{\kappa\omega}$.

The other example is of a more recursion-theoretic flavor. Let ω_1^{CK} denote the first non-recursive ordinal. That is, it denotes the first ordinal whose order type is not given by a recursive relation. Then, the set $L(\omega_1^{\mathrm{CK}})$ of all sets constructible before the ω_1^{CK}-th stage is an admissible set. In fact, it is the smallest admissible set containing ω. It is quite easy to see that no smaller set containing ω would be admissible. For a proof that it actually is admissible the reader should see Barwise [1975]. We note, for use later, that the subsets of ω in $L(\omega_1^{\mathrm{CK}})$ are exactly the hyperarithmetic sets.

An extremely important fact—and one about which we will have more to say in Section 5.4—is that for each set x there is a smallest admissible set containing x as an element. This set is denoted $\mathrm{HYP}(x)$. For B transitive, we let $o(B)$ denote the least ordinal not in B. Given an arbitrary set x—particularly if x happens to be some structure $\mathfrak{M}$—we can associate with x the ordinal $o(\mathrm{HYP}(x))$. As we shall see, this ordinal will have a strong model-theoretic relation to $\mathfrak{M}$.

5.3. Some Theorems of KP

KP is, of course, a weakened version of ZF, a version with separation and collection limited to Δ_0-formulas and power set totally eliminated. However, it turns out that collection actually follows for Σ-formulas, while separation holds for Δ subsets. In addition, replacement holds for Σ-formulas, as does the reflection principle; that is, if φ is a Σ-formula, then $\varphi \leftrightarrow \exists u \varphi^{(u)}$ is a theorem of KP. As a consequence, every Σ-formula is equivalent to a Σ_1-formula.

In KP we can show that for any set x, its transitive closure $\mathrm{TC}(x)$ exists, and then prove the following scheme for *definition by Σ recursion*:

5.3.1 Lemma. *Suppose G is an $(n + 2)$ place Σ-function. An $(n + 1)$ place Σ-function may be defined by*:

$$F(x_1, \ldots, x_n, y) = G(x_1, \ldots, x_n, y, \{\langle z, F(x_1, \ldots, x_n, z)\rangle : z \in \mathrm{TC}(y)\}).$$

$\square$

There is an analogous scheme for relations, and it is given in

5.3.2 Lemma. *Suppose P, Q are Δ-relations of $(n+1)$ and $(n+2)$ places, respectively. An n place Δ-relation may be defined by*:

$$R(x_1, \ldots, x_n, 0) \leftrightarrow P(x_1, \ldots, x_n)$$

$$R(x_1, \ldots, x_n, y) \leftrightarrow Q(x_1, \ldots, x_n, y, \{z \in \mathrm{TC}(y){:}\quad R(x_1, \ldots, x_n, z)\}). \quad \square$$

These schemes guarantee that certain important functions and relations are, respectively, Σ or Δ definable. For example, the usual operations of ordinal arithmetic or the rank of a set are Σ. In addition, by a straightforward argument it is possible to show that if $\langle X, \leq \rangle$ is a well-ordering of order type α and $\langle X, \leq \rangle$ is an element of the admissible set A, then $\alpha \in A$. Specifically, A can contain only well-orderings of order type $< o(A)$.

If $\langle B, E \rangle \models \mathrm{KP}$ and $b \in B$, then $\mathrm{TC}(b)$ will be well-founded in the sense of the real world V just in case the rank of b in the sense of $\langle B; E \rangle$ happens to be well-ordered in V. The set of all $a \in B$ which satisfy the condition (which is not expressible in $\langle B, E \rangle$, unless all elements of B satisfy the condition) is called the well-founded part of $\langle B, E \rangle$ and denoted $\mathrm{WF}(B, E)$. A result originating with Ville (see Barwise [1975]) states that if $\langle B, E \rangle \models \mathrm{KP}$, then $\mathrm{WF}(B, E)$ is isomorphic to an admissible set. This is often called the "truncation lemma".

Returning now to more model-theoretic concerns, suppose that A is admissible. Then, if the underlying vocabulary is Δ on A, so also will be the set of formulas of L_A and the set of sentences of L_A. The satisfaction relation will be Δ, while the quantifier rank of a formula will be given by a Σ-formula.

5.3.3 Application. Suppose $\mathfrak{M}$ is a structure, $m_1, \ldots, m_k \in M$, and α is an ordinal. It is quite easy to see that the function taking $(\mathfrak{M}, m_1, \ldots, m_k, \alpha)$ to $\sigma^{\alpha}_{\mathfrak{M}, m_1, \ldots, m_k}$ is defined by a Σ recursion; and so, in particular, the relation "$x = \sigma(\mathfrak{M})$" is Σ on any admissible set containing ω. Of course, this does not mean that an admissible set need be closed under σ.

Now, if $A = \mathrm{HC}$, and if φ is any sentence of $\mathscr{L}_{\omega_1\omega}$, then since, as was noted earlier, every countable structure $\mathfrak{M}$ (for a countable language) has its canonical Scott sentence $\sigma(\mathfrak{M})$ in $\mathscr{L}_{\omega_1\omega}$, the set $S = \{\sigma(\mathfrak{M}){:}\ \mathfrak{M} \models \varphi\}$ is Σ on HC and in one-to-one correspondence with the isomorphism types of countable models of φ. Now, by the general set-theoretic result of Mansfield [1975], S has cardinality $\leq \aleph_1$ or $= 2^{\aleph_0}$. This, of course, is simply the result of Morley [1970] on the weak form of Vaught's conjecture for $\mathscr{L}_{\omega_1\omega}$. The same argument also works for PC classes. On the other hand, it is known that the Vaught conjecture itself fails for PC classes. In fact, using the "truncation lemma", it is not difficult to see that the order types of the ordinals in countable models of KP must be of the form α or $\alpha + \eta \cdot \alpha$, where η is the order type of the rationals and α is a countable admissible ordinal. (To see that all the "non-standard" values are obtained one can appeal, for instance to Theorem 7.2.7. H. Friedman originally noted this for ZF in place of KP. However, by using KP, we get all the standard α immediately, which is what we need here). There are other proofs of this result and others relating to Vaught's conjecture. A good reference is Steel [1978]. More recently, Shelah (see Harrington–Makkai–Shelah [198?]) proved the Vaught conjecture for ω-stable theories in $\mathscr{L}_{\omega\omega}$.

5.3.4 Remark. A next step up from the theorems we have discussed would be Σ-separation. This principle is not provable in KP and is, in fact, quite strong. For example, if $\langle A, \in\rangle \models$ "Σ-separation", then it is easy to see that $\langle A, \in\rangle$ is a *β-model*, i.e., if $\langle A, \in\rangle \models$ "$\langle x, \leq\rangle$ is a well-ordering", then $\langle x, \leq\rangle$ really is a well-ordering. If $\langle A, \in\rangle \models$ "Σ-separation" and is *locally countable*. That is, if for each $a \in A$, there is some bijection from a into ω, then $\langle A, \in\rangle$ is *recursively inaccessible*, which means that if $a \in A$, then there is some admissible $\langle B, \in\rangle$ such that $a \in B \in A$. However, the smallest recursively inaccessible admissible set does not satisfy Σ-separation. For $\langle A, \in\rangle$ locally countable, it is shown in Nadel [1974b] that $\langle A, \in\rangle$ is a β-model iff $\langle A, \in\rangle$ is recursively inaccessible. Though the implication from right to left holds without local countability, there are β-models that are not recursively inaccessible; for example, consider HYP($L(\omega_1)$).

5.4. Urelements

When a model theorist studies a model $\mathfrak{M} = \langle M, R, \ldots\rangle$, the only structure he wants to consider is that imposed upon the elements of M by $R, \ldots$. The particular elements forming the universe M are irrelevant and regarded as atoms or urelements. Unfortunately, with ZF as metatheory, there are no urelements and M will consist of sets, each with its own internal structure. While this may not be aesthetically pleasing, in most instances the model theorist is able to simply ignore the fact. However, in the present rather sensitive context, this is not possible.

For the current purpose, there are two main considerations. First, the set HYP($\mathfrak{M}$) should depend only on the isomorphism type of $\mathfrak{M}$. In fact it would also be reasonable to expect that if $\mathfrak{M}$ and $\mathfrak{N}$ are potentially isomorphic, then so are $\langle \text{HYP}(\mathfrak{M}), \in\rangle$ and $\langle \text{HYP}(\mathfrak{N}), \in\rangle$. It should be apparent that even the first version would never be literally satisfied. One might then try to patch things up as follows: assign to each isomorphism type the intersection of all admissible sets containing models of that isomorphism type. This would work to some extent for countable structures (see Nadel–Stavi [1977]); but, as we shall mention later, even here there would be the difficulty that there need be no copy of $\mathfrak{M}$ in the intersection. However, suppose we consider even the simplest example of a structure, a set M with no relations or functions at all. Suppose M has cardinality $\beth_{\omega_1}$. Then any admissible set containing M must contain an uncountable ordinal. Clearly this would violate the stronger version.

The second consideration is that by allowing urelements, there are more admissible sets; and, consequently, a finer classification becomes possible. For example, so far HF is the only admissible set with ordinal ω. Allowing urelements will provide many others, and these will turn out to be a significant class which will be considered in more detail in Section 7.4.

Having presented some reasons why doing without urelements would cause problems, we go ahead and permit them from now on. This requires some changes in terminology and a slight modification of the axioms of KP to form the analogous theory KPU. We omit the precise details, all of which are carefully presented in Barwise [1975]. We also omit the precise construction of HYP($\mathfrak{M}$), which is via the Gödel operations beginning with $\mathfrak{M}$, a structure on urelements. Suffice it to say that HYP($\mathfrak{M}$) is the smallest admissible set containing $\mathfrak{M}$ as an element, and that the first consideration mentioned above holds in the strong version.

Now, having insisted on the need for urelements, we must confess that in terms of our presentation here—because we are considering admissibility more from the model-theoretic point of view than from the recursion theoretic, and we will be omitting most of the details—urelements will really not play a significant rôle, except in Section 5.5 and in our discussion of recursively saturated structures in Section 7.4. The results we will be considering usually carry over from admissible sets without urelements to the more general setting allowing urelements with little or no change. Thus, we will simply suppress mention of urelements except where they really do make a difference. However, there is one restriction that we should make clear at this point. In exchange for having additional admissible sets with ordinal ω, it is sometimes necessary to restrict the underlying vocabulary to be finite.

5.5. *The Pure Part of* HYP(M)

In this section we discuss some results concerning admissible sets with and without urelements. Assume that $\langle A, \in\rangle$ is an admissible set which may contain urelements. Those elements of A other than the urelements are called *sets*. Among the sets are distinguished the *pure sets* whose transitive closures do not contain urelements. We call admissible sets without urelements (that is, those containing only pure sets) *pure admissible sets.*

One urelement is like any other. And that is just the point. Consequently, distinct sets may only be distinguishable by reference to the specific urelements involved and might even be images of each other under some $\in$-automorphism. This cannot happen to pure sets. In some sense, then, pure sets have a real identity while arbitrary sets need not. This is especially evident in comparing elements from different admissible sets. For this reason, the set of pure sets in an admissible set A, denoted pp(A), the *pure part of* A, plays a special rôle. For example, the set of sentences of L_A would be taken to be a subset of pp(A), so that these sentences would form a subset of the sentences of $\mathscr{L}_{\infty\omega}$ as viewed from "the real world" where we need not have urelements.

The following easy result is from Barwise [1975].

5.5.1 Theorem. *If* $\langle A, \in\rangle$ *is admissible, then* $\langle \mathrm{pp}(A), \in\rangle$ *is a pure admissible set.* □

The next result, which is due to Makkai (see Nadel–Stavi [1977]), gives some idea of the "internal" relation between pure sets and sets of urelements.

5.5.2 Theorem. *Let M be a countable structure on urelements. Then the following are equivalent*:

(i) $\mathfrak{M}$ *has only countably many automorphisms.*
(ii) $\mathrm{HYP}(\mathfrak{M})$ *contains a pure structure* $\mathfrak{N}$ *which is an isomorphic copy of* $\mathfrak{M}$ *and an isomorphism between* $\mathfrak{M}$ *and* $\mathfrak{N}$. $\square$

In Theorem 5.5.2, $\mathrm{HYP}(\mathfrak{M})$ might contain an isomorphic pure copy, but not an isomorphism. Moreover, $\mathrm{HYP}(\mathfrak{M})$ could be replaced by the class of sets constructible from $\mathfrak{M}$, or even hereditarily symmetric over $\mathfrak{M}$. The next result from Nadel–Stavi [1977] shows how $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$ can be described without reference to urelements.

5.5.3 Theorem. $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$ *is the smallest admissible set containing* $\sigma^{\beta}_{\mathfrak{M}}$, *for each* $\beta \in \mathrm{HYP}(\mathfrak{M})$.

A case can be made for using $\mathrm{pp}(\mathrm{HYP}(x))$ as a measure of the information contained in x. If we begin with a pure set x, rather than with a structure on urelements, then we denote by x^+ the smallest pure admissible set containing x as an element. The next result, which may be appreciated more after considering canonical Scott sentences again in Section 7.1, shows that $\mathfrak{M}$ and $\sigma(\mathfrak{M})$ contain about the same information.

5.5.4 Corollary. (i) *If* $\sigma(\mathfrak{M}) \in \mathrm{HYP}(\mathfrak{M})$, *then* $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M})) = (\sigma(\mathfrak{M}))^+$;
(ii) *If* $\sigma(\mathfrak{M}) \notin \mathrm{HYP}(\mathfrak{M})$, *then* $(\mathrm{pp}(\mathrm{HYP}(\mathfrak{M})) \cup \{\sigma(\mathfrak{M})\})^+ = (\sigma(\mathfrak{M}))^+$. $\square$

Since admissible sets of the form $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$ might have special properties, it is natural to ask which pure admissible sets can be represented as $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$ for some $\mathfrak{M}$. First, some terminology is needed. An *ordinal* α is called *admissible* if $L(\alpha)$ is admissible. This is the same as saying that $\alpha = o(A)$ for some admissible set A. Sacks (see Friedman–Jensen [1968]) showed that a countable admissible ordinal is of the form ω_1^x for some $x \subseteq \omega$, where ω_1^x denotes Church–Kleene ω_1 relativized to x. In a similar spirit, Nadel–Stavi [1977] showed that every admissible $L(\alpha)$ is of the form $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$ for some $\mathfrak{M}$, as well as some other representation theorems. Not all pure admissible sets could be represented as $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$. An admissible set A is said to be *resolvable* iff there is a function $F\colon o(A) \to A$ such that $A = \bigcup_{\beta \in A} F(\beta)$, and $\langle A, \in, F\rangle$ is admissible. It is not difficult to see that if A is resolvable, we can always find F, such that, for each $\alpha < \beta \in A$, $F(\alpha) \in F(\beta)$, and $F(\alpha)$ is transitive. If F can be chosen $\mathbf{\Delta}$ on A, we call A, $\mathbf{\Delta}$-resolvable. Clearly $\mathrm{pp}(\mathrm{HYP}(\mathfrak{M}))$ is resolvable, using $F(\beta) = \mathrm{pp}(L(\beta, \mathfrak{M}))$, but there are non-resolvable countable admissible sets. Nadel–Stavi [1977] asked if this is the only constraint. Using structures M motivated by Steel forcing in place of the simpler structures used in the partial result of Nadel–Stavi [1977], S. Friedman [1982a] has shown this to be the case.

5.6. *Barwise Compactness*

As we remarked earlier, compactness fails for $\mathscr{L}_{\omega_1\omega}$ even for the simplest infinitary fragments. However, the following variant of compactness does hold.

5.6.1 Theorem (Barwise Compactness Theorem). *Let $\mathfrak{A}$ be a countable admissible structure and let T be a set of L_A sentences Σ definable on $\mathfrak{A}$. Suppose that each $T' \subseteq T$, $T' \in A$, has a model. Then T has a model.* $\square$

This result can be proved directly using the model existence theorem, or it can be obtained as a corollary to the extended Barwise completeness theorem which will be treated in Section 6.1. Barwise compactness resembles ordinary compactness, except that the theory T is restricted to be Σ on A, rather than arbitrary, while the hypothesis requires more than just finite sets being satisfiable. Nonetheless, Barwise compactness is a very powerful and important tool. It is safe to say that this result is what established admissible sets as an ongoing feature of model theory and started a second wave of interest in infinitary logic.

5.6.2 Remarks. It is easy to see that ordinary compactness for $\mathscr{L}_{\omega\omega}$ follows from Barwise compactness. The restriction to Σ-theories is really no restriction here since, for any set $X \subseteq \mathrm{HF}$, $\langle \mathrm{HF}, \in, X\rangle$ is admissible.

We will have more to say about Barwise compactness in Section 6.2 and will end this chapter with a brief application of it.

5.7. *An Application of Barwise Compactness*

In this section we will give a simple example of how Barwise compactness may be used. There are numerous applications to model theory. For a striking example of a more set-theoretic nature the reader should see Barwise [1971]. Barwise compactness is an especially potent tool used in conjunction with the omitting types theorem, as, for example, in Keisler [1971a, p. 58]. We will give a simple recursion-theoretic application which we will use later for model-theoretic purposes.

Kleene [1955b] gave an explicit definition of a recursive linear ordering that is well-ordered with respect to hyper-arithmetic subsets, but is not really well-ordered. Later, in Section 7.1 we will be interested in the canonical Scott sentence of such an ordering. We now will use Barwise compactness to show that such an ordering indeed exists. Once that is established, it is relatively simple to see what its order type could be. The object we construct is, by model-theoretic standards, quite refined, since we are insisting that it be recursive. Although Barwise compactness may seem at first glance to be much more restricted than ordinary compactness, the far greater expressive power of $\mathscr{L}_{\omega_1\omega}$ allows Barwise compactness to provide more subtle models than can be obtained from ordinary compactness.

Now, to begin the argument, let $A = L(\omega_1^{\mathrm{CK}})$. We will use a language with a binary relation symbol $\in$, a constant symbol $\mathbf{a}$ for each $a \in A$, and an additional constant symbol $\mathfrak{M}$ (the symbols $\mathbf{a}$ are really expendable). Consider now a theory T in L_A that expresses the following:

(i) KP;
(ii) atomic diagram of $\langle A, \in \rangle$;
(iii) "every ordinal is recursive";
(iv) "$\mathfrak{M}$ is a recursive binary relation on ω which is a well-ordering";
(v) "$\mathfrak{M}$ has an initial segment of type $\boldsymbol{\alpha}$", $\alpha \in A$.

It is not difficult to see that T could be chosen to be Σ on A. It is also easy to see that every subset $T' \subseteq A$, $T' \in A$ has a model. Thus, T has a model $\mathscr{L} = \langle B, a, \mathfrak{M} \rangle_{a \in A}$. Finally, there is sufficient absoluteness to guarantee that M really is a recursive linear ordering with initial segment of type ω_1^{CK} and is also such that every hyper-arithmetic subset of ω has a least element. $\mathfrak{M}$ cannot really be well-ordered, since, if it were, it would be of order type some non-recursive ordinal.

6. *General Model-Theoretic Properties with Admissibility*

In this section we will deal with aspects of the model theory of L_A, for A admissible, where the syntax is somehow bound to the set A, but the models involved need not be.

6.1. *Barwise Completeness*

In Section 3.2 we introduced the notion of provability $\vdash_{L_B}$, and stated a completeness theorem for it in Theorem 3.2.1. Now, we would like to use a stronger notion of provability, a notion in which the proof itself—as well as the formulas in the proof—are elements of an admissible set A. In order for this stronger notion to be complete, however, we will need to modify the definition of proof slightly. Without going into all the details here (these can be found, for instance, in Barwise [1975]), we modify the clause for conjunctions by taking as a proof of $\psi \to \bigwedge \Phi$ a function f with domain Φ such that for each $\varphi \in \Phi$, $f(\varphi)$ is a set of proofs of $\psi \to \varphi$. Basically, this change is necessary because the axiom of choice need not hold within an admissible set. Let us denote this new notion of proof by $\vdash'_{L_A}$. It is then quite easy to see (using the axiom of choice in the universe) that for any sentence of L_A, $\vdash'_{L_A} \varphi$ iff $\vdash_{L_A} \varphi$. Finally, let $\vdash_A \varphi$ mean that there is some proof in A, in the sense of $\vdash'_{L_A}$, of φ. This is equivalent to saying "$\langle A, \in \rangle \models \vdash'_{L_A} \varphi$" since the notion that p is a proof of φ in the sense of $\vdash'_{L_A}$ is absolute for admissible sets. In particular, if T is a Σ_A-theory, that is, a theory in L_A Σ-definable on A, then $\{\varphi: T \vdash_A \varphi\}$ is Σ_A.

Barwise [1967] (also Barwise [1969b]) was able to prove

6.1.1 Theorem. *For any admissible A, and φ a sentence of L_A, $\vdash_{L_A} \varphi$ iff $\vdash_A \varphi$. Moreover, if T is a Σ_A-theory, then $T \vdash_{L_A} \varphi$ iff $T \vdash_A \varphi$.* ☐

Now, as an immediate consequence of Theorem 6.1.1 and the Karp completeness theorem (3.2.1) we have the following sharpening.

6.1.2 Theorem (Barwise Completeness Theorem). *Let A be a countable admissible set and φ a sentence of L_A, then $\models \varphi$ iff $\vdash_A \varphi$. Moreover, if T is a Σ_A-theory, then $T \models \varphi$ iff $T \vdash_A \varphi$.* ☐

We now obtain the following generalization of the fact that the set of valid sentences of $L_{\omega\omega}$ is r.e.

6.1.3 Corollary. *Let A be countable admissible and T a Σ_A-theory. Then $\{\varphi: \varphi$ is a sentence of L_A and $T \models \varphi\}$ is Σ_A.* ☐

6.1.4 Remarks. The Barwise completeness theorem must clearly fail in general for uncountable A, since the Karp completeness theorem already fails. In fact, the extended version is easily seen to fail, even for subsets of HC of power $\aleph_1$. The fact that Theorem 6.1.1 holds without cardinality restriction does show that provability is absolute for models of ZFC, and this allows us to finish the argument that was begun in the remarks of Section 3.2.2 that provability is equivalent to validity in boolean-valued extensions of the universe. If φ is boolean-valid, we simply pass to a universe in which φ is countable. In this universe φ is valid, and we now appeal to the Barwise completeness theorem.

Corollary 6.1.3 also fails for uncountable A. We will consider this subject further in Section 6.3.

There is a converse to the Barwise completeness theorem due to Stavi [1973] and extending partial results of Barwise [1967]. It is stated in reference to Theorem 6.1.1 instead, since, it then may hold for all cardinalities.

6.1.5. *Let B be a transitive primitively recursively closed set such that if $\vdash_{L_B} \varphi$, then $\vdash_B \varphi$. Then B is a union of admissible sets.* ☐

This result could be stated in a more general framework using certain classes of abstract provability predicates rather than the particular ones we have used. In contrast to Theorem 6.1.5, however, Stavi [1973] has shown that there is a countable transitive primitive recursively closed set A such that the set of valid sentences of L_A is Σ_1 on A, but A is not the union of admissible sets.

6.2. *Barwise Compactness* (Continued)

Recall that for A admissible, $S \subseteq A$, S is Σ on A iff S is Σ_1 on A. We say that a transitive set A is *Σ_1-compact* if L_A satisfies the Barwise compactness theorem for Σ_1 sets of sentences (rather than Σ). There are also relativized notions where additional predicates are mentioned. The next result is due to Barwise [1967] and shows that admissibility is the weakest assumption one can make to get Σ_1-compactness.

6.2.1 Theorem. *Suppose A is rudimentary. Then if A is Σ_1-compact, A is admissible.* $\square$

The subject of compactness for admissible fragments of $\mathscr{L}_{\infty\omega}$ will be considered in Section 6.3.

6.3. *Uncountable Admissible Sets*

In considering Barwise compactness on uncountable admissible sets, or in trying to determine the uncountable admissible fragments L_A for which the L_A validities are Σ_1 on A, there are basically two sorts of results. The first sort involves implications between these properties and other conditions that seek to strengthen the notion of admissibility. We will not pursue this line here. The interested reader should consult Barwise [1975] for an introduction to these matters. The second sort establishes the existence (in a "concrete" way) of uncountable admissible sets satisfying Barwise compactness, or on which the validities are Σ_1. Specialized results in this direction were obtained earlier by Barwise [1968], Chang–Moschovakis [1970], Green [1974], Karp [1972], Makkai [1974b], Nyberg [1974, 1976] and perhaps others. More recently, S. Friedman [1981] and Magidor–Shelah–Stavi [1984] have obtained more general treatments. Our presentation here is based upon the latter of these. The idea is simply to assume that in some reasonably nice way, the admissible set in question is the union of countably many "small" sets. For simplicity, we will assume our admissible sets are pure and give

6.3.1 Definition. Suppose $\mathfrak{A}$ is admissible. $S \subseteq A$, is said to be a *smallness predicate* for $\mathfrak{A}$ if

(i) S is Σ_1 on A;
(ii) if $x \in S$, then $\mathscr{P}(x) \in A$;
(iii) the relation $\{(x, \mathscr{P}(x)) \colon x \in S\}$ is Σ_1 on A.

A is said to have the *first decomposition property* (DP1) if for some smallness predicate S for $\mathfrak{A}$, every member of A is a countable union of members of S (in the real world).

6.3.2 Definition. A binary relation R on A is a *decomposition relation for* A if

(i) R is Σ_1 on A;
(ii) $\forall X \, \exists Y R(X, Y)$;
(iii) whenever $R(X, Y)$, then for some sequence $\langle X_n : n \in \omega \rangle$ such that $X_n \in Y$, and $\mathscr{P}(X_n) \subseteq Y$ for $n \in \omega$, $X = \bigcup \{X_n : n \in \omega\}$.

A is said to have the *second decomposition property* (DP2) if it has a decomposition relation. $\mathfrak{A}$ is said to have the decomposition property (DP) if $\mathfrak{A}$ has (DP1) and (DP2). A set $B \subseteq A$ is said to be *σ-small* if it is a countable union of elements of A. If $\mathfrak{A}$ is σ-small and has (DP), it is said to be *countably decomposable.*

Using the above notions, Magidor–Shelah–Stavi [1984] obtain their main result in the next theorem and its corollary.

6.3.3 Theorem. *Let A satisfy* (DP) *and assume $T \subseteq L_A$ is σ-small and Σ_1 on A_1. Then*

(i) $\{\varphi \in L_A : T \models \varphi\}$ *is* Σ_1 *on* $\mathfrak{A}$;
(ii) *If T has no model, then some $T_0 \subseteq T$, $T_0 \in A$ has no model.* □

6.3.4 Corollary. (i) *If A satisfies* (DP) *then* $\{\varphi \in L_A : \models \varphi\}$ *is* Σ_1 *on* A.
(ii) *If A is countably decomposable then A satisfies Barwise compactness and for a theory T, Σ_1 on A, $\{\varphi \in L_A : T \models \varphi\}$ is Σ_1 on A.* □

All of the specialized results on Barwise compactness and completeness alluded to above are consequences of Theorem 6.3.3 and Corollary 6.3.4, including the original results for A countable.

6.3.5 Examples. (i) If A is closed under (real) power set and the relation $\{\langle x, \mathscr{P}(x)\rangle : x \in A\}$ is Σ_1 on A, then, letting $S = A$ and $R = \{\langle x, \mathscr{P}(x)\rangle : x \in A\}$, we have the Barwise–Karp cofinality ω compactness theorem (see Barwise [1968] and Karp [1972]).

(ii) Let A be an admissible set containing some element b, such that in the sense of A, every element of A has cardinality at most the cardinality of b, and that for some sequence $\langle b_n : n \in \omega \rangle \in A$, where $\bigcup \{\mathscr{P}(b_n) : n \in \omega\} \in A$, $b = \bigcup \{b_n : n \in \omega\}$. Then, if we take $S = \{x \in A : x$ has cardinality at most b_n in the sense of A, for some $n \in \omega\}$ and $R = \{(X, Y) : \exists f \in A\, [f$ is a function from a subset of b onto X and $Y = \{f''Z : Z \in \bigcup \{\mathscr{P}(b_n) : n \in \omega\}\}$ we obtain Makkai's compactness theorem, Makkai [1974b], which generalizes Green [1974].

To what extent is the above decomposition property necessary? Magidor–Shelah–Stavi [1984] gives the following partial converse to Corollary 6.3.4(i).

6.3.6 Theorem. *Assume that $V = L$, then for $\alpha > \omega$ admissible, $\{\varphi \in L_\alpha : \models \varphi\}$ is Σ_1 on $\langle L_\alpha, \in\rangle$ iff $\langle L_\alpha, \in\rangle$ satisfies* (DP).

The situation for Barwise compactness is more complicated. Results of Barwise [1975] and Stavi [1978] show, for example, that for κ regular, there is a closed unbounded subset of $\alpha < \kappa$ such that $\langle L_\alpha, \in\rangle$ satisfies Barwise compactness. The idea here is that, for "soft" reasons, there are many $\langle L_\alpha, \in\rangle$ satisfying Barwise compactness, and some of these will not be countably decomposable. Magidor–Shelah–Stavi [1984] realized that by strengthening the notion of Barwise compactness to *stable* Σ_1*-compactness*, where we call $\mathfrak{A}$ stably Σ_1-compact if all admissible expansions of $\mathfrak{A}$ satisfy Barwise compactness, a result would be forthcoming. And this result we give in

6.3.7 Theorem. *Assume that* $V = L$. *Let* A *be an admissible structure of the form* $\langle L_\alpha, \in, R_1, \ldots, R_n\rangle$. *Then* $\mathfrak{A}$ *is stably* Σ_1*-compact iff either* A *is countably decomposable or* α *is a weakly compact cardinal.* $\square$

There is an analogous result for the second part of Corollary 6.3.4(ii) and other interesting results which the reader can find in Magidor–Shelah–Stavi [1984].

6.4. Interpolation

In Section 3.2 we mentioned that $\mathscr{L}_{\omega_1\omega}$ satisfies interpolation. However, countable fragments L_B of $L_{\omega_1\omega}$ do not, in general, satisfy interpolation. Barwise [1967] has nevertheless shown that for A countable admissible, L_A does satisfy interpolation; and, hence, its consequences such as Beth definability. His proof in Barwise [1975] is similar to the consistency property proof of the Lopez-Escobar interpolation theorem for $L_{\omega_1\omega}$, except that in order to show the set under consideration is a consistency property, it is necessary to appeal to the Barwise completeness theorem. We point out here that no analogous appeal is needed in the earlier result.

There is a converse result due to H. Friedman (see Makowsky–Shelah–Stavi [1976]).

6.4.1. Theorem. *Let* A *be a transitive primitive recursively closed set. If* L_A *is* Δ*-closed, then* A *is the union of admissible sets.* $\square$

6.5. Hanf Numbers

Barwise [1967] was able to obtain a finer Hanf number result for countable admissible fragments L_A. The results we state here are for Σ_1-theories rather than single sentences and originate in Barwise–Kunen [1971].

6.5.1 Theorem. *Let* A *be countable admissible and* T *a* Σ_A*-theory. If, for each* $\beta < o(A)$, T *has a model of cardinality at least* $\beth_\beta$, *then* T *has a model of each infinite cardinality.* $\square$

There is a generalization of Theorem 6.4.1 to arbitrary admissible sets, but for these some preliminary discussion is required.

6.5.2 Definition. Let T be a Σ_A-theory for some admissible fragment A. Assume the vocabulary has among its relation symbols a binary relation symbol $<$. T is said to *pin down the ordinal* α if

(i) For any model M of T, $<^{\mathfrak{M}}$ is a well-ordering of its field and
(ii) T has a model with $<^{\mathfrak{M}}$ of order type α.

The least ordinal not pinned down by some Σ_A-theory T is denoted $h_\Sigma(A)$.

6.5.3 Theorem. *Let A be admissible and $\kappa = |A|$. The Hanf number (for Σ_A-theories) of L_A is* $\sup\{\beth_\beta(\kappa)\colon \beta < h_\Sigma(A)\}$.

In Section 7.2 we will give a very short proof of the following important fact.

6.5.4 Theorem. *Let A be countable admissible, then $h_\Sigma(A) = o(A)$.* $\square$

See Chapter IX for information about the size of $h_\Sigma(A)$ for A uncountable.

6.6. Global Definability

In Theorem 4.2.2 we mentioned an interesting local definability result. Here, we give an important global definability result of Makkai [1977b]. The version we will give first appeared in Barwise [1975], and it involves Σ_1^1-sentences of $\mathscr{L}_{\omega_1\omega}$ which are simply sentences of the form $\exists\bar{Q}\varphi$ where $\bar{Q}$ is a set of symbols and φ is a sentence of $\mathscr{L}_{\omega_1\omega}$. The semantics is the obvious one.

6.6.1 Theorem. *Let $\exists\bar{Q}\varphi(P,\bar{Q})$ be a Σ_1^1-sentence of the countable admissible fragment $L_A(\tau)$. For a countable structure $\mathfrak{M}$ define $S(\mathfrak{M}) = \{P\colon \mathfrak{M} \models \exists\bar{Q}\varphi(P,\bar{Q})\}$. The following are equivalent:*

(i) *For each countable $\mathfrak{M}$, $|S(M)| = \aleph_0$.*
(ii) *For each countable $\mathfrak{M}$, $|S(M)| < 2^{\aleph_0}$.*
(iii) *There is a sentence ψ of $L_A(\tau)$ of the form*

$$\bigvee \exists y_1 \ldots y_{j_i} \forall x_1, \ldots, x_k[P(x_1, \ldots, x_k) \leftrightarrow \psi_i(x_1, \ldots, x_k, y_1, \ldots, y_{j_i})],$$

which is a logical consequence of $\varphi(P,\bar{Q})$, where each ψ_i contains only symbols of τ not in $\bar{Q} \cup \{P\}$. $\square$

The proof of this result is somewhat involved and uses the interpolation theorem. It has a number of important corollaries, all of which can be found in Barwise [1975].

6.7. Omitting Types Revisited

We now continue the thread that we began spinning in Section 1.5. Barwise [1981] has shown that the facts that $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}_{\omega_1\omega}(Q_1)$ each satisfies an omitting types theorem as well as Barwise completeness and compactness results are not isolated events, but rather are part of a general result of the type described at the end of Section 1.5. Recall that we assume that the logic $\mathscr{L}^*$ satisfies the substitution axiom and admits only first-order variables. In the subsequent discussion we will assume that $\mathfrak{R}, \mathfrak{S}_1, \ldots, \mathfrak{S}_m$ are relation symbols, and $c_1, \ldots, c_n$ are constant symbols not contained in the vocabulary under consideration, while "true" is some valid sentence in that vocabulary.

Barwise [1981] generalizes the notion of an omitting types theorem by the following string of definitions.

6.7.1 Definition. A sentence $\varphi(R)$ of L^* is said to be a *test sentence* if for all structures $\mathfrak{M}$, $(\mathfrak{M}, \bigcup_{n<\omega} R_n) \models^* \varphi(R)$ implies there is some $n < \omega$ such that $(\mathfrak{M}, \mathfrak{R}_n) \models^* \varphi(R)$. A *test set* is a set of test sentences.

For $\mathscr{L}_{\omega\omega}$ the relevant test set is just the set of all sentences of the form $\exists\vec{x}[\varphi(\vec{x})$ & $R(\vec{x})]$, for $\varphi \in \mathscr{L}_{\omega\omega}$, while for $\mathscr{L}(Q_1)$ it is the set of sentences of the form $S\, \exists\vec{x}(\varphi(\vec{y}, \vec{x})$ & $R(\vec{x}))$ where S is a string of $\exists y_i$'s and Qy_j's and φ is a sentence of $\mathscr{L}(Q_1)$.

6.7.2 Definition. (i) For any theory T of $\mathscr{L}^*$ and set $\Sigma(c_1, \ldots, c_n)$ of $\mathscr{L}^*$-sentences, we say that T *accepts* $\Sigma(c_1, \ldots, c_n)$ if there is a model of $T \cup \{\forall x_1 \ldots x_n \bigvee \Sigma\}$.

(ii) T *locally accepts* $\Sigma(c_1, \ldots, c_n)$ with respect to a test set $\mathscr{T}$ if for all $\varphi(R) \in \mathscr{T}$, if $T \cup \{\varphi(\text{true}/R)\}$ has a model, so does $T \cup \{\varphi(\sigma/R)\}$ for some $\sigma \in \Sigma$.

(iii) $\mathscr{L}^*$ has the *Omitting Types Property* (OTP) with respect to a test set $\mathscr{T}$ if for all theories T of $\mathscr{L}^*$ and all countable sets $\{\Sigma_i(c_i, \ldots, n_i): i < \omega\}$, if T locally accepts each Σ_i, then T accepts all the Σ_i simultaneously; that is to say, there is some

$$M \models T \cup \{\forall x_1, \ldots, x_{n_i} \bigvee \Sigma_i(x_1, \ldots, x_{n_i}): i < \omega\}.$$

(iv) $\mathscr{L}^*$ has the OTP if $\mathscr{L}^*$ has the OTP for some test set $\mathscr{T}$.

We need one final definition before the results can be stated.

6.7.3 Definition. Let $\mathscr{T}$ be a test set. By a *$\mathscr{T}$-closed fragment* of $\mathscr{L}^*_{\omega_1\omega}$ we mean a sublogic L^*_B which contains $\mathscr{L}^*$, is closed under subformulas, satisfies the substitution axiom, if $\bigwedge \Phi \in L^*_B$ so is $\bigvee \{\neg\varphi: \varphi \in \Phi\}$, and such that if $\varphi(R) \in \mathscr{T}$ and $\varphi(\bigvee \{\psi_i : i < \omega\}/R) \in L^*_B$ then $\bigvee \{\varphi(\psi_i/R): i < \omega\} \in L^*_B$. A $\mathscr{T}$-closed fragment L^*_B is said to be *countable* if for each countable vocabulary τ, $\mathscr{L}^*_B(\tau)$ is countable.

6.7.4 Theorem. *Let $\mathscr{L}^*$ be $\aleph_0$-compact and have the* OTP *with respect to $\mathscr{T}$. Let L_B^* be a countable $\mathscr{T}$-closed fragment of $\mathscr{L}^*_{\omega_1\omega}$. Then $\mathscr{L}_B^*$ has the* OTP *with respect to the set $\mathscr{T}_B$ of $\mathscr{L}_B^*$ sentences of the form $\varphi(R, \psi_1/S_1, \ldots, \psi_n/S_n)$, where $\varphi(R, S_1, \ldots, S_n) \in \mathscr{T}$, and $\psi_1, \ldots, \psi_n$ are sentences of L_B^*.* $\square$

This result follows easily from the proof of the next completeness result, a result which gives an alternate axiomatization for $\mathscr{L}_{\omega_1\omega}$.

6.7.5 Theorem. *Let $\mathscr{L}^*$ be an $\aleph_0$-compact logic and $\mathscr{L}^*$ have the* OTP *with respect to the test set $\mathscr{T}$. Then the following proof system is complete for $\mathscr{L}^*_{\omega_1\omega}$:*

Axioms:

(A1) *For each $\varphi(R) \in \mathscr{T}$, all sentences of the form*

$$\varphi(\bigvee \{\psi_i : i < \omega\}/R) \to \bigvee \{\varphi(\psi_i/R) : i < \omega\}.$$

(A2) *All valid sentences of $\mathscr{L}^*$.*

(A3) *All sentences of $\mathscr{L}^*_{\omega_1\omega}$ of the form*

$$\bigwedge \{\psi_i : i < \omega\} \to \psi_j, \qquad j < \omega.$$

(A4) *All sentences of $\mathscr{L}^*_{\omega_1\omega}$ of the form*

$$\bigvee \{\psi_i : i < \omega\} \to \neg \bigwedge \{\neg \psi_i : i < \omega\}.$$

Rules:

(R1) *Modus ponens.*

(R2) *Generalization.*

(R3) *From $\varphi \to \psi_i$ for all $i < \omega$ infer $\varphi \to \bigwedge \{\psi_i : i < \omega\}$.*

(R4) *From $\varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k)$ infer $\varphi(\sigma_1/\mathfrak{R}_1, \ldots, \sigma_k/\mathfrak{R}_k)$, for all formulas $\varphi(\mathfrak{R}_1, \ldots, \mathfrak{R}_k) \in \mathscr{L}^*$, $\sigma_1, \ldots, \sigma_k \in \mathscr{L}_{\omega_1\omega}$.* $\square$

We will not give a complete proof of Theorem 6.6.5 (the reader should consult Barwise [1981] for this), but will merely sketch the main lines of argument. The proof is based on an idea from Keisler [1970]. Beginning with L_B^*, we first form $L_B^*(\tau')$ in which we allow finitely many occurrences of some countable set of new constants. Then, for each infinite disjunction $\bigvee \Phi_i(c_1, \ldots, c_{n_i})$ of $L_B^*(\tau')$, we add a new unary relation symbol R_i which will interpret $\bigvee \Phi_i(c_1, \ldots, c_{n_i})$. The vocabulary obtained by adding on these R_i's will be called τ''. The idea is that for each ψ of $L_B^*(\tau')$ we will define some $\psi^\#$ of $L^*(\tau'')$ which will play the part of ψ and will be "finitary" also. Specifically, $\psi^\#$ is defined inductively by the following four clauses:

(i) if θ is in $\mathscr{L}^*(\tau')$, $\theta^\# = \theta$.

(ii) $(\bigvee \Phi_i(c_1, \ldots, c_n))^\# = R_i(c_1, \ldots, c_n)$.

(iii) $(\bigwedge \Phi)^\# = \neg(\bigvee \{\neg\varphi : \varphi \in \Phi\})^\#$.

(iv) $\varphi(\sigma_1/R_1, \ldots, \sigma_k/\mathfrak{R}_k)^\# = \varphi(\sigma_1^\#/\mathfrak{R}_1, \ldots, \sigma_k^\#/\mathfrak{R}_k)$.

An appeal to the $\aleph_0$-compactness of $\mathscr{L}_B^*$ is then made in order to prove the key lemma to follow, where $T \cup \{\varphi\}$ is a set of sentences of $\mathscr{L}_B(\tau')$, $T^\# = \{\theta^\# : \theta \in T\}$, and $\vdash$ denotes provability in the above system:

$$(\#) \qquad T^\# \models \varphi^\# \quad \text{iff} \quad T \vdash \varphi.$$

Next, by making use of $(\#)$ and the fact that $\mathscr{L}^*$ satisfies the OTP with respect to $\mathscr{T}$ (recall that $\vdash$ depends on $\mathscr{T}$), we can then prove Theorem 6.7.5. Finally, by adding admissibility, Barwise [1981] obtains the result given in.

6.7.6 Theorem. *Let $\mathscr{L}^*$ be $\aleph_0$-compact and have the* OTP *with respect to the test set $\mathscr{T}$. Let $L_{\mathbb{A}}^*$ be a countable admissible fragment where the admissible structure $\mathbb{A} = (A, \in, \ldots)$ has the property that the set of valid sentences of L^* and the set $\mathscr{T}$ are each Σ_1 definable on $\mathbb{A}$. Then:*

(i) *The set of valid sentences of $L_{\mathbb{A}}^*$ is Σ_1 on $\mathbb{A}$;*
(ii) *$L_{\mathbb{A}}^*$ is Σ_1-compact; that is to say, if $T \subseteq L_{\mathbb{A}}^*$ is Σ on $\mathbb{A}$, and if every $T_0 \subseteq T$ with $T_0 \in \mathbb{A}$ has a model, then T has a model;*
(iii) *If $\varphi \in L_{\mathbb{A}}^*$, then the least ordinal not pinned down by φ is in $\mathbb{A}$.*

The proofs of all parts of the above follow from Theorem 6.7.5 in the same way that the analogous results for $\mathscr{L}_A$ follow from the Karp completeness theorem for $\mathscr{L}_{\omega_1\omega}$, the hypothesis on the validities of $\mathscr{L}^*$ being required to manage the axioms of the form (A2).

7. *"Harder" Model Theory with Admissibility*

In this section we will consider aspects of the model theory of countable admissible fragments L_A in which the structures themselves are restricted to the set A or its environs.

7.1. Scott Sentences and Admissible Sets

Suppose A is admissible and $\mathfrak{M}$ is a structure with $\mathfrak{M} \in A$. How much can we say about $\mathfrak{M}$ or its complete $\mathscr{L}_{\infty\omega}$ theory $\mathrm{th}_{\infty\omega}(\mathfrak{M})$ by just knowing its complete L_A-theory, $\mathrm{th}_A(\mathfrak{M})$? The first result asserts that $\mathrm{th}_A(\mathfrak{M})$ tells you all you need to know to distinguish $\mathfrak{M}$ from other structures $\mathfrak{N} \in A$.

7.1.1 Theorem. *Suppose A is admissible and $\mathfrak{M}$, $\mathfrak{N} \in A$ with $\mathfrak{M} \equiv_{L_A} \mathfrak{N}$. Then $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$.* $\square$

7.1.2 Corollary. *Suppose A is countable admissible and $\mathfrak{M}$, $\mathfrak{N} \in A$ with $M \equiv_{L_A} \mathfrak{N}$. Then $\mathfrak{M} \cong \mathfrak{N}$.* $\square$

For the easy proof of Theorem 7.1.1 see Nadel [1974b] where a slightly weaker hypothesis is used. Scott's theorem then easily gives Corollary 7.1.2. From Theorem 7.1.1 we can obtain the better bounds promised in Section 4.2. Specifically, the formula φ in Theorems 4.2.1 and 4.2.2 can be taken to be in HYP($\mathfrak{M}$). For the remainder of this section let us ignore all structures $\mathfrak{M}$ such that $o(\mathrm{HYP}(\mathfrak{M})) = \omega$ since the questions we consider are of no interest for them.

Can Theorem 7.1.1 be improved by dropping the restriction that $\mathfrak{N} \in A$, or—even better—by showing that $\mathfrak{M}$ has a Scott sentence in A; or—still better—that the canonical Scott sentence $\sigma(A)$ is in A? Any of the possibilities would actually imply Vaught's conjecture for $\mathscr{L}_{\omega_1\omega}$. The results are due to Sacks, Harnik-Makkai [1976], Makkai [1977b], and Steel [1978] who showed that Vaught's conjecture holds for sentences whose models have these properties.

However, all of these possible strengthenings fail to hold, as the following example will show. It has long been known (see Nadel [1974b]) that there is a recursive ordering $\mathfrak{M}$ of order type $\omega_1^{\mathrm{CK}} + \omega_1^{\mathrm{CK}} \cdot \eta$. (In fact, the example in Section 5.7 can be strengthened to provide this.) $\mathfrak{M}$ is obviously an element of $L(\omega_1^{\mathrm{CK}})$; and, moreover, it can be shown that $\mathfrak{M} \equiv_{\omega_1^{\mathrm{CK}}} (\omega_1^{\mathrm{CK}}, <)$. This latter fact follows from general results found in Karp [1965] or from a more specialized argument given in Nadel [1974b], an argument which is based on the fact that $L(\omega_1^{\mathrm{CK}})$ "thinks" that $\mathfrak{M}$ is well-ordered.

On the positive side, by applying Theorem 7.1.1 to expansions of $\mathfrak{M}$ by finitely many constants, we easily obtain the following result of Nadel [1974b] on Scott heights.

7.1.3 Theorem. *Let A be admissible and suppose $\mathfrak{M} \in A$. Then* SH($\mathfrak{M}$) $\leq o(A)$, *whence $\sigma(\mathfrak{M})$ has quantifier rank at most $o(A) + \omega$, and is in* HYP(A). ☐

Let us call a structure $\mathfrak{M}$ such that SH($\mathfrak{M}$) $< o(\mathrm{HYP}(\mathfrak{M}))$ *tame*. Otherwise, they will be termed, *wild*. It is easy to see that $\mathfrak{M}$ is tame iff $\sigma(\mathfrak{M}) \in \mathrm{HYP}(\mathfrak{M})$. Practically speaking, one has to go out of one's way to find a wild structure. On the other hand, there are not many positive results saying that various types of structures are tame. We mention three. Nadel [1974b] shows that every scattered linear ordering is tame. Nadel [1974a] shows that if $\varphi \in L_A$, A countable, and φ has only finitely many non-isomorphic countable models, then, for every $\mathfrak{M} \models \varphi$, SH($\mathfrak{M}$) $< o(A)$; and, if φ is countable in the sense of A, then $\sigma(\mathfrak{M}) \in A$. Thus, in the above situation, if $o(A) = \omega_1^{\mathrm{CK}}$, then every model of φ is tame. Finally, if $\mathfrak{M}$ is countable and has $< 2^{\aleph_0}$ automorphisms, then $\mathfrak{M}$ is tame (see Nadel [1974b]).

Now we state the result of Nadel [1974b], a result which was alluded to earlier in Section 4.1 and which, in some sense, helps justify the choice of $\sigma(\mathfrak{M})$ as the "canonical" Scott sentence of $\mathfrak{M}$.

7.1.4 Theorem. *Let A be countable admissible with ω, $\mathscr{L} \in A$. Suppose $\varphi \in A$ is a Scott sentence of some model $\mathfrak{M}$ (not necessarily in A). Then $\sigma(\mathfrak{M}) \in A$. In fact,* SH($\mathfrak{M}$) *is at most the quantifier rank of $\varphi + \omega$.*

We will return for additional comments on wild structures after discussing Gregory's result on uncountable models in Section 7.3.

7.2. Löwenheim–Skolem Results and Σ_A-saturated Models

In this section we briefly treat some downward Löwenheim–Skolem or, alternatively, "basis"—results, that are more subtle than the standard results dealing only with cardinality. We assume all theories T mentioned are consistent.

7.2.1 Theorem. *Let A be admissible and L_B a countable fragment of $L_{\infty\omega}$ in the sense of A. Let $T \in A$ be a complete L_B theory. Then T has a model $\mathfrak{M} \in A$.* $\square$

The proof of the above result is straightforward and can be found in Nadel [1974b]. If L_B is not required to be countable in the sense of A the result does not hold, nor does it hold if T is not required to be complete. In the latter case, a model can always be found in A^+, even if T is a theory in L_A which is Σ on A, so long as A is countable in A^+. If the theory T in Theorem 7.2.1 happens to have a prime model, then a prime model can be found in A. Instead of looking for a model in a set A, we can also try to find one in a "fattening" of A, that is to say, in a set $B \supseteq A$ such that $o(A) = o(B)$. The next result, which is in this direction, is due to Barwise–Schlipf [1976], Nadel [1974a], and Ressayre [1977] and is only one aspect of an equivalence we shall discuss later.

7.2.2 Theorem. *Suppose A is countable admissible and T is a Σ_A-theory. Then there is a countable admissible set $B \supseteq A$, with $o(A) = o(B)$ and a model $\mathfrak{M} \models T$, with $\mathfrak{M} \in B$.* $\square$

In the special case that $o(A) = \omega$, $\mathfrak{M}$ will be a model on urelements. Otherwise, $\mathfrak{M}$ could be composed of urelements or sets. (The results in Nadel [1974a] and Ressayre [1977] were formulated before the re-introduction of urelements.) We can give now a very short proof of Theorem 6.4.3 as we promised earlier.

Proof of Theorem 6.5.4. First, modifying the example we discussed in Section 1.3, we define by induction formulas $\psi_\alpha(x)$ in the vocabulary of linear orderings that express that the predecessors of x have order type α. Note that the formulas ψ_α can be found in L_A whenever $\alpha \in A$. This already shows that $h_\Sigma(A) \geq o(A)$.

Now, suppose $h_\Sigma(A) > o(A)$. In particular, suppose the Σ_A-theory T pins down some $\alpha > o(A)$. Consider the Σ_A-theory $T' = T \cup \{\exists x \psi_\beta(x) \colon \beta < o(A)\}$. T' is clearly consistent by Barwise compactness and by Theorem 7.2.2. T' has a model $\mathfrak{M}$ in some admissible set B with $o(B) = o(A)$. Now, if $<^{\mathfrak{M}}$ is a well-ordering, then T' insists it have type at least $o(B)$. However, we observed in Section 5.3 that an admissible set C cannot contain a well-ordering of order type $\geq o(C)$, and so $<^{\mathfrak{M}}$ is not well-ordered. $\square$

The three papers Barwise–Schlipf [1976], Nadel [1974a] and Ressayre [1977] were written with different purposes in mind and employed different terminology. We will try to employ the terminology that seems to be in current use. The next definition, which is due to Ressayre [1977], with modifications by Harnik and Makkai, appears, at first glance to be more complicated than one might expect. We will point out the reason for this presently.

7.2.3 Definition. Let A be admissible. A structure $\mathfrak{M}$ is said to be Σ_A-*saturated* if, for each $m_1, \ldots, m_k \in M$ it satisfies

(i) if $\Gamma(x_1, \ldots, x_k, v)$ is a Σ_A type in L_A, then

$$\mathfrak{M} \models \left(\bigwedge_{\Gamma' \subseteq \Gamma, \Gamma' \in A} \exists v \Gamma'(m_1, \ldots, m_k, v) \right)$$
$$\to \exists v \bigwedge \Gamma(m_1, \ldots, m_k, v);$$

(ii) if $I \in A$, q is Σ_A, and for each $i \in I$, q_i denotes $\{\varphi\colon \varphi$ is a formula of L_A in the free variables $x_1, \ldots, x_k$, such that $(i, \varphi) \in q\}$, then

$$\mathfrak{M} \models \left(\bigwedge_{(q' \subseteq q, q' \in A)} \bigvee_{i \in I} \bigwedge q'_i \right) \to \bigvee_{i \in I} \bigwedge q_i.$$

Condition (i) alone is what one might expect as the definition. Models satisfying (i) alone are sometimes called Σ_A-compact. Both conditions are needed, however, to prove Theorem 7.2.6, which explains much of the importance of Σ_A-saturated models.

7.2.4 Theorem. *Let A be countable admissible and let T be a consistent Σ_A-theory. Then T has a Σ_A-saturated model.*

The proof of this result can be obtained from the proof of Lemma 8.2.2. ▯

7.2.5 Definition. Let A be admissible and suppose $(\mathfrak{M}, m_1, \ldots, m_k)$ is a structure for a vocabulary $\tau \in A$. $\mathfrak{M}$ is said to be Σ_A-*resplendent* if, whenever $\tau' \supseteq \tau$, $\tau' \in A$ and T is a Σ_A-theory in $L_A(\tau')$ consistent with the $L_A(\tau)$ theory of $(\mathfrak{M}, m_1, \ldots, m_k)$, then $(\mathfrak{M}, m_1, \ldots, m_k)$ can be expanded to a model of T. If the expansion can always be taken to be itself Σ_A-saturated we say $\mathfrak{M}$ is *strongly* Σ_A-*resplendent.*

7.2.6 Theorem. *Let A be countable admissible. If $\mathfrak{M}$ is a countable Σ_A-saturated structure, then $\mathfrak{M}$ is Σ_A-resplendent. In fact, $\mathfrak{M}$ is strongly Σ_A-resplendent.*

In building the expansion, condition (i) in the definition of Σ_A-saturation is used to realize types and witness existential formulas, while condition (ii) is needed to handle disjunctions. We can obtain, with little difficulty, the converse of Theorem 7.2.6, which holds without any cardinality restrictions. Now, we can relate Σ_A-saturated structures to the earlier Löwenheim–Skolem results. This result was first obtained by Ressayre [1977], with Schlipf [1977] examining the case in which $A = \mathrm{HF}$.

7.2.7 Theorem. *Let A be admissible and $\mathfrak{M}$ a Σ_A-saturated model. Then there is some admissible $B \supset A$ with $o(B) = o(A)$ such that $\mathfrak{M} \in B$.* ▯

To prove Theorem 7.2.7 we use, for the countable case, strong Σ_A-resplendency to build a model of KP around $\mathfrak{M}$, with standard ordinals the same as A, and then use Ville's result to take its well-founded part. Lévy's absoluteness gives the general result.

Does Theorem 7.2.7 have a converse? The answer is "almost". Nadel [1974a] and Ressayre [1977] were able to show that countable $\mathfrak{M}$ satisfying the conclusion of Theorem 7.2.7 were almost Σ_A-resplendent. The problem occurs because Σ_A sets need not be Σ_B sets. This does not occur if A is Σ_B; for instance, if $A = L(\alpha)$, for some α. More recently, Adamson [1978] has been able to find a complete converse, by slightly strengthening the notion of "fattening" used.

Most often in practice, rather than use the property of Σ_A-saturation directly, we use instead the properties given in Theorems 7.2.6 and 7.2.7. However, Σ_A-saturation has a distinct advantage over the other two notions: It is easy to see that Σ_A-saturation is preserved under the union of an L_A-elementary chain. This point is quite important for the proof of the main result of the next section.

7.3. Uncountable Models

As we noted earlier, a consistent sentence of $L_{\omega_1\omega}$ with an infinite model need not have an uncountable model. The following important result is from Gregory [1973] and it tells us when certain countable theories have uncountable models.

7.3.1 Theorem. *Let A be countable admissible and suppose T is a Σ_A-theory of L_A. Then the following are equivalent:*

(i) *T has an uncountable model*
(ii) *There are models of T $\mathfrak{M}$, $\mathfrak{N}$ such that $\mathfrak{M} \prec_{\neq L_A} \mathfrak{N}$.*

Using the results of the previous section Ressayre was able to give a proof of Theorem 7.3.1, a proof which was much simpler that Gregory's original argument and which we can present quite briefly. The difficult direction is in showing that (ii) implies (i). The idea here is to build an L_A-elementary chain of countable models whose union will be the desired uncountable model. Using (ii), the fact that T is Σ_A and the appropriate expansion theory, it is possible to find Σ_A-saturated models $\mathfrak{M}_0 \models T$ and $\mathfrak{M}_1$ such that $\mathfrak{M}_0 \prec_{\neq L_A} \mathfrak{M}_1$. Now, using strong resplendency, we can find a Σ_A-saturated $\mathfrak{M}_2$ such that $\mathfrak{M}_1 \prec_{\neq L_A} \mathfrak{M}_2$. This shows how to take care of any successor stage in the chain. To manage limit stages, we need only use the fact that the union of an L_A-elementary chain of Σ_A-saturated models is Σ_A-saturated.

The requirement that T is Σ_A in Theorem 7.3.1 is necessary, as was shown by an example of Gregory mentioned in Gregory [1970].

We will now return to the subject of Scott sentences for a few additional remarks. Since most familiar structures were tame, and wild structures were only found with difficulty, various conjectures concerning wild structures naturally arose from this limited experience.

Of the original wild structures $\mathfrak{M}$, each had a proper $L_{\mathrm{HYP}(\mathfrak{M})}$ elementary submodel, and so $\mathrm{Th}_{L_{\mathrm{HYP}(\mathfrak{M})}}(\mathfrak{M})$ had an uncountable model. It was thought that this might always be the case. However, Makkai [1981] has given a counterexample. He also gives an example of a sentence φ of $\mathscr{L}_{\omega_1\omega}$ with models of Scott height cofinal in ω_1, but no uncountable model.

Again, for all the original wild structures it was the case that $\bigwedge \mathrm{Th}_{L_{\mathrm{HYP}(\mathfrak{M})}}(\mathfrak{M})$ was not a Scott sentence. Makkai [1981] also gives a counterexample to the obvious conjecture here as well. It should be pointed out that the examples mentioned above, even with the alternate proofs by Shelah, are quite complicated.

7.4. *Recursively Saturated Models*

We now specialize our consideration of Σ_A-saturated models to the case in which A = HF, the case originally considered by Barwise–Schlipf [1976]. In particular, τ will now be finite.

Here, condition (ii) in the definition holds automatically, since I must be finite, and so the definition looks more like what we might have first guessed. Furthermore, Σ_{HF} is essentially the same as r.e. in the sense of ordinary recursion theory. Thus, a structure is Σ_{HF}-saturated iff every r.e. 1-type over the model is realized. By Craig's theorem, this becomes no weaker if we restrict to recursive types. In fact, such models are called *recursively saturated*. On the other hand, a recursively saturated model will realize every type over the model r.e. in the complete theory of any simple expansion of the model by finitely many constants.

The notion of Σ_{HF}-resplendent is actually equivalent to the weaker looking condition on $\mathfrak{M}$, that if $\mathfrak{M}$ is a τ-structure and R is relation symbol not in τ such that for some $\mathfrak{N} \succ \mathfrak{M}$, $\mathfrak{N} \models \exists R\varphi(R)$, then $\mathfrak{M} \models \exists R\varphi(R)$, where φ is any sentence, possibly with parameters from M. Without admitting parameters, the notion becomes strictly weaker for $\mathfrak{M}$ uncountable. For $\mathfrak{M}$ countable, the parameters are not necessary.

The corresponding condition on fattenings is that $o(\mathrm{HYP}(\mathfrak{M})) = \omega$, and so, of course, we must have $\mathfrak{M}$ a model on urelements.

Finally, it follows from our earlier discussion, that these three conditions are equivalent. We should also point out that, from Theorem 7.1.1, it follows that recursively saturated models are ω-homogeneous.

It has been noticed that the class of recursively saturated models appears in certain natural applied situations. For example, Barwise–Schlipf [1975] showed that the recursively saturated models of Peano arithmetic are exactly those models that can be expanded to models of Δ_1^1-PA, a certain natural fragment of analysis. Lipshitz–Nadel [1978] show that if $\langle A, +, \cdot\rangle$ is a model of Peano arithmetic, then both $\langle A, +\rangle$ and $\langle A, \cdot\rangle$ must be recursively saturated. If $\langle A, +\rangle$ is a countable recursively saturated model of Presburger arithmetic, then resplendency allows us to expand it to a model of Peano. This is not true in the uncountable case; but, as shown in Nadel [1980b] for groups of cardinality $\aleph_1$, recursive saturation together with a simple group theoretic condition is enough, at least for the "integer" version of Presburger arithmetic, and is also necessary.

The notion of recursive saturation has already become an object of great interest and many results have been forthcoming concerning it. While space does not permit its further consideration here, it is safe to say that recursive saturation seems likely to enter the permanent repertory of the model theorist.

It seems especially fitting to end our study of $\mathscr{L}_{\omega_1\omega}$ with the topic of recursive saturation, which, after all, can be expressed quite simply in $\mathscr{L}_{\omega\omega}$. The investigation of finitary logic led to the investigation of infinitary logic, which in turn engendered the study of admissible sets, a study which has since come back to enrich the study of $\mathscr{L}_{\omega\omega}$.

8. *Extensions of $\mathscr{L}_{\omega_1\omega}$ by Propositional Connectives*

The objective of this concluding section is threefold. First, there is the matter of considering propositional connectives other than simple conjunction and disjunction. The second objective will be achieved as a by-product. In the course of obtaining the results we will have occasion to employ techniques which help to illustrate some of the ideas of the earlier sections. The third objective, which we will consider first, involves more abstract considerations, namely the problem of characterizing $\mathscr{L}_{\omega_1\omega}$.

The reader has no doubt been already struck by Lindström's characterizations of $\mathscr{L}_{\omega\omega}$ as a maximal logic satisfying various sets of conditions in Chapter II. $\mathscr{L}_{\infty\omega}$ can also be characterized as a maximal logic in several different ways, ways that are described in Chapters III and XVII. Can $\mathscr{L}_{\omega_1\omega}$ be characterized in this way? It is obvious how to characterize $\mathscr{L}_{\omega_1\omega}$ as a minimal logic, but not as a maximal logic. A natural question to ask would be whether $\mathscr{L}_{\omega_1\omega}$ is the maximal logic whose syntax lives on HC and which satisfies certain basic model theoretic properties, such as interpolation, some natural completeness result, and perhaps some others. The results of Section 8.3 will show that this would not seem to be the case.

8.1. *Propositional Connectives*

Our presentation in the remainder of this section is based on Harrington [1980] which continues earlier work of H. Friedman [1977] and unpublished work of Kunen. We will be concerned with the logic obtained by adding to $\mathscr{L}_{\omega_1\omega}$ a new countable propositional connective.

First, we add to the definition of the formulas of $\mathscr{L}_{\omega_1\omega}$ the clause

(*) if φ_i is a formula for $i < \omega$, in some fixed finite set of free variables, then so is $C(\langle\varphi_i : i \in \omega\rangle)$.

The semantics corresponding to this clause depends on the choice of a fixed function $P\colon \mathscr{P}(\omega) \to \{0, 1\}$. We denote the resulting logic by $\mathscr{L}(P)$. Specifically, we have the clause

$$(*) \qquad \mathfrak{M} \models_P C(\langle \varphi_i : i \in \omega\rangle) \quad \text{iff} \quad P(\{i : \mathfrak{M} \models_P \varphi_i\}) = 1.$$

Though the syntax of $\mathscr{L}(P)$ looks rather different, it is easy to see that $\mathscr{L}(P)$ is a sublogic of $\mathscr{L}_{\infty\omega}$. In fact, it is a sublogic of $\mathscr{L}_{(2^\omega)^+\omega}$.

There is a natural proof system for $\mathscr{L}(P)$ which is obtained from the usual Hilbert-style proof system for $\mathscr{L}_{\omega_1\omega}$ by adding the following axioms:

1a. $\bigwedge(\{\varphi_i : i \in X\} \cup \{\neg\varphi_i : i \in \omega\backslash X\}) \to C(\langle \varphi_i : i \in \omega\rangle)$, for each $X \subseteq \omega$ such that $P(X) = 1$;
1b. $\bigwedge(\{\varphi_i : x \in X)\} \cup \{\neg\varphi_i : i \in \omega\backslash X\}) \to \neg C(\langle \varphi_i : i \in \omega\rangle)$, for each $X \subseteq \omega$ such that $P(X) = 0$;
2. $\bigwedge\{\varphi_i \leftrightarrow \varphi_i' : i \in \omega\} \to (C(\langle \varphi_i : i \in \omega\rangle) \leftrightarrow C(\langle \varphi_i' : i \in \omega\rangle))$, for each pair of sequences $\langle \varphi_i : i \in \omega\rangle$, $\langle \varphi_i' : i \in \omega\rangle$ of formulas.

We write $\vdash_P$ for provability in this system and reserve $\vdash$ for provability in our standard system for $\mathscr{L}_{\omega_1\omega}$. We use $\models_P$ and $\models$ in a similar way for validity in the two logics as well as for satisfaction. We say that P—or, more properly $\mathscr{L}(P)$—*is complete* if for every sentence φ of $\mathscr{L}(P)$, $\vdash_P \varphi$ iff $\models_P \varphi$.

Just as for $\mathscr{L}_{\omega_1\omega}$, since each rule of proof has only countably many hypotheses, if $\vdash_P \varphi$, then φ has a countable proof. This point will be essential for what comes later and so we simply require that proofs be countable. As usual, one direction in completeness is easy to verify, that is, that, $\vdash \varphi$ implies $\models \varphi$.

It will be necessary to consider *partial proposition connectives*, which are simply (partial) functions from a subset of $\mathscr{P}(\omega)$ to $\{0, 1\}$. If D is a derivation in the above system, then there is a natural associated partial propositional connective P_D defined so that $P_D(X) = 1$ if some axiom of type 1a for X is used in D, and $P_D(X) = 0$ if some axiom of type 1b for X is used. Otherwise, $P_D(X)$ is undefined.

Very much as in Section 6.6, the general technique employed here will be to treat the extra connective as a new atomic formula. Specifically, writing $\langle \varphi_i \rangle$ in place of the longer $C(\langle \varphi_I : i \in \omega\rangle)$, for each $\langle \varphi_i \rangle$ we introduce a new relation symbol $\mathfrak{R}_{\langle \varphi_i \rangle}$ of the appropriate number of places. Given a formula ψ of $\mathscr{L}(P)$, we define ψ^* in such a way that $\psi^* = \psi$ for ψ atomic, $(C(\langle \varphi_i : i \in \omega\rangle))^* = \mathfrak{R}_{\langle \varphi_i \rangle}$, and so that $*$ commutes with the other connectives and quantifiers. Notice also that ψ^* is always a formula in $\mathscr{L}_{\omega_1\omega}$. A structure for the new relation symbols will be called an *expanded structure*.

There is a small technical problem which must be overcome before we proceed: Not every $\mathscr{L}_{\omega_1\omega}$ formula in the new symbols is of the form ψ^*, for some formula of $\mathscr{L}(P)$ (in the original symbols). This arises because if, for example, $\mathfrak{R}_{\langle \varphi_i \rangle}(x)$ is 1-place and τ is a term we may form $\mathfrak{R}_{\langle \varphi_i \rangle}(\tau)$ and this will not be of the form ψ^*. However, $\mathfrak{R}_{\langle \varphi_i \rangle}(\tau)$ "ought to be equivalent" to $R_{\langle \varphi_i(\tau) \rangle}(x)$. We make this official by adding a set of axioms Γ to this effect, for each $\mathfrak{R}_{\langle \varphi_i \rangle}$ and appropriate sequence of terms. Then, relative to Γ, each φ in $\mathscr{L}_{\omega_1\omega}$ is equivalent to some ψ^*, where the $\mathscr{L}(P)$-formula ψ is found by tracing back through the recursive definition of $*$.

We denote this ψ by $\varphi^\#$; and, similarly, we let $T^\# = \{\theta^\#: \theta \in T\}$, for a set T of $\mathscr{L}_{\omega_1\omega}$ formulas. We use a similar convention for derivations.

By a *fragment* we will mean a subclass $\mathscr{F}$ of the formulas of $\mathscr{L}(P)$ that is closed under subformulas such that if $C(\langle\varphi_i: i \in \omega\rangle)$ and $C(\langle\varphi_i': i \in \omega\rangle)$ are in $\mathscr{F}$, so is the corresponding axiom of type 2. Now, given a fragment $\mathscr{F}$, we let $S(\mathscr{F})$ be the collection of all ψ^* such that ψ in $\mathscr{F}$ is an instance of the axiom scheme 2. An expanded structure is called an *$\mathscr{F}$-structure* if it is a model of $S(\mathscr{F})$. If P is a partial propositional connective, we let $S(P, \mathscr{F})$ be the collection of all ψ^* such that ψ in $\mathscr{F}$ is an instance of the axiom scheme 1. An $\mathscr{F}$-structure $\mathfrak{M}$ gives rise to a partial propositional connective $P_\mathfrak{M}$ as follows: Suppose $\mathfrak{M} \models \varphi_i[\bar{a}]$ iff $i \in X$. Then let $P_\mathfrak{M}(X) = 1$ if $\mathfrak{M} \models \mathfrak{R}_{\langle\varphi_i\rangle}(\bar{a})$, and let $P_\mathfrak{M}(X) = 0$ if $\mathfrak{M} \models \neg\mathfrak{R}_{\langle\varphi_i\rangle}(\bar{a})$. Since $\mathfrak{M}$ is an $\mathscr{F}$-structure, $P_\mathfrak{M}$ is well-defined. The next result mentions some basic facts about the notions we have just introduced. These facts are easy to check.

8.1.1 Lemma. *Let $\mathscr{F}$ be a fragment and $\mathfrak{M}$ an $\mathscr{F}$-structure, then*

(i) *suppose $T \subseteq \mathscr{F}$ is a set of sentences and $\mathfrak{M} \models T^*$. Then, for any propositional connective $P \supseteq P_\mathfrak{M}$, $\mathfrak{M} \models_P T$;*
(ii) *if P is a partial propositional connective, then P and $P_\mathfrak{M}$ are compatible; that is to say, $P \cup P_\mathfrak{M}$ is a partial propositional connective, iff $\mathfrak{M} \models S(P, \mathscr{F})$;*
(iii) *for $T \subseteq \mathscr{F}$, and P a partial propositional connective, if D is a derivation in $\mathscr{L}_{\omega_1\omega}$ from $T^* \cup S(\mathscr{F}) \cup S(P, \mathscr{F})$, then $D^\#$ is a derivation from T in $\mathscr{L}(P)$, with $P_D^\# \subseteq P$;*
(iv) *if D is a derivation in $\mathscr{L}(P)$ using axioms $\alpha_0, \alpha_1, \ldots$, then D^* is a derivation in $L_{\omega_1\omega}$ from $\alpha_0^*, \alpha_1^*, \ldots$.* ☐

8.2. The Main Lemma

The next result deals with $\mathscr{L}_{\omega_1\omega}$ and is the main lemma we will need to derive the desired results about $\mathscr{L}(P)$. It mixes omitting types with Σ_A-saturated models and its proof—which we will only sketch here—will nevertheless fill in some earlier omissions.

8.2.1 Definition. Let A be an admissible structure and let Φ be a type over L_A. We say that Φ is *semi-complete over A* iff $\Phi \cup \{\neg\varphi: \varphi \in \Phi\}$ is Δ on A.

It is obvious that complete types are semi-complete. If a semi-complete type Φ is principal over a Σ_A-theory, then Φ is Δ on A.

8.2.2 Lemma. *Let $\mathfrak{A}$ be a countable admissible structure, T a consistent Σ_A-theory, and Γ a collection of L_A types, each semi-complete over A, such that no member of Γ is Δ on $\mathfrak{A}$ and $|\Gamma| < 2^{\aleph_0}$. Then there is a Σ_A-saturated countable model of T which omits all the types in Γ.*

Proof. For each $f \in 2^\omega$, we build a countable Σ_A-saturated model $\mathfrak{M}_f$ of T such that for $f \neq g$, the only semi-complete types realized in both $\mathfrak{M}_f$ and $\mathfrak{M}_g$ are Δ

over A (and hence not in Γ). Thus, since $|\Gamma| < 2^{\aleph_0}$ and any $\Phi \in \Gamma$ can be realized in at most one $\mathfrak{M}_f$, some $\mathfrak{M}_f$ must omit all types in Γ.

Let D be a countable set of new constant symbols to use in the ensuing Henkin construction. For each $\alpha \in 2^{<\omega}$, we construct by induction a theory T_α satisfying the following conditions:

(i) T_α is a consistent Σ_A-theory, involving only finitely many constants from D;
(ii) $T_\phi = T$ and for $\alpha \subseteq \beta$, $T_\alpha \subseteq T_\beta$;
(iii) For each step of a complete Henkin construction, there is some $n \in \omega$ such that for all $\alpha \in 2^n$, T_α has carried out this step.
(iv) For each Σ_A-type $\Phi(\vec{x})$ that mentions only finitely many constants from D, there is an $n \in \omega$ such that for all $\alpha \in 2^n$, if $T_\alpha \cup \Phi(\vec{x})$ is consistent, then there are constants $\mathbf{d}_1, \ldots, \mathbf{d}_k \in D$ such that $\Phi(\mathbf{d}_1, \ldots, \mathbf{d}_k) \in T_\alpha$.
(v) For each $I \in A$, q, and i as in Definition 7.2.3(ii), there is some $n \in \omega$ such that for all $\alpha \in 2^n$, if for some $i \in I$, $T_\alpha \cup \{\bigwedge q_i\}$ is consistent, then there are constants $\mathbf{d}_1, \ldots, \mathbf{d}_k \in D$ such that $q_i(\mathbf{d}_1, \ldots, \mathbf{d}_k) \subseteq T_\alpha$.
(vi) For each sequence of variables $\vec{x} = x_1, \ldots, x_k$ and collection F of formulas in the free variables $\vec{x}$ closed under negation and Δ on $\mathfrak{A}$, and each $\mathbf{c}_1, \ldots, \mathbf{c}_k$, $\mathbf{d}_1, \ldots, \mathbf{d}_k$ from D, there are infinitely many $n \in \omega$ such that for all $\alpha, \beta \in 2^n$, if $\alpha \neq \beta$, then either (1) for all $\varphi \in F$, $T_\alpha \models \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ or $T_\alpha \models \neg\varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ or (2) for some $\varphi \in F$, $T_\alpha \models \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ but $T_\beta \models \neg\varphi(\mathbf{d}_1, \ldots, \mathbf{d}_k)$.

Using the fact that T and the types in Γ are Σ, Barwise completeness allows us to carry through a construction with the above properties. Now, for each $f \in 2^\omega$, $\bigcup \{T_{f \restriction n} : n \in \omega\}$ is a complete Henkin theory by (iii) and so gives rise to a countable model $\mathfrak{M}_F$ of T. Conditions (iv) and (v) guarantee that $\mathfrak{M}_F$ is Σ_A-saturated. (Observe that to verify part (ii) in the definition of Σ_A-saturation, we must appeal to some property of admissibility such as Σ-refection). Finally, condition (vi) guarantees that if $f \neq g$, and $\mathfrak{M}_f$ and $\mathfrak{M}_g$ realize some type Φ semi-complete over $\mathfrak{A}$, then Φ is Δ on $\mathfrak{A}$. ☐

8.3. $\mathscr{L}(P)$'s with Nice Properties

Armed with Lemma 8.2.2, we are now able to begin our construction of logics $\mathscr{L}(P)$ which are complete and enjoy other desirable properties. For P a partial propositional connective and $\mathfrak{A} = \langle A, \in, P \restriction A\rangle$ a countable admissible structure, $\mathscr{L}(P) \cap A$ is a fragment and $S(\mathscr{L}(P) \cap A)$ and $S(P \restriction A, \mathscr{L}(P) \cap A)$ are each $\Sigma_\mathfrak{A}$.

8.3.1 Lemma. *Let P be a partial propositional connective with $|P| < 2^{\aleph_0}$ and let $\mathfrak{A} = \langle A, \in, P \restriction A\rangle$ be a countable admissible structure. Suppose T is a set of $\mathscr{L}(P)$ sentences of A, Σ on $\mathfrak{A}$. Then either*

(i) *there is an $\mathscr{L}(P)$ derivation $D \in A$ of a contradiction from T with $P_D \subseteq P$ or*
(ii) *there is a countable $\Sigma_\mathfrak{A}$-saturated $\mathscr{L}(P) \cap A$-structure $\mathfrak{M}$ such that $\mathfrak{M} \models T^*$ and $P_\mathfrak{M}$ and P are compatible.*

Proof. Suppose that the $\Sigma_{\mathfrak{A}}$-theory $T' = T^* \cup S(\mathscr{L}(P) \cap A) \cup S(P \restriction A, \mathscr{L}(P) \cap A)$ is consistent. Let Γ be the set of all types Φ of the form $\Phi = \{\varphi_i^* : i \in X\} \cup \{\neg\varphi_i^* : i \in \omega \backslash X\}$ where $X \in (\text{dom } P)\backslash A$ and $\langle \varphi_i \rangle \in A$. Then each Φ is semi-complete but not Δ on A, since $X \notin A$. $|\Gamma| < 2^{\aleph_0}$ since $|P| < 2^{\aleph_0}$. Now, by Lemma 8.2.2, we obtain $\mathfrak{M}$ as in option (ii) since our choice of Γ prevents $P_{\mathfrak{M}}$ from clashing with P.

If, on the other hand, T' is inconsistent, then since T' is just a Σ_A-theory of $\mathscr{L}_{\omega_1\omega}$, we may apply Barwise compactness to obtain an $\mathscr{L}_{\omega_1\omega}$-derivation D in A of a contradiction from T'. Now, Lemma 8.1.1(iii) gives us option (i). $\square$

8.3.2 Theorem. *There is a complete $\mathscr{L}(P)$.*

Proof. We will build an increasing chain of partial propositional connectives P_ζ, $\zeta < 2^{\aleph_0}$ such that $P_0 = \phi$, $P_\lambda = \bigcup \{P_\zeta : \zeta < \lambda\}$ for λ a limit, and such that $|P_\zeta| \le |\zeta \cdot \omega|$ for all $\zeta < 2^{\aleph_0}$. P will then be $\bigcup \{P_\zeta : \zeta < 2^{\aleph_0}\}$.

First, we enumerate all sentences of $\mathscr{L}(P)$ as $\langle \varphi_{\zeta+1} : \zeta < 2^{\aleph_0} \rangle$. Suppose we have already constructed P_ζ. Choose a countable A such that $\varphi_{\zeta+1} \in A$ and $(A, \in, P_\zeta \restriction A)$ is admissible (this is no problem using, for example, the downward Löwenheim–Skolem theorem). Now, applying Lemma 8.3.1 there is a partial propositional connective P' compatible with P_ζ such that either $P' \supseteq P_D$ for some $\mathscr{L}(P)$-derivation D of $\neg\varphi_{\zeta+1}$, or $P' \supseteq P_{\mathfrak{M}}$ for some $\mathscr{L}(P) \cap A$-structure $\mathfrak{M} \models \varphi_{\zeta+1}$. We then take $P_{\zeta+1} = P_\zeta \cup P'$. It is easy to see that $P = \{P_\zeta : \zeta < 2^{\aleph_0}\}$ will be complete. $\square$

8.3.3 Remarks. At each successor step $\zeta + 1$ of the construction there would be no problem in fixing P' arbitrarily on some X not in the domain of P_ζ. This would allow us to construct $2^{(2^{\aleph_0})}$ different complete P's.

Now that we know complete P's exist, the next result sheds a great deal of light on the problem of characterizing $\mathscr{L}_{\omega_1\omega}$ as a maximal "nice" logic whose syntax "lives" on HC, a goal that we mentioned at the outset of this section.

8.3.4 Theorem. *Let P be a complete propositional connective and let $\mathfrak{A} = \langle A, \in, P \restriction A \rangle$ be a countable admissible structure. Then $\mathscr{L}(P) \cap A$ satisfies each of the following:*

(i) *Extended Barwise completeness.*
(ii) *Barwise compactness.*
(iii) *Interpolation.*

Proof. Suppose $T \subseteq \mathscr{L}(P) \cap A$ is Σ_1 on A and inconsistent. Then, since P is complete, there is some derivation D in $\mathscr{L}(P)$ of a contradiction from T. Now, if we apply Lemma 8.3.1 to $P \restriction A \cup P_D$ then option (i) must hold, since otherwise we have the contradictory situation that $\mathfrak{M} \models T^*$ and P_D and $P_{\mathfrak{M}}$ are compatible, whence $\mathfrak{M} \models S(P_D, \mathscr{L}(P) \cap A)$, and $\mathfrak{M} \models \neg \bigwedge T^*$ since D was a derivation of a contradiction from T.

(ii) Barwise compactness for $\mathscr{L}(P) \cap A$ now follows immediately as usual from extended Barwise completeness.

(iii) Harrington [1980] describes two different proofs of interpolation for $\mathscr{L}(P) \cap A$.

The first is via a cut-free proof system for $\mathscr{L}(P)$. The second also gives a new proof of interpolation for $\mathscr{L}_A$ as well. It makes heavy use of the details of the construction of HYP($\mathfrak{M}$) and so is beyond the scope of our presentation here. In particular, it uses the fact that every element of HYP($\mathfrak{M}$) is denoted by some "term" with parameters from M and, furthermore, that the behavior of Δ_0-formulas over HYP($\mathfrak{M}$) is already "mirrored" back in $\mathfrak{M}$. A rather detailed treatment of these matters can be found in the final section of Nadel–Stavi [1977]. □

8.3.5 Remark. An alternate approach to $\mathscr{L}(P)$ was given earlier (although not published) by Kunen. It involves the notion of a *selective ultrafilter* on ω which is an ultrafilter $\mathscr{U}$ on ω having the property that if $f: \omega \to \omega$ then either $f^{-1}(n) \in \mathscr{U}$ for some $n \in \omega$ or $f \upharpoonright X$ is 1–1 on some $X \in \mathscr{U}$. Though the existence of selective ultrafilters on ω follows from the continuum hypothesis or Martin's axiom, Kunen has shown that it is independent of ZFC.

Given an ultrafilter $\mathscr{U}$ on ω, we define a propositional connective $P_{\mathscr{U}}$ by $P_{\mathscr{U}}(X) = 1$ iff $X \in \mathscr{U}$. Kunen strengthens the usual proof system for $L_{\omega_1\omega}$ by adding the following axioms where $\langle \varphi_i : i \in \omega \rangle$ and $\langle \varphi_i^j : i \in \omega \rangle$ are any sequences of $\mathscr{L}(P)$ formulas:

1′. $(\bigwedge \{\bigvee \{\varphi_i^j : j \le n\} : i \in X\}) \to \bigvee \{C(\langle \varphi_i^j : i \in \omega \rangle) : j \le n\}$, for each $X \in \mathscr{U}$ and $n \in \omega$; and
2′. $\neg C(\langle \varphi_i : i \in \omega \rangle) \leftrightarrow C(\langle \neg \varphi_i : i \in \omega \rangle)$.

Now, with respect to this proof system, Kunen shows that $P_{\mathscr{U}}$ is complete iff $\mathscr{U}$ is selective. Kunen is also able to prove a Barwise completeness and compactness theorem, as well as interpolation for admissible fragments $\mathscr{L}(P) \cap A$, but only under the added hypothesis that every member of $\mathscr{U}$ that is Σ on $\mathfrak{A}$ has a subset in $\mathscr{U}$ that is Δ on $\mathfrak{A}$. This extra hypothesis is actually necessary. Kunen's proof is naturally more set-theoretic, and we will not go into it here. It can, however, be found in Harrington [1980].

8.3.6 Exercise. A good review of the material in this section, as well as of much that is in the entire chapter, can be had by working out the following problems. It is assumed that A is as in Theorem 8.3.4.

(i) Prove that the Hanf number for $\mathscr{L}(P) \cap A$ is $\beth_{o(A)}$.
(ii) State and prove an omitting types theorem for $\mathscr{L}(P) \cap A$.

Appendix

In this short section we will briefly note some of the major omissions of our article and give some references for each.

We have said nothing at all about the work done on categoricity theory for $\mathscr{L}_{\omega_1\omega}$. The interested reader should consult Keisler [1971a], Kierstead [1980], and Shelah [1975c].

Some work has been done on model-theoretic forcing in $\mathscr{L}_{\omega_1\omega}$. The reader who is interested in this aspect of the subject might want to consult Keisler [1973] and Lee–Nadel [1977].

Game sentences are closely connected to the subject of this article. Relevant information is available in Vaught [1973b], Harnik–Makkai [1976] and, to some extent, in Chapter X of the present volume. The reader should also be aware of Makkai [1977a] in which game sentences play a very basic rôle in the presentation of the general theory of admissible fragments. Another important connection involved here is that between $\mathscr{L}_{\omega_1\omega}$ and descriptive set theory.

Venturing off more in the direction of recursion theory proper, we come to the subject to inductive definability, the study of which could naturally be begun with Chapter X of the present volume. More "classical" recursion theory on admissible sets has become an object of much interest, and a study of this area might well begin by consulting Barwise [1975] and Shore [1977].

Finally, information about the "soft model-theoretic" aspects of the logics we have considered, including the relevant Lindström type results, can be found in Chapters III and XVII of the present work.

Chapter IX
Larger Infinitary Languages

by M. A. Dickmann

1. The Infinitary Languages $\mathscr{L}_{\kappa\lambda}$ and $\mathscr{L}_{\infty\lambda}$

The motivations underlying the study of infinitary languages which are given in the introduction to Chapter VIII will also serve well here, thereby relieving us of the need to make further comments.

Recall that for infinite cardinals κ, λ, with $\kappa \geq \lambda$, the language $\mathscr{L}_{\kappa\lambda}$ is constructed by prescribing a stock of individual variables of cardinality κ and a list τ of finitary non-logical symbols called the *vocabulary*. Furthermore, $\mathscr{L}_{\kappa\lambda}$ contains connectives and quantifiers permitting the formation of:

(i) the negation of any expression;
(ii) conjunctions and disjunctions of any number (strictly) fewer than κ expressions;
(iii) existential and universal quantifications over any set of fewer than λ variables.

The formal definition of the set of *expressions* of $\mathscr{L}_{\kappa\lambda}$ is left as an exercise. *Formulas* will be expressions containing less than λ free variables. This restriction is made in order to provide the means for "quantifying out" all free variables in a formula.

The class-language $\mathscr{L}_{\infty\lambda}$ will have as its formulas those formulas of all the languages $\mathscr{L}_{\kappa\lambda}$, for $\kappa \geq \lambda$ (with the same vocabulary); that is, $\mathscr{L}_{\infty\lambda}$ allows conjunctions and disjunctions of any *set* of its *formulas* but permits quantifications only over fewer than λ variables. The language $\mathscr{L}_{\infty\infty}$ contains as formulas those formulas of the languages $\mathscr{L}_{\infty\lambda}$ for all infinite cardinals λ.

The semantics of $\mathscr{L}_{\kappa\lambda}$, $\mathscr{L}_{\infty\lambda}$ and $\mathscr{L}_{\infty\infty}$ are defined by straightforward extrapolation of the first-order definition of satisfaction, for instance, by declaring that $\bigwedge_{i\in I}\phi_i$ is true iff each ϕ_i is true, etc..

In the remainder of this section, we will present a number of examples illustrating the use and the expressive power of the languages we have just introduced. They were chosen so as to provide a foretaste of what general results we may or may not expect from the model theory of these larger infinitary languages. Indeed, some of the model-theoretic results in Section 3 are elaborations on some of the examples which follow.

1.1 The Notion of Cardinality

It is well known that in the first-order language for $\tau = \emptyset$ a fixed finite cardinal can be characterized by a single sentence while, by compactness, no characterization of the notion of finiteness is possible by any set of sentences.

In $\mathscr{L}_{\kappa^+\kappa}(\emptyset)$ we can express the notion of cardinality less than κ by the sentence:

$$\sigma_\kappa\colon \bigvee_{\substack{\lambda \in CN \\ \lambda < \kappa}} (\exists v \restriction \lambda)\Bigg[\bigwedge_{\substack{\gamma,\delta<\lambda \\ \gamma\neq\delta}} (v_\gamma \neq v_\delta) \wedge \forall y \bigvee_{\gamma<\lambda} (y = v_\gamma)\Bigg].$$

$\lambda \in \mathrm{CN}$ means that λ is a cardinal, and $v \restriction \lambda = \langle v_\gamma | \gamma < \lambda\rangle$ denotes a block of λ variables. This sentence is in $\mathscr{L}_{\kappa^+\kappa}$, because the number of cardinals $< \kappa$ is at most κ. Whenever this number is strictly smaller than κ, for instance, when $\kappa = \omega_1$ or $\kappa = \omega_\omega$, σ_κ is in $\mathscr{L}_{\kappa\kappa}$.

This example shows that an infinitary formula may not have a prenex normal form. Indeed, if κ is a limit cardinal, then σ_κ is not even equivalent to a conjunction of prenex formulas of $\mathscr{L}_{\infty\kappa}$. This follows from the following simple fact.

1.1.1 Fact. *A pure equality sentence of $\mathscr{L}_{\infty\lambda}$ either holds in all structures of power $\geq \lambda$, or it holds in none.* □

For the proof of this statement, see Dickmann [1975, p. 139]. The reader should also see Theorem 4.3.1.

Assume now that $\{\phi_i | i \in I\}$ is a set of prenex $\mathscr{L}_{\infty\kappa}(\emptyset)$ formulas, say:

$$\phi_i\colon (Q_1 v_1^i \restriction \lambda_1^i) \ldots (Q_{n_i} v_{n_i} \restriction \lambda_{n_i}^i)\psi_i,$$

where each Q is $\forall$ or $\exists$, and ψ_i is quantifier-free. Let λ_i be the largest of $\lambda_1^i, \ldots, \lambda_{n_i}^i$. Since κ is a limit cardinal, then $\lambda_i^+ < \kappa$. By its very definition, σ_κ has a model of power λ_i^+; hence, if $\models \sigma_\kappa \leftrightarrow \bigwedge_{i\in I}\phi_i$, so does each ϕ_i. By Fact 1.1.1, ϕ_i is true in all structures of power $\geq \lambda_i^+$. Hence, σ_κ has a model of power $\geq \kappa$, which, of course, is absurd. □

This example leaves undecided the question of the validity of a prenex normal form theorem for $\mathscr{L}_{\kappa\kappa}$, when κ is a successor cardinal, for example, for $\mathscr{L}_{\omega_1\omega_1}$. But this is false too, as has been proven by M. Jones. Roughly speaking. Jones' argument runs as follows: He gives a coding of $\mathscr{L}_{\omega_1\omega_1}$-formulas on one binary relation symbol $\in$ by hereditarily countable sets; and, using this, he then defines, for each $n \in \omega$, a formula $T_n(z, y)$ which expresses the notion

> "z is (the code of) a prenex $\mathscr{L}_{\omega_1\omega_1}(\in)$-formula with n alternations of quantifiers satisfied by y in $R(\omega_2)$".

A standard diagonal argument then shows that the formula $\bigvee_{n\in\omega} T_n(z, y)$ cannot have a prenex form. For details, see Dickmann [1975, Appendix B].

1.2. Well-orderings

(1) We leave as an exercise for the reader to construct an $\mathscr{L}_{\omega_1\omega_1}(<)$-sentence axiomatizing the class of non-empty well-orderings. We will, however, observe that the description of well-orderings needed here uses the axiom of choice.

(2) What of well-orderings in $\mathscr{L}_{\kappa\omega}$? Consider the following formulas $\phi_\alpha(v)$, $(\alpha < \kappa)$, which contain only the symbol $<$ and are defined by transfinite induction:

$$\phi_0(v)\colon \neg\exists w(w < v) \wedge \sigma,$$

$$\phi_\alpha(v)\colon \forall w(w < v \leftrightarrow \bigvee_{\xi<\alpha} \phi_\xi(w)) \wedge \sigma \qquad \text{for} \quad \alpha > 0,$$

where σ stands for the (first-order) axioms for linear order. The reader can easily verify that for $a \in A$:

$$\langle A, <\rangle \models \phi_\alpha[a] \quad \text{iff} \quad < \text{ is a total order on } A \text{ and } \{x \in A \mid x < a\} \text{ is of type } \alpha.$$

Let

$$\theta_\alpha\colon \neg\exists x\phi_\alpha(x) \to \forall y \bigvee_{\xi<\alpha} \phi_\xi(y).$$

1.2.1 Exercise. If A is of power $< \kappa$, then $\langle A, <\rangle$ is a model of the $\mathscr{L}_{\kappa\omega}$-theory $\{\theta_\alpha \mid \alpha < \kappa\}$ iff it is well-ordered. $\square$

In particular, the proper class $\{\theta_\alpha \mid \alpha \in \mathrm{ON}\}$ of sentences does characterize well-orderings. On the other hand, $\{\theta_\alpha \mid \alpha < \kappa\}$ has non-well-ordered models in every cardinal $\geq \kappa$ (Exercise).

As a matter of fact, López-Escobar showed (Theorem 3.2.20 below) that there is no set of sentences in any language $\mathscr{L}_{\kappa\omega}$—that is to say, no single sentence of $\mathscr{L}_{\infty\omega}$—which characterize well-orderings. This remains true if by characterizing is meant not simply being an elementary class in $\mathscr{L}_{\kappa\omega}(<)$ but also the much more comprehensive notion of being a relativized projective class in $\mathscr{L}_{\kappa\omega}(<)$; see Chapter II, Definition 3.1.1 for more on this notion.

(3) We want to have at hand the notion of *η_λ-set* (or *set of type η_λ*) for later use. These are totally ordered sets $\langle A, <\rangle$ with the following property: Whenever X, Y are subsets of A of cardinality $< \lambda$ such that each member of X is smaller than every member of Y, $X < Y$, there is an $a \in A$ such that $X < a < Y$. Observe that here X or Y may be empty. If $\lambda = \aleph_\alpha$, sets of type η_λ are frequently called η_α-sets.

1.2.2 Exercise. Show that the notion of η_λ-set is axiomatizable by an $\mathscr{L}_{\lambda\lambda}(<)$-sentence if $\lambda < \aleph_\lambda$, and by an $\mathscr{L}_{\lambda^+\lambda}(<)$-sentence otherwise. $\square$

1.3. Some Infinitary Theories of Trees

(1) The notion of a (well-ordered) tree is axiomatizable in $\mathscr{L}_{\omega_1\omega_1}$ by the sentence:

$$\forall x\, \forall v \restriction \omega\left[\bigwedge_{n\in\omega}(v_n \leq x) \to \bigvee_{n\in\omega}(v_n \leq v_{n+1})\right] \wedge \text{“} \leq \text{ is a partial order”}.$$

Various special notions of tree of mathematical interest admit natural infinitary axiomatizations; following are some examples:

(2) κ-Souslin trees, that is, trees of power κ in which every chain and every antichain is of power $< \kappa$, can be characterized in $\mathscr{L}_{\kappa^+\kappa^+}$:

$$(\exists v \restriction \kappa)\left[\bigwedge_{\substack{\alpha,\beta\in\kappa\\ \alpha\neq\beta}}(v_\alpha \neq v_\beta) \wedge \forall y \bigvee_{\alpha\in\kappa}(y = v_\alpha)\right],$$

$$(\forall v \restriction \kappa)\left[\bigwedge_{\alpha,\beta\in\kappa}(v_\alpha \leq v_\beta \vee v_\beta \leq v_\alpha) \to \bigvee_{\substack{\alpha,\beta\in\kappa\\ \alpha\neq\beta}}(v_\alpha = v_\beta)\right],$$

$$(\forall v \restriction \kappa) \neg \bigwedge_{\substack{\alpha,\beta\in\kappa\\ \alpha\neq\beta}}(v_\alpha \nleq v_\beta \wedge v_\beta \nleq v_\alpha).$$

Based on these examples, the reader might try to find appropriate axioms for the kinds of trees given in

1.3.1 Exercise. (a) Trees in which all branches have power $< \kappa$, and each element has $< \kappa$ immediate successors (in $\mathscr{L}_{\kappa^+\kappa^+}$).

(b) κ-Aronszajn trees, that is trees of height κ in which every level and every branch have power $< \kappa$ (in $\mathscr{L}_{\kappa^+\kappa^+}$).

(c) Trees with only one root, finite branching, and all branches of length $\leq \omega$, in the language having an individual constant 0 for the root, and the function $P(x)$ giving the node preceding x (in $\mathscr{L}_{\omega_1\omega}$). This example was proposed by López-Escobar. $\square$

1.4. Examples From Set Theory

Certain set-theoretical notions can be formulated in the infinitary languages we are dealing with.

(1) Transitive sets (or, rather, structures isomorphic to them), coincide with the models of an $\mathscr{L}_{\omega_1\omega_1}(E)$-sentence expressing extensionality and well-foundedness; this follows from the Shepherdson–Mostowski collapsing theorem (see Dickmann [1975, Appendix A]). We leave as an exercise for the reader to write out this sentence.

(2) The class of sets hereditarily of power $\leq \kappa$ can be characterized in $\mathscr{L}_{\kappa^+\kappa^+}$ by the sentence of (1) in conjunction with:

$$(\forall v \upharpoonright \kappa)\, \exists y\, \forall z\Big(z \mathrel{E} y \leftrightarrow \bigvee_{\xi<\kappa} (z = v_\xi)\Big),$$

$$\forall y\Big[\exists z(z \mathrel{E} y) \rightarrow (\exists v \upharpoonright \kappa)\, \forall z\Big(z \mathrel{E} y \leftrightarrow \bigvee_{\xi<\kappa} (z = v_\xi)\Big)\Big].$$

(3) Certain substructures of $\langle R(\alpha), \in \upharpoonright R(\alpha)\rangle$, $\alpha \in \mathrm{ON}$, can be axiomatized in $\mathscr{L}_{\kappa\omega}$, where κ is the first cardinal larger than α. [Recall that $R(0) = \varnothing$ and $R(\alpha) = \bigcup_{\xi<\alpha} \mathbb{P}(R(\xi))$, for $\alpha > 0$.] Indeed, if we set

$$V_0(x)\colon x \neq x,$$

$$V_\alpha(x)\colon \forall y\Big(y \mathrel{E} x \rightarrow \bigvee_{\xi<\alpha} V_\xi(y)\Big),$$

and

$$\sigma_\alpha\colon \forall xy[\forall z(z \mathrel{E} x \leftrightarrow z \mathrel{E} y) \rightarrow x = y] \wedge \forall x \bigvee_{\beta<\alpha} V_\beta(x),$$

then any model of σ_α can be isomorphically embedded in $R(\alpha)$ [Exercise: Use the Shepherdson–Mostowski collapsing theorem]. In particular, any such model has cardinality $\leq \beth_\alpha$ (= the cardinality of $R(\alpha)$, for α infinite).

This example is interesting, since it sets some limits on the possibility of extending the upward Löwenheim–Skolem theorem to the languages $\mathscr{L}_{\kappa\omega}$. Recall that a set of first-order ($=\mathscr{L}_{\omega\omega}$) sentences which has an infinite model or models of arbitrarily large finite cardinality, also has models of arbitrarily large cardinalities. Naively, we may try to generalize this to $\mathscr{L}_{\kappa\omega}$ by replacing "infinite" for "power $\geq \kappa$"; the preceding example shows that one ought to go as high as $\beth_\kappa$. We will see later (Section 3.2) that, in general, we ought to go considerably beyond this cardinal, although, in the important case in which $\kappa = \omega_1$, we need not do so.

Incidentally, questions of this type and many other model-theoretic problems concerning the languages $\mathscr{L}_{\kappa\omega}$ are of interest only when κ is a regular cardinal. For, if κ is singular and $\lambda \leq \kappa$, then the languages $\mathscr{L}_{\kappa\lambda}$ and $\mathscr{L}_{\kappa^+\lambda}$ have the same power of expression: Every $\mathscr{L}_{\kappa^+\lambda}$-formula can be converted into an $\mathscr{L}_{\kappa\lambda}$-formula with the same meaning by transforming, for example, a conjunction of κ formulas, say $\bigwedge_{i\in I}\phi_i$, into an iterated conjunction of $< \kappa$ formulas, $\bigwedge_{\alpha<\mathrm{cf}(\kappa)}\bigwedge_{i\in I_\alpha}\phi_i$, where $\langle I_\alpha | \alpha < \mathrm{cf}(\kappa)\rangle$ is a decomposition of I in $\mathrm{cf}(\kappa)$-many sets, each of power $< \kappa$. For more details on this, see Dickmann [1975, p. 85].

1.5. Examples From Algebra

We will only mention here that many widely used algebraic structures and notions can be axiomatized or treated in various other ways in the infinitary logics $\mathscr{L}_{\kappa\lambda}$ and $\mathscr{L}_{\infty\lambda}$ although they cannot be treated in the same way in first-order logic. Some outstanding examples of this are shown in Chapter XI.

For instance, common algebraic structures such as torsion groups, simple groups, characteristically simple groups, finitely generated algebras, archimedean fields, etc., can be axiomatized in $\mathscr{L}_{\omega_1\omega}$. For more on this, see Dickmann [1975, pp. 74, 78–82].

The most important application to date of infinitary model theory to algebra is a far-reaching extension of Ulm's theorem on the classification of abelian p-groups, due to Barwise–Eklof [1970]. Due attention is given to this application in Chapter XI, Section 4. The technique employed—the so-called back-and-forth method—is treated in detail in Section 4 of the present chapter, where other relevant algebraic examples (for instance, real closed fields) and the infinitary behaviour of some algebraic constructions are also discussed.

1.6. Examples From Topology

There are several possible ways of formalizing the notion of a topological space in a language. Here we shall regard them as structures of the form $\langle X \cup T, X, T, E\rangle$, each of which is isomorphic to a structure $\langle Y \cup \mathscr{T}, Y, \mathscr{T}, \in\rangle$, where $\mathscr{T}$ is a topology on the set Y and $\in$ is the standard membership relation. The corresponding vocabulary, v, will have unary predicates Pt (for "point"), Op (for "open"), and a binary predicate E.

The following topological notions, among others, can be expressed in this formalism:

The class of spaces with a countable base (= separable) is axiomatized by the conjunction of the following sentences of $\mathscr{L}_{\omega_1\omega_1}(v)$:

$$\forall xy[x\ E\ y \to \mathrm{Pt}(x) \wedge \mathrm{Op}(y)];$$

$$\forall yz\ \exists w[\mathrm{Op}(y) \wedge \mathrm{Op}(z) \to \mathrm{Op}(w) \wedge \forall u[u\ E\ w \leftrightarrow u\ E\ y \wedge u\ E\ z]],$$

$$(\exists v \upharpoonright \omega)\left[\bigwedge_{i\in\omega} \mathrm{Op}(v_i) \wedge \forall y\left[\mathrm{Op}(y) \leftrightarrow \forall x\left(x\ E\ y \to \bigvee_{i\in\omega} (x\ E\ v_i \wedge \forall u(u\ E\ v_i \to u\ E\ y))\right)\right]\right],$$

together with the (first-order) extensionality axiom for the relation E. Indeed, by extensionality, a model $\mathscr{X} = \langle X \cup T, X, T, E\rangle$ is isomorphic to $\langle X \cup \mathscr{T}, X, \mathscr{T}, \in\rangle$, where $\mathscr{T} = \{O_y \mid y \in T\}$ and $O_y = \{x \in X \mid \mathscr{X} \models x\ E\ y\}$, and the first three axioms guarantee that $\mathscr{T}$ is a topology on the set X.

Further topological notions axiomatizable in this formalism are given in

1.6.1 Exercise. (a) Write down an axiom for compact, separable spaces in the vocabulary v (but not necessarily in $\mathscr{L}_{\omega_1\omega_1}$).

(b) Show that the complete, separable metric (= Polish) spaces form a PC-class in the vocabulary v, for an appropriate $\mathscr{L}_{\kappa\lambda}$. [*Hint*: For each positive rational number, q, use a new binary predicate R_q with the meaning:

$$R_q(x, y) \leftrightarrow d(x, y) < q.]$$

1.7. Counterexamples From Topology

In all the preceding examples, a second-order quantifier which only needs to range over sets of some bounded cardinality has been axiomatized in an infinitary language. *A priori*, there is no reason for this to be true of other topological notions which have an unbounded second-order definition, such as those of topological space, compact space or, say Hausdorff or regular space. In Section 3.1, we shall apply the infinitary downward Löwenheim–Skolem theorem to show that these and many other classes of topological spaces are not characterizable by infinitary sentences. Indeed, they are not even RPC in $\mathscr{L}_{\kappa\lambda}(v)$, for any κ, λ; and, therefore, they are not RPC in $\mathscr{L}_{\infty\infty}(v)$ either. Among such classes we have the following:

Topological spaces.
Compact spaces.
Discrete spaces.
T_i spaces ($i = 0, \ldots, 5$).
Regular, completely regular, normal, completely normal spaces.
Compact and any of the preceding separation axioms.
Metrizable spaces.
Stone spaces, extremally disconnected spaces.
Complete uniform spaces.

Similar non-axiomatizability results hold for certain algebraic-topological notions such as topological groups, rings, modules, etc.

1.8. Further Counterexamples

(1) Variants of the general method used to prove the preceding results can be used to prove that the following second-order notions are not RPC in any infinitary language $\mathscr{L}_{\kappa\lambda}$:

Complete partial and linear orderings.
Complete lattices and complete distributive lattices.
Complete boolean algebras and complete atomic boolean algebras.
Completely distributive boolean algebras.

The general method used to prove these results as well as those of Section 1.7, is due to Cole–Dickmann [1972].

(2) Let us briefly reconsider the last example. Saying that a boolean algebra B is completely distributive involves, *a priori*, two different second-order assertions:

(a) (completeness): For every subset $X \subseteq B$, the supremum $\bigvee X$ exists;
(b) (complete distributivity): For every family $\{X_i \mid i \in I\}$ of subsets of B,

$$\bigwedge_{i \in I} (\bigvee X_i) = \bigvee_{f \in \prod_i X_i} \left(\bigwedge_{i \in I} f(i) \right),$$

and dually.

A result of Ball [1984] shows that only the first is genuinely a second-order assertion. Let us call a lattice *relatively completely distributive* if only condition (b) is required to hold, and this when all the indicated suprema and infima exist.

1.8.1 Proposition (Ball). *Relative complete distributivity is expressible in the first-order language of lattice theory.* $\square$

Ball proves similar results for other forms of (relative) infinite distributivity as well.

(3) As a last counterexample, we mention the class of free abelian groups, a class which is not axiomatizable by any class of $\mathscr{L}_{\infty\omega}$-sentences in the vocabulary for groups (this result is due to Kueker and Keisler). However, this class is PC in $\mathscr{L}_{\omega_1\omega}$. For details, see Dickmann [1975, pp. 379–384]. Further ramifications of this example are treated in Chapter XI, Section 4.

1.9. Omitting First-Order Types

In the introduction to Chapter VIII it is noted that $\mathscr{L}_{\omega_1\omega}$ can express in a single sentence the realization or omission of a first-order type—indeed, even of countably many of them. Likewise, $\mathscr{L}_{\kappa^+\omega}$ can express the realization or omission of up to κ first-order types.

An interesting result of Chang [1968c] shows that a kind of converse holds as well. To be precise, we have

1.9.1 Proposition. *Given a sentence ϕ of $\mathscr{L}_{\kappa^+\omega}(\tau)$, where τ has cardinality $\leq \kappa$, there is an enrichment τ' of τ, also of cardinality $\leq \kappa$, and a set S of power $\leq \kappa$ of $\mathscr{L}_{\omega\omega}(\tau')$-types such that for every structure $\mathfrak{A}$,*

$$\mathfrak{A} \models \phi \quad \textit{iff} \quad \text{there is an expansion } \mathfrak{A}' \text{ of } \mathfrak{A} \text{ to } \tau' \text{ such that } \mathfrak{A}' \text{ omits } S.$$

That is to say, the result asserts that "satisfaction in $\mathscr{L}_{\kappa^+\omega}$ is PC in the omission of up to κ first-order types".

Proof of Proposition 1.9.1. We proceed in two steps:

(1) We construct τ' and a particularly simple formula ϕ' of $\mathscr{L}_{\kappa^+\omega}(\tau')$ such that

$$\mathfrak{A} \models \phi \quad \text{iff} \quad \text{there is an expansion } \mathfrak{A}' \text{ of } \mathfrak{A} \text{ to } \tau'$$
$$\text{so that } \mathfrak{A}' \models \phi';$$

and then,

(2) We construct the required set S of types so that

$$\mathfrak{B} \models \phi' \quad \text{iff} \quad \mathfrak{B} \text{ omits } S,$$

for every τ'-structure $\mathfrak{B}$.

Construction (1). In order to get τ', we add to τ a new n-ary relation symbol R_σ for each subformula σ of ϕ with n freee variables. This is possible since each subformula of ϕ has finitely many variables. Since there are $\leq \kappa$ such subformulas, τ' has cardinality $\leq \kappa$. If σ has no free variables, then we regard R_σ as a propositional variable. If we do not like these (I personally do not!), then we take R_σ to be a unary predicate, being careful to add the clause

$$\forall x\, R_\sigma(x) \leftrightarrow \exists x\, R_\sigma(x),$$

where ϕ is constructed so that R_σ takes only two values in each model.

As we want R_σ to reflect the structure of σ, we prescribe:

(i) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \sigma(\mathbf{v}))$, if σ is atomic;
(ii) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \neg R_\psi(\mathbf{v}))$, if σ is $\neg\psi$;
(iii) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \bigwedge_{\xi<\kappa} R_{\psi_\xi}(\mathbf{v}))$, if σ is $\bigwedge_{\xi<\kappa} \psi_\xi$;
(iv) $\forall \mathbf{v}(R_\sigma(\mathbf{v}) \leftrightarrow \exists y\, R_\psi(\mathbf{v}, y))$, if σ is $\exists y\psi$.

If σ does not have free variables, replace (i) and (iv) by:

(i′) $\sigma \leftrightarrow \forall x\, R_\sigma(x)$,

and

(iv′) $\forall x\, R_\sigma(x) \leftrightarrow \exists y\, R_\psi(y)$.

Finally, we set

(v) $\forall x\, R_\phi(x)$.

Let ϕ be the conjunction of all these formulas; it is routine to check that (1) holds.

Construction (2). The set S contains a one-formula type for each axiom of the form (i), (ii), (iv) or (v): the negation of the axiom with the outer quantifiers erased. Furthermore, for each axiom of the form (iii), we throw into S the following κ types:

$$\{\neg(R_\sigma(\mathbf{v}) \rightarrow R_{\psi_\xi}(\mathbf{v}))\} \quad \text{for each} \quad \xi < \kappa,$$

and

$$\{R_{\psi_\xi}(\mathbf{v}) \wedge \neg R_\sigma(\mathbf{v}) \mid \xi < \kappa\}.$$

The verification of (2) is easy and is left as an exercise. □

2. *Basic Model Theory: Counterexamples*

We will now begin to examine the model-theoretical behaviour of the larger infinitary logics. As a first step, we will want to analyze the validity or the failure of the most important properties arising from first-order model theory. By Lindström's theorem (see Chapter III) we cannot expect too many of these properties to hold simultaneously in any one of our languages. In fact, while some of them fail very badly throughout the hierarchy of the larger infinitary logics, there is a reasonable generalization of some of the others.

The present section collects those model-theoretic properties which tend to fail in the infinitary context. From an organizational point of view, the more optimistic side of the picture is left for the next section, and the heart of the subject is postponed until the final section. In spite of the essentially negative tone of the panorama we have given here, not everything is lost. Occasionally, something can be salvaged by moderating the level of our ambitions.

2.1. *Completeness and Definability of Truth*

In the most general terms, the completeness problem for a language $\mathscr{L}$ is the question of knowing whether there is a Hilbert-type system of axioms and rules of inference so that for any set $\Sigma \cup \{\phi\}$ of $\mathscr{L}$-sentences the following are equivalent:

(a) ϕ holds in all models of Σ; and
(b) ϕ can be deduced, using the axioms and rules of the system, from the set Σ of premises.

Let us say that a system is *adequate for deductions* if the equivalence between (a) and (b) holds for all ϕ and Σ. It is well known that one can construct such systems for first-order logic. But this is not possible for $\mathscr{L}_{\omega_1\omega}$—and, *a fortiori*, for any of the larger infinitary logics $\mathscr{L}_{\kappa\lambda}$—even if the rules allow inferences from any number of premises smaller than κ, as does the rule:

$$\frac{\phi_0, \phi_1, \ldots, \phi_\xi, \ldots (\xi < \delta)}{\bigwedge_{\xi<\delta} \phi_\xi} \; (\delta < \kappa).$$

The impossibility of constructing such a system follows at once from the existence of sets Σ of $\mathscr{L}_{\omega_1\omega}$-sentences which have no model, but every countable subset of which does have a model (setting ϕ to be any false statement violates the implication (a) $\Rightarrow$ (b)). We give a simple example: The vocabulary has individual constants c_α, for all $\alpha < \omega_1$, and a unary function symbol F, and the set Σ is:

(i) $c_\alpha \neq c_\beta$ for $\alpha < \beta < \omega_1$,
(ii) F is an injection of the universe into $\{c_n | n \in \omega\}$.

The reader should consult Dickmann [1975, p. 136] for more details and other examples.

In view of this situation, one possible line of retreat is to ask only for an axiomatic system *adequate for proofs*, that is, such that a sentence ϕ is valid iff it is a theorem of the system. In other words, the equivalence between (a) and (b) above holds for arbitrary ϕ, but only for $\Sigma = \varnothing$ (equivalently, for any Σ of cardinality $< \kappa$, if we are dealing with the logic $\mathscr{L}_{\kappa\lambda}$). Deductive systems with this weaker property do exist for various $\mathscr{L}_{\kappa\lambda}$. The known results are as follows, and all are due to Karp [1964], who first examined the matter in that book.

2.1.1 Completeness Results. (1) $\mathscr{L}_{\omega_1\omega}$ *admits an axiomatic system adequate for proofs.* ☐

This system is a straightforward extrapolation of the usual deductive systems for first-order logic and is discussed in Chapter VIII, Section 3.2. Keisler [1971a, Lecture 4] gives a nice proof of the theorem.

(2) *For the logics listed below there are deductive systems of axioms and rules of inference adequate for proofs:*
   (a) *For $\mathscr{L}_{\kappa^+\lambda}$, whenever $\kappa^{<\lambda} = \kappa$ (the exponent denotes weak cardinal exponentiation); note that this includes the case $\mathscr{L}_{\kappa^+\omega}$.*
   (b) *For $\mathscr{L}_{\kappa\lambda}$, whenever,* (i) *$\kappa$ is strongly inaccessible, or* (ii) *κ is weakly inaccessible, λ is regular and $\mu^\nu < \kappa$ for all cardinals $\mu < \kappa$, $\nu < \lambda$. This applies, in particular, to $\mathscr{L}_{\kappa\omega}$ with κ (strongly or weakly) inaccessible.* ☐

These deductive systems are all built by taking as axioms the version for $\mathscr{L}_{\kappa\lambda}$ of the basic deductive system of (1), the axiom-schemes expressing certain infinite distributive laws, and a combination of rules of inferences expressing various principles of choice and of dependent choices. In particular, this means that as soon as we go beyond the countable level, non-trivial set-theoretical principles are needed to deal with the elementary infinitary predicate calculus.

These completeness results imply that the corresponding set Val($\mathscr{L}$) of valid $\mathscr{L}$-sentences lies low in an appropriate hierarchy of definable sets. The situation is quite analogous to that of first-order logic, where the Gödel completeness theorem implies that Val($\mathscr{L}_{\omega\omega}$) is recursively enumerable.

In order to make sense of this assertion, we need a coding machinery for $\mathscr{L}_{\kappa\lambda}$-formulas. The simplest and most natural *coding structure* is the structure $\langle H(\kappa), \in \upharpoonright H(\kappa)\rangle$, of all sets hereditarily of power less than κ. This reflects the idea of conceiving of $\mathscr{L}_{\kappa\lambda}$-formulas as set-theoretical objects, rather than as (linear) strings of symbols. We also need a *coding map*, that is, a one-one map from formulas into the coding structure which, moreover, should satisfy some requirements of simplicity (we want to avoid complications due to a bad choice of the coding map). A reasonable requirement is that the set of codes of formulas (that is, the range of the coding map) be a Δ-definable subset of the coding structure. Here, and in the rest of this section, "definable" means definable in the language of set theory (by a formula of the indicated complexity).

Fortunately, such simple coding maps do exist—the reader might try to construct one as an exercise. In any case, he will find such constructions described in detail in Dickmann [1975, pp. 412–413]; the reader should see also Keisler [1971a, pp. 40–41].

2.1.2. Definability Results. (1′) $\mathrm{Val}(\mathscr{L}_{\omega_1\omega})$ *is* Σ_1*-definable over* $\langle H(\omega_1), \in \upharpoonright H(\omega_1)\rangle$.

(2′) *In the cases* 2(a) *and* 2(b) (ii) *of Section* 2.1.1, $\mathrm{Val}(\mathscr{L}_{\kappa\lambda})$ *is* Σ_2*-definable over* $\langle H(\kappa), \in \upharpoonright H(\kappa)\rangle$.

(2″) *In the case* 2(b)(i) *of Section* 2.1.1, $\mathrm{Val}(\mathscr{L}_{\kappa\lambda})$ *is* Σ_1*-definable over* $\langle H(\kappa), \in \upharpoonright H(\kappa)\rangle$. □

The result given in (1′) is proven in Keisler [1971a, Lecture 9]; the result in (2′) can be proven by methods similar to those presented in lectures 8 and 9 of that book. It is the presence of the infinite distributive laws among the axioms which forces the use of Σ_2 ($=\exists\forall$) formulas in (2′). In (2″) the strong inaccessibility of κ makes it possible to bound the universal quantifier and, hence, to go down again to Σ_1-definability.

The positive results discussed above leave open the question whether a Hilbert-style system adequate for proofs exists for the language $\mathscr{L}_{\kappa\kappa}$, when κ is a successor cardinal. The impossibility of constructing such systems was shown by Scott in 1960, although it first appeared in print in Karp [1964, Chapter 14]. The method consists in proving that the set $\mathrm{Val}(\mathscr{L}_{\kappa^+\kappa^+})$ is not definable in any reasonable way over the coding structure $\langle H(\kappa^+), \in \upharpoonright H(\kappa^+)\rangle$. Since a completeness result would imply some kind of definability of $\mathrm{Val}(\mathscr{L}_{\kappa^+\kappa^+})$, this will suffice to establish that the logics $\mathscr{L}_{\kappa^+\kappa^+}$ do not admit a satisfactory complete axiomatization.

2.1.3 Scott's Undefinability Theorem. *Let E be a binary relation symbol. The set* $\mathrm{Val}(\mathscr{L}_{\kappa^+\kappa^+}(E))$ *is not definable over* $\langle H(\kappa^+), \in \upharpoonright H(\kappa^+)\rangle$ *by any formula of* $\mathscr{L}_{\kappa^+\kappa^+}(E)$. □

The method of proof is an adaptation of Tarski's argument proving that the set of sentences of first-order arithmetic valid in the standard model $\langle \mathbb{N}, +, \cdot, 0, 1\rangle$ is not first-order definable over the coding structure $\langle \mathbb{N}, +, \cdot, 0, 1\rangle$. However, there is one crucial difference: While, in the arithmetical case, the coding structure $\langle \mathbb{N}, +, \cdot, 0, 1\rangle$ is not characterizable up to isomorphism by any set of first-order sentences, in the infinitary case, the coding structure $\langle H(\kappa^+), \in \upharpoonright H(\kappa^+)\rangle$ is characterized up to isomorphism by the $\mathscr{L}_{\kappa^+\kappa^+}$-sentence of Example 1.4(2). This observation accounts for the additional strength of Scott's theorem—it applies to *all* valid sentences, not just the arithmetical ones. The proof of this theorem is given in Dickmann [1975, pp. 425–430].

2.2. *The Failure of Compactness*

Any reasonable analogue of the compactness theorem of first-order logic fails very badly in all infinitary languages. Let us begin with some simple examples.

2.2.1 Example (Propositional incompactness). Consider the following propositional formulas of $\mathscr{L}_{\kappa\omega}$, where κ is such that $\kappa \leq \lambda^{\mu}$ for some cardinals $\mu, \lambda < \kappa$:

$$(1)\qquad \bigwedge_{\xi<\mu} \bigvee_{\eta<\lambda} p_{\xi\eta};$$

$$(2)\qquad \neg \bigwedge_{\xi<\mu} p_{\xi f(\xi)} \quad \text{for each map} \quad f\colon \mu \to \lambda.$$

If (1) holds, let $f_0(\xi)$ be the smallest η which makes $p_{\xi\eta}$ true. Then $\bigwedge_{\xi<\mu} p_{\xi f_0(\xi)}$ is true, that is, (2) fails for $f = f_0$. But there is a model for all sentences of form (2), for we can make $p_{\xi\eta}$ false for all ξ, η; and, if we omit just one sentence of form (2), then the remaining sentences (of both forms) also have a model. Observe here that if the omitted sentence is given by the map f_1, we can make $p_{\xi\eta}$ true if $\eta = f_1(\xi)$ and false otherwise. □

This example takes care of the case when κ is a successor cardinal ($\mu = \lambda =$ the predecessor of κ). However, it does not exclude the possibility of compactness holding for a set of $\mathscr{L}_{\kappa\omega}$-sentences *of power exactly* κ (unless some set-theoretical assumption is made). Consider then the following:

2.2.2 Example (An incompact set of $\mathscr{L}_{\kappa\omega}$-sentences of power κ, when κ is a singular cardinal). Let κ be the limit of the sequence $\langle \gamma_\xi \mid \xi < \mathrm{cf}(\kappa)\rangle$ of smaller ordinals, and set

$$(3)\qquad < \quad \text{is a total ordering,}$$

$$(4)\qquad \bigvee_{\xi<\mathrm{cf}(\kappa)} \forall x\left[P(x) \to \bigvee_{\eta<\gamma_\xi} \phi_\eta^P(x)\right],$$

$$(5)\qquad \exists x(\phi_\eta(x) \wedge P(x)) \quad \text{for } \eta < \kappa,$$

where P is an additional predicate, ϕ_η are the formulas of Example 1.2(2), and the superscript denotes relativization to P.

In any model $\mathfrak{A}$ of (3) through (5), $P^{\mathfrak{A}}$ contains elements determining an initial section (of $\mathfrak{A}$) of any given order type $< \kappa$. Hence, $P^{\mathfrak{A}}$ has power $\geq \kappa$. But (4) above asserts that for some $\xi < \mathrm{cf}(\kappa)$, the subset $P^{\mathfrak{A}}$ is well-ordered in type $\leq \gamma_\xi$. Hence, $P^{\mathfrak{A}}$ has power $< \kappa$, a contradiction. Thus, (3) through (5) do not have a model. We leave to the reader to construct a model of (3) and (4) and an arbitrary subset of (5) of power $< \kappa$. □

2.2.3 Example (An incompact set of $\mathscr{L}_{\kappa\omega}$-sentences of power κ, when κ is a successor cardinal). Let $\kappa = \lambda^+$. Consider then the language containing the symbols P, $<$ (as before) and a new binary relation symbol, F. The required set of sentences consists of (3) above and

(6) F is a one–one function with domain containing $\{x \mid P(x)\}$,

(7) $$\forall x(P(x) \to \bigvee_{\eta<\lambda} \phi_\eta(x)),$$

(8) $$\exists x(P(x) \wedge \forall y(F(x, y) \to \phi_\eta(y))) \quad \text{for} \quad \eta < \kappa.$$

Let $\mathfrak{A}$ be a model of these sentences. By (7), each element of $P^{\mathfrak{A}}$ determines (in $\mathfrak{A}$) a section of type $< \lambda$. Hence, $P^{\mathfrak{A}}$ has cardinality $\leq \lambda < \kappa$. But (8) asserts that for every $\eta < \kappa$ there is an element of Range($F^{\mathfrak{A}} \restriction P^{\mathfrak{A}}$) which determines a section of type η. Hence, Range($F^{\mathfrak{A}} \restriction P^{\mathfrak{A}}$) has cardinality $\geq \kappa$; and, by (6), $P^{\mathfrak{A}}$ also has cardinality $\geq \kappa$. As an exercise, the reader might try to construct a model for (3), (6), and (7) as well as any subset of (8) of power $< \kappa$. Should this not be successful, he can fall back on Dickmann [1975, pp. 163–164]. ▯

The Failure of Compactness for Inaccessible Cardinals

The preceding examples show that the only possible chance for $\mathscr{L}_{\kappa\lambda}$ to be compact is that κ be (at least) weakly inaccessible. For some time, there was hope that a restricted form of compactness could hold in $\mathscr{L}_{\kappa\kappa}$ for at least some reasonably sized inaccessible cardinals (for example, for the first such κ). In a celebrated paper W. Hanf [1964] crushed any such hope. He showed that the compactness theorem for sets of $\mathscr{L}_{\kappa\kappa}$-sentences of size κ is false whenever κ belongs to any one of a whole panoply of ever increasing classes of (strongly) inaccessible cardinals.

Let us briefly describe the extent and the significance of Hanf's results. He considers the inaccessible cardinals which belong to some member of a certain increasing transfinite sequence $\langle M^\alpha \mid \alpha \in \mathrm{ON}\rangle$ of classes of cardinals. In a certain sense, $M^{\alpha+1}$ is "constructibly defined" from M^α. This method of construction of larger and larger classes of inaccessible cardinals was invented by Mahlo in 1911–1913. Hanf proves:

2.2.4 Theorem. *If $\kappa \in M^\alpha$ for some $\alpha < \kappa$, then $\mathscr{L}_{\kappa\kappa}$ contains a set of sentences of power κ for which compactness fails.* ▯

In order to give an idea of the comprehensiveness of the classes M^α, we will consider the following hierarchy of inaccessible cardinals: We will say that κ is *hyperinaccessible of type* 1 if it is inaccessible and there are κ inaccessibles below κ. In other words, if $\langle \theta_\alpha \mid \alpha \in \mathrm{ON}\rangle$ enumerates the inaccessibles in increasing order, then the hyperinaccessibles of type 1 are the θ_α's such that $\theta_\alpha = \alpha$. We can iterate this definition into the transfinite by saying that κ is *hyperinaccessible of type* $\alpha + 1$ iff it is hyperinaccessible of type α and there are κ hyperinaccessibles of type α below κ, and taking κ to be *hyperinaccessible of type* α, for α limit > 0, iff it is hyperinaccessible of type β for all $\beta < \alpha$. The hyperinaccessibles of type 0 are simply the inaccessibles. Now, *all the hyperinaccessibles* κ *of some type* $\alpha < \kappa$ *are in the first Mahlo class* M^1.

As a matter of fact, it is impossible to find inaccessible cardinals outside the classes $M^1, M^2, \ldots$, unless a very powerful axiom of infinity is added to the axioms of ZFC, namely:

> "Every normal function has a regular fixed point".

The reader may try to convince himself that this is a very powerful axiom indeed, by deriving, as an exercise, the following consequences:

> "The class of inaccessible cardinals is cofinal with the ordinal numbers",

and also:

> "For every ordinal α, the class of hyperinaccessible cardinals of type α is cofinal with the class of all ordinal numbers."

Hanf's counterexample can be adapted so as to show the incompactness of inaccessible cardinals belonging to even larger classes. Thus, if we set

$$\kappa \in M^{\Delta} \quad \text{iff} \quad \kappa \in \bigcup_{\alpha < \kappa} M^{\alpha}$$

(so that Theorem 2.2.4 holds for all $\kappa \in M^{\Delta}$), we can start iterating the operation M on the class M^{Δ} again to get $(M^{\Delta})^{\Delta} = M^{(\Delta, 2)}$, then $M^{(\Delta, 3)}, \ldots$. We obtain, then

2.2.5 Theorem. *If* $\kappa \in M^{(\Delta, \alpha)}$ *for some* $\alpha < \kappa$, *then the compactness theorem fails for some set of* $\mathscr{L}_{\kappa\kappa}$*-sentences of size* κ. □

The process of diagonalization sketched above can be iterated without an end, producing larger and larger classes of inaccessible cardinals κ for which $\mathscr{L}_{\kappa\kappa}$ will be incompact. However, this does not suffice to prove that compactness fails *for all* $\mathscr{L}_{\kappa\kappa}$. But the cardinals κ for which $\mathscr{L}_{\kappa\kappa}$ does have compactness (for sets of sentences of size κ)—the so-called *weakly compact* cardinals, if any—must be of a size defying imagination. Incidentally, observe that we will not be any better off by reducing the length of quantifications; the compactness property for sets of $\mathscr{L}_{\kappa\kappa}$-sentences of size κ is equivalent to the same property for sets of $\mathscr{L}_{\kappa\omega}$-sentences of size κ (see Dickmann [1975, p. 185]).

After Hanf's work the study of the compactness property for infinitary logic departed the realm of the model-theorist to enter that of the set-theorist, or rather—that of the mystic.

2.2.6 Comment on Bibliography. There is a vast literature concerning weakly compact cardinals. The equivalences of this notion with many other properties appear in Dickmann [1975, Chapter 3, Section 3C] where we tried to adhere to the model-theoretic aspect of the question, in Drake [1974, Chapter 10, Section 2], and in Keisler–Tarski [1964]. The fastest road to weak compactness is via the equivalent notion of Π_1^1-indescribability. This road can be followed in Drake [1974, Chapters 9, 10], which also contains a thorough study of the hierarchies of Π_m^n- and Σ_m^n-indescribable cardinals; Devlin [1975] is also devoted to this subject. The most important classes of large cardinals studied to date—Ramsey, measurable, compact, etc.—all find their place in this hierarchy.

The reader wanting to proceed along the set-theoretic road is urged to consult Drake's excellent book [1974] and the very readable and witty survey paper of Kanamori–Magidor [1978]. Devlin [1975] and Boos [1975] are also good sources of information.

2.3. *Interpolation and (Beth-) Definability*

The interpolation and (Beth) definability properties of a logic have been defined in Chapter II, Sections 1 and 7. Among the infinitary languages, these properties hold only for $\mathscr{L}_{\omega_1\omega}$ and the countable admissible fragments of $\mathscr{L}_{\infty\omega}$ (see Chapter VIII, Sections 3.3 and 6.3.8). They fail rather badly for all the others, as we shall soon see.

In order to capture the exact extent of this failure (and then save what is left), we will consider *relative* notions of interpolation and definability. A logic $\mathscr{L}'$ *allows interpolation for* $\mathscr{L}$ if every valid sentence $\sigma_0 \to \sigma_1$ of $\mathscr{L}$ has an interpolant in $\mathscr{L}'$. Here, the definition of interpolant is as usual, and we are implicitly assuming that $\mathscr{L}'$ is at least as strong as $\mathscr{L}$. Modifying in a similar way the definition of the (Beth) definability property (see Chapter II, Definition 1.2.4(i)) we arrive at the notion of $\mathscr{L}'$ *allows (Beth) definability for* $\mathscr{L}$. The usual proof of "interpolation implies definability" also works in this relativized context.

We will begin with a simple example which due to Malitz [1971] and which shows:

2.3.1 Example (The failure of the interpolation property in $\mathscr{L}_{\kappa\omega}$, for $\kappa > \omega_1$). Furthermore, we will exhibit a valid $\mathscr{L}_{\kappa\omega}$-sentence $\sigma_0 \to \sigma_1$ which does not have an interpolant in any language $\mathscr{L}_{\infty\lambda}$ with $\lambda^+ < \kappa$. To this end, let

$$\tau = \{c_\alpha | \alpha < \lambda^+\},$$

$$\sigma_0 : \forall v \bigvee_{\alpha < \lambda} (v = c_\alpha),$$

$$\sigma_1 : \bigvee_{\lambda < \beta < \gamma < \lambda^+} (c_\beta = c_\gamma).$$

Since any model of σ_0 has power $\leq \lambda$, the sentence $\sigma_0 \to \sigma_1$ is valid. An interpolant σ for this implication has to be a pure equality formula. Then, $\models \sigma_0 \to \sigma$ would imply that σ holds in some structure of power λ, and by Fact 1.1.1 of this chapter, σ would hold in all structures of power $\geq \lambda$. From $\models \sigma \to \sigma_1$, the same would be true of σ_1, which is obviously absurd. □

This counterexample shows that in order to get a relative interpolation result for $\mathscr{L}_{\kappa\omega}$, we must allow interpolants having quantifiers of length close to κ. As a matter of fact, there are some positive results in this direction:

2.3.2 Theorem. (a) (Malitz [1971]). *If κ is regular, then $\mathscr{L}_{(2^{<\kappa})^+\kappa}$ allows interpolation for $\mathscr{L}_{\kappa\omega}$.*

(b) (Chang [1971]). *If $\mathrm{cf}(\kappa) = \omega$, then $\mathscr{L}_{(2^{<\kappa})^+\kappa}$ allows interpolation for $\mathscr{L}_{\kappa^+\omega}$. In particular, we have*

(c) *For any infinite κ, $\mathscr{L}_{(2^\kappa)^+\kappa^+}$ allows interpolation for $\mathscr{L}_{\kappa^+\omega}$.*

(d) *If κ is strongly inaccessible, then $\mathscr{L}_{\kappa\kappa}$ allows interpolation for $\mathscr{L}_{\kappa\omega}$.*

(e) *If κ is a strong limit cardinal of cofinality ω, then $\mathscr{L}_{\kappa^+\kappa}$ allows interpolation for $\mathscr{L}_{\kappa^+\omega}$.* □

Of course, corresponding statements for relative definability follow automatically. Counterexample 2.3.1 leaves open the possibility of an interpolation result for $\mathscr{L}_{\kappa^+\omega}$ in $\mathscr{L}_{\infty\kappa}$, for successor κ. Since a counterexample to (relative) definability is also a counterexample to (relative) interpolation, Example 2.3.12 below will dispose of this possibility also. Moreover, it will also show that the preceding theorem is best possible as far as the length of quantifications is concerned.

In order to deal with the definability property, we need some information about

The Preservation of Infinitary Equivalence by Sum and Product Operations

We state here a few results which we will use, without touching the wider chapter of model theory which deals with generalized product operations. We consider only binary operations # which assign to each pair of (possibly disjoint) structures $\mathfrak{A}$, $\mathfrak{B}$, with (possibly distinct) vocabularies τ_1, τ_2, a new structure $\mathfrak{A} \# \mathfrak{B}$ with a vocabulary τ. We have in mind—and will use—the following:

2.3.3 Example. (1) *Disjoint Sum* (simple cardinal sum; disjoint union). Here $\tau_1 = \tau_2 = \tau$ is a vocabulary containing only relation symbols, and $\mathfrak{A}$, $\mathfrak{B}$ are disjoint. The operation is defined by:

$$|\mathfrak{A} \oplus \mathfrak{B}| = |\mathfrak{A}| \cup |\mathfrak{B}|,$$

$$R^{\mathfrak{A}\oplus\mathfrak{B}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}} \quad \text{for each } R \in \tau.$$

(2) *Full Cardinal Sum* (extended cardinal sum). Here τ_1, τ_2 do not contain function symbols. By renaming, we can also assume that τ_1, τ_2 are disjoint. The vocabulary τ contains $\tau_1 \cup \tau_2$ and two extra unary predicates P_1, P_2. $\mathfrak{A}$, $\mathfrak{B}$ are supposed to be disjoint. The operation is defined by:

$$|\mathfrak{A} + \mathfrak{B}| = |\mathfrak{A}| \cup |\mathfrak{B}|,$$

$$P_1^{\mathfrak{A}+\mathfrak{B}} = |\mathfrak{A}|,$$

$$P_2^{\mathfrak{A}+\mathfrak{B}} = |\mathfrak{B}|,$$

and, for $R \in \tau_1 \cup \tau_2$, $R^{\mathfrak{A}+\mathfrak{B}}$ is $R^{\mathfrak{A}}$ or $R^{\mathfrak{B}}$, depending on whether $R \in \tau_1$ or $R \in \tau_2$.

(3) *Direct Product*. This is a well-known construction. ☐

The preservation result which we shall need is due to Malitz [1971] and takes the following form:

2.3.4 Theorem. *Let # denote any one of the operations on structures described in Example* 2.3.3. *Then the following is true for any cardinal λ:*

$(\dagger)_\lambda$ *for every $\kappa \geq \lambda$ and every sentence σ of $\mathscr{L}_{\kappa\lambda}(\tau)$, there is a cardinal $\theta \geq \kappa$ such that, for all structures $\mathfrak{A}_i$ and $\mathfrak{B}_i$ with vocabulary τ_i $(i = 1, 2)$.*

$$\mathfrak{A}_1 \equiv_{\theta\lambda} \mathfrak{B}_1 \quad \textit{and} \quad \mathfrak{A}_2 \equiv_{\theta\lambda} \mathfrak{B}_2 \textit{ imply}$$

$$\mathfrak{A}_1 \# \mathfrak{A}_2 \models \sigma \quad \textit{iff} \quad \mathfrak{B}_1 \# \mathfrak{B}_2 \models \sigma. \quad ☐$$

Note that this result immediately implies the following

2.3.5 Corollary. *The operations of disjoint sum, full cardinal sum and direct product preserve $\mathscr{L}_{\infty\lambda}$-equivalence.* ☐

It is very easy to prove this corollary, using the back-and-forth criterion for $\mathscr{L}_{\infty\lambda}$-equivalence given in Theorem 4.3.1 below. The proof of Theorem 2.3.4 is syntactical (see Dickmann [1975, Chapter 5, Section 2]) and gives additional information such as, for example, that the cardinal θ in $(\dagger)_\kappa$ is of the order 2^κ. This yields:

2.3.6 Corollary. *The operations of disjoint sum, full cardinal sum, and direct product preserve $\mathscr{L}_{\kappa\lambda}$-equivalence, if κ is strongly inaccessible.* ☐

This result has an interesting converse which is due to Malitz [1971], namely,

2.3.7 Theorem. *If $\mathscr{L}_{\kappa\omega}$-equivalence is preserved by any one of the operations of Example* 2.3.3, *then κ is strongly inaccessible.* ☐

For a more detailed account of preservation results of this type, see Dickmann [1975, Chapter 5, Section 2].

The Beth-Definability Property in Infinitary Logic

We begin this discussion with the following result.

2.3.8 Example (Failure of the definability property in $\mathscr{L}_{\omega_1\omega_1}$). We shall exhibit an $\mathscr{L}_{\omega_1\omega_1}$-sentence which implicitly defines a relation which is itself not explicitly defined by any formula of $\mathscr{L}_{\infty\infty}$. This drastic counterexample shows that there is no definability result for infinite-quantifier logics relative to any other logic of the same sort. Its basic ingredients are that $\mathscr{L}_{\omega_1\omega_1}$ expresses well-order and that there is at most one isomorphism between well-ordered structures.

Let σ be the $\mathscr{L}_{\omega_1\omega_1}$-sentence on a unary predicate U and two-binary predicates F, $<$, which says:

(a) $< \restriction U$ is a (non-empty) well-ordering,
(b) $< \restriction \neg U$ is a (non-empty) well-ordering,
(c) F is an isomorphism of $\langle U, < \restriction U\rangle$ onto $\langle U, < \restriction \neg U\rangle$.

Here $\neg U$ stands for $\{x \mid \neg U(x)\}$.

Note that $<$ may not be an order of the universe, and that if for (isomorphic) well-orders $\langle A, <\rangle \cong^f \langle B, \prec\rangle$ we set:

$$\mathfrak{A} = \langle A, A, <\rangle, \qquad \mathfrak{B} = \langle B, \varnothing, \prec\rangle,$$

then $\langle \mathfrak{A} \oplus \mathfrak{B}, f\rangle$ is a model of σ.

As the isomorphism between two well-ordered sets is unique if it exists, it follows that the relation $F(\cdot, \cdot)$ is *implicitly* defined by σ.

Now assume that there is a formula $\phi(\cdot\,, \cdot)$ in $\mathscr{L}_{\infty\infty}(U, <)$ *explicitly* defining the relation $F(\cdot\,, \cdot)$ relative to σ. If σ^* denotes the substitution of ϕ for F in σ, then σ^* is in $\mathscr{L}_{\kappa\kappa}(U, <)$, for some $\kappa \geq \omega_1$, and for disjoint, non-empty well-orders $\langle A, <\rangle$, $\langle B, \prec\rangle$, and $\mathfrak{A}$, $\mathfrak{B}$ defined as above, we have:

(*) $\quad \mathfrak{A} \oplus \mathfrak{B} \models \sigma^*$ implies that $\phi^{\mathfrak{A}\oplus\mathfrak{B}}(\cdot\,, \cdot)$ is an isomorphism between $\langle A, <\rangle$ and $\langle B, \prec\rangle$.

Applying Theorem 2.3.4 to the sentence σ^* gives a cardinal $\theta \geq \kappa$ such that $(\dagger)_\kappa$ holds. Consider the following structures:

$$\langle A_1, <_1\rangle = \langle B_1, \prec_1\rangle = \langle 2^{2^\theta}, \in\rangle,$$

$$\langle A_2, <_2\rangle = \text{a disjoint copy of } \langle A_1, <_1\rangle,$$

and, using the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\theta\theta}$ (Theorem 3.1.2 below), get $\langle B_2, \prec_2\rangle \prec_{\theta\theta} \langle A_2, <_2\rangle$ such that B_2 has cardinality 2^θ. Since $\mathfrak{A}_1 \oplus \mathfrak{A}_2 \models \sigma^*$ by $(\dagger)_\kappa$ it follows that $\mathfrak{B}_1 \oplus \mathfrak{B}_2 \models \sigma^*$, and by (*) above we conclude that $\langle B_1, \prec_1\rangle$ is isomorphic to $\langle B_2, \prec_2\rangle$, which is absurd for cardinality reasons. $\square$

Gregory [1974] settled the question for finite-quantifier languages beyond $\mathscr{L}_{\omega_1\omega}$ in a negative way by the use of rigid structures—that is, structures having the identity as their only automorphism—instead of well-ordered sets. Extending certain counterexamples due to Morley and Tait (see Section 4.3.6), he proved

2.3.9 Theorem. *Let κ be a regular uncountable cardinal. There are rigid structures $\mathfrak{A}$, $\mathfrak{B}$ of power κ in a purely relational vocabulary involving $\leq \kappa$ symbols, such that*

$$\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B} \quad \textit{and} \quad \mathfrak{A} \not\cong \mathfrak{B}. \quad \square$$

Any such example has the following special feature:

2.3.10 Lemma. *Let $\mathfrak{A}$, $\mathfrak{B}$ be structures with the properties of the preceding theorem. Then $\mathfrak{B}$ contains an $\mathscr{L}_{\infty\kappa}$-undefinable element, that is, an element b such that for each $\mathscr{L}_{\infty\kappa}$-formula $\phi(x)$,*

$$\mathfrak{B} \models \phi[b] \quad \textit{and} \quad \mathfrak{B} \models \exists v(v \neq b \wedge \phi(v)).$$

Proof. Let τ be the vocabulary of $\mathfrak{B}$. If the conclusion is false, then every $b \in |\mathfrak{B}|$ is definable by an $\mathscr{L}_{\infty\kappa}(\tau)$-formula, say $\phi_b(x)$. Let ψ be the conjunction of

(i) $\forall v \bigvee_{b \in |\mathfrak{B}|} \phi_b(v)$,
(ii) $\exists v_1 \ldots v_n[\bigwedge_{i=1}^{n} \phi_{b_i}(v_i) \wedge \sigma(v_1, \ldots, v_n)]$, for each $b_1, \ldots, b_n \in |\mathfrak{B}|$ and each atomic or negated atomic formula σ such that $\mathfrak{B} \models \sigma[b_1, \ldots, b_n]$.

Obviously ψ is in $\mathscr{L}_{\infty\kappa}(\tau)$ and $\mathfrak{B} \models \psi$. Since $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B}$, then $\mathfrak{A} \models \psi \wedge \bigwedge_{b \in |\mathfrak{B}|} \exists!\, v\phi_b(v)$. And this implies that the map of $\mathfrak{B}$ into $\mathfrak{A}$ defined by

$$b \mapsto \text{“the unique element of } \phi_b\text{”}$$

is an isomorphism, a contradiction. $\square$

We shall also need the result given in

2.3.11 Lemma. *An element $b \in |\mathfrak{B}|$ is not $\mathscr{L}_{\infty\kappa}$-definable iff there is $a \in |\mathfrak{B}|$, $a \neq b$, such that $\langle \mathfrak{B}, a \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b \rangle$.*

Proof. The implication from right to left is obvious. For the other direction, by Theorem 4.4.6 below, there is an $\mathscr{L}_{\infty\kappa}$-sentence $\phi_{\langle \mathfrak{B}, b \rangle} = \phi(c)$ involving a new individual constant c, such that for any structure $\mathfrak{A}$ of the appropriate vocabulary and any $a \in |\mathfrak{A}|$,

$$(*) \qquad \langle \mathfrak{A}, a \rangle \models \phi(c) \quad \text{iff} \quad \langle \mathfrak{B}, b \rangle \equiv_{\infty\kappa} \langle \mathfrak{A}, a \rangle.$$

Since b is not $\mathscr{L}_{\infty\kappa}$-definable, then $\mathfrak{B} \models \neg\exists!\, v\phi(v)$, that is, $\mathfrak{B} \models \phi[a]$ for some $a \neq b$. And, by $(*)$, $\langle \mathfrak{B}, a \rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b \rangle$. $\square$

Now we can give Gregory's counterexample:

2.3.12 Example (Failure of the Beth-definability property for $\mathscr{L}_{\kappa^+\omega}$, when $\kappa > \omega_1$). In fact, we will show that $\mathscr{L}_{\infty\kappa}$ does not allow definability for $\mathscr{L}_{\kappa^+\omega}$.

Let $\mathfrak{B}$ be a structure (with vocabulary τ) meeting the conditions of Theorem 2.3.9. By Lemma 2.3.10, let $b_0 \in |\mathfrak{B}|$ be an $\mathscr{L}_{\infty\kappa}(\tau)$-undefinable element. Let τ'' contain the vocabulary τ' appropriate for full cardinal sums of structures with vocabularies $\tau_1 = \tau \cup \{c_b | b \in |\mathfrak{B}|\}$ and $\tau_2 = \tau$ (see Example 2.3.3(2)), and a new binary predicate F. Consider the conjunction σ of the following $\mathscr{L}_{\kappa^+\omega}(\tau'')$-sentences:

(a) $\forall v(P_1(v) \leftrightarrow \bigvee_{b\in|\mathfrak{B}|} (v = c_b))$,
(b) the elementary diagram of $\mathfrak{B}$ (in the vocabulary τ_1),
(c) F is an isomorphism between $\langle\{x | P_1(x)\}, \tau\rangle$ and $\langle\{x | P_2(x)\}, \tau\rangle$.

Thus, if $\mathfrak{B}'$ is a disjoint copy of $\mathfrak{B}$ and f denotes the copying isomorphism, we must have

$$(*) \qquad \langle \mathfrak{C}, f\rangle \models \sigma,$$

where $\mathfrak{C} = \langle \mathfrak{B}, b\rangle_{b\in|\mathfrak{B}|} + \mathfrak{B}'$.

Since $\mathfrak{B}$ is rigid, the sentence σ implicitly defines the relation $F(\cdot,\cdot)$. Assume then that there is a formula $\phi(\cdot,\cdot)$ of $\mathscr{L}_{\infty\kappa}(\tau')$ which explicitly defines this relation relative to σ:

$$(**) \qquad \sigma \models \forall xy[F(x, y) \leftrightarrow \phi(x, y)].$$

Since b_0 is $\mathscr{L}_{\infty\kappa}(\tau)$-undefinable, by Lemma 2.3.11 there is $b_1 \in |\mathfrak{B}|$, $b_1 \neq b_0$, such that:

$$\langle \mathfrak{B}, b_0\rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b_1\rangle;$$

and, since f is an isomorphism, we also have

$$\langle \mathfrak{B}', f(b_0)\rangle \equiv_{\infty\kappa} \langle \mathfrak{B}', f(b_1)\rangle.$$

As $\mathscr{L}_{\infty\kappa}$-equivalence is preserved under full cardinal sums (Corollary 2.3.5), we conclude that

$$\langle \mathfrak{B}, b\rangle_{b\in|\mathfrak{B}|} + \langle \mathfrak{B}', f(b_0)\rangle \equiv_{\infty\kappa} \langle \mathfrak{B}, b\rangle_{b\in|\mathfrak{B}|} + \langle \mathfrak{B}', f(b_1)\rangle.$$

In the terminology introduced above, this can be rephrased as:

$$(***) \qquad \langle \mathfrak{C}, f(b_0)\rangle \equiv_{\infty\kappa} \langle \mathfrak{C}, f(b_1)\rangle.$$

Now, (*) and (**) imply:

$$\mathfrak{C} \models \phi[b_0, f(b_0)],$$

which, by (***), yields

$$\mathfrak{C} \models \phi[b_0, f(b_1)].$$

But, as (**) and (c) imply that $\phi(\cdot, \cdot)$ is a function, we can conclude $f(b_0) = f(b_1)$; and, hence, $b_0 = b_1$. This is a contradiction. □

This counterexample does not settle the following questions, which, to our knowledge are still

2.3.13 Open Problems. (a) Can one prove in ZFC that $\mathscr{L}_{\kappa^+\kappa}$ (or $\mathscr{L}_{\kappa^+\kappa^+}$) allows interpolation (definability) for $\mathscr{L}_{\kappa^+\omega}$, whenever $\mathrm{cf}(\kappa) = \omega$ and $\kappa > \omega$?

(b) Does $\mathscr{L}_{\infty\kappa}$ allow interpolation (definability) for $\mathscr{L}_{\kappa^+\omega}$ when κ is a singular cardinal of uncountable cofinality? □

In connection with question (a) above, Gregory [1974, p. 22] mentions that Friedman had shown that $\mathscr{L}_{\kappa^+\kappa^+}$ does not allow definability for $\mathscr{L}_{\kappa^+\omega}$, whenever $\mathrm{cf}(\kappa) > \omega$.

3. *Basic Model Theory: The Löwenheim–Skolem Theorems*

In this section we will deal with the infinitary analogs of the Löwenheim–Skolem theorems. These basic results of first-order model theory do admit reasonable generalizations. However, in the case of the upward theorem, these are neither naive nor immediate.

3.1. The Downward Löwenheim–Skolem Theorem

This is one of the few results from first-order model theory which generalizes practically without restrictions to the infinitary languages $\mathscr{L}_{\kappa\lambda}$—although not to $\mathscr{L}_{\infty\lambda}$. Since the proof is a straightforward generalization of the first-order argument, we will only state the results and provide some counterexamples and applications.

The following is a very general form of the theorem; it implies all the known forms and is useful in its own right.

3.1.1 Main Theorem. *Let $\mathfrak{B}$ be an infinite structure with vocabulary τ, $X \subseteq |\mathfrak{B}|$, and Γ a set of $\mathscr{L}_{\kappa\lambda}(\tau)$-formulas closed under subformulas. Furthermore, let:*

ρ = *the supremum of $\aleph_0$ and the number of free variables of formulas in Γ,*

μ = *an arbitrary cardinal ≥ 2,*

ν = *an arbitrary cardinal $\geq \rho$.*

Assume that one of the following alternatives hold:

(1) $$\max\{\overline{\overline{X}}, \overline{\overline{\tau}}, \overline{\overline{\Gamma}}\} \leq \mu^{\nu} \leq \overline{\overline{\mathfrak{B}}},$$

or

(2) *ρ is larger than the number of variables of formulas in Γ, $\rho < \mathrm{cf}(\nu)$ and*

$$\max\{\overline{\overline{X}}, \overline{\overline{\tau}}, \overline{\overline{\Gamma}}\} \leq \mu^{<\nu} \leq \overline{\overline{\mathfrak{B}}}.$$

Then there is a structure $\mathfrak{A}$ such that:

(a) *$X \subseteq |\mathfrak{A}|$ and $\mathfrak{A} \subseteq \mathfrak{B}$;*
(b) *For every $\phi \in \Gamma$ and every assignment g for ϕ in $\mathfrak{A}$,*

$$\mathfrak{A} \models \phi[g] \Leftrightarrow \mathfrak{B} \models \phi[g];$$

(c) *$\overline{\overline{\mathfrak{A}}} = \mu^{\nu}$ in case (1), and $\overline{\overline{\mathfrak{A}}} = \mu^{<\nu}$ in case (2).* □

Observe that condition (b) is stronger than $\mathfrak{A} \prec_{\Gamma} \mathfrak{B}$, which for an arbitrary set of formulas Γ, only requires the implication from left to right to hold. As a consequence, we have.

3.1.2 Corollary. *Let $\mathfrak{B}$, τ, X be as in Theorem* 3.1.1 *and assume that*

$$\max\{\overline{\overline{X}}, \overline{\overline{\tau}}\} \leq \lambda = \lambda^{\kappa} \leq \overline{\overline{\mathfrak{B}}}.$$

Then there is a structure $\mathfrak{A}$ such that $X \subseteq |\mathfrak{A}|$, $\mathfrak{A} \prec_{\kappa\kappa} \mathfrak{B}$ and $\overline{\overline{\mathfrak{A}}} = \lambda$. If, in addition, κ is regular, then the same conclusion follows from the weaker assumption $\lambda = \lambda^{<\kappa}$. Under the GCH, *and if $\kappa \leq \mathrm{cf}(\lambda)$, the assumption $\lambda = \lambda^{\kappa}$ is superfluous.*

Proof. Set Γ = the set of all $\mathscr{L}_{\kappa\kappa}(\tau)$-formulas and, then perform the necessary cardinal computations. □

3.1.3 Corollary. *Let $\lambda < \kappa$ be regular cardinals satisfying that*

$$\mu < \kappa \text{ and } \nu < \lambda \text{ imply } \mu^{\nu} < \kappa.$$

Then every sentence of $\mathscr{L}_{\kappa\lambda}$ which has a model, also has a model of power $< \kappa$. If, in addition, $\overline{\overline{\tau}} < \kappa$, then the latter can be chosen to be a (first-order) elementary substructure of the former.

Proof. Set $\Gamma =$ the set of all subformulas of the given sentence. For the last assertion, we also include in Γ all $\mathscr{L}_{\omega\omega}(\tau)$-formulas. $\square$

3.1.4 Corollary. *Let κ be a regular uncountable cardinal. Then any $\mathscr{L}_{\kappa\omega}$-sentence having a model, has a model of power $< \kappa$.* $\square$

3.1.5 Corollary. *If κ is strongly inaccessible $> \omega$, then every sentence of $\mathscr{L}_{\kappa\kappa}$ having a model, has a model of power $< \kappa$.* $\square$

The smallest cardinal for which Theorem 3.1.1 proves the existence of a model, is 2^{ρ} in case (1) and $2^{<\rho}$ in case (2). In general, these bounds cannot be improved.

3.1.6 Counterexamples. (1′) In case (1), take ϕ to be the $\mathscr{L}_{\kappa^+\kappa^+}$-sentence axiomatizing the notion of η_{κ^+}-set (see Example 1.2(3)) and let Γ be the set of all subformulas of ϕ. This is a counterexample, because a set of type η_{κ^+} has cardinality $\geq 2^{\kappa}$ (Gillman [1956]).

(2′) In case (2), take κ to be a singular beth number and ϕ the $\mathscr{L}_{\kappa\kappa}$-sentence of Example 1.4(2) which characterize, up to isomorphism, the structure $\langle H(\kappa), \in \upharpoonright H(\kappa)\rangle$ of all sets hereditarily of power $< \kappa$. Details are left to the reader; (see Dickmann [1975, pp. 213–214]).

Application. As an application of the infinitary downward Löwenheim–Skolem theorem, we shall prove one of the nonaxiomatizability results from topology that were announced in Example 1.7. The idea is to consider a class $\mathbb{K}$ of structures—topological spaces in the present situation—containing a member of sufficiently large cardinality which is "generated" by a set of smaller cardinality. If $\mathbb{K}$ were RPC in some $\mathscr{L}_{\kappa\lambda}$, then an application of Corollary 3.1.2 to $\mathscr{L}_{\kappa\lambda}$ would quickly produce a contradiction.

3.1.7 Theorem. *Let $\mathbb{K}$ be a class of topological spaces (viewed as structures with vocabulary v, as in Section 1.6) which contains discrete spaces of arbitrarily large cardinality. Then $\mathbb{K}$ is not RPC in $\mathscr{L}_{\kappa\lambda}$, for any κ, λ.*

Proof. Suppose that the contrary holds. Then there are a vocabulary $\tau \supseteq v$, a set Σ of $\mathscr{L}_{\kappa\lambda}(\tau)$-sentences, and an $\mathscr{L}_{\kappa\lambda}(\tau)$-formula $\phi(x)$ such that for any v-structure $\mathfrak{A}$,

(∗) $\mathfrak{A} \in \mathbb{K}$ iff there is a τ-structure $\mathfrak{A}'$ such that $\mathfrak{A}' \models \Sigma$ and $\mathfrak{A} = (\mathfrak{A}' \upharpoonright \phi^{\mathfrak{A}'}) \upharpoonright v$.

Let μ be a cardinal $\geq \bar{\bar{\tau}}$ such that $\mu^{\kappa} = \mu$ (for example, $\mu = 2^{\rho}$ with $\rho = \max\{\bar{\bar{\tau}}, \kappa\}$). Let $\mathfrak{A} = \langle Y \cup \mathbb{P}(Y), Y, \mathbb{P}(Y), \in\rangle$ be a discrete space in $\mathbb{K}$ of cardinality $\geq \mu$. By (∗) above, $\mathfrak{A} = (\mathfrak{A}' \upharpoonright \phi^{\mathfrak{A}'}) \upharpoonright v$ for some $\mathfrak{A}' \models \Sigma$. Let $Y' \subseteq Y$, $\bar{\bar{Y}}' = \mu$, and $X = Y' \cup \{\{y\} \mid y \in Y'\}$. Thus, $\bar{\bar{X}} = \mu$. Now apply Corollary 3.1.2 to get $\mathfrak{B}' \prec_{\kappa\lambda} \mathfrak{A}'$ such that $\bar{\bar{\mathfrak{B}}}' = \mu$ and $X \subseteq |\mathfrak{B}'|$. Set $\mathfrak{B} = (\mathfrak{B}' \upharpoonright \phi^{\mathfrak{B}'}) \upharpoonright v$. Since $\mathfrak{B}' \models \Sigma$, then $\mathfrak{B} \in \mathbb{K}$ by (∗). Hence, $\mathfrak{B} \cong \langle Z \cup \mathscr{T}, Z, \mathscr{T}, \in\rangle$ for a topology $\mathscr{T}$ on some set Z, and we identify these structures. Also, we have that $\mathfrak{B}' \prec_{\kappa\lambda} \mathfrak{A}'$ implies that $\phi^{\mathfrak{B}'} = \phi^{\mathfrak{A}'} \cap |\mathfrak{B}'| = |\mathfrak{A}| \cap |\mathfrak{B}'|$. Hence, $X \subseteq |\mathfrak{B}| = Z \cup \mathscr{T}$ and we get $Y' \subseteq Z$, and $\{\{y\} \mid y \in Y'\} \subseteq \mathscr{T}$. Since

$\mathscr{T}$ is closed under arbitrary unions, it follows that $\mathbb{P}(Y') \subseteq \mathscr{T}$, and therefore we must have

$$\overline{\overline{\mathfrak{B}}}' \geq \overline{\overline{\mathfrak{B}}} = \overline{\overline{Z \cup \mathscr{T}}} \geq 2^{\mu}.$$

But this contradicts the choice of $\mathfrak{B}'$. □

The classes of topological spaces, discrete spaces, T_i-spaces ($i = 0, \ldots, 5$), regular spaces, etc., obviously satisfy the assumptions of the theorem. But the class of compact spaces certainly does not. In order to deal with this case, we use the same method, letting the Stone space of a power-set algebra play the crucial rôle, instead of a discrete space. For the details of these and other applications of this method, see Cole–Dickmann [1972] or Dickmann [1975, pp. 219–223].

An application of the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega_1\omega}$ (Corollary 3.1.4, with $\kappa = \omega_1$) to group theory is given in Chapter XI, at the end of Section 7.

3.2. *The Upward Löwenheim–Skolem Theorem and Hanf Numbers*

Example 1.4(3) revealed some of the constraints on possible generalizations of the upward Löwenheim–Skolem theorem to infinitary languages. A further constraint stems from the existence of infinitary sentences which do not have models of some specific but arbitrarily large cardinalities, such as in the following:

3.2.1 Exercise. Construct an $\mathscr{L}_{\omega_1\omega_1}$-sentence having models of cardinality κ iff $\mathrm{cf}(\kappa) \neq \omega$. □

These examples are about the strongest obstacle—at least, in principle—to the existence of some sort of extension of the upward theorem to infinitary logic as well as to any language whose sentences form a *set*. This is shown by a simple but astute remark, which shows that the *Hanf number* of any such language exists. For the sake of easy reference, we include

3.2.2 Definition. Given a set X of sentences of an arbitrary language $\mathscr{L}$, we define its *Hanf number*, $h(X)$, to be the smallest cardinal λ such that any sentence of X which has a model of power $> \lambda$, has model of arbitrarily large cardinality. If the sentences of $\mathscr{L}$ form a set, its Hanf number is called *the Hanf number of* $\mathscr{L}$ and is denoted by $h(\mathscr{L})$. □

See Chapter II, Theorem 6.1.4 for the existence of $h(X)$. Note that in the above definition "language" means "syntactical structure + vocabulary". Thus, $h(\mathscr{L}_{\omega\omega}(\tau)) = \max\{\aleph_0, \overline{\overline{\tau}}\}$. In order to get a more invariant notion, we shall be rather concerned with

$$h(\mathscr{L}_{\kappa\lambda}) = \sup\{h(\mathscr{L}_{\kappa\lambda}(\tau)) \mid \overline{\overline{\tau}} < \kappa\}.$$

The panorama concerning the values of the Hanf numbers $h(\mathscr{L}_{\kappa\lambda})$ is very different, depending on whether we are dealing with finite or infinite quantifier languages.

The Hanf Number of Finite Quantifier Languages: An Introduction

(1) In this case, it is possible to give upper and lower bounds for $h(\mathscr{L}_{\kappa^+\omega})$ in terms of the cardinal arithmetical operations of ZFC, namely

$$\beth_{\kappa^+} \leq h(\mathscr{L}_{\kappa^+\omega}) < \beth_{(2^\kappa)^+},$$

and in some cases, to even give its exact value

$$h(\mathscr{L}_{\omega_1\omega}) = \beth_{\omega_1}.$$

Assuming the generalized continuum hypothesis, we also have

$$h(\mathscr{L}_{\kappa^+\omega}) = \beth_{\kappa^+} \quad \text{for all } \kappa \text{ of cofinality } \omega.$$

Furthermore, when $\mathrm{cf}(\kappa) > \omega$, the following holds:

$$\beth_{\kappa^+} < h(\mathscr{L}_{\kappa^+\omega}).$$

(2) Along these same lines, it was shown by Barwise, Kunen, and Morley that we can express the exact value of $h(\mathscr{L}_{\kappa^+\omega})$, for all κ, in terms of certain recursive operations on ordinals depending on κ^+. This, shows (in ZFC) that whenever $\mathrm{cf}(\kappa) > \omega$, the value of $h(\mathscr{L}_{\kappa^+\omega})$ is much larger than $\beth_{\kappa^+}$—larger, for example, than $\beth_\alpha$, where α is the 1st, 2nd, ..., nth, ... iteration of ordinal exponentation on κ^+; and, even more generally, it is larger than $\beth_{f(\kappa^+)}$, where f is any recursive function on ordinals.

(3) However, this does not mean that the axioms of ZFC suffice to give a precise location for the value of $h(\mathscr{L}_{\kappa^+\omega})$ in the hierarchy of the beth numbers no more than they suffice to locate the value of $2^{\aleph_\alpha}$ in the hierarchy of the aleph numbers. Indeed, by using forcing techniques, Kunen proved that by making 2^κ large with respect to κ, we can consistently make $h(\mathscr{L}_{\kappa^+\omega})$ small or large within the interval $(\beth_{\kappa^+}, \beth_{(2^\kappa)^+})$. More precisely, we have

3.2.3 Theorem. *Assume that* ZFC *is consistent and let* κ, θ *be regular cardinals such that* $\omega < \kappa < \theta$. *Then there are models* $\mathfrak{M}$, $\mathfrak{N}$ *of* ZFC *in which the values of the continuum function are as follows*:

$$2^\lambda = \lambda^+ \quad \textit{for} \quad \omega \leq \lambda < \kappa,$$

and

$$2^\lambda = \max\{\lambda^+, \theta\} \quad \textit{for} \quad \lambda \geq \kappa;$$

but

$$\mathfrak{M} \models h(\mathscr{L}_{\kappa^+\omega}) < \beth_{\kappa^{++}},$$

while

$$\mathfrak{N} \models h(\mathscr{L}_{\kappa^+\omega}) > \beth_\theta. \quad \square$$

The Hanf Number of Infinite Quantifier Languages

The situation is much more hopeless in this case. For, although the Hanf number of these languages has been proven to exist in ZFC, the expedient of giving bounds for them in terms of the cardinal arithmetical operations of ZFC fails. The mere possibility of expressing the size of $h(\mathscr{L}_{\kappa\lambda})$, $\lambda \geq \omega_1$, in terms of known set-theoretical notions seems to require the adjunction to ZFC of extremely powerful—hence, rather dubious—set-theoretical axioms. But, whatever these additional axioms may be, all known results underline the fact that the size of $h(\mathscr{L}_{\kappa\lambda})$ for uncountable λ is extremely large.

We remark, in passing, that Barwise [1972b] and Friedman [1974] have analyzed the strength of the set-theoretical axioms needed to prove the existence of the Hanf number $h(\mathscr{L})$ and to express bounds for it in set-theoretical terms for various logics $\mathscr{L}$, including $\mathscr{L}_{\omega_1\omega_1}$.

(1) Upper Bounds. The only upper bounds for the Hanf number of infinite quantifier languages provable in ZFC are the following, and they are obtained by very simple compactness arguments:

3.2.4 Proposition. *Assume that there is a strongly compact cardinal κ (that is, a compact cardinal for which $\mathscr{L}_{\kappa\kappa}$ has the compactness property for sets of sentences of any size). Then, we have*

$$h(\mathscr{L}_{\lambda\lambda}) \leq \kappa \quad \textit{for any} \quad \lambda \leq \kappa;$$

and

$$h(\mathscr{L}_{\lambda\lambda}) < \kappa \quad \textit{for} \quad \lambda < \kappa.$$

In particular, $h(\mathscr{L}_{\omega_1\omega_1})$ is smaller than the first strongly compact cardinal, and

$$h(\mathscr{L}_{\kappa\kappa}(\tau)) = \kappa \quad \textit{for any vocabulary } \tau. \quad \square$$

Some relative consistency results for upper bounds for the Hanf number of $\mathscr{L}_{\omega_1\omega_1}$ are also known. In the first place, Magidor [1976] proved that the equality "first strongly compact cardinal = first measurable cardinal" is consistent with ZFC, provided there is a strongly compact cardinal. Together with the preceding bounds, this immediately yields

3.2.5 Proposition. *If* ZFC + "*there is a strongly compact cardinal*" *is consistent, then so is* ZFC + "*$h(\mathscr{L}_{\omega_1\omega_1})$ is smaller than the first measurable cardinal*". $\square$

Starting from a different assumption, Väänänen [1980c] proves another relative consistency result, namely

3.2.6 Proposition. *If* ZFC + "*there is a proper class of measurable (respectively, weakly compact and strongly inaccessible) cardinals*" *is consistent, then so is* ZFC + "*$h(\mathscr{L}_{\omega_1\omega_1})$ is smaller than the first measurable (respectively, weakly compact and strongly inaccessible) cardinal*". □

Of course, these results do not exclude the possibility of obtaining much smaller upper bounds for the Hanf number of smaller, but interesting, sets of infinitary sentences. This question has hardly been investigated. Nevertheless, there is the following result of Silver [1971a], a result which uses the construction of models from indiscernibles.

3.2.7 Proposition. *The Hanf number of the set of all prenex-universal sentences of $\mathscr{L}_{\lambda\lambda}(\tau)$ does not exceed the first cardinal μ with the partition property $\mu \to (\lambda)_{2^\nu}^{<\omega}$, where $\nu = \max\{\aleph_0, \bar{\bar{\tau}}\}$, provided such μ exists. For $\lambda = \omega_1$ and countable τ, this bound can be reduced to the first μ such that $\mu \to (\omega_1)_2^{<\omega}$.* □

(2) Lower Bounds. Following is a brief account on the results concerning lower bounds for the Hanf number of infinite quantifier languages which have been obtained under additional set-theoretical assumptions. For the sake of simplicity, we confine ourselves to $\mathscr{L}_{\omega_1\omega_1}$.

3.2.8 Theorem (Kunen [1970]). *If* ZFC + "*there is a measurable cardinal*" *is consistent, then so is* ZFC + "*$h(\mathscr{L}_{\omega_1\omega_1})$ exceeds the first measurable cardinal*". □

In particular, this result implies that no upper bound for $h(\mathscr{L}_{\omega_1\omega_1})$ can be expressed in ZFC exclusively in terms of the partition cardinals used in Proposition 3.2.7. Propositions 3.2.6 and 3.2.8 imply

3.2.9 Theorem. *The statement* "*$h(\mathscr{L}_{\omega_1\omega_1})$ is smaller than the first measurable cardinal*" *is independent of* ZFC + "*there is a proper class of measurable cardinals*". □

3.2.10 Theorem (Silver [1971a]). ZFC + "*there is a cardinal κ such that $\kappa \to (\omega)_2^{<\omega}$*" *proves: the Hanf number of the set of all prenex-universal sentences of $\mathscr{L}_{\omega_1\omega_1}$—hence also $h(\mathscr{L}_{\omega_1\omega_1})$—exceeds the first weakly compact, strongly inaccessible cardinal.* □

A similar result holds for any $\mathscr{L}_{\lambda^+\lambda^+}$.

3.2.11 Theorem (Silver [1971a]). ZFC + $V = L$ + "*there is a cardinal which is Π_m^n-indescribable for all $n, m \in \omega$*" *proves: $h(\mathscr{L}_{\omega_1\omega_1})$ is larger than the first such cardinal.* □

Since Π_1^1-indescribable cardinals are just the same as weakly compact, strongly inaccessible cardinals, and this notion relativizes to L, from Theorem 3.2.11 and Proposition 3.2.6 we may infer

3.2.12 Theorem. *The statement "$h(\mathscr{L}_{\omega_1\omega_1})$ is smaller than the first weakly compact, strongly inaccessible cardinal" is independent of* ZFC + "*there is a proper class of weakly compact, strongly inaccessible cardinals*". □

The Hanf Number of Finite Quantifier Languages (Continued)

The remainder of this section is devoted to a sketch of the main ideas and techniques used in the computation of the Hanf number of finite quantifier languages.

Example 1.4(3) shows directly that $\beth_{\kappa^+} \leq h(\mathscr{L}_{\kappa^+\omega})$. The remaining results are more difficult by at least an order of magnitude. Some of the steps that lead to them are more easily visualized in the terminology of omitting (first-order) types which exploits the equivalence proved in Proposition 1.9.1. In these terms, the analogue of the Hanf number $h(\mathscr{L}_{\kappa^+\omega})$ is given in

3.2.13 Definition. The *Morley number* m_κ is the least cardinal λ such that every set of $\leq \kappa$ first-order types which is omitted in some model of power $\geq \lambda$ is also omitted in models of arbitrarily large cardinality. □

Proposition 1.9.1 implies immediately that we have

3.2.14 Proposition. $h(\mathscr{L}_{\kappa^+\omega}) = m_\kappa$. □

Another basic tool in this theory is an elaboration on Example 1.4(3). Since this has been treated with some detail in Chapter II, we will merely state the result, referring the reader to Definition 5.2.1 of that chapter for the definition of the expression "a sentence pins down an ordinal", and to Theorem 6.1.6 for the proof itself.

3.2.15 Theorem. *Assume that an ordinal α is pinned down by an $\mathscr{L}_{\kappa^+\omega}$-sentence; then $h(\mathscr{L}_{\kappa^+\omega}) > \beth_\alpha$.* □

There is an omitting-types version of this theorem which it is obtained by replacing in the definition of pinning down the words "model of an $\mathscr{L}_{\kappa^+\omega}$-sentence" by the words "model of a first-order theory T omitting a set S of $\leq \kappa$ types", and changing the conclusion to read "$m_\kappa > \beth_\alpha$".

A first application of Theorem 3.2.15 is given in

3.2.16 Theorem. *If* $\mathrm{cf}(\kappa) > \omega$, *then κ^+ is pinned down by an $\mathscr{L}_{\kappa^+\omega}$-sentence. Hence, $\beth_{\kappa^+} < h(\mathscr{L}_{\kappa^+\omega})$.*

Hint of Proof. Although pinning down ordinals $< \kappa$ is easy—the reader can convince himself of this by using the sentences θ_α of Example 1.2(2)—pinning down ordinals larger than κ requires a subtle argument, the gist of which is as follows.

Recall that every $\alpha \in \kappa$ is a subset of κ. Let $r \subseteq \kappa \times \kappa$ be a linear order of κ. If r happens to be a well-order (although not necessarily the canonical one), then $r \restriction \alpha$ is also a well-order of α, for all $\alpha \in \kappa$. Since $\bar{\bar{\alpha}} < \kappa$, there is a $\beta \in \kappa$ such that $\langle \alpha, r \restriction \alpha \rangle \cong \langle \beta, \in \restriction \beta \rangle$.

This shows that (a) implies (b), where

(a) r well-orders κ; and
(b) for every $\alpha \in \kappa$, there is $\beta \in \kappa$ such that $\langle \alpha, r \restriction \alpha \rangle \cong \langle \beta, \in \restriction \beta \rangle$.

If $\mathrm{cf}(\kappa) > \omega$, then the converse is also true. For, if r does not well-order κ, there is an infinite r-descending sequence

$$\ldots r\, \alpha_n\, r\, \alpha_{n-1}\, r \ldots r\, \alpha_1\, r\, \alpha_0$$

of ordinals $\alpha_n \in \kappa$. Let $\alpha \in \kappa$ be such that $\alpha_n < \alpha$ for all n. Then r does not well-order α and $\langle \alpha, r \restriction \alpha \rangle$ cannot be isomorphic to any $\langle \beta, \in \restriction \beta \rangle$.

The point here is that (b) can be "said" by a first-order theory and the omission of κ types, thus allowing us to single out well-orderings of κ—that is, ordinals below κ^+—amongst linear orderings. The details of this part of the proof are given in Dickmann [1975, pp. 241–242]. ☐

The foregoing argument is due to Chang [1968c], although the result was first proven by Morley–Morley [1967], using $V = L$.

The inequality $h(\mathscr{L}_{\omega_1\omega}) \leq \beth_{\omega_1}$—and hence the equality—was proved by Morley [1965b] by a very subtle combination of the construction of models from indiscernibles (Ehrenfeucht–Mostowski [1956]) and the Erdös–Rado [1956] theorem of partition calculus as a device for producing sets of indiscernibles of large cardinality. His proof was later extended by Chang [1968c] to obtain the inequality $h(\mathscr{L}_{\kappa^+\omega}) < \beth_{(2^\kappa)^+}$ for all κ, and by Helling [1964] to obtain the inequality $h(\mathscr{L}_{\kappa^+\omega}) \leq \beth_{\kappa^+}$ and, hence, the equality when $\mathrm{cf}(\kappa) = \omega$.

The details of these proofs go far beyond the scope of this guide to the subject, and they can be consulted in the original papers or in Dickmann [1975, Chapter 4, Section 3]. The basic result is

3.2.17 Theorem. *Let T be a first-order theory and S a set of (first-order) types. If for every $\zeta < (2^\kappa)^+$ there is a model of T of power $> \beth_\zeta$ omitting S, then there are models of T of arbitrarily large cardinality omitting S.* ☐

The statement is independent of the cardinality of S. However, it gives us the following

3.2.18 Corollary. $m_\kappa < \beth_{(2^\kappa)^+}$.

Proof. There are 2^κ sets of types of power $\leq \kappa$ in a language with $\leq \kappa$ symbols, say $\langle S_\xi \mid \xi < 2^\kappa \rangle$. Let μ_ξ be the omitting-types cardinal (as defined in Definition 3.2.13) of the set S_ξ; then $m_\kappa \leq \sup\{\mu_\xi \mid \xi < 2^\kappa\}$. Note that $\mu_\xi < \beth_{(2^\kappa)^+}$; for if S_ξ is omitted by a structure of power $\geq \beth_{(2^\kappa)^+}$, then by Theorem 3.2.17 it is omitted by structures of arbitrary large cardinality. And, by the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega\omega}$, it is also omitted by a model of power $< \beth_{(2^\kappa)^+}$. Since

$$\mathrm{cf}(\beth_{(2^\kappa)^+}) = (2^\kappa)^+ > 2^\kappa,$$

it follows that

$$m_\kappa \leq \sup\{\mu_\xi \mid \xi < 2^\kappa\} < \beth_{(2^\kappa)^+}. \quad \square$$

Helling's result is similar to Theorem 3.2.17; however, it is assumed that $\bar{\bar{S}} < \kappa$ and that $\kappa = \beth_\alpha$, with $\mathrm{cf}(\alpha) = \omega$, and 2^κ is replaced by κ. Since ω is of this form and, under the GCH, every cardinal is a beth number, we immediately have

3.2.19 Corollary. (a) $m_\omega = h(\mathscr{L}_{\omega_1\omega}) = \beth_{\omega_1}$. $\square$
(b) (GCH) *If* $\mathrm{cf}(\kappa) = \omega$, *then* $m_\kappa = h(\mathscr{L}_{\kappa^+\omega}) = \beth_{\kappa^+}$. $\square$

An outstanding corollary of these upper bounds is the following theorem due to López-Escobar [1966a, b].

3.2.20 Theorem. *The class of all (nonempty) well-orderings is not* RPC *in any finite quantifier language* $\mathscr{L}_{\kappa\omega}$.

Hint of Proof. If this class were RPC in, say, $\mathscr{L}_{\kappa^+\omega}$, then by using a few additional predicates and constants, we could easily manufacture another $\mathscr{L}_{\kappa^+\omega}$-sentence which pins down the cardinal $\lambda = 2^{2^{2^\kappa}}$. By Theorem 3.2.15, this would force $h(\mathscr{L}_{\kappa^+\omega}) > \beth_\lambda$, which manifestly contradicts Corollary 3.2.18. $\square$

In order to complete this account, let us briefly look at the argument leading to the computation of the exact value of $h(\mathscr{L}_{\kappa^+\omega})$. This argument was discovered by Barwise–Kunen [1971] and, independently, by Morley (an unpublished result). The techniques reviewed above are all used here along with a number of other key refinements.

Let $P(\kappa^+)$ denote the class of all ordinals pinned down by some $\mathscr{L}_{\kappa^+\omega}$-sentence; it has the following properties:

(a) $P(\kappa^+)$ is an initial segment of ordinals without last element (see the remarks following Definition 5.2.1 in Chapter II)
(b) $P(\kappa^+) \subset (2^\kappa)^+$, by Theorem 3.2.15 and Corollary 3.2.18;
(c) $\kappa^+ \subseteq P(\kappa^+)$, by Example 1.2(2);
(d) If $\mathrm{cf}(\kappa) > \omega$, then $\kappa^+ \in P(\kappa^+)$, by Theorem 3.2.16;
(e) (Karp–Jensen): $P(\kappa^+)$ is closed under primitive recursive operations on ordinals.

3.2.21 Exercise. Prove (e) above for ordinal addition and multiplication. $\square$

Let $a(\kappa^+)$ be the first ordinal not in $P(\kappa^+)$. By Theorem 3.2.15, $\beth_{a(\kappa^+)} \leq h(\mathscr{L}_{\kappa^+\omega})$. The converse is also true, although it is a much more delicate matter.

The notion of pinning down considered above is too coarse for our purposes. A more manageable notion along the same lines is obtained, first, by relaxing the well-orderedness requirement to well-foundedness; and, second, by adding the metatheoretic requirement that the well-founded structures under consideration be reasonably well-behaved set-theoretical objects. A first, nontrivial step consists of proving that the new notion coincides with the older one. See Dickmann [1975, Chapter 4, Section 5C]. Once this is done, we then prove

3.2.22 Theorem. *Let ϕ be an $\mathscr{L}_{\kappa^+\omega}$-sentence whose models have bounded cardinality. Then there is a well-founded structure $\langle \mathscr{T}, \prec \rangle$ definable in* ZFC *by a bounded quantifier formula with parameters from $H(\kappa^+)$, such that if α denotes its height, then all models of ϕ have power $\leq \beth_{\delta+\omega\cdot(\alpha+1)}$, for some $\delta < \kappa^+$.*

Dénouement. By the remarks preceding the statement, $\alpha \in P(\kappa^+)$; by (c) and (e), $\delta + \omega(\alpha + 1) \in P(\kappa^+)$, and hence this ordinal is smaller than $a(\kappa^+)$. By the definition of the Hanf number, the inequality $h(\mathscr{L}_{\kappa^+\omega}) \leq \beth_{a(\kappa^+)}$ thus follows. ☐

Concerning the Proof of Theorem 3.2.22. A few remarks on this argument's main ingredients are destined (at least, we hope) to sharpen the reader's appetite for more on this subject. In fact, the full meal is served up in Barwise-Kunen [1971] and in Dickmann [1975, pp. 274–281].

(1) The members of $\mathscr{T}$ are certain sets of sentences belonging to a fragment Ψ of $\mathscr{L}_{\kappa^+\omega}$ contained in $H(\kappa^+)$. All of these sets contain ϕ and are chosen in such a way that they are rich enough to make the following work:

(i) an analogue of the model existence theorem of Chapter VIII, Section 3.1;
(ii) the essentials of the indescernibility arguments involved in the proof of Theorem 3.2.17.

The order $\prec$ of $\mathscr{T}$ is reverse (proper) inclusion.

(2) If $\mathscr{T}$ had an infinite $\prec$-decreasing sequence, $\Sigma_1 \succ \Sigma_2 \succ \ldots$, the indescernibility arguments mentioned in (ii) above can be used to produce models of $\bigcup_n \Sigma_n$—hence also of ϕ—of arbitrarily large cardinalities, contrary to the assumption on the cardinalities of the models of ϕ.

(3) An induction argument on the foundation rank of members of $\langle \mathscr{T}, \prec \rangle$ (involving also the Erdös–Rado theorem to get sets of indiscernibles of large cardinality) is used to show that the models of any $\Sigma \in \mathscr{T}$ have power $< \beth_{\omega\cdot(\beta+1)}(\lambda)$, where β is the $\langle \mathscr{T}, \prec \rangle$-rank of Σ and $\lambda = 2^{\overline{\overline{\Psi}}}$. In particular, every model of ϕ has power $< \beth_{\omega\cdot(\alpha+1)}(\lambda)$. Since $\Psi \in H(\kappa^+)$, there is $\gamma < \kappa^+$ such that $\Psi \in R(\gamma)$, so that $\overline{\overline{\Psi}} \leq \beth_\gamma$ and $\lambda \leq \beth_{\gamma+1}$. The conclusion follows by setting $\delta = \gamma + 1$. ☐

4. *The Back-and-Forth Method*

4.1. *Introduction and History*

The method of extension of partial isomorphisms originated with Cantor who used it for the stepwise construction of an isomorphism between any two countable dense linear orderings without endpoints. Since then, this type of argument has been used to construct isomorphisms in a large variety of mathematical contexts. For example, some celebrated uses of this method are:

- The proof that a countable reduced abelian p-group is characterized up to isomorphism by its Ulm invariants (see Kaplansky [1969, Theorem 14]).
- Hausdorff's generalization of Cantor's theorem showing that two η_λ-sets of cardinality λ are isomorphic, for regular cardinals λ.
- The proof that two real closed fields of cardinality $\aleph_1$ whose underlying orders are of type η_{ω_1} are isomorphic as fields (Erdös–Gillman–Henriksen [1955]).
- The proof that two saturated, elementarily equivalent structures of the same cardinality are isomorphic.

These examples illustrate two rather different situations. In the countable case (Cantor's and Ulm's examples), the method produces an isomorphism rather naturally and without additional assumptions. In the uncountable case (the three last examples), one frequently needs to introduce cardinality hypotheses extraneous to the problem in order to end up with an isomorphism (for example, in the two examples involving η_λ-sets, the assumption is vacuously verified unless GCH is used). The theory developed in this section gives a very satisfactory explanation for this state of affairs. Moreover, it provides a machinery which renders the exact content of the proofs, thus eliminating the extraneous cardinality assumptions in the problematic cases.

A different use of the back-and-forth method was inaugurated by Langford [1926]. He used it to show that any two dense linear orderings without endpoints are elementarily equivalent, regardless of their cardinalities. Fraïssé [1955a] and Ehrenfeucht [1961] generalized Langford's use of the method (and result as well) by giving a purely algebraic characterization of elementary equivalence in terms of families of partial isomorphisms with the back-and-forth properties (Theorem 4.3.4). Furthermore, Ehrenfeucht gave a game-theoretical interpretation of the method which subsequently became very popular. However, it was Karp [1965] who conclusively showed that the mathematical framework where the basic ("one-at-a-time") back-and-forth technique is naturally expressed is infinitary, rather than first-order, logic. More precisely, it is the class-logic $\mathscr{L}_{\infty\omega}$. Karp's results tied neatly in with Scott's earlier characterization of the countable isomorphism type of a countable structure by a single $\mathscr{L}_{\omega_1\omega}$-sentence (see the end of Section 4.4 below). This connection was developed and generalized by Chang [1968c]. Barwise–Eklof [1970] and Barwise [1973b] gave a unified form to all these arguments and provided the basis for a more general treatment. Benda [1969] and Calais [1972] generalized the work of Karp to the class-logics $\mathscr{L}_{\infty\lambda}$. The general theory of back-and-forth arguments is presented in Dickmann [1975, Chapter 5]. This is the subject matter of Sections 4.3 and 4.4 below.

A third and quite different use of partial isomorphisms is for building embeddings—rather than isomorphisms—or even other kinds of maps, as in the following examples:

- The proof that every λ-saturated structure is λ-universal, that is, it contains an embedded copy of every structure of power $\leq \lambda$ with the same first-order theory.

– The so-called countable embedding theorem (Barwise [1969c]) which shows that for any two countable structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}$ can be embedded in $\mathfrak{B}$ iff every universal $\mathscr{L}_{\omega_1\omega}$-sentence which holds in $\mathfrak{B}$ also holds in $\mathfrak{A}$.

This kind of use hardly falls under the denomination "back-and-forth"; for, frequently one moves in only one direction. However, it fits very naturally into the general setting developed in Section 4.4 below.

4.2. *Basic Facts*

4.2.1 Definition. (a) Let $\mathfrak{A}$, $\mathfrak{B}$ be structures with the same vocabulary. A map f from a subset of $\mathfrak{A}$ into a subset of $\mathfrak{B}$ will be called a *partial isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$ iff either:

(i) f is the empty map and $\mathfrak{A}$, $\mathfrak{B}$ satisfy the same *atomic* sentences; or,
(ii) $\mathrm{Dom}(f)$ is a substructure of $\mathfrak{A}$, $\mathrm{Range}(f)$ is a substructure of $\mathfrak{B}$, and f is a monomorphism, that is, for every atomic formula $\phi(v_1 \ldots v_n)$ and every $x_1, \ldots, x_n \in \mathrm{Dom}(f)$,

$$\mathfrak{A} \models \phi[x_1 \ldots x_n] \quad \text{iff} \quad \mathfrak{B} \models \phi[f(x_1), \ldots, f(x_n)].$$

(b) Given a cardinal λ, a *λ-partial isomorphism* is a partial isomorphism, where $\mathrm{Dom}(f)$ is generated (as a substructure of $\mathfrak{A}$) by fewer than λ elements.

Notice that in other chapters (for example, in Chapter II, Section 4.2) the domains of partial isomorphisms need not be substructures of $\mathfrak{A}$; the difference is not essential, because if $\mathrm{Dom}(f) \neq \varnothing$, or if the language has at least one individual constant, then f extends to the substructure of $\mathfrak{A}$ generated by $\mathrm{Dom}(f)$.

The extension relation between maps will (also) be denoted by $\subseteq$. As a motivation for later arguments, we prove the theorem of Hausdorff that was mentioned in the introduction.

4.2.2 Example and Theorem. *Let λ be a regular infinite cardinal. Then any two η_λ-sets of cardinality λ are isomorphic.*

Proof. The argument separates into two parts, and we first consider

Part I: Let $\langle A, <\rangle$, $\langle B, \prec\rangle$ be η_λ-sets of power λ, and consider the set $\mathbb{I}$ of all λ-partial isomorphisms (that is, in this case, order-preserving maps with $\overline{\overline{\mathrm{Dom}(f)}} < \lambda$). $\mathbb{I}$ has the following properties:

(i) *λ-extension property*: Any subfamily of $\mathbb{I}$ of power $< \lambda$, totally ordered under the order of extension, has an upper bound in $\mathbb{I}$.

(ii) One-at-a-time back-and-forth properties:
(a) *Forth property*: For every $f \in \mathbb{I}$ and $a \in A$, there is $g \in \mathbb{I}$ such that $f \subseteq g$ and $a \in \mathrm{Dom}(g)$;
(b) *Back property*: For every $f \in \mathbb{I}$ and $b \in B$, there is $g \in \mathbb{I}$ such that $f \subseteq g$ and $b \in \mathrm{Range}(g)$.

Condition (i) is clear by the regularity of λ, but (ii) is more delicate. We will do (b), the proof of (a) being symmetric. Thus, assume that $b \notin \mathrm{Range}(f)$, and let (Y, Z) be the cut of $\mathrm{Range}(f)$ determined by b:

$$Y = \{y \in \mathrm{Range}(f) \mid y \prec b\}, \qquad Z = \{z \in \mathrm{Range}(f) \mid b \prec z\}.$$

Since f is order-preserving, we have $f^{-1}[Y] < f^{-1}[Z]$ (see Example 1.2(3) for this notation); and, since $\overline{\overline{\mathrm{Dom}(f)}} < \lambda$, these sets have power $< \lambda$. Since $\langle A, < \rangle$ is of type η_λ, there is $a \in A$ such that $f^{-1}[Y] < a < f^{-1}[Z]$. Thus, the map

$$\mathrm{Dom}(g) = \mathrm{Dom}(f) \cup \{a\},$$

$$g \restriction \mathrm{Dom}(f) = f,$$

$$g(a) = b,$$

does the job. Part I now established, we turn to

Part II. Given a nonempty family $\mathbb{I}$ of partial isomorphisms with properties (i) and (ii), we construct an isomorphism of $\langle A, < \rangle$ onto $\langle B, \prec \rangle$. To this purpose, we enumerate A and B without repetitions:

$$A = \langle a_\alpha \mid \alpha < \lambda \rangle, \qquad B = \langle b_\alpha \mid \alpha < \lambda \rangle.$$

Starting with any $f_0 \in \mathbb{I}$, we now construct a sequence

$$f_0 \subseteq f_1 \subseteq \cdots \subseteq f_\alpha \subseteq \cdots (\alpha < \lambda)$$

of partial isomorphisms by taking f_α to be:

(i′) If α is a limit ordinal, then $f_\alpha =$ any map $g \in \mathbb{I}$ extending all f_δ, $\delta < \alpha$. Here we use (i),
(ii′) If α is a successor ordinal, then $f_\alpha =$ any map $g \in \mathbb{I}$ extending $f_{\alpha-1}$, and such that:
(a′) $a_{\beta+n} \in \mathrm{Dom}(g)$, if $\alpha = \beta + 2n + 1$, β limit,
(b′) $b_{\beta+n} \in \mathrm{Range}(g)$, if $\alpha = \beta + 2n + 2$, β limit.

Here we use (ii)(a) and (ii)(b), respectively.

As an exercise, the reader might check that $f = \bigcup_{\alpha<\lambda} f_\alpha$ is an isomorphism, as is required. $\square$

Clearly, Part II is a general theorem which has nothing to do with orderings (Proposition 4.2.5). In order to analyze the situation, it is convenient to introduce

4.2.3 Definition and Notation. The notation

$\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$	means that	$\mathbb{I}$ is a nonempty family of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ with the back-and-forth properties given in (ii)(a) and (ii)(b) of Theorem 4.2.2 (caution: property (i) is not required to hold).
$\mathfrak{A} \simeq_p \mathfrak{B}$	means that	there is an $\mathbb{I}$ such that $\mathbb{I}: \mathfrak{A} \cong_p \mathfrak{B}$.
$\mathbb{I}: \mathfrak{A} \simeq_\lambda^{p,e} \mathfrak{B}$	means that	$\mathbb{I}$ is a nonempty family of partial isomorphisms with the back-and-forth-properties of Theorem 4.2.2 and the extension property *for* $\subseteq$*-chains of power* $< \lambda$.
$\mathbb{I}: \mathfrak{A} \simeq_\lambda^p \mathfrak{B}$	means that	$\mathbb{I}$ is a nonempty family of partial isomorphisms with the *fewer than* λ *at a time back-and-forth properties*; that is, for every $f \in \mathbb{I}$ and $A \subseteq |\mathfrak{A}|$, $\bar{\bar{A}} < \lambda$, there is $g \in \mathbb{I}$ such that $f \subseteq g$ and $A \subseteq \mathrm{Dom}(g)$, and similarly for the "back" part.

Observe that the extension property and the back-and-forth properties do not always occur together as they do in Theorem 4.2.2. The following connections between the notions just introduced are easily proven and are left as an exercise.

4.2.4 Fact. *For any family* $\mathbb{I}$ *of partial isomorphisms, the following holds*:

(a) $\mathbb{I}: \mathfrak{A} \simeq_\lambda^{p,e} \mathfrak{B}$ *implies* $\mathbb{I}: \mathfrak{A} \simeq_\lambda^p \mathfrak{B}$ *implies* $\mathbb{I}: \mathfrak{A} \simeq_\kappa^p \mathfrak{B}$ *for any* $\kappa < \lambda$, *implies* $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$;
(b) $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ *iff* $\mathbb{I}: \mathfrak{A} \simeq_\omega^p \mathfrak{B}$;
(c) *If* $\mathfrak{A} \cong^f \mathfrak{B}$, *then* $\{f\}: \mathfrak{A} \simeq_\lambda^{p,e} \mathfrak{B}$ *for any* λ. □

With this notation the second part of Theorem 4.2.2 becomes

4.2.5 Proposition. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are of power* $\leq \lambda$, *or generated by sets of power* $\leq \lambda$, *then*

$$\mathfrak{A} \cong \mathfrak{B} \quad \textit{iff} \quad \mathfrak{A} \simeq_\lambda^{p,e} \mathfrak{B}.$$

If, in addition, $\mathrm{cf}(\lambda) = \omega$, *then*

$$\mathfrak{A} \cong \mathfrak{B} \quad \textit{iff} \quad \mathfrak{A} \simeq_\lambda^p \mathfrak{B}.$$

Hence, all four relations $\cong$, $\simeq_p$, $\simeq^p_\omega$ *and* $\simeq^{p,e}_\omega$ *are equivalent on countably generated structures.* $\square$

Later we will see that for regular uncountable cardinals λ, the relation $\simeq^{p,e}_\lambda$ is much stronger than the relation $\simeq^p_\lambda$.

4.3. Partial Isomorphisms and Infinitary Equivalence

The fundamental result of the theory presented in this section is due to Karp [1965]; it shows that the relation $\simeq^p_\lambda$ of partial isomorphism is identical with the relation $\equiv_{\infty\lambda}$ of $\mathscr{L}_{\infty\lambda}$-equivalence. We will give a sketch of its proof.

4.3.1 Theorem. *For all structures* $\mathfrak{A}$ *and* $\mathfrak{B}$, $\mathfrak{A} \simeq^p_\lambda \mathfrak{B}$ *is equivalent to* $\mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$.

Proof. For the sake of notational simplicity, we will assume that $\lambda = \omega$ (hence, $\simeq^p_\omega$ becomes $\simeq_p$) and that the vocabulary has only relation symbols.

(1) We assume that $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ and prove that $\mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$. By induction on the structure of $\mathscr{L}_{\infty\omega}$-formulas, one shows that any $f \in \mathbb{I}$ is an $\mathscr{L}_{\infty\omega}$-map, that is, for any ϕ with $\leq n$ free variables and any $a_1, \ldots, a_n \in \mathrm{Dom}(f)$,

$$(*) \qquad \mathfrak{A} \models \phi[a_1, \ldots, a_n] \Leftrightarrow \mathfrak{B} \models \phi[f(a_1), \ldots, f(a_n)].$$

This is quite immediate except, possibly, in the case in which $\phi = \exists y \psi$ where the following sequence of equivalences settles the matter:

For some $a \in |\mathfrak{A}|$,

$$\mathfrak{A} \models \psi[a_1, \ldots, a_n, a] \Leftrightarrow \text{(Forth Property)}.$$

For some $a \in |\mathfrak{A}|$ and $g \in \mathbb{I}$ such that $f \subseteq g$ and $a \in \mathrm{Dom}(g)$.

$$\mathfrak{A} \models \psi[a_1, \ldots, a_n, a] \Leftrightarrow \text{(Induction Hypothesis)}.$$

For some $a \in |\mathfrak{A}|$ and $g \in \mathbb{I}$ such that $f \subseteq g$ and $a \in \mathrm{Dom}(g)$,

$$\mathfrak{B} \models \psi[g(a_1), \ldots, g(a_n), g(a)] \Leftrightarrow \text{(Back Property)}.$$

For some $b \in |\mathfrak{B}|$,

$$\mathfrak{B} \models \psi[f(a_1), \ldots, f(a_n), b].$$

(2) We prove now the converse. The preceding implication tells us that the members of any back-and-forth set $\mathbb{I}: \mathfrak{A} \simeq_p \mathfrak{B}$ are necessarily $\mathscr{L}_{\infty\omega}$-maps. Let $\mathbb{I}$ be the family of all such maps with finite domain. Since $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$, the empty map is in $\mathbb{I}$, and $\mathbb{I} \neq \varnothing$. Let us prove, for example, that $\mathbb{I}$ has the forth property.

To this end, let $f \in \mathbb{I}$, with $\mathrm{Dom}(f) = \{a_1, \ldots, a_n\}$, and $a \in |\mathfrak{A}|$, $a \neq a_i$ $(i = i, \ldots, n)$. If we find $b \in |\mathfrak{B}|$ such that

$$(**) \qquad \mathfrak{A} \models \phi[a_1, \ldots, a_n, a] \Rightarrow \mathfrak{B} \models \phi[f(a_1), \ldots, f(a_n), b]$$

holds for every $\mathscr{L}_{\infty\omega}$-formula ϕ with $\leq n + 1$ free variables, then the map

$$\mathrm{Dom}(g) = \mathrm{Dom}(f) \cup \{a\},$$

$$g \upharpoonright \mathrm{Dom}(f) = f,$$

$$g(a) = b,$$

would solve the problem, because (**) implies its own converse. If this is not the case, then for each $b \in |\mathfrak{B}|$, there would be an $\mathscr{L}_{\infty\omega}$-formula ϕ_b such that $\mathfrak{A} \models \phi_b[a_1, \ldots, a_n, a]$ but $\mathfrak{B} \models \neg\phi_b[f(a_1), \ldots, f(a_n), b]$. Set

$$\psi(v_1, \ldots, v_{n+1})\colon \bigwedge_{b \in |\mathfrak{B}|} \phi_b(v_1, \ldots, v_{n+1}).$$

Then, $\mathfrak{A} \models \psi[a_1, \ldots, a_n, a]$, and this, of course, implies that

$$\mathfrak{A} \models (\exists v_{n+1}\psi)[a_1, \ldots, a_n],$$

while

$$\mathfrak{B} \models (\forall v_{n+1}\neg\psi)[f(a_1), \ldots, f(a_n)].$$

But this clearly contradicts the definition of f. $\square$

Show as an exercise that for a fixed map $h\colon \mathfrak{A} \to \mathfrak{B}$, the condition "$h$ is an $\mathscr{L}_{\infty\lambda}$-embedding" can be characterized in a similar manner.

A minor modification of the same argument gives a back-and-forth characterization of the important notion of *$\mathscr{L}_{\infty\lambda}$-equivalence up to bounded quantifier-rank.*

4.3.2 Definition. (a) To each $\mathscr{L}_{\infty\lambda}$-formula ϕ, we inductively assign an ordinal $\mathrm{qr}(\phi)$ called its *quantifier rank*:

$$\mathrm{qr}(\phi) = 0 \quad \text{if} \quad \phi \text{ is atomic;}$$

$$\mathrm{qr}(\phi) = \mathrm{qr}(\psi) \quad \text{if} \quad \phi \text{ is } \neg\psi;$$

$$\mathrm{qr}(\phi) = \sup\{\mathrm{qr}(\psi_i) \mid i \in I\} \quad \text{if} \quad \phi \text{ is } \bigwedge_{i \in I} \psi_i \text{ or } \bigvee_{i \in I} \psi_i;$$

$$\mathrm{qr}(\phi) = \mathrm{qr}(\psi) + 1 \quad \text{if} \quad \phi \text{ is } (\forall x)\psi \text{ or } (\exists X)\psi.$$

When $\lambda = \omega$ the proviso $\overline{\overline{X}} = 1$ is frequently added to the last clause.

(b) By $\mathfrak{A} \equiv^{\beta}_{\infty\lambda} \mathfrak{B}$, we mean that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same $\mathscr{L}_{\infty\lambda}$-sentences of quantifier-rank $\leq \beta$. ☐

4.3.3 Theorem (Karp). *If $\mathfrak{A}$ and $\mathfrak{B}$ have the same vocabulary, and β is an ordinal, then the following are equivalent:*

(1) $\mathfrak{A} \equiv^{\beta}_{\infty\lambda} \mathfrak{B}$;
(2) *there is a sequence $\mathscr{I} = \langle \mathbb{I}_\alpha | \alpha \leq \beta \rangle$ with the properties:*
   (a) *each $\mathbb{I}_\alpha$ is a nonempty family of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$;*
   (b) *$\mathbb{I}_\alpha \subseteq \mathbb{I}_\gamma$, for $\gamma \leq \alpha \leq \beta$;*
   (c) *Back-and-forth property: if $\alpha + 1 \leq \beta$, then*
      (i) *for every $f \in \mathbb{I}_{\alpha+1}$ and $A \subseteq |\mathfrak{A}|$, $\overline{\overline{A}} < \lambda$, there is $g \in \mathbb{I}_\alpha$ such that $f \subseteq g$ and $A \subseteq \mathrm{Dom}(g)$;*
      (ii) *for every $f \in \mathbb{I}_{\alpha+1}$ and $B \subseteq |\mathfrak{B}|$, $\overline{\overline{B}} < \lambda$, there is $g \in \mathbb{I}_\alpha$ such that $f \subseteq g$ and $B \subseteq \mathrm{Range}(g)$.* ☐

As was remarked in the introduction to this chapter, the back-and-forth characterization of elementary equivalence is another important result along these lines. Thus, we have

4.3.4 Theorem (Ehrenfeucht–Fraïssé). *If $\mathfrak{A}$ and $\mathfrak{B}$ are structures in a finite vocabulary without function symbols, then the following are equivalent:*

(1) $\mathfrak{A} \equiv \mathfrak{B}$;
(2) *there is a sequence of length ω, $\mathscr{I} = \langle \mathbb{I}_n | n \in \omega \rangle$, with properties of Theorem* 4.3.3(a)–(c), *where the sets A, B of power $< \lambda$ in* (c)(i) *and* (c)(ii) *are replaced by one-element sets.*

Gist of Proof. For the proof that (2) implies (1), we proceed as in the first half of the proof of Theorem 4.3.1, showing by induction on n that the maps in $\mathbb{I}_n$ preserve first-order formulas of quantifier-rank $\leq n$.

For the argument that (1) implies (2), we put in $\mathbb{I}_k$ all partial isomorphisms preserving formulas of quantifier-rank $\leq k$. Observe that the infinitely many formulas $\{\phi_b(v_1, \ldots, v_{n-1}) | b \in |\mathfrak{B}|\}$—all of quantifier-rank $\leq k$—separate into only finitely many classes modulo (logical) equivalence. By selecting representatives of these classes, ψ then becomes a first-order formula. This is because in a finite vocabulary without function symbols there are only finitely many classes modulo (logical) equivalence of formulas of bounded quantifier-rank with a fixed finite number of free variables (Exercise and Hint: Use induction on the quantifier-rank). ☐

4.3.5 Remark. The restriction to a finite vocabulary without function symbols is unavoidable. For more on this, see Dickmann [1975, Example 5.3.12]. We note that in the proof given in that book (Theorem 5.3.11, pp. 321–322) the clause "without function symbols" was inadvertently omitted.

4.3.6 $\mathscr{L}_{\infty\lambda}$-Equivalence and Isomorphism. The fact that partial isomorphism implies isomorphism for structures of power $\leq \lambda$, when $\mathrm{cf}(\lambda) = \omega$ (see Proposition 4.2.5), does not extend to other values of λ. The first examples of, say, non-isomorphic $\mathscr{L}_{\infty\omega_1}$-equivalent structures of power $\aleph_1$ were constructed by Morley (see Chang [1968c, p. 45], Nadel–Stavi [1978]), and Tait (see Dickmann [1975, pp. 350–360]). The same construction applies to any regular uncountable cardinal, but not to singular cardinals of cofinality $> \omega$. For the latter the problem is still open. Gregory [1974] showed how to transform any example with these properties into one which, in addition, is *rigid*—and this even for any infinite cardinal.

The example of Morley and Tait is a tree. Later on, Paris [1972, unpublished] gave an example of a total ordering with the same property. More recently, Shelah [1981b, 1982b] has made a more conclusive study of the number of structures $\mathscr{L}_{\infty\lambda}$-equivalent to a given structure of power λ. His results are given in

4.3.7 Theorem and Example. *Let λ be a regular cardinal.*

(1) *Under the assumption that $V = L$, if λ is not weakly compact, then the number of isomorphism types of models of cardinality λ which are $\mathscr{L}_{\infty\lambda}$-equivalent to a given structure of cardinality λ is either* 1 *or* 2^{λ}.

(2) *If λ is weakly compact, then for any cardinal $1 \leq \kappa \leq \lambda$ there is a structure of cardinality λ which, up to isomorphism has exactly κ structures of cardinality λ that are $\mathscr{L}_{\infty\lambda}$-equivalent to it. This construction also applies to any supercompact cardinal κ such that $\lambda < \kappa < 2^{\lambda}$.* $\square$

A recent paper by Kueker [1981] investigates the ways in which $\mathscr{L}_{\infty\omega_1}$-equivalent structures of power $\aleph_1$ can be built up from increasing, continuous chains of isomorphic countable substructures.

4.3.8 The Strong Partial Isomorphism Relation. As the relation $\simeq_{\lambda}^{p,e}$ of strong partial isomorphism arises spontaneously in mathematical practice as much as the relation $\simeq_{\lambda}^{p}$ of partial isomorphism does, it is natural to ask whether it also has a metamathematical interpretation. This question was posed, independently, by Dickmann [1975, p. 316] and Kueker [1975, pp. 34–35]. Nevertheless it remains largely open—even to the point that we do not yet know whether or not the relation $\simeq_{\lambda}^{p,e}$ is transitive.

However, Karttunen [1979] has made a partial step in this direction, by giving a back-and-forth characterization of equivalence in infinitary languages of a different type, which was first introduced—rather informally, too—in Hintikka–Rantala [1976]. These are the languages $N_{\infty\lambda}$. A precise definition of these and the corresponding languages $N_{\kappa\lambda}$ is to be found in Rantala [1979] and in Karttunen's paper. Roughly speaking, their characteristic feature is that formulas are defined by giving the tree of their subformulas, and that this tree may have branches of infinite height (contrary to the case of $\mathscr{L}_{\infty\lambda}$-formulas, where the tree of subformulas is well-founded; see Dickmann [1975, pp. 87–88]).

Karttunen characterizes $N_{\infty\lambda}$-equivalence in terms of a certain relation $\simeq_{\lambda}^{w,e}$; this being *a priori* weaker than $\simeq_{\lambda}^{p,e}$. Briefly stated, $\mathfrak{I}: \mathfrak{A} \simeq_{\lambda}^{w,e} \mathfrak{B}$ holds iff the family of partial isomorphisms $\mathfrak{I}$ has a tree order $\leq$ finer than the extension order $\subseteq$, and the (same) back-and-forth and extension properties hold for the order $\leq$, instead of that of extension. It is not known how much weaker is the relation $\simeq_{\lambda}^{w,e}$. However, Karttunen does show

(1) $\simeq_{\lambda}^{w,e}$ implies $\simeq_{\lambda}^{p}$, and
(2) for structures of cardinality $\leq \lambda$, we have that $\simeq_{\lambda}^{w,e}$ implies isomorphism.

Hence, in view of the comments in Section 4.3.6, the relation $\simeq_{\lambda}^{w,e}$ is seen to be much stronger than $\simeq_{\lambda}^{p}$.

4.4. A General Setting for Back-and/or-Forth Arguments

In this section we will consider the problem of using extensions of partial isomorphisms as a tool for constructing maps other than isomorphisms. This kind of application ties in with the question—*a priori* a different one—of whether there are back-and-forth characterizations of semantical relations between structures other than $\mathscr{L}_{\infty\lambda}$-equivalence. What we have in mind are semantical relations induced by classes of infinitary formulas other than the class of all such formulas. A first example, the relation of $\mathscr{L}_{\infty\lambda}$-equivalence up to bounded quantifier-rank, was already considered in Theorem 4.3.3. As a matter of fact, both these problems have a common solution; the connecting thread is the countable embedding theorem stated at the end of Section 4.1.

Let us begin by properly defining the semantical relation $\mathfrak{A}\ (\Phi)\ \mathfrak{B}$ induced by a class Φ of $\mathscr{L}_{\infty\lambda}$-formulas. In the examples that we already know, the relations $\equiv_{\infty\lambda}$ and $\equiv_{\infty\lambda}^{\beta}$ are induced by classes Φ of formulas closed under negation, so that the condition

$$(\dagger) \qquad \text{for every sentence} \quad \phi \in \Phi, \quad \mathfrak{A} \models \phi \quad \text{implies} \quad \mathfrak{B} \models \phi,$$

entails its own converse. This is not true of other classes of formulas (for instance, $\Phi =$ the existential $\mathscr{L}_{\infty\lambda}$-formulas). This indicates that (†) defines the appropriate semantical relation between $\mathfrak{A}$ and $\mathfrak{B}$, which we will denote $\mathfrak{A}\ (\Phi)\ \mathfrak{B}$.

We should also expect that if an appropriate characterization of the relation $\mathfrak{A}\ (\Phi)\ \mathfrak{B}$ is to exist, the class Φ ought to have some closure properties. It turns out that these requirements are very mild.

4.4.1 Definition. A class Φ of $\mathscr{L}_{\infty\lambda}$-formulas is *normal* if it satisfies the following requirements:

(N1) $v_0 = v_0$ is in Φ;
(N2) If ϕ is in Φ, then some reduced form of ϕ is also in Φ (*a reduced form* of ϕ is obtained by "pushing" all negation symbols to their innermost places);
(N3) Φ is closed under subformulas;
(N4) Φ is closed under conjunctions and disjunctions of *sets* of its formulas;
(N5) Φ is closed under substitutions of (*some* occurrences of) variables by terms;
(N6) For every ordinal α, if Φ contains a formula of quantifier-rank $\alpha + 1$, beginning with $\exists$ (respectively, $\forall$), then any quantification of the same type on a formula in Φ of quantifier-rank $\leq \alpha$ is in Φ. $\square$

The clause (N2) is designed to allow inductions on the structure of formulas in Φ.

4.4.2 Examples. (a) The following classes of $\mathscr{L}_{\infty\lambda}$-formulas are normal: all formulas, all reduced formulas, all quantifier-free formulas [(N6) holds vacuously]; all existential formulas [(N6) holds vacuously for $\forall$], all universal formulas, all positive formulas. Furthermore, if Φ is normal, then the class Φ^{β} of all formulas in Φ of quantifier-rank $\leq \beta$ is also normal.

(b) The following classes are not normal: all prenex $\mathscr{L}_{\infty\lambda}$-formulas [(N4) fails], all $\mathscr{L}_{\kappa\omega}$-formulas [(N4) fails]. ☐

The notion of partial isomorphism must also be adapted to the present setting, and the appropriate notion for this is that of a (partial) Φ_0-*morphism*, that is, of a map preserving all quantifier-free formulas ϕ in Φ:

$$\mathfrak{A} \models \phi[g] \quad \text{implies} \quad \mathfrak{B} \models \phi[f \circ g],$$

for every assignment g in $\mathrm{Dom}(f)$.

The following result is a common generalization of Theorems 4.3.1 and 4.3.3, and its proof is similar to that of the latter.

4.4.3 Theorem. *For any normal class Φ of $\mathscr{L}_{\infty\lambda}$-formulas and for any structures $\mathfrak{A}$, $\mathfrak{B}$ with the appropriate vocabulary, the following are equivalent*:

(1) $\mathfrak{A}\ (\Phi)\ \mathfrak{B}$;

(2) *There is a sequence $\langle \mathbb{I}_\alpha \mid \alpha \in \mathrm{ON} \rangle$ of nonempty families of partial Φ_0-morphisms from $\mathfrak{A}$ to $\mathfrak{B}$, such that*;
   (a) *if $\alpha \leq \gamma$, then $\mathbb{I}_\gamma \subseteq \mathbb{I}_\alpha$*;
   (b) *for every $\alpha \in \mathrm{ON}$,*
      (i) *if Φ contains a formula of quantifier-rank $\alpha + 1$ beginning with an existential quantifier, then the forth property holds: For every $f \in \mathbb{I}_{\alpha+1}$ and $A \subseteq |\mathfrak{A}|$, $\bar{\bar{A}} < \lambda$, there is $g \in \mathbb{I}_\alpha$ such that $f \subseteq g$ and $A \subseteq \mathrm{Dom}(g)$*;
      (ii) *if Φ contains a formula of quantifier-rank $\alpha + 1$ beginning with a universal quantifier, then the back property holds: For every $f \in \mathbb{I}_{\alpha+1}$ and $B \subseteq |\mathfrak{B}|$, $\bar{\bar{B}} < \lambda$, there is $g \in \mathbb{I}_\alpha$ such that $f \subseteq g$ and $B \subseteq \mathrm{Range}(g)$.* ☐

4.4.4 Some Important Remarks. (a) If Φ has the additional property that, whenever it contains one formula of quantifier-rank > 0 beginning with $\exists$ (respectively $\forall$), then it contains formulas or arbitrary large quantifier-rank beginning with $\exists$ (respectively, $\forall$), then the sequence $\langle \mathbb{I}_\alpha \mid \alpha \in \mathrm{ON} \rangle$ can be replaced by just one family of partial morphisms. Obviously, this is the case if Φ is any one of the following classes: All formulas (see Theorem 4.3.1), existential formulas, universal formulas, positive formulas.

(b) For normal classes of the form Φ^β, where β is an ordinal—see Example 4.4.2(a)—the sequence $\langle \mathbb{I}_\alpha | \alpha \in \mathrm{ON} \rangle$ can be cut down to $\langle \mathbb{I}_\alpha | \alpha \leq \beta \rangle$. In this way, a generalization of Theorem 4.3.3 can be obtained.

These and other variants are discussed in detail in Dickmann [1975, Chapter 5, Section 3.C]. $\square$

An argument of this type leads to results such as:

4.4.5 Proposition. *Let λ be a fixed infinite cardinal. To every cardinal μ there corresponds a cardinal κ which depends only μ and λ such that if $\overline{\overline{\mathfrak{B}}} \leq \mu$, then for arbitrary $\mathfrak{A}$ the following holds:*

(a) (Chang) $\mathfrak{A}\,(\mathrm{Ex}_{\kappa\lambda})\,\mathfrak{B}$ *implies that* $\mathfrak{A}\,(\mathrm{Ex}_{\infty\lambda})\,\mathfrak{B}$;
(b) (Kueker) $\mathfrak{A} \equiv_{\kappa\lambda} \mathfrak{B}$ *implies that* $\mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$;
(c) (Chang) $\mathfrak{B}\,(\mathrm{Un}_{\kappa\lambda})\,\mathfrak{A}$ *implies that* $\mathfrak{B}\,(\mathrm{Un}_{\infty\lambda})\,\mathfrak{A}$;
(d) (Chang) *If also $\overline{\overline{\mathfrak{A}}} \leq \mu$, then*

$$\mathfrak{A}\,(\mathrm{Pos}_{\kappa\lambda})\,\mathfrak{B} \quad \textit{implies} \quad \mathfrak{A}\,(\mathrm{Pos}_{\infty\lambda})\,\mathfrak{B}.$$

Here, the symbols Ex, Un, Pos, *respectively denote the classes of existential, universal, and positive formulas of the corresponding languages.* $\square$

See Chang [1968c] or Dickmann [1975, pp. 335–339] for proofs of this.

These results hold regardless the number of symbols in the vocabulary. Bringing this parameter into consideration yields a generalization of Scott's famous countable isomorphism theorem.

4.4.6 Theorem. *Given a vocabulary τ and cardinals μ and λ, where λ is infinite, let $\kappa = \max\{\mu^{<\lambda}, \overline{\overline{\tau}}\}^+$. Then for each τ-structure $\mathfrak{A}$ of cardinality $\leq \mu$, there is an $\mathscr{L}_{\kappa\lambda}(\tau)$-sentence $\phi_{\mathfrak{A}}$ such that*

$$\mathfrak{B} \models \phi_{\mathfrak{A}} \quad \textit{iff} \quad \mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$$

holds for any structure $\mathfrak{B}$. $\square$

When $\mu = \lambda = \overline{\overline{\tau}} = \aleph_0$, $\phi_{\mathfrak{A}}$ is in $\mathscr{L}_{\omega_1\omega}$. If, in addition, $\mathfrak{B}$ is also countable, then Theorem 4.3.1 and Proposition 4.2.5 give

$$\mathfrak{B} \models \phi_{\mathfrak{A}} \quad \text{iff} \quad \mathfrak{A} \cong \mathfrak{B}.$$

This is Scott's isomorphism theorem (see also Chapter VIII, Section 4.1).

Theorem 4.4.6 holds for relations which are slightly (?) more general than $\mathscr{L}_{\infty\lambda}$-equivalence. However, we do not know of any interesting application of this additional information; see Dickmann [1975, p. 340].

The connection between isomorphism and $\mathscr{L}_{\infty\lambda}$-equivalence given by Theorem 4.3.1 and Proposition 4.2.5 when $\mathrm{cf}(\lambda) = \omega$ does have analogs in the present general setting:

4.4.7 Proposition (Chang). *Let* $\mathrm{cf}(\lambda) = \omega$. *Then*

(a) $\overline{\overline{\mathfrak{A}}} \leq \lambda$ *implies that* $\mathfrak{A}\ (\mathrm{Ex}_{\infty\lambda})\ \mathfrak{B}$ *iff* $\mathfrak{A} \lesssim \mathfrak{B}$.
(b) $\overline{\overline{\mathfrak{B}}} \leq \lambda$ *implies that* $\mathfrak{A}\ (\mathrm{Un}_{\infty\lambda})\ \mathfrak{B}$ *iff* $\mathfrak{B} \lesssim \mathfrak{A}$.
(c) $\overline{\overline{\mathfrak{A}}}, \overline{\overline{\mathfrak{B}}} \leq \lambda$ *imply that* $\mathfrak{A}\ (\mathrm{Pos}_{\infty\lambda})\ \mathfrak{B}$ *iff* $\mathfrak{B}$ *is a homomorphic image of* $\mathfrak{A}$. ▯

Propositions 4.4.5 and 4.4.7 can be combined in an obvious way to improve the left-hand side of the result, when $\mathfrak{A}$ and/or $\mathfrak{B}$ are of bounded cardinality. We immediately obtain a proof of the countable embedding theorem (see the end of Section 4.1).

4.5. *Some Applications*

No account of the back-and-forth method would be complete without at least some mention of concrete mathematical applications. And such we will briefly give here. Further examples will be found in Chapter XI, where several important applications to algebra are discussed—especially in Sections 1–5.

The Functoriality of Back-and-Forth Methods

A little practice with the application of the techniques presented in this section reveals that some of the back-and-forth relations we have considered (such as, $\simeq^{\beta}_{\lambda}$) tend to be preserved by many standard algebraic constructions. As an example of this, the reader may try

4.5.1 Exercise. Using Theorem 4.4.3, prove that if Φ normal class and $\mathfrak{A}_i\ (\Phi)\ \mathfrak{B}_i$ for all $i \in I$, then

$$\prod_{i \in I} \mathfrak{A}_i\ (\Phi) \prod_{i \in I} \mathfrak{B}_i \quad \text{and} \quad \bigoplus_{i \in I} \mathfrak{A}_i\ (\Phi) \bigoplus_{i \in I} \mathfrak{B}_i.$$

Warning: Direct sums only make sense if the vocabulary contains an individual constant, 0, such that $F(0, \ldots, 0) = 0$ for every operation F. ▯

This exercise should convince the reader that only "general nonsense" arguments are used, which is an indication of some kind of functoriality. The extent of it was worked out by Feferman [1972], who showed:

4.5.2 Theorem. *If F is a λ-local functor (see below), then F preserves $\mathscr{L}_{\infty\lambda}$-equivalence and also $\mathscr{L}_{\infty\lambda}$-equivalence up to quantifier-rank β for any ordinal β.* ▯

A *λ-local functor* is an operation on structures (of possibly infinitely many arguments) and on maps between them, satisfying:

(i) in each coordinate, its domain of definition is closed under substructures;
(ii) F preserves inclusion (of both structures and maps);
(iii) for every subset $Z \subseteq |F(\langle\mathfrak{A}_i | i \in I\rangle)|$ of power $< \lambda$ there are substructures $\mathfrak{B}_i \subseteq \mathfrak{A}_i$ $(i \in I)$, each generated by $< \lambda$ elements, such that

$$Z \subseteq |F(\langle\mathfrak{B}_i | i \in I\rangle)|.$$

Granted properties (i) and (ii), one variable ω-local functors are precisely those which preserve direct limits.

From Theorem 4.5.2 we thus infer

4.5.3 Corollary. *For the indicated values of λ, $\mathscr{L}_{\infty\lambda}$-equivalence is preserved by the following algebraic and model-theoretic constructions (among others):*

(1) *The polynomial ring in one indeterminate over a ring (any λ);*
(2) *The ring of formal power series in one indeterminate over a ring (any $\lambda \geq \aleph_1$);*
(3) *The field of fractions of an integral domain (any λ);*
(4) *The free group generated by a set (any λ);*
(5) *Tensor products of modules (any λ);*
(6) *Generalized product operations; including direct products, direct sums, and the various cardinal sum operations considered in Section 2.3 (any λ);*
(7) *The structure $\mathfrak{H}_\Sigma(X, <)$, with blueprint Σ, generated by the set of order-indiscernibles $\langle X, <\rangle$ (any λ).* $\square$

Several warnings ought to be sounded here. In particular, we caution.

(a) That these preservation results are derived by explicit description of each of the operations, rather than by use of their universal properties. This relates to Hodges' *λ-word constructions*, constructions which also preserve $\mathscr{L}_{\infty\lambda}$-equivalence (see Chapter XI, Section 6).

(b) That in (1) and (7) the functor is ω-local (hence, it is λ-local for every $\lambda \geq \omega$), while in (2) it is ω_1-local. In the other cases it is not local as it stands. However, the construction can be put in equivalent form as a composition of a local functor and other operations which preserve $\mathscr{L}_{\infty\lambda}$-equivalence.

(c) That although the method is quite general, there are certain forms of the back-and-forth argument to which it does not apply. For example, elementary equivalence is not preserved by the operations (1) and (4). See Feferman [1972, pp. 92–93].

(d) That the construction in (7) is delicate, and we will refer the reader to Morley [1968] for the details. However, the preservations results are quite general in this case. For example, this construction preserves the strong partial isomorphism relation $\simeq_\lambda^{p,e}$ (see Dickmann [1975, pp. 393–397]). As a matter of fact, an analysis of these results will show that we obtain the following extension of Theorem 4.5.2.

4.5.4 Theorem. *Let F be a unary λ-local functor sending τ-structures into τ'-structures. Let Φ and Ψ be normal classes of $\mathscr{L}_{\infty\lambda}$-formulas with vocabularies τ, τ',* *respectively. Assume that F transforms partial Φ_0-morphisms into partial Ψ_0-morphisms. Furthermore, let us assume that these classes are correlated in the following way: For every ordinal α, if Ψ contains a formula of quantifier-rank α beginning with $\exists$ (respectively, $\forall$), then Φ contains at least one formula of the same type, also of quantifier-rank α. Then, for structures $\mathfrak{A}$, $\mathfrak{B}$ in the domain of F,*

$$\mathfrak{A}\,(\Phi)\,\mathfrak{B} \quad \textit{implies} \quad F(\mathfrak{A})\,(\Psi)\,F(\mathfrak{B}). \quad \square$$

As an exercise, the reader may try to derive some consequences of this theorem in the style of Theorem 4.5.3.

Partial Isomorphisms and Reduced Products

The foregoing results also apply to the operation of reduced product modulo a filter. Indeed, it is easy to verify that we have

4.5.5 Exercise. The reduced product operation (modulo a fixed filter) is an ω-local functor. [Certain precautions will be observed in defining the reduced product of a family of maps.] $\square$

However, Benda [1969] proved a much stronger result whenever the filter satisfies some mild conditions:

4.5.6 Theorem. *Suppose we are given an infinite set I, a fixed infinite cardinal λ, a vocabulary τ of power $\leq \lambda$ and a λ-regular filter $\mathscr{F}$ on I. If the τ-structures $\{\mathfrak{A}_i \mid i \in I\}$ and $\{\mathfrak{B}_i \mid i \in I\}$ satisfy:*

$$\mathfrak{A}_i \equiv \mathfrak{B}_i \quad \textit{for each} \quad i \in I,$$

then

$$\prod_{i \in I} \mathfrak{A}_i/\mathscr{F} \equiv_{\infty\lambda^+} \prod_{i \in I} \mathfrak{B}_i/\mathscr{F}. \quad \square$$

In other words, elementary equivalence is *strengthened* to $\mathscr{L}_{\infty\lambda^+}$-equivalence.

A filter $\mathscr{F}$ is *λ-regular* just in case it contains a family of λ sets such that the intersection of any infinite number of them is empty. For more information on this matter, see Chang–Keisler [1973, Section 4.3]. The theory developed there shows that λ-regularity is a rather mild condition. For example, ω-regular and ω-incomplete (obviously) coincide, and the notions of non-principal and ω-regular ultrafilters are coextensive on sets of power less than the first measurable cardinal. This proves at once:

4.5.7 Corollary. *If $\mathscr{F}$ is an ω-incomplete filter or if $\mathscr{F}$ is a non-principal ultrafilter and $\bar{\bar{I}}$ is smaller than the first measurable cardinal, then $\mathfrak{A}_i \equiv \mathfrak{B}_i$ for all $i \in I$, implies*

$$\prod_{i \in I} \mathfrak{A}_i/\mathscr{F} \equiv_{\infty\omega_1} \prod_{i \in I} \mathfrak{B}_i/\mathscr{F},$$

for structures with a countable vocabulary. $\square$

Since a single $\mathscr{L}_{\lambda^+\lambda^+}$-sentence involves at most λ symbols, Theorem 4.5.6 also yields the conclusion

$$\prod_{i\in I} \mathfrak{A}_i/\mathscr{F} \equiv_{\lambda^+\lambda^+} \prod_{i\in I} \mathfrak{B}_i/\mathscr{F},$$

for structures with a vocabulary of arbitrary cardinality.

Benda [1972] proved that the conclusion of Theorem 4.5.6 can be strengthened to $\mathscr{L}_{\infty\lambda^{++}}$-equivalence for filters with additional properties. A proof of Theorem 4.5.6 which is, we think, easier than Benda's and which is more in keeping with the spirit of the theory that has been developed here can be found in Dickmann [1975, Theorem 5.4.15].

Real Closed Fields

As a final example, let us consider the following classical theorem of Erdös–Gillman–Henriksen [1955].

4.5.8 Theorem. *Any two real closed fields of cardinality $\aleph_1$ whose underlying orders are of type η_{ω_1}, are isomorphic.* ▯

This statement has a major drawback: It is totally vacuous unless the continuum hypothesis holds (see Gillman [1956]). An analysis of the proof reveals, however, that something is proven which has nothing to do with the cardinality of the fields, let alone with the continuum hypothesis. As in other situations, the machinery developed in this section makes it possible to formulate a statement which renders the exact content of the proof.

4.5.9 Theorem. *Let λ be a regular cardinal and F, F' two real closed fields of type η_λ (no restriction are placed on their cardinalities). Then $F \simeq_\lambda^{p,e} F'$.*

Hint of Proof. This combines the argument used in Theorem 4.2.2, together with:

(i) The fundamental result of Artin–Schreier that an isomorphism between ordered fields extends uniquely to their real closures (see Jacobson [1964; pp. 285–286]); and
(ii) the fact that if f is a partial isomorphism from F to F', $x \in F$, $y \in F'$ are transcendental over Dom(f) and Range(f) respectively, and, for all $z \in$ Dom(f),

$$x > z \quad \text{iff} \quad y > f(z),$$

then f can be uniquely extended to the subfield generated by Dom(g) $\cup$ $\{x\}$ in such a way that x is sent onto y. ▯

For details on this line of inquiry, the reader should see Dickmann [1977].

Chapter X
Game Quantification

by Ph. G. Kolaitis

Game quantification interacts with the model theory of infinitary logics, abstract model theory, generalized recursion theory, and descriptive set theory. The aim of this chapter is to examine these connections and give some applications of the game quantifiers to the above areas of mathematical logic.

The chapter is divided into four sections. The first presents the basic notions and the interpretation of infinite strings of quantifiers via two-person infinite games. Section 2 deals with the interaction between game quantification and global definability theory, the main theme being that certain second-order statements can be reduced to formulas involving the game quantifiers which can, in turn, be approximated by formulas of $L_{\infty\omega}$. This section also includes a proof of Vaught's covering theorem, as well as applications of game quantification to the model theory of $L_{\omega_1\omega}$ and admissible fragments. In Section 3, we show that the game logics are absolute and unbounded, and most of the model-theoretic properties of these logics will then follow from this fact. Section 4, the final section, discusses the interaction with local definability theory. Here we consider the basic relation of the game quantifiers to inductive definability and higher recursion theory, and give some of their uses in descriptive set theory.

1. *Infinite Strings of Quantifiers*

This section presents the main definitions and basic results about infinite strings of quantifiers $(Q_0x_0Q_1x_1Q_2x_2\ldots)$ where, for each $i = 0, 1, 2, \ldots, Q_i$ is the *existential quantifier* $\exists$ or the *universal quantifier* $\forall$ on a set A. The interpretation of such strings is via two-person infinite games of perfect information. We first describe the interpretation in an informal way and indicate the expressive power of certain infinite strings. The precise definitions involve the notions of a winning strategy and a winning quasistrategy. The Gale–Stewart theorem is then proven and used to push negation through infinite strings in certain cases.

Throughout this section, A is a non-empty infinite set, $A^{<\omega} = \bigcup_{n\in\omega} A^n$ is the set of all finite sequences from A, and A^ω is the collection of all infinite sequences of elements of A. We use variables $x, y, z, \ldots$ to denote elements of A, variables

$s, t, u, \ldots$ to represent elements of $A^{<\omega}$, and variables $\alpha, \beta, \ldots$ to denote the members of A^ω. The empty sequence is denoted by $(\ )$, while $s^\frown t$ denotes the *concatenation* of two elements s, t of $A^{<\omega}$. Finally, if $\alpha \in A^\omega$ and $n \in \omega$, then $\alpha \restriction n$ is the restriction of α to n, that is, $\alpha \restriction n = (\alpha(0), \alpha(1), \ldots, \alpha(n-1)) \in A^n$.

1.1. Iterating the Existential and the Universal Quantifier Infinitely Often

1.1.1. The most natural infinite strings of quantifiers are obtained by iterating the existential quantifier or the universal quantifier—or, alternatively, the existential and the universal quantifier. If $R \subseteq A^\omega$ is a non-empty set of infinite sequences from A, then three infinite strings that result in this way are:

(1) $$(\exists x_0\, \exists x_1\, \exists x_2 \cdots) R(x_0, x_1, x_2, \ldots),$$

(2) $$(\forall x_0\, \forall x_1\, \forall x_2 \cdots) R(x_0, x_1, x_2, \ldots),$$

(3) $$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots) R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots).$$

The first two strings, (1) and (2), respectively express existential and universal quantification over the set A^ω of infinite sequences from A. In order to interpret the infinite string given in (3), we associate it with the following two-person game $G(\exists\forall, R)$ of perfect information:

> A round of the game $G(\exists\forall, R)$ is played by players I and II alternatively choosing elements from A:
>
> | | | | | | | | |
> |---|---|---|---|---|---|---|---|
> | I | x_0 | | x_1 | | x_2 | | $\cdots$ |
> | II | | y_0 | | y_1 | | y_2 | $\cdots$ |
>
> *Player* I *wins the above round* if $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots) \in R$, otherwise *Player* II *wins the round.*

We say that *Player* I *wins the game* $G(\exists\forall, R)$ if I has a systematic way to win every round of the game. Similarly, we say that *Player* II *wins the game* $G(\exists\forall, R)$ if II has a systematic way to win every round of the game. Finally, we put

$$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots) R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$$
$$\text{iff} \quad \text{Player I wins the game } G(\exists\forall, R).$$

In general, if $\bar{Q} = (Q_0, Q_1, Q_2, \ldots, Q_i, \ldots)$ is an arbitrary infinite string such that each Q_i is the existential or the universal quantifier, then the interpretation of the statement

(4) $$(Q_0 x_0 Q_1 x_1 Q_2 x_2 \cdots Q_i x_i \cdots) R(x_0, x_1, x_2, \ldots, x_i, \ldots)$$

is entirely analogous to the preceding one for (3). More specifically, we associate with $\bar{Q}$ and R a two-person infinite game $G(\bar{Q}, R)$ in a round of which, for each $i = 0, 1, 2, \ldots$, an element x_i in A is picked by Player I if $Q_i = \exists$ and by Player II if $Q_i = \forall$. At the end of the round, *Player* I *wins the round* if the infinite sequence $(x_0, x_1, x_2, \ldots, x_i, \ldots)$ is an element of R. Otherwise, *Player* II *wins the round.* We say that *Player* I *wins the game* $G(\bar{Q}, R)$ if I has a systematic way to win every round of it. Similarly, we say that *Player* II *wins the game* $G(\bar{Q}, R)$ if II has a systematic way to win every round of it. As before, we put

$$(Q_0 x_0 Q_1 x_1 Q_2 x_2 \cdots Q_i x_i \cdots) R(x_0, x_1, x_2, \ldots, x_i, \ldots)$$
$$\text{iff} \quad \text{Player I wins the game } G(\bar{Q}, R).$$

1.1.2 Remark. Often the infinite strings given in (1), (2), (3), and (4) are not applied to arbitrary relations $R \subseteq A^\omega$, but rather to relations which are either *open* or *closed.*

A relation $R \subseteq A^\omega$ is *open*, if it can be written as the infinitary disjunction of finitary relations; that is, if there are relations $R_n \subseteq A^n$, $n \in \omega$, such that

$$R(x_0, x_1, \ldots, x_{n-1}, x_n, \ldots) \Leftrightarrow \bigvee_{n \in \omega} R_n(x_0, x_1, \ldots, x_{n-1}).$$

Similarly, we say that a relation $R \subseteq A^\omega$ is *closed* if it can be written as the infinitary conjunction of finitary relations; that is, if there are relations $R_n \subseteq A^n$, for each $n \in \omega$, such that

$$R(x_0, x_1, \ldots, x_{n-1}, x_n, \ldots) \Leftrightarrow \bigwedge_{n \in \omega} R_n(x_0, x_1, \ldots, x_{n-1}).$$

This terminology is justified by the fact that a relation R is open (or closed) if it is an open set (or, respectively, a closed set) in the product topology on A^ω, where A is equipped with the discrete topology.

If the infinite strings in (1), (2), and (3) are applied to relations on A^ω which are open or closed, they can then be identified with certain monotone quantifiers on the set $A^{<\omega}$ of finite sequences from A. In order to make this idea precise, we introduce the following notions, which will be also used in Section 4 of this chapter.

1.1.3 Definitions. A *monotone quantifier Q on a set A* is a collection Q of subsets of A such that:

(i) Q is non-trivial; that is, $\varnothing \subsetneq Q \subsetneq \mathscr{P}(A)$;
(ii) Q has the *monotonicity property*, that is, if $X \in Q$ and $X \subseteq Y$, then $Y \in Q$.

Interchangeably, we write

$$QxR(x) \quad \text{iff} \quad R \in Q \quad \text{iff} \quad \{x \in A : R(x)\} \in Q.$$

The *dual* of a monotone quantifier Q is the collection $\breve{Q}$, where

$$X \in \breve{Q} \quad \text{iff} \quad (A - X) \notin Q.$$

It is quite clear that $\breve{Q}$ is also a monotone quantifier and that $(\breve{Q})^{\smile} = Q$.

Under these definitions, the *existential quantifier* $\exists$ *on* A is identified with the collection of non-empty subsets of A, and we write

$$\exists = \{X \subseteq A : X \neq \varnothing\},$$

while the *universal quantifier* $\forall$ *on* A is the singleton given by

$$\forall = \{A\}.$$

We obviously have that

$$\breve{\exists} = \forall \quad \text{and} \quad \breve{\forall} = \exists.$$

By iterating the existential and the universal quantifier on A infinitely often, we obtain the following interesting quantifiers on the set $A^{<\omega}$ of finite sequences from A:

(5) *The Suslin quantifier* $\mathscr{S}$

$$\mathscr{S} = \left\{X \subseteq A^{<\omega} : (\forall x_0\, \forall x_1\, \forall x_2 \cdots) \bigvee_n ((x_0, x_1, x_2, \ldots, x_{n-1}) \in X)\right\}.$$

(6) *The classical quantifier* $\mathscr{A}$

$$\mathscr{A} = \left\{X \subseteq A^{<\omega} : (\exists x_0\, \exists x_1\, \exists x_2 \cdots) \bigwedge_n ((x_0, x_1, x_2, \ldots, x_{n-1}) \in X)\right\}.$$

Here it is obvious that $\mathscr{A}$ is the dual of the Suslin quantifier.

(7) *The open game quantifier* $\mathscr{G}$,

$$\mathscr{G} = \left\{X \subseteq A^{<\omega} : (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots)\right.$$
$$\left.\bigvee_n ((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in X)\right\}.$$

(8) *The closed game quantifier* $\breve{\mathscr{G}}$

$$\breve{\mathscr{G}} = \left\{ X \subseteq A^{<\omega} : (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)\right.$$
$$\left.\bigwedge_n ((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in X)\right\}.$$

It will follow from results in Section 1.2 that the closed game quantifier is the dual of the open game quantifier.

1.1.4 Remark. The Suslin quantifier $\mathscr{S}$, the classical quantifier $\mathscr{A}$, and the two-game quantifiers can capture properties which are not, in general, expressible using the infinitary logic $L_{\omega_1\omega}$ or even the logic $L_{\infty\infty}$. The following examples indicate the expressive power of these quantifiers.

(i) The notion of well-foundedness can be expressed using the Suslin quantifier $\mathscr{S}$. Indeed, if R is a binary relation on a set A, then

$$R \text{ is } \textit{well-founded} \text{ iff } \quad (\forall x_0\, \forall x_1\, \forall x_2 \cdots) \bigvee_n (\neg(x_{n+1} R x_n)).$$

It is well known, of course, that this property is not expressible in the infinitary logic $L_{\omega_1\omega}$.

(ii) If $\mathfrak{A}$ is a structure which possesses a first-order definable coding machinery of finite sequences, then the Suslin quantifier and the classical quantifier $\mathscr{A}$ can be identified with monotone quantifiers on the universe A of the structure $\mathfrak{A}$. For example, this is the case with the structure $\mathbb{N} = \langle \omega, +, \cdot \rangle$ of natural numbers. On this structure, the Suslin quantifier and the classical quantifier $\mathscr{A}$ can capture second-order statements. This follows from the fact that on $\mathbb{N}$ every Π^1_1 relation $R(\bar{z})$ can be written in the form

$$R(\bar{z}) \Leftrightarrow (\forall x_0\, \forall x_1\, \forall x_2 \cdots)\left(\bigvee_n \psi(\langle x_0, x_1, \ldots, x_{n-1}\rangle, \bar{z})\right),$$

where ψ is a first-order formula and $\langle x_0, x_1, \ldots, x_{n-1}\rangle$ is an element of ω coding the sequence $(x_0, x_1, \ldots, x_{n-1})$.

The above is a rather special property of the structure $\mathbb{N}$ of natural numbers. At the other extreme, if $\mathbb{R} = \langle \omega^\omega \cup \omega, \omega, +, \cdot, Ap\rangle$, where $Ap(\alpha, n) = \alpha(n)$, is the structure of real numbers, then the Suslin quantifier and the classical quantifier $\mathscr{A}$ coincide respectively with the universal and the existential quantifier on the reals. This is a consequence of the fact that we can code infinitely many reals by a real in a first-order definable way.

(iii) The open game and the closed game quantifier have, in general, higher expressive power than the Suslin and the classical quantifier $\mathscr{A}$. If a structure $\mathfrak{A}$ possesses a first-order coding machinery of finite sequences, then the relation of satisfaction "$\mathfrak{A} \models \varphi$", where φ is a sentence of the first-order logic of the vocabulary of $\mathfrak{A}$, can be shown to be expressible in terms of the open game or the closed game quantifier, while this relation is not first-order definable on such structures. In particular, on the structure $\mathbb{R}$ of the real numbers the game quantifiers properly transcend the Suslin and the classical $\mathscr{A}$ quantifier.

The connections between local definability theory and game quantification will be investigated in Section 4 of this chapter.

(iv) Consider a vocabulary τ consisting of two binary relation symbols $<_1$, $<_2$ and the equality symbol $=$. Using the infinite string $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)$ and countable disjunctions and conjunctions, we can write a statement $\varphi(u, v, <_1, <_2)$ expressing that:

"$<_1$ and $<_2$ are well-orderings

and

u is in the field of $<_1$, v is in the field of $<_2$

and

the order type $|u|_1$ of u in $<_1$ is less than or equal to the order type $|v|_2$ of v in $<_2$."

The crucial property $|u|_1 \leq |v|_2$ is then expressed as follows:

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots)\left[\left(\bigwedge_n (x_n <_1 u) \leftrightarrow \bigwedge_n (y_n <_2 v)\right)\right.$$
$$\left. \wedge \bigwedge_{m,n} (x_m <_1 x_n \leftrightarrow y_m <_2 y_n) \wedge \bigwedge_{m,n} (x_m = x_n \leftrightarrow y_m = y_n)\right].$$

The proof that this statement works can be obtained by induction on $|u|_1$.

From the above, it easily follows that using the infinite string $(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots)$ and countable disjunctions and conjunctions, we can write a statement $\psi(<)$ asserting that

"$<$ is a well-ordering of order type $\gamma + \gamma$ for some ordinal γ".

Malitz [1966] has shown, however, that this statement is not expressible by any formula of the infinitary logic $L_{\infty\infty}$. Thus, game quantification can give rise to infinitary logics which are different from the usual infinitary logics $L_{\kappa\lambda}$. These new infinitary logics will be introduced and studied in Section 3 of this chapter, while in Section 2 we will pursue the relationship between game quantification and global definability theory.

1.2. Winning Strategies and Winning Quasistrategies

Assume that $\bar{Q} = (Q_0, Q_1, Q_2, \ldots, Q_i, \ldots)$ is an infinite string such that for each $i = 0, 1, 2, \ldots$ Q_i is the existential or the universal quantifier on a set A. In the preceding section the interpretation of the statement

$$(Q_0 x_0 Q_1 x_1 Q_2 x_2 \cdots Q_i x_i \cdots) R(x_0, x_1, x_2, \ldots, x_i, \ldots)$$

was given in a rather informal way, since we defined the concept "Player I wins the game $G(\bar{Q}, R)$" by saying simply that "Player I has a systematic way to win every round of the game $G(\bar{Q}, R)$." This definition is intuitive, but not very precise. We will now give precise definitions of these concepts in a set-theoretic framework. It actually turns out that we can give at least two interpretations for infinite strings of quantifiers which are equivalent in the presence of the full axiom of choice, but which may nevertheless be different if only weaker choice principles are available. For the sake of clarity, we give the definitions and then state the results only for the infinite string $(\exists, \forall, \exists, \forall, \ldots, \exists, \forall, \ldots)$. However, these notions will generalize to arbitrary strings $\bar{Q} = (Q_0, Q_1, Q_2, \ldots, Q_i, \ldots)$ with only notational changes in the definitions or the proofs.

1.2.1. Let $R \subseteq A^\omega$ be a relation on the set of infinite sequences from A, and let $G(\exists\forall, R)$ be the two-person infinite game associated with the statement

$$(9) \qquad (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots).$$

A *strategy* σ *for Player* I *in the game* $G(\exists\forall, R)$ *is a function* $\sigma\colon \bigcup_{n\in\omega} A^{2n} \to A$ from the set of finite sequences of even length into A.

Intuitively, a strategy σ for I provides him with a next move. We say that I *follows the strategy* σ *in a round* $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of the game $G(\exists\forall, R)$ if $x_0 = \sigma(())$ and $x_n = \sigma((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}))$, for all $n = 1, 2, 3, \ldots$. We call σ *a winning strategy for* I *in the game* $G(\exists\forall, R)$ if I wins every round of the game in which he follows σ.

In an analogous way, we define a *strategy* τ *for Player* II *in* $G(\exists\forall, R)$ to be a function $\tau\colon \bigcup_{n\in\omega} A^{2n+1} \to A$. *Player* II *follows* τ *in a round* $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of the game if $y_n = \tau((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n))$ for all $n = 0, 1, 2, \ldots$. We say that τ *is a winning strategy for* II *in* $G(\exists\forall, R)$ if II wins every round of the game in which he follows τ.

Using the above notions, we rigorously interpret the statement given in (9) as follows:

$$(10) \qquad (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall x_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$$
$$\text{iff} \quad \text{Player I has a winning strategy for the game } G(\exists\forall, R).$$

In practice, when we prove theorems about infinite strings of quantifiers, we must often use the axiom of choice to exhibit a winning strategy for one of the players in the game associated with the infinite string. There are situations, however, in which one is working in a set theory where the full axiom of choice is not available. In such cases, we can still prove the results about the infinite strings of quantifiers by reformulating the interpretation of the infinite string given in (9). The idea here is to replace the notion of a strategy by that of a quasistrategy, a quasistrategy being essentially a multiple-valued strategy that provides the player with a non-empty set of possible next moves instead of a single move.

A *quasistrategy* Σ *for Player* I *in the game* $G(\exists\forall, R)$ is a set $\Sigma \subseteq A^{<\omega}$ of finite sequences from A such that:

(i) there is some $x_0 \in A$ for which $(x_0) \in \Sigma$;
(ii) if $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in \Sigma$, then there is some $x \in A$ for which $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x) \in \Sigma$;
(iii) if $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x) \in \Sigma$, then for every $y \in A$
$$(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y) \in \Sigma.$$

Player I *follows the quasistrategy* Σ *in a round* $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of $G(\exists\forall, R)$ if every initial segment of the round is in Σ. Furthermore, we say that Σ *is a winning quasistrategy for* I *in the game* $G(\exists\forall, R)$ if I wins every round of the game in which he follows Σ.

We define also the notions of *quasistrategy for* II *and winning quasistrategy for* II *in the game* $G(\exists\forall, R)$ in an analogous dual way.

We can now interpret the statement in (9) in an alternative way as follows:

(11) $$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$$
iff Player I has a winning quasistrategy in the game $G(\exists\forall, R)$.

It is quite obvious that if Player I has a winning strategy in the game $G(\exists\forall, R)$, then I also has a winning quasistrategy in this game. If, in addition, the set A can be well-ordered, then every winning quasistrategy for I in $G(\exists\forall, R)$ gives rise to a winning strategy for I in this game. We therefore see that, in the presence of the axiom of choice, the two interpretations given by (10) and (11) of the statement in (9) are equivalent. This equivalence, however, depends on the axiom of choice in an essential way.

If we interpret the infinite string $(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots)$ via quasistrategies, then most theorems about this string can be proved using the axiom of dependent choices. A weaker principle than the full axiom of choice, *the axiom of dependent choices* states that, for every non-empty set B and for every binary relation $P \subseteq B \times B$ on B,

$$(\forall x \in B)(\exists y \in B)P(x, y) \Rightarrow (\exists f : \omega \to B)(\forall n)P(f(n), f(n + 1)).$$

Observe that we used the axiom of dependent choices implicitly, when we asserted in Section 1.1.4 that the Suslin quantifier can express the notion of well-foundedness. Indeed, this axiom is precisely the choice principle needed to show that a relation is well-founded if and only if it has no infinite descending chains.

We will now investigate some simple properties of strategies and quasistrategies, beginning with

1.2.2 Lemma. *Let* $R \subseteq A^\omega$ *be a relation on the set of infinite sequences from* A. *Then,*

(i) *It is not possible that both Players* I *and* II *have winning strategies in the game* $G(\exists\forall, R)$.
(ii) (*Assuming the axiom of dependent choices*). *It is not possible that both Players* I *and* II *have winning quasistrategies in the game* $G(\exists\forall, R)$.

Proof. Part (i) is obvious and requires no choice principles. To prove part (ii) we will assume, towards a contradiction, that Player I has a winning quasistrategy Σ in $G(\exists\forall, R)$ and that II also has a winning quasistrategy T in this same game. Using dependent choices, we can then produce a round $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of the game $G(\exists\forall, R)$ every initial segment of which is in both Σ and T. But then the round $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ is in both R and $\neg R$. This is a contradiction. $\square$

If $R \subseteq A^{\omega}$ is a relation on the set of infinite sequences from A, and if $G(\exists\forall, R)$ is the game associated with the statement

(9) $$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots) R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots),$$

then $G(\forall\exists, \neg R)$ is the game associated with the statement

(12) $$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1\, \forall x_2\, \exists y_2 \cdots) \neg R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots).$$

It is clear from the definitions that a winning strategy (respectively, quasistrategy) for II in the game $G(\exists\forall, R)$ is a winning strategy (respectively quasistrategy) for I in the game $G(\forall\exists, \neg R)$. We therefore have the following

1.2.3 Lemma. *Let $R \subseteq A^{\omega}$ be a relation on the set of infinite sequences from A. Then,*

(i) *Player* II *has a winning strategy (respectively quasistrategy) in $G(\exists\forall, R)$ if and only if Player* I *has a winning strategy (respectively quasistrategy) in $G(\forall\exists, \neg R)$.*
(ii) *Player* I *has a winning strategy (respectively quasistrategy) in $G(\exists\forall, R)$ if and only if Player* II *has a winning strategy (respectively quasistrategy) in $G(\forall\exists, \neg R)$.* □

Assume now that $R \subseteq A^{\omega}$ is a relation such that Player I or Player II has a winning strategy (respectively a winning quasistrategy) in the game $G(\exists\forall, R)$. Combining this with Lemmas 1.2.2 and 1.2.3, we obtain the equivalence:

(13) $$\neg(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) R(x_0, y_0, x_1, y_1, \ldots) \Leftrightarrow (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \neg R(x_0, y_0, x_1, y_1, \ldots),$$

where the interpretation of the statements given in (9) and (12) is via winning strategies as in (10) (respectively via winning quasistrategies as in (11)).

We say that the game $G(\exists\forall, R)$ is *determined* if Player I or Player II has a winning strategy in this game. We also say that $G(\exists\forall, R)$ is *weakly determined* if Player I or Player II has a winning quasistrategy in the game. The preceding facts show that if the game $G(\exists\forall, R)$ is determined or weakly determined, then to negate the statement given in (9), we can push the negation through the infinite string $(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots)$ and apply it to the relation R. Although this manipulation is always true for finite strings and all relations R, it is not true for infinite strings and arbitrary relations $R \subseteq A^{\omega}$. Indeed using the axiom of choice, Gale and Stewart [1953] showed that there is a relation $R \subseteq 2^{\omega}$ such that the game $G(\exists\forall, R)$ is not determined. It turns out, however, that if the relation R is open or closed, then the associated game $G(\exists\forall, R)$ is determined.

1.2.4 Theorem (Gale–Stewart [1953]). *Let $R \subseteq A^{\omega}$ be a relation on the set of infinite sequences from A which is either open or closed. Then,*

(i) *(Assuming the axiom of choice). Player I or Player* II *has a winning strategy in the game $G(\exists\forall, R)$;*
(ii) *Player* I *or Player* II *has a winning quasistrategy in the game $G(\exists\forall, R)$.*

Proof. The first part of the theorem follows from the second by well-ordering the set A. Moreover, in view of Lemma 1.2.3, it is enough to establish the result for the case of a closed relation $R \subseteq A^\omega$. Therefore, assume that there are finitary relations $R_n \subseteq A^{2n+2}$, for each $n = 0, 1, 2, \ldots$, such that

$$R(x_0, y_0, x_1, y_1, \ldots, x_n, y_n, \ldots) \Leftrightarrow \bigwedge_{n \in \omega} R_n(x_0, y_0, x_1, y_1, \ldots, x_n, y_n).$$

We will show that Player I or Player II has a winning quasistrategy in the game $G(\exists\forall, R)$. The winning quasistrategy will be obtained by using an inductive analysis for the set of "winning positions" for Player I in the open game $G(\forall\exists, \neg R)$. More precisely, consider the following monotone operator $\varphi(u, S)$, where u ranges over the elements of $A^{<\omega}$ and S over the subsets of $A^{<\omega}$:

$$\varphi(u, S) \Leftrightarrow (u \text{ has even length}) \,\&\, \Big(\text{if } u = (x_0, y_0, \ldots, x_n, y_n),$$

$$\text{then } \bigvee_{m \le n} \neg R_m(x_0, y_0, \ldots, x_m, y_m)\Big) \vee (\forall x\, \exists y)(u^\frown(x, y) \in S).$$

By induction on the ordinals define a sequence $\{\varphi^\xi\}_\xi$ of subsets of $A^{<\omega}$, where

$$u \in \varphi^0 \Leftrightarrow \varphi(u, \varnothing),$$
$$u \in \varphi^\xi \Leftrightarrow \varphi(u, \bigcup_{\eta < \xi} \varphi^\eta),$$

and let $\varphi^\infty = \bigcup_\xi \varphi^\xi$. Intuitively, the set φ^∞ consists of all "winning positions" for Player I in the game $G(\forall\exists, \neg R)$, since (using the axiom of dependent choices) we can show that

$$(14) \qquad (x_0, y_0, \ldots, x_n, y_n) \in \varphi^\infty$$
$$\Leftrightarrow (\forall x_{n+1}\, \exists y_{n+1}\, \forall x_{n+2}\, \exists y_{n+2} \cdots) \bigvee_{m \in \omega} \neg R_m(x_0, y_0, \ldots, x_m, y_m).$$

In completing the proof of the theorem, we will not use the above equivalence, but have included it in order to make the role of φ^∞ transparent.

We claim now that if the empty sequence $(\ )$ is not in φ^∞, then Player I has a winning quasistrategy in the game $G(\exists\forall, R)$, while if $(\ ) \in \varphi^\infty$, then Player II has a winning quasistrategy in $G(\exists\forall, R)$. Indeed, if $(\ ) \notin \varphi^\infty$, then it can be easily verified that the set

$$\Sigma = \{u \in A^{<\omega}\colon (u \text{ has even length and } u \notin \varphi^\infty)$$
$$\vee\ (u \text{ has odd length and } (\forall y)(u^\frown(y) \notin \varphi^\infty)\}$$

is a winning quasistrategy for I in $G(\exists\forall, R)$. On the other hand, if $(\ ) \in \varphi^\infty$, then for $u \in \varphi^\infty$, we first put $|u|_\varphi$ = least ordinal ξ such that $u \in \varphi^\xi$, and then let

$$\begin{aligned} T = \{u \in A^{<\omega}:\ & \text{for every } v \in A^{<\omega} \text{ if } v = (x_0, y_0, \ldots, x_i, y_i, x_{i+1}, y_{i+1}) \\ & \text{is an initial segment of } u \text{ of even length, then } v \in \varphi^\infty \\ & \text{and} \\ & |(x_0, y_0, \ldots, x_i, y_i)|_\varphi = 0 \quad \text{or} \\ & |(x_0, y_0, \ldots, x_i, y_i)|_\varphi \\ & \qquad > |(x_0, y_0, \ldots, x_i, y_i, x_{i+1}, y_{i+1})|_\varphi\}. \end{aligned}$$

It is now quite easy to show that T is a winning quasistrategy for II in $G(\exists\forall, R)$. □

Combining the Gale–Stewart theorem with Lemmas 1.2.2 and 1.2.3 we have the following:

1.2.5 Corollary. *Let $R \subseteq A^\omega$ be a relation which is open or closed. Then,*

(i) *(Assuming the axiom of choice). Player* I *does not have a winning strategy in $G(\exists\forall, R)$ if and only if Player* II *has a winning strategy in $G(\exists\forall, R)$.*
(ii) *(Assuming the axiom of dependent choices). Player* I *does not have a winning quasistrategy in $G(\exists\forall, R)$ if and only if Player* II *has a winning quasistrategy in $G(\exists\forall, R)$.* □

The above corollary allows us to push the negation through the infinite string Thus, if $R \subseteq A^\omega$ is open or closed, then

$$\neg(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots)R(x_0, y_0, x_1, y_1, \ldots) \Leftrightarrow (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)\, \neg R(x_0, y_0, x_1, y_1, \ldots). \tag{13}$$

□

1.2.6 Corollary. *The closed game quantifier $\breve{\mathcal{G}}$ is the dual of the open game quantifier $\mathcal{G}$.* □

As was mentioned in the introduction to this section, all the preceding results extend to arbitrary infinite strings. In general, if $\overline{Q} = (Q_0, Q_1, \ldots, Q_i, \ldots)$ is an infinite string such that for each $i = 0, 1, 2, \ldots$ Q_i is the existential or the universal quantifier on A, then the *dual string* $\overline{Q^\cup}$ is defined by

$$\overline{Q^\cup} = (\breve{Q}_0, \breve{Q}_1, \ldots, \breve{Q}_i, \ldots).$$

If a relation $R \subseteq A^\omega$ is open or closed, then we have the equivalence

$$\neg(Q_0 x_0 Q_1 x_1 \cdots Q_i x_i \cdots)R(x_0, x_1, \ldots, x_i, \ldots) \Leftrightarrow (\breve{Q}_0 x_0 \breve{Q}_1 x_1 \cdots \breve{Q}_i x_i \cdots)\, \neg R(x_0, x_1, \ldots, x_i, \ldots). \tag{15}$$

Proof of the above equivalence requires the full axiom of choice if the interpretation is via winning strategies and the axiom of dependent choices if the interpretation is via winning quasistrategies.

1.2.7. In view of the preceding results for the open and the closed games, it is natural to ask whether or not there are other relations $R \subseteq A^\omega$ for which the game $G(\exists\forall, R)$ is determined. We say that the game $G(\exists\forall, R)$ is *Borel* if the relation R is a Borel set in the product topology on A^ω, where A discrete. Martin [1975] proved that in ZFC every Borel game is determined. His proof actually established that in ZF + axiom of dependent choices (DC) every Borel game is weakly determined; that is, that, one of the two players has a winning quasistrategy in such a game. The question of determinacy for games $G(\exists\forall, R)$, where R has higher complexity, is independent of ZF and leads into strong set-theoretic hypotheses.

1.2.8 Remarks. We have two reasons in mind for making explicit the distinction between winning quasistrategies and winning strategies. The first, is that it is often important to know the weakest possible metatheory in which we can formulate and prove results about infinite strings of quantifiers. This will be useful, in Section 3 of this chapter; for there we discuss the set-theoretic definability of the infinitary logics built by using the game quantifiers. The second reason is the connection between game quantification and descriptive set theory, a connection which will be briefly pursued in Section 4. Much of the current research in descriptive set theory is carried in ZF together with the axiom of dependent choices (DC) and the hypothesis that certain infinite games are weakly determined.

From now on, we will distinguish explicitly between strategies and quasistrategies in only a very few cases. Instead, we will use the statement "*Player* I *wins the game* $G(\exists\forall, R)$" for both interpretations, i.e., depending on the context or on the metatheory available, this means that Player I has a winning strategy or a winning quasistrategy in the game $G(\exists\forall, R)$.

1.2.9. We should point out that finite strings of quantifiers at the beginning can always be absorbed inside an infinite string. More precisely, for any relation $R \subseteq A^\omega$, we have the equivalence

$$(16)\qquad \begin{aligned}&(Q_0 x_0)(Q_1 x_1)\cdots(Q_n x_n)\{(Q_{n+1}x_{n+1})(Q_{n+2}x_{n+2})\cdots\}\\ &\quad R(x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots)\\ &\quad \Leftrightarrow (Q_0 x_0 Q_1 x_1 \cdots Q_n x_n Q_{n+1} x_{n+1}\cdots)R(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots),\end{aligned}$$

where $Q_i = \exists$ or $Q_i = \forall$, for each $i = 0, 1, 2, \ldots$.

In general, if the relation R is arbitrary, the proof of the above equivalence requires the axiom of choice, even though the interpretation may be via winning quasistrategies. However, in the case where R is open or closed, no choice principles are required in the proof, since there are canonical quasistrategies for such games.

We end this section with two simple propositions. These will provide a first insight into the relationship between game quantification and second-order logic.

If $R \subseteq A^{<\omega}$ is a relation on the set of finite sequences from A, then R gives rise to an *open* relation $\bigvee R$ and a *closed* relation $\bigwedge R$ on the set A^ω of infinite sequences from A, where

$$\bigvee R = \{\alpha \in A^\omega : \text{there is some } n \in \omega \text{ such that } (\alpha \restriction n) \in R\}$$

and

$$\bigwedge R = \{\alpha \in A^\omega : (\alpha \restriction n) \in R \text{ for all } n \in \omega\}.$$

1.2.10 Proposition. *Let $R \subseteq A^{<\omega}$ be a relation on the set of finite sequences from A. Then,*

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots \forall x_n\, \exists y_n \cdots) \bigwedge_n R(x_0, y_0, x_1, y_1, \ldots, x_n, y_n)$$
$$\text{iff} \quad (\exists T)(T \text{ is a winning quasistrategy for I in } G(\forall\exists, \textstyle\bigwedge R) \text{ and } T \subseteq R).$$

Proof. The result follows immediately from the observation that if T is a winning quasistrategy for Player I in $G(\forall\exists, \bigwedge R)$, then, using dependent choices, we see that any sequence $u = (x_0, y_0, \ldots, x_n, y_n)$ in T can be extended to a round $(x_0, y_0, \ldots, x_n, y_n, x_{n+1}, y_{n+1}, \ldots)$ of $G(\forall\exists, \bigwedge R)$ in which I follows T. □

The closed game quantifier can be expressed using second-order existential quantification. This is the content of the next proposition, a result that we will use repeatedly in the sequel.

1.2.11 Proposition. *Let $R \subseteq A^{<\omega}$ be a relation on the set of finite sequences from A. Then,*

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots \forall x_n\, \exists y_n \cdots) \bigwedge_n R(x_0, y_0, x_1, y_1, \ldots, x_n, y_n)$$

$$\begin{aligned}\text{iff} \quad (\exists T_1\, \exists T_2 \cdots \exists T_n \cdots)\Big\{ \bigwedge_n\, & (T_n \subseteq A^{2n} \;\&\; T_1 \subseteq R \\ & \&\; (\forall x_0\, \exists y_0)((x_0, y_0) \in T_1) \\ & \&\; (\forall x_0\, \forall y_0 \cdots \forall x_{n-1}\, \forall y_{n-1})[(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in T_n \\ & \quad \to (R(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \\ & \&\; (\forall x_n\, \exists y_n)(T_{n+1}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n, y_n))] \Big\}.\end{aligned}$$

Proof. In view of Proposition 1.2.10, it is enough to consider a winning quasistrategy T for I in the game $G(\forall\exists, \bigwedge R)$ and to put

$$T_n = \{(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in A^{2n} : (x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in T\}. \quad □$$

2. *Projective Classes and the Approximations of the Game Formulas*

In this section we will study the interactions between game quantification and global definability theory. The first basic result to be presented here is Svenonius' theorem which establishes that on countable structures the relations definable by the closed game quantifier coincide with the Σ_1^1 relations. Following this theorem, we will show that the game quantifier formulas can be approximated by formulas of the infinitary logic $L_{\omega_1\omega}$. These two results make it possible to analyze certain second-order statements, such as Σ_1^1 and Π_1^1 formulas, by the use of methods and techniques from the model theory of $L_{\omega_1\omega}$. As an illustration of these ideas, we will here outline a proof of Vaught's covering theorem. The section will end with applications of the approximations of the game formulas to descriptive set theory and to the model theory of $L_{\omega_1\omega}$ and admissible fragments.

2.1. *Game Quantification and Projective Classes*

Throughout this section we will be working with vocabularies which contain only relation and constant symbols. If τ is such a vocabulary, then $L_{\omega\omega}[\tau]$ is the set of all first-order formulas of vocabulary τ. As usual, $L_{\omega_1\omega}$ is the infinitary logic which allows for countable disjunctions and conjunctions, while $L_{\omega_1\omega}[\tau]$ is the set of all formulas of $L_{\omega_1\omega}$ of vocabulary τ. If the vocabulary is either fixed or understood from the context, then we will often write $L_{\omega\omega}$ and $L_{\omega_1\omega}$ instead of $L_{\omega\omega}[\tau]$ and $L_{\omega_1\omega}[\tau]$.

In what follows *countable* means of cardinality less than or equal to ω; that is, the cardinality is either finite or denumerably infinite. Moreover, we write HF for the set of hereditarily finite sets and HC for the set of hereditarily countable sets, so that

$$\mathrm{HF} = \{x\colon |Tc(x)| < \omega\} \quad \text{and} \quad \mathrm{HC} = \{x\colon |Tc(x)| < \omega_1\}.$$

All the vocabularies to be considered here are countable. If τ is such a countable vocabulary, then we can identify the formulas of $L_{\omega_1\omega}[\tau]$ with set-theoretic objects, so that if φ is in $L_{\omega_1\omega}[\tau]$, then $Tc(\{\varphi\}) \subseteq \mathrm{HC}$. In particular, we have that

$$L_{\omega\omega}[\tau] = L_{\omega_1\omega}[\tau] \cap \mathrm{HF} \quad \text{and} \quad L_{\omega_1\omega}[\tau] = L_{\omega_1\omega}[\tau] \cap \mathrm{HC}.$$

If A is an admissible set (possibly with urelements) and $\tau \in A$, then

$$L_A[\tau] = L_{\infty\omega}[\tau] \cap A$$

denotes the *admissible fragment of* $L_{\infty\omega}[\tau]$ *associated with* A, *where* $L_{\infty\omega}$ *is the* infinitary logic which allows for arbitrary disjunctions and conjunctions, but which only allows for finite strings of quantifiers.

2.1.1 Definitions. Let τ be a countable vocabulary containing only relation and constant symbols.

(i) We say that a *second-order formula* φ *is* $\mathrm{PC}_\Delta[\tau]$ (or simply PC_Δ) if it is of the form

$$\exists \bar{R} \bigwedge_{n \in \omega} \psi_n(\bar{R}),$$

where $\bar{R}$ is a countable set of relation symbols $\bar{R} = (R_1, R_2, \ldots)$ not in the vocabulary τ and where, for each $n \in \omega$, we have that $\psi_n(\bar{R})$ is a formula of $L_{\omega\omega}[\tau']$, with $\tau' = \tau \cup \bar{R}$.

(ii) We say that a second-order formula φ is Σ_1^1 *over* $L_{\omega_1\omega}[\tau]$, and we write φ is $\Sigma_1^1(L_{\omega_1\omega}[\tau])$ or simply $\Sigma_1^1(L_{\omega_1\omega})$ if it is of the form

$$\exists \bar{R} \psi(\bar{R}),$$

where $\bar{R}$ is a countable set of relation symbols not in τ and $\psi(\bar{R})$ is a formula of $L_{\omega_1\omega}[\tau']$, with $\tau' = \tau \cup \bar{R}$.

(iii) If A is an admissible set and $\tau \in A$, then we say that a formula φ is Σ_1^1 *over* $L_A[\tau]$, and we write φ is $\Sigma_1^1(L_A[\tau])$ or simply $\Sigma_1^1(L_A)$, in case φ is of the form

$$\exists \bar{R} \psi(\bar{R}),$$

where $\bar{R}$ is a countable set of relation symbols not in τ such that $\bar{R} \in A$ and $\psi(\bar{R})$ is a formula of the admissible fragment $L_A[\tau']$, with $\tau' = \tau \cup \bar{R}$.

We now introduce the notions of a *closed game formula* and an *open game formula*, which are obtained by applying the closed and the open game quantifier to formulas of the first-order logic $L_{\omega\omega}$.

2.1.2 Definitions. Let τ be a vocabulary which is countable and contains only relation and constant symbols.

(i) We say that $\Phi(\bar{z})$ is a *closed game formula* if it is of the form

$$(1) \qquad (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_{n < \omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where φ_n is a formula of $L_{\omega\omega}[\tau]$ in the displayed free variables, for each $n \in \omega$.

(ii) We say that $\Phi(\bar{z})$ is an *open game formula* if it is of the form

$$(2) \qquad (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_{n < \omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where φ_n is a formula of $L_{\omega\omega}[\tau]$ in the displayed free variables, for each $n \in \omega$.

The Gale–Stewart theorem (1.2.4) implies that the negation of a closed game formula is always logically equivalent to an open game formula, and vice-versa. It actually turns out that there is a strong connection between PC_Δ formulas and closed game formulas. However, in order to analyze $\Sigma_1^1(L_{\omega_1\omega})$ formulas we must consider the following generalization of the game formulas, a generalization introduced by Vaught [1973b].

(iii) A *closed Vaught formula* $\Phi(\bar{z})$ is one of the form

$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right) \bigwedge_{n<\omega} \varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}), \tag{3}$$

where I is a countable set and, for each $(i_0, j_0, \ldots, i_{n-1}, j_{n-1}) \in I^{2n}$, we have that $\varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a formula of $L_{\omega_1\omega}[\tau]$ in the displayed free variables.

(iv) An *open Vaught formula* $\Phi(\bar{z})$ is one of the form

$$\left(\exists x_0 \bigvee_{i_0 \in I} \forall y_0 \bigwedge_{j_0 \in I} \exists x_1 \bigvee_{i_1 \in I} \forall y_1 \bigwedge_{j_1 \in I} \cdots\right) \bigvee_{n<\omega} \varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}), \tag{4}$$

where I is a countable set and each $\varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a formula of $L_{\omega_1\omega}[\tau]$ in the displayed free variables.

To simplify the already cumbersome notation, we will henceforth write

$$\bar{\imath},\ \bar{\jmath} \quad \text{for the sequence} \quad (i_0, j_0, \ldots, i_{n-1}, j_{n-1}) \text{ in } I^{2n}$$

and

$$\bar{x},\ \bar{y} \quad \text{for the sequence of variables} \quad (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

so that

$$\varphi^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}) \quad \text{denotes the formula}$$

$$\varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}).$$

(v) We say that $\Phi(\bar{z})$ is a *game formula* if it is either an open or a closed game formula. Similarly, a *Vaught formula* is one which is either an open or a closed Vaught formula.

(vi) If $\Phi(\bar{z})$ is either a game formula or a Vaught formula and if A is an admissible set, then we say that $\Phi(\bar{z})$ *is in* A just in case the family of formulas $\{\varphi^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}): (\bar{\imath}, \bar{\jmath}) \in I^{2n}, n < \omega\}$ is an element of A.

2.1.3. If $\mathfrak{A}$ is a structure of vocabulary τ, then the interpretation of a Vaught formula on $\mathfrak{A}$ is via a two-person infinite game in a round of which Player I and Player II take turns and each chooses an element from the universe A of the structure $\mathfrak{A}$ and an index from the set I. The definition of a winning strategy and a winning quasistrategy in this game is analogous to that given in Section 1.2. The Gale–Stewart theorem extends to Vaught formulas by essentially the same proof, so that the negation of a closed Vaught formula is logically equivalent to an open Vaught formula, and vice-versa.

In general, game formulas cannot capture statements expressible by formulas of the weak second-order logic L_{wII}. On the other hand, the infinitary logic $L_{\omega_1\omega}$ is stronger than L_{wII}, so that if we hope to study $\Sigma_1^1(L_{\omega_1\omega})$ formulas using some infinitary logic, then we must consider a logic which is at least as strong as L_{wII}. These comments provide a first justification for introducing the Vaught formulas. We should also point out here that if $I = \omega$ and $\mathfrak{A} = \langle A, \ldots \rangle$ is a structure of vocabulary τ such that $\omega \subseteq A$ and $\mathfrak{A}$ possesses a first-order coding machinery of finite sequences, then the open and the closed Vaught formulas have no more expressive power than the formulas obtained by applying the open and the closed game quantifier to formulas of $L_{\omega_1\omega}$. Of course, over such structures the weak second-order logic L_{wII} is subsumed by the first-order logic $L_{\omega\omega}$.

We now proceed to investigate the connections between PC_Δ and $\Sigma_1^1(L_{\omega_1\omega})$ formulas on the one hand and closed game and Vaught formulas on the other. All the results refer to a fixed vocabulary τ which is countable and contains only relation and constant symbols.

2.1.4 Proposition. (i) *Any closed game formula is logically equivalent to a* PC_Δ *formula.*

(ii) *Any closed Vaught formula* $\Phi(\bar{z})$ *is logically equivalent to a* $\Sigma_1^1(L_{\omega_1\omega})$ *formula. Moreover, if* A *is an admissible set and* $\Phi(\bar{z})$ *is in* A, *then* $\Phi(\bar{z})$ *is logically equivalent to a* $\Sigma_1^1(L_A)$ *formula.*

Proof. The first part of this proposition follows immediately from Proposition 1.2.11. On the other hand, the extension of this proposition to closed Vaught formulas gives easily the second part. ▯

Svenonius [1965] established a partial converse to Proposition 2.1.4. More specifically, he showed that over countable models the closed game formulas have the same expressive power as the PC_Δ formulas. Vaught [1973b] obtained a generalization of this result by introducing the class of formulas which here we call closed Vaught formulas and by showing that over countable structures they are equivalent to the $\Sigma_1^1(L_{\omega_1\omega})$ formulas. Before presenting these results, we will introduce the following notation:

$\models' \varphi$ means that the sentence φ is true in all countable structures.

Notice that if φ is a sentence of $L_{\omega_1\omega}[\tau]$, then

$$\models' \varphi \quad \text{if} \quad \models \varphi,$$

because the Skolem–Löwenheim theorem holds for the infinitary logic $L_{\omega_1\omega}$.

2.1.5 Theorem. (i) (Svenonius [1965]). *For any* PC_Δ *formula* $\exists\bar{R} \bigwedge_{n<\omega} \psi_n(\bar{z}, \bar{R})$, *there is a sequence of quantifier-free formulas* $\varphi_n(\bar{z}, \bar{x}, \bar{y})$ *of* $L_{\omega\omega}[\tau]$ *such that if* $\Phi(\bar{z})$ *is the closed game formula* $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_n \varphi_n(\bar{z}, \bar{x}, \bar{y})$, *then*

(a) $\models \exists\bar{R} \bigwedge_{n<\omega} \psi_n(\bar{z}, \bar{R}) \to \Phi(\bar{z})$;

(b) $\models' \Phi(\bar{z}) \to \exists\bar{R} \bigwedge_{n<\omega} \psi_n(\bar{z}, \bar{R})$; *and hence*

(c) $\models' \Phi(\bar{z}) \leftrightarrow \exists\bar{R} \bigwedge_{n<\omega} \psi_n(\bar{z}, \bar{R})$.

Moreover, the quantifier-free formulas $\varphi_n(\bar{z}, \bar{x}, \bar{y})$ *can be obtained recursively from n,* $\bar{R}$ *and the sequence* $\{\psi_n(\bar{z}, \bar{R})\}$.

(ii) (Vaught [1973b]). *For any* $\Sigma_1^1(L_{\omega_1\omega})$ *formula* $\exists\bar{R}\psi(\bar{z}, \bar{R})$, *there is a closed Vaught formula* $\Phi(\bar{z})$ *which does not contain symbols from* $\bar{R}$ *and such that*

(a) $\models \exists\bar{R}\psi(\bar{z}, \bar{R}) \to \Phi(\bar{z})$;

(b) $\models' \Phi(\bar{z}) \to \exists\bar{R}\psi(\bar{z}, \bar{R})$; *and hence*

(c) $\models' \Phi(\bar{z}) \leftrightarrow \exists\bar{R}\psi(\bar{z}, \bar{R})$.

Moreover, the formulas $\{\varphi^{\bar{i},\bar{j}}(\bar{z}, \bar{x}, \bar{y})\colon (\bar{i}, \bar{j}) \in I^{2n}, n < \omega\}$, *which determine* $\Phi(\bar{z})$, *can be chosen to be in* $L_{\omega\omega}[\tau]$ *and to depend on* $\exists\bar{R}\psi(\bar{z}, \bar{R})$ *and* ω *in a primitive recursive way. In particular, if A is an admissible set,* $\omega \in A$ *and* $\exists\bar{R}\psi(\bar{z}, \bar{R})$ *is* $\Sigma_1^1(L_A)$, *then the closed Vaught formula* $\Phi(\bar{z})$ *can be chosen in A.*

Sketch of Proof. In what follows we merely outline a proof of part (i) and give a hint of the proof of part (ii) of the theorem.

If we add new constant symbols, it will suffice to prove the result for a PC_Δ sentence $\exists\bar{R} \bigwedge_{n<\omega} \psi_n(\bar{R})$, where $\psi_n(\bar{R})$ is a sentence of $L_{\omega\omega}[\tau \cup \bar{R}]$, for each $n \in \omega$. Moreover, using the Skolem normal form, we may assume without loss of generality that the PC_Δ sentence $\exists\bar{R} \bigwedge_{n<\omega} \psi_n(\bar{R})$ is actually of the form

$$\exists\bar{R} \bigwedge_{n<\omega} (\forall x_1 \cdots \forall x_{k_n})(\exists y_1 \cdots \exists y_{l_n})\chi_n(x_1, \ldots, x_{k_n}, y_1, \ldots, y_{l_n}, \bar{R}),$$

where $\chi_n(x_1, \ldots, x_{k_n}, y_1, \ldots, y_{l_n}, \bar{R})$ is a quantifier-free formula of $L_{\omega\omega}[\tau \cup \bar{R}]$, for each $n \in \omega$.

To make the game-theoretic argument involved transparent, we will also assume that we have only one quantifier-free formula $\chi(x, y, \bar{R})$ in the variables x and y, so that the original PC_Δ sentence is

$$\exists\bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R}).$$

It is easy to show that for any quantifier-free formula $\theta(\bar{w}, \bar{R})$ in $L_{\omega\omega}[\tau \cup \bar{R}]$ one can find, recursively from θ, a quantifier-free formula $\theta^*(\bar{w})$ in $L_{\omega\omega}[\tau]$ such that

$$\models \exists\bar{R}\theta(\bar{w}, \bar{R}) \leftrightarrow \theta^*(\bar{w}).$$

Using the above fact, we let $\varphi_n(x_0, y_0, \ldots, x_n, y_n)$ be a quantifier free formula of $L_{\omega\omega}[\tau]$ which is logically equivalent to

$$\exists \bar{R} \bigwedge_{m \le n} \chi(x_m, y_m, \bar{R})$$

and then consider the closed game sentence Φ:

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_n \varphi_n(x_0, y_0, x_1, y_1, \ldots, x_n, y_n).$$

We claim that this closed game sentence has the required properties, namely

(a) $\models \exists \bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R}) \to \Phi$; and
(b) $\models' \Phi \leftrightarrow \exists \bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R})$.

It is clear that if $\mathfrak{A}$ is a structure of vocabulary τ such that

$$\mathfrak{A} \models \exists \bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R}),$$

then the set

$$\Sigma = \{u \in A^{<\omega}\colon \text{if } (x_0, y_0, \ldots, x_n, y_n) \subseteq u, \text{ then } (\mathfrak{A}, \bar{R}^{\mathfrak{A}}, x_n, y_n) \models \chi(x, y, \bar{R})\}$$

is a winning quasistrategy for Player I in the game associated with Φ.

Assume now that $\mathfrak{A}$ is a countable structure such that $\mathfrak{A} \models \Phi$. Consider a round of the game associated with Φ in which Player II enumerates the universe A of $\mathfrak{A}$ and Player I answers using his winning quasistrategy; that is, the round looks like:

II	a_0		a_1		a_2		$\cdots$
I		b_0		b_1		b_2	$\cdots$

with $A = \{a_0, a_1, a_2, \ldots\}$.

Since I follows his winning quasistrategy in this round, we have that

$$\mathfrak{A} \models \exists \bar{R} \bigwedge_{m \le n} \chi(a_m, b_m, \bar{R}), \quad \text{for all} \quad n \in \omega.$$

Let $\mathbf{a}_m$, $\mathbf{b}_m$, for $m < \omega$, be new constant symbols not in τ and consider the set of quantifier-free sentences T, where

$$T = \text{Diagram}(\mathfrak{A}) \cup \{\chi(\mathbf{a}_m, \mathbf{b}_m, \bar{R})\colon m < \omega\}.$$

T is finitely satisfiable; and, hence, by the compactness theorem it has a model. Since each sentence $\chi(\mathbf{a}_m, \mathbf{b}_m, \bar{R})$ is quantifier-free, this implies that there is a set $\bar{R}^{\mathfrak{A}}$ of relations on A such that

$$\mathfrak{A}, \bar{R}^{\mathfrak{A}} \models \chi(\mathbf{a}_m, \mathbf{b}_m, \bar{R}) \quad \text{for each} \quad m < \omega.$$

However, the sequence $\{a_0, a_1, a_2, \ldots\}$ exhausts the universe A of the structure $\mathfrak{A}$, and therefore we have

$$\mathfrak{A}, \bar{R}^{\mathfrak{A}} \models (\forall x)(\exists y)\chi(x, y, \bar{R}).$$

The main argument remains the same in the general case where we have infinitely many quantifier-free formulas $\chi_n(x_1, \ldots, x_{k_n}, y_1, \ldots, y_{l_n}, \bar{R})$ for $\eta < \omega$. There are only minor combinatorial complications which can be handled by enumerating the tuples $\bar{x}$, $\bar{y}$ of variables in such a way that the variables occurring at stage m of the enumeration have indices $\leq m$. This completes the proof of the first part of the theorem.

In order to establish part (ii) of our result we show first that a $\Sigma_1^1(L_{\omega_1\omega}[\tau])$ formula $\Psi(\bar{z})$ is equivalent to a PC_Δ formula $\Psi'(\bar{z})$ over an expanded vocabulary τ' which contains τ and subsumes weak second-order logic. By applying part (i) of the above, we can find a closed game formula $\Phi'(\bar{z})$ over τ' which is logically equivalent to $\Psi'(\bar{z})$ on countable structures. The closed game formula $\Phi'(\bar{z})$ over τ' can, in turn, be translated to a closed Vaught formula $\Phi(\bar{z})$ over τ. In such a translation the propositional part of the Vaught formula is used to capture the expanded vocabulary.

We should point out that Harnik [1974] and Makkai [1977a] gave direct proofs of part (ii) by associating an appropriate countable admissible fragment with the $\Sigma_1^1(L_{\omega_1\omega}[\tau])$ formula $\Psi(\bar{z})$. The proof is analogous to the one we gave for part (i) with the model existence theorem for fragments used in place of the compactness theorem. □

2.2. *The Approximations of the Game and the Vaught Formulas*

In Section 1 we pointed out that game formulas can be used to capture statements which are not expressible in $L_{\infty\infty}$. We will see here however that the Vaught formulas in general and the game formulas in particular can be approximated by formulas of $L_{\infty\omega}$. This result combined with Theorem 2.1.5 (the theorems of Svenonius and of Vaught) makes it possible to analyze $\Sigma_1^1(L_{\omega_1\omega})$ and $\Pi_1^1(L_{\omega_1\omega})$ formulas via $L_{\omega_1\omega}$ formulas.

2.2.1 Definition (Vaught [1973a]). Assume that $\Phi(\bar{z})$ is a closed Vaught formula of the form

$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right) \bigwedge_n \varphi^{i,j}(\bar{z}, \bar{x}, \bar{y}).$$

Then, for any $n < \omega$, any $(\bar{i}, \bar{j}) = (i_0, j_0, \ldots, i_{n-1}, j_{n-1}) \in I^{2n}$, and any ordinal α, by induction on α simultaneously define a formula

$$\delta_\alpha^{\bar{i},\bar{j}}(\bar{z}, \bar{x}, \bar{y}) \equiv \delta_\alpha^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

as follows:

(1) $\delta_0^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is

$$\bigwedge_{m \leq n} \varphi^{i_0 j_0 \cdots i_{m-1} j_{m-1}}(\bar{z}, x_0, y_0, \ldots, x_{m-1}, y_{m-1});$$

(2) $\delta_{\alpha+1}^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is

$$\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I} \delta_\alpha^{i_0 j_0 \cdots i_n j_n}(\bar{z}, x_0, y_0, \ldots, x_n, y_n);$$

(3) $\delta_\alpha^{\bar{i},\bar{j}}(\bar{z}, \bar{x}, \bar{y})$ is $\bigwedge_{\beta < \alpha} \delta_\beta^{\bar{i},\bar{j}}(\bar{z}, \bar{x}, \bar{y})$, if α is a limit ordinal.

We write $\delta_\alpha(\bar{z})$ for the formula $\delta_\alpha^{(\ )}(\bar{z})$, where $(\ )$ is the empty sequence, and we call $\delta_\alpha(\bar{z})$ the *α-th approximation of* $\Phi(\bar{z})$. For each ordinal α, we let $\rho_\alpha(\bar{z})$ be the formula

(4)
$$\bigwedge_{n<\omega} \left[\left(\forall x_0 \bigwedge_{i_0 \in I} \forall y_0 \bigwedge_{j_0 \in I} \cdots \forall x_{n-1} \bigwedge_{i_{n-1} \in I} \forall y_{n-1} \bigwedge_{j_{n-1} \in I} \right) (\delta_\alpha^{\bar{i},\bar{j}}(\bar{z}, \bar{x}, \bar{y}) \rightarrow \delta_{\alpha+1}^{\bar{i},\bar{j}}(\bar{z}, \bar{x}, \bar{y})) \right].$$

2.2.2. It is clear that for each ordinal α and each $(\bar{i}, \bar{j}) \in I^{2n}$, where $n < \omega$, the formulas $\delta_\alpha^{\bar{i},\bar{j}}(\bar{z})$ and $\rho_\alpha(\bar{z})$ are formulas of $L_{\infty\omega}$. Moreover, if $\alpha < \omega_1$, then they are actually formulas of $L_{\omega_1\omega}$.

It is also quite easy to verify that the formulas $\delta_\alpha^{\bar{i},\bar{j}}(\bar{z})$ can be defined by a Σ-recursion as a function of the Vaught formula $\Phi(\bar{z})$, the sequence $\bar{i}, \bar{j}$ and the ordinal α. Consequently, if A is an admissible set having ordinal $o(A)$ and if the Vaught formula $\Phi(\bar{z})$ is in A, then for every ordinal $\alpha < o(A)$, the formulas $\delta_\alpha(\bar{z})$ and $\rho_\alpha(\bar{z})$ are elements of A.

2.2.3. If $\Phi(\bar{z})$ is a closed game formula, then the approximations of $\Phi(\bar{z})$ are defined in an analogous way, although they are actually of a simpler form. More specifically, if $\Phi(\bar{z})$ is the closed game formula

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \bigwedge_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

then

(5) $$\delta_0^n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is} \quad \bigwedge_{m \le n} \varphi_m(\bar{z}, x_0, y_0, \ldots, x_{m-1}, y_{m-1}),$$

(6) $$\delta_{\alpha+1}^n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is} \quad \forall x_n \, \exists y_n \delta_\alpha^{n+1}(\bar{z}, x_0, y_0, \ldots, x_n, y_n),$$

(7) $$\delta_\alpha^n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is} \quad \bigwedge_{\beta < \alpha} \delta_\beta^n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

for α limit.

We write $\delta_\alpha(\bar{z})$ for the formula $\delta_\alpha^0(\bar{z})$ and call it the α-th *approximation of the closed game formula* $\Phi(\bar{z})$.

Also, we put $\rho_\alpha(\bar{z})$ for the formula

(8) $$\bigwedge_{n < \omega} [(\forall x_0 \, \forall y_0 \cdots \forall x_{n-1} \, \forall y_{n-1})(\delta_\alpha^n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \to \delta_{\alpha+1}^n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}))].$$

If $\Phi(\bar{z})$ is an open Vaught formula (or an open game formula), then we define the approximations

$$\varepsilon_\alpha^{i,j}(\bar{z}, \bar{x}, \bar{y}) \quad (\text{respectively, } \varepsilon_\alpha^n(\bar{z}, \bar{x}, \bar{y}))$$

of $\Phi(\bar{z})$ in a dual way, so that if

$$\delta_\alpha^{i,j}(\bar{z}, \bar{x}, \bar{y}) \quad (\text{respectively, } \delta_\alpha^n(\bar{z}, \bar{x}, \bar{y}))$$

are the approximations of the closed Vaught formula (or the closed game formula) which is logically equivalent to $\neg\Phi(\bar{z})$, then

$$\varepsilon_\alpha^{i,j}(\bar{z}, \bar{x}, \bar{y}) \quad \text{is logically equivalent to} \quad \neg\delta_\alpha^{i,j}(\bar{z}, \bar{x}, \bar{y})$$

$$(\text{respectively, } \varepsilon_\alpha^n(\bar{z}, \bar{x}, \bar{y}) \text{ is logically equivalent to } \neg\delta_\alpha^n(\bar{z}, \bar{x}, \bar{y})).$$

2.2.4 Example. Let $<$ be a binary relation symbol in the vocabulary τ and let Φ be the open game sentence which asserts that $<$ is well-founded; that is to say, Φ is the sentence

$$(\forall x_0 \, \forall x_1 \, \forall x_2 \cdots)\left(\bigvee_{n \in \omega} \neg(x_{n-1} < x_{n-2})\right).$$

Below we compute the approximations ε_α of Φ and find their meaning:

(i) if $m < \omega$, then $\varepsilon_m = \varepsilon_m^0$ is the sentence

$$(\forall x_0 \, \forall x_1 \cdots \forall x_{m-1})\left(\bigvee_{k \le m} \neg(x_{k-1} < x_{k-2})\right).$$

(ii) $\varepsilon_\omega = \varepsilon_\omega^0$ is the sentence

$$\bigvee_{m<\omega} \varepsilon_m^0 = \bigvee_{m<\omega} \left[(\forall x_0 \, \forall x_1 \cdots \forall x_{m-1})\left(\bigvee_{k\leq m} \neg(x_{k-1} < x_{k-2})\right)\right].$$

Notice that ε_ω asserts that, for some $m < \omega$, there is no descending chain with m elements in $<$. Therefore, ε_ω states that $<$ is a well-founded relation of finite rank.

(iii) $\varepsilon_{\omega+1} = \varepsilon_{\omega+1}^0$ is the sentence

$$\forall x_0\left(\bigvee_{m<\omega} \forall x_1 \, \forall x_2 \cdots \forall x_{m-1}\left(\bigvee_{k\leq m} \neg(x_{k-1} < x_{k-2})\right)\right).$$

This sentence asserts that, for every element x in the field of $<$, the set of predecessors of x has finite rank. Therefore, $\varepsilon_{\omega+1}$ is equivalent to the assertion that $<$ is a well-founded relation with rank $\leq \omega < \omega + 1$.

The pattern revealed in (i), (ii), and (iii) holds in general. Indeed, by induction on α, we can show that for any ordinal α

ε_α asserts that " $<$ is a well-founded relation of rank less than α".

It follows, therefore, that if $\mathfrak{A}$ is a structure of cardinality $\leq k$, then

$$\mathfrak{A} \models (\forall x_0 \, \forall x_1 \, \forall x_2 \cdots)\left(\bigvee_{n<\omega} \neg(x_{n-1} < x_{n-2})\right) \quad \text{iff} \quad \mathfrak{A} \models \bigvee_{\alpha<\kappa^+} \varepsilon_\alpha.$$

Later on we will show that the above equivalence holds for arbitrary open games or for open Vaught formulas. Before developing the general theory of the approximations, we will present the main properties of the finite approximations of game formulas on saturated structures. Consequently, we now consider

2.2.5 Theorem. *Let $\Phi(\bar{z})$ be the closed game formula*

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \bigwedge_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

and let $\mathfrak{A}$ be a structure of vocabulary τ.

(i) *If $\mathfrak{A}$ is ω-saturated, then*

$$\mathfrak{A} \models \forall \bar{z}\left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{m<\omega} \delta_m(\bar{z})\right);$$

(ii) *If $\mathfrak{A}$ is recursively saturated and the sequence $\{\varphi_n(\bar{z}, \bar{x}, \bar{y}) : n < \omega\}$ is recursive, then again*

$$\mathfrak{A} \models \forall \bar{z}\left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{m<\omega} \delta_m(\bar{z})\right).$$

Proof. We outline the argument for part (ii) since that part is the effective version of part (i).

Let $\mathfrak{A}$ be a recursively saturated structure and assume that the sequence $\{\varphi_n(\bar{z}, \bar{x}, \bar{y}): n < \omega\}$ is recursive. It is clear from the definition of the finite approximations that for any structure $\mathfrak{A}$

$$\mathfrak{A} \models \forall\bar{z}(\Phi(\bar{z}) \to \delta_m(\bar{z})) \quad \text{for all} \quad m < \omega.$$

Thus, it remains to show that, under the above hypotheses,

$$\mathfrak{A} \models \forall\bar{z}\left(\bigwedge_{m<\omega} \delta_m(\bar{z}) \to \Phi(\bar{z})\right).$$

The main idea comes from the proof of the Gale–Stewart theorem in Section 1. More specifically, as in Theorem 1.2.4, we consider the monotone operator $\varphi(\bar{z}, u, S)$, where

$$\begin{aligned}\varphi(\bar{z}, u, S) \Leftrightarrow (u \in A^{<\omega} &\text{ and } u \text{ has even length})\\ &\& \text{ (if } u = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}),\\ &\text{then } (\bigvee_{k\le n} \neg\, \varphi_k(\bar{z}, x_0, y_0, \ldots, x_{k-1}, y_{k-1})\\ &\vee (\exists x\, \forall y)((\bar{z}, u^\frown(x, y)) \in S)).\end{aligned}$$

Let φ^α be the stages of the inductive definition generated by φ. That is,

$$\varphi^0 = \{(\bar{z}, u) : \varphi(\bar{z}, u, \varnothing)\}, \quad \text{and} \quad \varphi^\alpha = \left\{(\bar{z}, u) : \varphi\left(\bar{z}, u, \bigcup_{\beta<\alpha} \varphi^\beta\right)\right\}.$$

From this, it is easy to show that, for any $m < \omega$ and any $n < \omega$, we have

$$(1) \qquad (\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in \varphi^m \quad \text{iff} \quad (\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \notin \delta_m^n.$$

Since the sequence $\{\varphi_n(\bar{z}, \bar{x}, \bar{y}): n < \omega\}$ is recursive, we can view $\varphi(\bar{z}, u, S)$ as a Σ_1 monotone inductive definition on $\mathrm{HYP}_{\mathfrak{A}}$. But $\mathfrak{A}$ is recursively saturated and so $o(\mathrm{HYP}_{\mathfrak{A}}) = \omega$. Therefore, by Gandy's theorem, (see Barwise [1975]) the inductive definition must close off at ω steps, so that we then have

$$(2) \qquad \varphi^\infty = \bigcup_{m<\omega} \varphi^m.$$

Assume now that $\mathfrak{A}, \bar{z} \models \bigwedge_{m<\omega} \delta_m(\bar{z})$. Then $\bar{z} \notin \varphi^m$ for all $m < \omega$ by the equivalence given in (1). Hence, $\bar{z} \notin \varphi^\infty$ by (2). The proof of the Gale–Stewart theorem implies then that Player I has a winning quasistrategy in the closed game $G(\forall\exists, \bigwedge_{n<\omega} \varphi_n)$. Hence, $\mathfrak{A}, \bar{z} \models \Phi(\bar{z})$. ☐

2.2.6. In many respects, the idea behind the approximations has its origins in classical descriptive set theory and the approximations of the operator $\mathscr{A}$ (see, for example, Kuratowski [1966]). The finite approximations of closed game formulas were introduced by Keisler [1965c], who established, among other results, the first part of Theorem 2.2.5. Moschovakis [1969, 1971, 1974a] developed the theory of positive elementary inductive definability on arbitrary structures $\mathfrak{A}$ which possess a first-order coding machinery of finite sequences. He obtained the basic connection between inductive definability and game quantification; and, in essence, discovered the properties of the approximations δ_α of closed game formulas.

However, Moschovakis' results were of a local nature, since they dealt with an arbitrary but fixed structure. In the abstracts Chang–Moschovakis [1968], Chang [1968a], and the paper by Chang [1971b], the approximations of the game formulas are used implicitly in the study of global definability. The approximations of the Vaught formulas were introduced by Vaught [1973b] who established their main properties and used them in the study of $\Sigma_1^1(L_{\omega_1\omega})$ and $\Pi_1^1(L_{\omega_1\omega})$ formulas.

2.2.7 Theorem (Vaught [1973b]). *Let $\Phi(\bar{z})$ be a closed Vaught formula of the form*

$$\left(\forall x_0 \bigwedge_{i_0\in I} \exists y_0 \bigvee_{j_0\in I} \forall x_1 \bigwedge_{i_1\in I} \exists y_1 \bigvee_{j_1\in I} \cdots\right) \bigwedge_n \varphi^{\bar{\imath},\bar{\jmath}}(\bar{z},\bar{x},\bar{y}).$$

Then, we have

(i) *for any ordinals α, β with $\alpha > \beta$ and for any $\bar{\imath}, \bar{\jmath}$,*

$$\models \delta_\alpha^{\bar{\imath},\bar{\jmath}}(\bar{z},\bar{x},\bar{y}) \to \delta_\beta^{\bar{\imath},\bar{\jmath}}(\bar{z},\bar{x},\bar{y});$$

(ii) *for any ordinal α,*

$$\models \Phi(\bar{z}) \to \delta_\alpha(\bar{z}) \text{ and } \models (\delta_\alpha(\bar{z}) \wedge \rho_\alpha(\bar{z})) \to \Phi(\bar{z});$$

(iii) *for any structure $\mathfrak{A}$ of cardinality $\leq \kappa$,*

(1) $$\mathfrak{A} \models \forall \bar{z}\left(\bigvee_{\alpha<\kappa^+} \delta_\alpha(\bar{z})\right);$$

(2) $$\mathfrak{A} \models \forall \bar{z}\left[\Phi(\bar{z}) \leftrightarrow \bigwedge_{\alpha<\kappa^+} \delta_\alpha(\bar{z})\right];$$

(3) $$\mathfrak{A} \models \forall \bar{z}\left[\Phi(\bar{z}) \leftrightarrow \bigvee_{\alpha<\kappa^+} (\rho_\alpha(\bar{z}) \wedge \delta_\alpha(\bar{z}))\right];$$

(iv) *Moreover, if M is an admissible set, $o(M) > \omega$, $\Phi(\bar{z})$ is in M and $\mathfrak{A} \in M$, then*

$$\mathfrak{A} \models \forall \bar{z} \rho_{o(M)}(\bar{z})$$

and hence

$$\mathfrak{A} \models \forall \bar{z}\left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{\alpha < o(M)} \delta_\alpha(\bar{z})\right).$$

Hint of Proof. Part (i) is proven by induction on the ordinal α. Part (ii) follows easily from the definitions of the formulas δ_α and ρ_α. For example, if $\mathfrak{A}$ is a structure such that $\mathfrak{A}, \bar{z} \models \delta_\alpha(\bar{z}) \wedge \rho_\alpha(\bar{z})$, then the set

$$\Sigma = \{u \in (A \times I)^{<\omega} : (\forall v)((v = (x_0, i_0, y_0, j_0, \ldots, x_{n-1}, i_{n-1}, y_{n-1}, j_{n-1}) \\ \& (v \subseteq u)) \rightarrow \mathfrak{A}, \bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1} \models \delta_\alpha^{\bar{i},\bar{j}})\}$$

is a winning quasistrategy for Player I in the game associated with $\Phi(\bar{z})$. Hence, $\mathfrak{A}, \bar{z} \models \Phi(\bar{z})$.

The proof of parts (iii) and (iv) requires the inductive analysis of the dual open game and is similar to the proof of Theorem 2.2.5. In (iii), a cardinality argument shows that the corresponding monotone operator closes off at some ordinal $\alpha < \kappa^+$. In (iv) this is proved using Gandy's theorem or directly using a boundedness argument. ☐

The following result is an immediate consequence of Theorem 2.2.7 in which we take $\kappa = \omega$ in part (iii). It has interesting applications in descriptive set theory.

2.2.8 Corollary. *Let $\Phi(\bar{z})$ be a closed Vaught formula. Then*

(i) $\models' (\forall \bar{z})\left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{\alpha < \omega_1} \delta_\alpha(\bar{z})\right)$

(ii) $\models' (\forall \bar{z})\left(\Phi(\bar{z}) \leftrightarrow \bigvee_{\alpha < \omega_1} (\delta_\alpha(\bar{z}) \wedge \rho_\alpha(\bar{z}))\right)$ ☐

Theorem 2.2.7 is the main result on the approximations of the closed Vaught and the closed game formulas. We can, of course, formulate and prove an analogous "dual" result on the approximations of the open Vaught and the open game formulas.

Burgess [1977] introduced a notion of approximations for formulas of abstract logics and showed that if $(L^*, \models^*)$ is an absolute logic, then the formulas of L^* can be approximated by formulas of $L_{\infty\omega}$. His proof makes use of Theorem 2.2.7, since he shows first that any formula of L^* can be approximated by formulas involving game quantification and arbitrary disjunctions and conjunctions. More about these results can be found in Chapter XVII of this volume.

In what follows we will combine the Svenonius–Vaught result which is given in Theorem 2.1.5 with the results on approximations in order to study properties of the $\Sigma_1^1(L_{\omega_1\omega})$ and the $\Pi_1^1(L_{\omega_1\omega})$ formulas. We begin by proving a strong version of the interpolation theorem for $L_{\omega\omega}$.

2.2.9 Theorem. *Let Φ, Ψ be $\Sigma_1^1(L_{\omega\omega})$ sentences and let $\delta_m^{\Psi^*}$, where $m < \omega$, be the finite approximations of the closed game sentence Ψ^* which is equivalent to Ψ on countable structures.*

If $\models \Phi \to \neg\Psi$, then there is some $m < \omega$ such that $\models \Phi \to \neg\delta_m^{\Psi^*}$.

Proof. In order to derive a contradiction, we assume that $\models \Phi \to \neg\Psi$, but for all $m < \omega$, the sentence $\Phi \wedge \delta_m^{\Psi^*}$ has a model. Consider then the closed game sentence Φ^* which is equivalent to Φ on countable structures and let $\delta_n^{\Phi^*}$, where $n < \omega$, be its finite approximations. Since $\models \Phi \to \Phi^*$, $\models \Phi^* \to \bigwedge_{n<\omega} \delta_n^{\Phi^*}$ and $\models \delta_m^{\Psi^*} \to \delta_{m'}^{\Psi^*}$, for $m > m'$, the set

$$T = \{\delta_n^{\Phi^*} \wedge \delta_m^{\Psi^*} : n, m < \omega\}$$

is finitely satisfiable. Let $\mathfrak{A}$ be a countable, recursively saturated model of T. Then $\mathfrak{A} \models (\bigwedge_{n<\omega} \delta_n^{\Phi^*}) \wedge (\bigwedge_{m<\omega} \delta_m^{\Psi^*})$. But by Theorem 2.2.5, we have

$$\mathfrak{A} \models \Phi^* \leftrightarrow \bigwedge_{n<\omega} \delta_n^{\Phi^*} \quad \text{and} \quad \mathfrak{A} \models \Psi^* \leftrightarrow \bigwedge_{m<\omega} \delta_m^{\Psi^*}$$

so that $\mathfrak{A} \models \Phi^* \wedge \Psi^*$. However, since $\mathfrak{A}$ is countable, $\mathfrak{A} \models (\Phi \leftrightarrow \Phi^*) \wedge (\Psi \leftrightarrow \Psi^*)$ and hence

$\mathfrak{A} \models \Phi \wedge \Psi$. But this is a condiction of the hypothesis that

$\models \Phi \to \neg\Psi$. ☐

The next result was established by Vaught [1973b] and has turned out to have many interesting consequences.

2.2.10 Vaught's Covering Theorem. *Let Φ, Ψ be $\Sigma_1^1(L_{\omega_1\omega})$ sentences and let $\delta_\alpha^{\Psi^*}$, for α an ordinal, be the approximations of the closed Vaught sentence Ψ^* which is equivalent to Ψ on countable structures.*

(i) *If $\models \Phi \to \neg\Psi$, then there is an ordinal $\beta < \omega_1$ such that $\models \Phi \to \neg\delta_\beta^{\Psi^*}$.*
(ii) *Moreover, if A is a countable admissible set, Φ and Ψ are $\Sigma_1^1(L_A)$ and $\models \Phi \to \neg\Psi$, then there is some ordinal $\beta < o(A)$ such that $\models \Phi \to \neg\delta_\beta^{\Psi^*}$.*

Proof. Here we give the proof for the case where Φ and Ψ are $\Sigma_1^1(L_{\omega_1\omega})$ sentences and, at the same time, point out the modifications that are needed if Φ and Ψ are $\Sigma_1^1(L_A)$.

Let Φ and Ψ be $\Sigma_1^1(L_{\omega_1\omega})$ sentences such that $\models \Phi \to \neg\Psi$ and let Φ^* and Ψ^* be the closed Vaught sentences which are respectively equivalent to Φ and Ψ on countable structures. The key idea is that if $\neg\Psi$ holds, then we can use the inductive analysis of the open Vaught formula which is equivalent to $\neg\Psi^*$ in order to extract a $\Sigma_1^1(L_{\omega_1\omega})$ sentence which pins down ordinals. But then the undefinability of well-order in $L_{\omega_1\omega}$ implies that all ordinals pinned down in this way are bounded by some ordinal $\beta < \omega_1$. From this, it will follow that $\models \Phi \to \neg\delta_\beta^{\Psi^*}$. We now provide some of the technical details there are necessary to make this idea precise.

The closed Vaught sentence Ψ^* is of the form

$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right) \bigwedge_{n<\omega} \psi^{\bar{\imath},\bar{\jmath}}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where I is a countable set and the $\psi^{\bar{\imath},\bar{\jmath}}(\bar{x}, \bar{y})$ are formulas of $L_{\omega_1\omega}$. It is easy to see that if $\delta_\alpha^{\Psi^*,\bar{\imath},\bar{\jmath}}$ are the approximations of Ψ^* for α an ordinal and $(\bar{\imath},\bar{\jmath}) \in I^{2n}$, then

$$(1) \qquad \delta_\alpha^{\Psi^*,\bar{\imath},\bar{\jmath}}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{iff}$$

$$\bigwedge_{\beta<\alpha}\left(\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I}\right)\delta_\beta^{\Psi^*,\bar{\imath},i_n,\bar{\jmath},j_n}(x_0, y_0, \ldots, x_n, y_n)$$
$$\wedge \bigwedge_{k\le n} \psi^{\bar{\imath}_k,\bar{\jmath}_k}(x_0, y_0, \ldots, x_{k-1}, y_{k-1}).$$

It is clear from the above equivalence that the approximations of Ψ^* would have the same meaning if, instead by induction on the ordinals, they were defined by induction on the rank of an arbitrary well-ordering $<$. We will now consider new relation symbols $<$, $P^{\bar{\imath},\bar{\jmath}}$ for $(\bar{\imath},\bar{\jmath}) \in I^{2n}$, $n < \omega$, and a new constant symbol c.

We claim that in the expanded vocabulary $\tau' = \tau \cup \{<, c\} \cup \{P^{\bar{\imath},\bar{\jmath}}: (\bar{\imath},\bar{\jmath}) \in I^{2n}, n < \omega\}$ we can find a sentence χ of $L_{\omega_1\omega}[\tau']$ which asserts that $<$ is a linear ordering and that the relations $P^{\bar{\imath},\bar{\jmath}}$ satisfy the equivalence given in (1) above along $<$. More precisely, we let χ be the conjunction of the following sentences of $L_{\omega_1\omega}[\tau']$:

(i) "$<$ is a linear ordering with greatest element c";
(ii) $P^{\langle\;\rangle}(c)$;
(iii) the universal closure of the formula,

$$P^{\bar{\imath},\bar{\jmath}}(u, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$
$$\leftrightarrow (\forall v < u)\left(\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I}\right) P^{\bar{\imath},i_n,\bar{\jmath},j_n}(v, x_0, y_0, \ldots, x_n, y_n)$$
$$\wedge \bigwedge_{k\le n} \psi^{\bar{\imath}_k,\bar{\jmath}_k}(x_0, y_0, \ldots, x_{k-1}, y_{k-1}),$$

for $(\bar{\imath},\bar{\jmath}) \in I^{2n}$, $n \in \omega$.

It follows from the preceding comments that if a structure $\mathfrak{A}$ is a model of χ and u is an element of $<^{\mathfrak{A}}$ of rank α, then for any $\bar{\imath}, \bar{\jmath}$, we have

$$\{(\bar{x}, \bar{y}): P^{\mathfrak{A},\bar{\imath},\bar{\jmath}}(u, \bar{x}, \bar{y})\} = \{(\bar{x}, \bar{y}): \mathfrak{A}, \bar{x}, \bar{y} \models \delta_\alpha^{\Psi^*,\bar{\imath},\bar{\jmath}}\}.$$

We will show now that the sentence $(\neg\Psi^*) \wedge \chi$ pins down ordinals. Indeed, we claim that:

(2) if $\mathfrak{A}$ is a structure of vocabulary τ' such that $\mathfrak{A} \models (\neg\Psi^*) \wedge \chi$, then $<^{\mathfrak{A}}$ is a well-ordering of its field.

Otherwise, let $\mathfrak{A} \models (\neg\Psi^*) \wedge \chi$ and let $c >^{\mathfrak{A}} v_1 >^{\mathfrak{A}} v_2 >^{\mathfrak{A}} \cdots >^{\mathfrak{A}} v_n >^{\mathfrak{A}} v_{n+1} >^{\mathfrak{A}} \cdots$ be an infinite descending chain in the field of $<^{\mathfrak{A}}$. Since $\mathfrak{A} \models \chi$, we can use then the conjucts given in (ii) and (iii) of χ and the infinite descending chain above to define a winning quasistrategy for Player I in the game associated with Ψ^*. Hence we have that $\mathfrak{A} \models \Psi^*$. But this is a contradiction.

In order to complete the proof of the theorem, we observe that since $\models \Phi \to \neg\Psi$ and $\models' \Psi \leftrightarrow \Psi^*$, we must have that $\models \Phi \to \neg\Psi^*$. It thus follows from (2) above that we have

(3) if $\mathfrak{A}$ is a structure of vocabulary τ' such that $\mathfrak{A} \models \Phi \wedge \chi$, then $<^{\mathfrak{A}}$ is a well-ordering of its field.

The undefinability of well-order in $L_{\omega_1\omega}$ now implies that there is an ordinal $\beta < \omega_1$ such that if $\mathfrak{A} \models \Phi \wedge \chi$, then $<^{\mathfrak{A}}$ has rank less than β. As a consequence, the sentence $\Phi \wedge \delta_\beta^{\Psi^*}$ has no model and therefore $\models \Phi \to \neg\delta_\beta^{\Psi^*}$.

If Φ and Ψ are $\Sigma_1^1(L_A)$, where A is a countable admissible set, then the result can be proved by an entirely analogous argument using the effective versions of Theorems 2.1.5 and 2.2.7, and the theorem for pinning down ordinals in admissible fragments (for the latter result, see Barwise [1975] or Chapter VIII of this volume). Notice also that if $A = \mathrm{HF}$, then the result was proved in Theorem 2.2.9. $\square$

Although Vaught's covering theorem is a generalization of Theorem 2.2.9, its proof appears to be quite different from the one given for Theorem 2.2.9. Therefore, it is natural to ask if Vaught's covering theorem can be proved by combining compactness results with recursive saturation. Harnik [1974] gave such a proof (his proof can be found also in Makkai [1977a]) using the Barwise compactness theorem for a countable admissible fragment A and the existence of Σ_A-saturated models. For the definition and related results about Σ_A-saturation, the reader should also see Section 7, Chapter VIII of this volume.

2.3. Some Applications of Game Quantification

The results in Sections 2.1 and 2.2 have many interesting applications to the model theory of $L_{\omega_1\omega}$ and admissible fragments L_A. It actually turns out that we can derive the main theorems about compactness, abstract completeness, and interpolation in $L_{\omega_1\omega}$ or in L_A from the Svenonius–Vaught theorem, the approximations and the covering theorem. Since these results are well known and are discussed in Chapter VIII of the present volume, we will here restrict ourselves to merely listing some of the applications and making occasional brief comments on the proofs.

2.3.1 Applications of the Svenonius–Vaught Theorem. Vaught [1973b] obtained a proof of the Barwise compactness theorem using tools from the theory of game quantification. His argument consists of the following two independent parts:

(i) Let A be an arbitrary admissible set such that $\omega \in A$ and consider the class of bounded open game formulas. These are game formulas for which the associated game is bounded for Player II in the sense that his next move must belong to the union of the moves played thus far. More precisely, a *bounded open game formula* $\Phi(z)$ is of the form

$$[(\exists x_0)(\forall y_0 \in z \cup x_0)(\exists x_1)(\forall y_1 \in z \cup x_0 \cup y_0 \cup x_1)\cdots] \bigvee_{n<\omega} \varphi^n(z, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where each $\varphi^n(z, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a Δ_0 formula.

Vaught [1973b] showed that every admissible set A with $\omega \in A$ *reflects* bounded open game formulas. That is, if $\Phi(z)$ is such a formula and $A, z \models \Phi(z)$, then there is a transitive set w such that $z \in w \in A$ and $\langle w, \in\rangle, z \models \Phi(z)$.

(ii) The proof of the Svenonius–Vaught theorem (2.1.5) can be easily adapted to show that if A is in addition countable, then every strict-Π^1_1 formula is equivalent on A to a bounded open game formula. It then follows from part (i) that if A is a countable admissible set with $\omega \in A$, then A satisfies strict Π^1_1-reflection, and hence A is Σ_1-compact.

2.3.2 Applications of the Approximations. (i) Every $\Sigma^1_1(L_{\omega_1\omega})$ class of countable models is the intersection of $\aleph_1$ $L_{\omega_1\omega}$-elementary classes.

(ii) Every $\Sigma^1_2(L_{\omega_1\omega})$ class of countable models is the union of $\aleph_1$ $L_{\omega_1\omega}$-elementary classes.

These two results are rather direct consequences of Corollary 2.2.8. The first result, in turn, implies that every analytic set of reals is the intersection of $\aleph_1$ Borel sets. On the other hand, the second result yields Scott's isomorphism theorem for countable structures, since if $\mathfrak{A}$ is countable, then the collection $\{\mathfrak{B}: \mathfrak{B} \approx \mathfrak{A}\}$ is a $\Sigma^1_1(L_{\omega_1\omega})$ class of countable models.

Other applications of the approximation theorem given in Section 2.2.7 include:

(iii) *The Reduction Principle for $\Pi^1_1(L_{\omega_1\omega})$ Classes of Countable Models.* This principle asserts that if $\mathscr{K}_1$, $\mathscr{K}_2$ are two $\Pi^1_1(L_{\omega_1\omega})$ classes of countable models, then we can find two other $\Pi^1_1(L_{\omega_1\omega})$ classes $\mathscr{K}'_1$, $\mathscr{K}'_2$ such that $\mathscr{K}_1 \cup \mathscr{K}_2 = \mathscr{K}'_1 \cup \mathscr{K}'_2$ and $\mathscr{K}'_1 \cap \mathscr{K}'_2 = \varnothing$.

(iv) *The Abstract Completeness Theorem.* This result states that if A is a countable admissible set, then the set of valid sentences in L_A is Σ_1 on A uniformly.

2.3.3 Applications of the Covering Theorem. In this discussion, we will examine:

(i) The interpolation theorem for $L_{\omega_1\omega}$ and countable admissible fragments.

(ii) The undefinability of well-order in $L_{\omega_1\omega}$ and the theorem on pinning down ordinals in countable admissible fragments.

The interpolation theorem follows immediately from the covering theorem. Actually, in addition we obtain some information about the interpolant. For the undefinability of well-order, we will assume that $\varphi(<)$ is a $\Sigma_1^1(L_{\omega_1\omega})$ sentence such that if $\mathfrak{A} \models \varphi(<)$, then $<^{\mathfrak{A}}$ is a well-ordering. Then $\models \varphi(<) \rightarrow \neg(\exists x_0 \exists x_1 \cdots) \bigwedge_{n<\omega} (x_{n+1} < x_n)$, hence there is an ordinal $\beta < \omega_1$ such that $\models \varphi(<) \rightarrow \neg\delta_\beta$, where δ_α are the approximations of $(\exists x_0 \exists x_1 \cdots) \bigwedge_{n<\omega} (x_{n+1} < x_n)$. It follows now immediately from Sections 2.2.3 and 2.2.4 that $\neg\delta_\beta$ asserts that the rank of $<$ is less than β.

The proof of the covering theorem we gave here makes use of the undefinability of well-order. However, Harnik's [1974] proof of this result does not depend on it, so that we can first prove the covering theorem and then establish the undefinability of well-order. This is, for example, the approach taken by Makkai [1977a].

Further applications of this material can be found in Makkai [1973b, 1974b], Vaught [1974], Harnik [1976] and Harnik–Makkai [1976].

2.4. *On the Connection with Invariant Descriptive Set Theory*

We have here tried to develop the theory of game quantification in a more or less self-contained way by using methods from the model theory of $L_{\omega_1\omega}$ and admissible fragments.

At this point we should mention that there is also a very interesting connection between game quantification and invariant descriptive set theory. It is part of the general interaction between infinitary logic and descriptive set theory, which arises by identifying countable structures with elements of a product of topological spaces of the form 2^{ω^n}, ω^{ω^n}, or ω^n. If φ is a sentence of some infinitary logic, then the collection of all countable models of φ can be viewed as a subset of such a product which is invariant under a certain action of the group $\omega!$ of the permutations on ω, or under a natural equivalence relation. Topological methods and results from invariant descriptive set theory can then be used to derive theorems of infinitary logic. In particular, some of the results we have presented here can be studied by these methods. This direction has been pursued with much success by Vaught [1974], Burgess–Miller [1975], Miller [1978] and others.

3. *Model Theory for Game Logics*

The aim of this section is to present an overview of the model theory for the infinitary logics $L_{\infty G}$ and $L_{\infty V}$ associated with game quantification. The main result is that the logics $L_{\infty G}$ and $L_{\infty V}$ are absolute in the sense of Barwise [1972a]. Many model-theoretic properties of $L_{\infty G}$ and $L_{\infty V}$ then follow from this result and from the fact that both of these logics can express the notion of well-foundedness.

3.1. The Infinitary Logics $L_{\infty G}$ and $L_{\infty V}$

We will begin our discussion with

3.1.1 Definition. The infinitary logic $(L_{\infty G}, \models_{L_{\infty G}})$ is determined by the class $L_{\infty G}[\tau]$ of *$L_{\infty G}$-formulas of vocabulary* τ and the *relation of satisfaction* $\models_{L_{\infty G}}$ between sentences of $L_{\infty G}[\tau]$ and structures of vocabulary τ. If τ is a vocabulary, then $L_{\infty G}[\tau]$ is the smallest class which:

(i) contains all atomic formulas over the vocabulary τ;
(ii) is closed under negation $\neg$;
(iii) is closed under single existential $\exists$ and single universal $\forall$ quantification;
(iv) if Φ is a set of formulas of $L_{\infty G}[\tau]$ with only finitely many free variables in Φ, then the conjunction $\bigwedge \Phi$ and the disjunction $\bigvee \Phi$ are also formulas of $L_{\infty G}[\tau]$;
(v) if $\{\varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}): n < \omega\}$ are formulas of $L_{\infty G}[\tau]$ in the displayed free variables, then the expressions

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

and

$$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

are also formulas of $L_{\infty G}[\tau]$ with $\bar{z}$ as free variables.

The *relation of satisfaction* "$\mathfrak{A} \models_{L_{\infty G}} \psi$" between sentences of $L_{\infty G}[\tau]$ and structures of vocabulary τ is defined inductively, using the game theoretic interpretation from Section 1 for the clause given in (v). It is understood that if the full axiom of choice is available in the metatheory, then the interpretation is via winning strategies. If one is working only with the axiom of dependent choices, then the interpretation of the clause in (v) is given using winning quasistrategies.

If τ is a vocabulary and HC is the set of hereditarily countable sets, then we put

$$L_{\omega_1 G}[\tau] = L_{\infty G}[\tau] \cap \mathrm{HC}.$$

Notice that the open game and closed game formulas that we considered in Section 2 are actually elements of $L_{\omega_1 G}[\tau]$.

3.1.2 Definition. The infinitary logic $(L_{\infty V}, \models_{L_{\infty V}})$ is defined as follows:
If τ is a vocabulary, then the collection $L_{\infty V}[\tau]$ is the smallest class of formulas which satisfies the closure properties (i), (ii), (iii), and (iv) in the previous definition and in addition is such that:

(v′) if I is a non-empty set and for every $n \in \omega$ and every $(\bar{\imath}, \bar{\jmath}) \in I^{2n}$ $\varphi^{\bar{\imath},\bar{\jmath}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a formula of $L_{\infty V}[\tau]$ in the displayed free variables, then the expressions

$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right)$$
$$\bigwedge_{n<\omega} \varphi^{i,\,j}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

and

$$\left(\exists x_0 \bigvee_{i_0 \in I} \forall y_0 \bigwedge_{j_0 \in I} \exists x_1 \bigvee_{i_1 \in I} \forall y_1 \bigwedge_{j_1 \in I} \cdots\right)$$
$$\bigvee_{n<\omega} \varphi^{i,\,j}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

are also formulas of $L_{\infty V}[\tau]$ with $\bar{z}$ as free variables.

The relation of satisfaction "$\mathfrak{A} \models_{L_{\infty V}} \psi$" between sentences of $L_{\infty V}[\tau]$ and structures of vocabulary τ is defined inductively, again associating a game with the formulas in (v′).

We put

$$L_{\omega_1 V}[\tau] = L_{\infty V}[\tau] \cap \mathrm{HC}$$

and observe that the open Vaught and closed Vaught formulas of Section 2 are elements of $L_{\omega_1 V}[\tau]$.

It is not hard to verify that the logic $L_{\omega_1 V}$ is stronger than the logic $L_{\omega_1 G}$. Indeed, $L_{\omega_1 V}$—and, of course, $L_{\infty V}$—can express infinitary connectives which cannot be captured by $L_{\omega_1 G}$ (nor by $L_{\infty G}$ for that matter).

Vaught [1974] pointed out that the weak second-order version of $L_{\omega_1 V}$ coincides with $L_{\omega_1 V}$, so that $L_{\omega_1 V}$ is invariant under passage to weak second-order logic, while $L_{\omega_1 G}$ is not. However, as we have mentioned before, over countable models possessing a first-order coding machinery of finite sequences, the infinitary logics $L_{\omega_1 G}$ and $L_{\omega_1 V}$ have the same expressive power.

3.1.3. We now recall the definition of an absolute logic from Chapter XVII, a definition which was originally given in Barwise [1972a].

Let T be a set theory at least as strong as the admissible set theory KP and let $(L, \models_L)$ be an abstract logic. We say that the logic $(L, \models_L)$ is *absolute relative to* T if:

(i) The relation "φ is a sentence of $L[\tau]$" is a Σ_1^T predicate of φ and the vocabulary τ; and
(ii) if φ is a sentence of $L[\tau]$ and $\mathfrak{A}$ is a structure of vocabulary τ, then the predicate "$\mathfrak{A} \models_L \varphi$" is a Δ_1^T predicate of $\mathfrak{A}$, φ and τ.

A logic $(L, \models_L)$ is *strictly absolute* if it is absolute relative to the admissible set theory KP.

One of the main results of Barwise [1972a] (see also Chapter XVII of the present volume) asserts that if $(L, \models_L)$ is a strictly absolute logic, then $L \leq L_{\infty\omega}$. However, we showed in Section 1.1.4 that there is a formula of $L_{\omega_1 G}$ which asserts that:

"$<$ is a well-ordering of order type $\gamma + \gamma$ for some ordinal γ."

Since the above statement is not expressible in $L_{\infty\infty}$, we obtain the following

3.1.4 Theorem. *The infinitary logics $L_{\omega_1 G}, L_{\omega_1 V}, L_{\infty G}, L_{\infty V}$ are not strictly absolute.* □

It is now natural to ask whether or not the game logics are absolute relative to some true set theory. The answer to this question is provided by the following result of Barwise [1972a].

3.1.5 Theorem. *The infinitary logics $L_{\omega_1 G}$, $L_{\omega_1 V}$, $L_{\infty G}$ and $L_{\infty V}$ are all absolute relative to the theory* KP + Σ_1*-separation + Axiom of Dependent Choices.*

Sketch of Proof. Once more the main idea comes from the inductive analysis of the open games, which was given in the proof of the Gale–Stewart theorem. An inspection of the proof given there reveals, first of all, that the Gale–Stewart theorem is itself provable in KP + Σ_1-separation + axiom of dependent choices. To establish that satisfaction is absolute for, say, the infinitary logic $L_{\omega_1 G}$, we define by induction on the construction of the $L_{\omega_1 G}[\tau]$-formulas a Σ_1 predicate $P(\tau, \mathfrak{A}, \psi, i)$ such that if $\mathfrak{A}$ is a structure of vocabulary τ, then

$$P(\tau, \mathfrak{A}, \psi, i) \quad \text{iff} \quad (i = 0 \;\&\; \mathfrak{A} \models \psi) \vee (i = 1 \;\&\; \mathfrak{A} \not\models \psi).$$

This automatically takes care of the negations, while for the crucial clause given in (v) of Definition 3.1.1 we use the Gale–Stewart theorem and Σ_1-separation. More precisely, if ψ is the sentence

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_{n<\omega} \psi_n(x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

then

$$P(\tau, \mathfrak{A}, \psi, 0) \Leftrightarrow \text{Player I has a winning quasistrategy in } G\left(\forall\exists, \bigwedge_{n<\omega} \psi_n\right)$$

$$\Leftrightarrow (\;) \notin \varphi^\infty,$$

and

$$P(\tau, \mathfrak{A}, \psi, 1) \Leftrightarrow \text{Player I has a winning quasistrategy in } G\left(\exists\forall, \bigvee_{n<\omega} \neg\psi_n\right)$$

$$\Leftrightarrow (\;) \in \varphi^\infty,$$

where φ^∞ is the smallest fixed point of the monotone operator $\varphi(u, S)$ associated with the open game $G(\exists\forall, \bigvee_{n<\omega} \neg\psi_n)$, just as in the proof of the Gale–Stewart theorem given in Section 1.2.4. $\square$

3.2. *Model-theoretic Properties of the Logics $L_{\infty G}$ and $L_{\infty V}$*

The following model-theoretic properties of the infinitary logic $L_{\infty V}$ follow from its absoluteness and the results in Chapter XVII of this volume.

3.2.1 Theorem. (i) *The logic $L_{\infty V}$ has the downward Skolem–Löwenheim property to ω. That is, if a sentence φ of $L_{\infty V}[\tau]$ has a model, then it has a countable model.*

(ii) *The logic $L_{\infty V}$ has the Karp property. That is to say, if $\mathfrak{A}$, $\mathfrak{B}$ are structures of vocabulary τ which satisfy the same sentences of $L_{\infty\omega}[\tau]$, then they satisfy the same sentences of $L_{\infty V}[\tau]$.* $\square$

Barwise [1972a] showed that these properties are shared by any abstract logic which is absolute. Moreover, Barwise [1972a] and Burgess [1977] established certain negative results about logics which are absolute and *unbounded*. That is, the collection of well-founded structures is a PC class. Since the infinitary logics $L_{\omega_1 G}$, $L_{\omega_1 V}$, $L_{\infty G}$ and $L_{\infty V}$ can all express the notion of well-foundedness, we have

3.2.2 Theorem. (i) (Failure of the Abstract Completeness Theorem). *The set of valid sentences of the infinitary logic $L_{\omega_1 G}$ is a complete Π_1 set on* HC. *The same is true for the validities of the infinitary logic $L_{\omega_1 V}$.*

(ii) *The infinitary logics $L_{\omega_1 G}$ and $L_{\omega_1 V}$ do not satisfy: the Craig interpolation theorem, the Δ-interpolation theorem, the Beth definability theorem, and the weak Beth definability theorem.* $\square$

The reader is referred to Chapter II for the definitions of these notions and to Chapter XVII for the proof of the above theorem.

3.2.3. The approximation theory for Vaught formulas, which was developed in Section 2, can be easily extended to arbitrary formulas of $L_{\infty V}$, the main result being that with any sentence ψ of $L_{\infty V}$ we can associate sentences δ_α^ψ of $L_{\infty\omega}$, for α an ordinal, such that

$$\models \psi \leftrightarrow \bigwedge_\alpha \delta_\alpha^\psi.$$

Green [1979] used these approximations to introduce consistency properties for $L_{\infty V}$ and obtained a model existence theorem for game logics. As we mentioned in

Corollary 2.2.8, Burgess [1977] extended the approximation theory to any absolute logic. Finally, Harnik [1976], using the approximations and model theoretic forcing, established certain strong preservation theorems for $L_{\infty V}$ which partially compensate for the failure of interpolation.

We conclude this section by pointing out that certain sublogics and extensions of the game logics $L_{\infty G}$ and $L_{\infty V}$ have also been studied. For example, Ellentuck [1975], Burgess [1978b] and Green [1978] have investigated the *Suslin logics* which can be described intuitively as the propositional part of $L_{\infty V}$, since they allow for infinite alternations of the connectives $\bigwedge$ and $\bigvee$, but not of the quantifiers $\forall$ and $\exists$. Burgess [1977] introduced the *Borel-game logic* $L_{\infty B}$, an extension of $L_{\infty V}$. In this logic, the infinite strings of quantifiers and connectives are applied not only to matrices which are open or closed, but also to matrices which can be coded by a Borel set. Of course, it takes Martin's [1975] theorem on Borel determinacy to show that negations can be pushed inside. The Borel-game logic is absolute relative to ZF + axiom of dependent choices.

4. *Game Quantification and Local Definability Theory*

This section contains the connections between game quantification, generalized recursion theory, and descriptive set theory. The first basic result asserts that on structures with a first-order coding machinery, the (positive elementary) inductive relations coincide with the ones that are explicitly definable using the open game quantifier. This result is due to Moschovakis [1972] and constitutes an absolute version of Svenonius' theorem (see Theorem 2.1.5). Aczel [1975] generalized this result and showed that the Q-inductive relations on a structure can be characterized using infinite strings $(Qx_0 Qx_1 Qx_2 \cdots)$, where Q is an arbitrary monotone quantifier. To present these theorems, we introduce infinite strings $(Qx_0 Qx_1 Qx_2 \cdots)$ and interpret them via two-person infinite games. We will pursue the study of the Q-inductive relations and state their characterizations in terms of functional recursion, representability in stronger logics, and admissible sets with quantifiers. We will also briefly indicate some of the tools of inductive definability which are used to derive local versions of the global results given in Section 2. That done, we will discuss the connections with non-monotone inductive definitions and the recursion-theoretic difference between the open game and the closed game quantifier. The chapter will end with some results and comments concerning the interactions of game quantification with descriptive set theory.

Because of the limitations of space, most of the results in this section will be given without proofs. However, we have included the definitions of the basic notions as well as all the relevant references to the literature.

4.1. Iterating a Monotone Quantifier Infinitely Often

4.1.1. Assume that Q is a monotone quantifier on a set A; that is, suppose that Q is a non-empty, proper subset of $\mathscr{P}(A)$ which is closed under supersets. In order to iterate the quantifier Q infinitely often, we must give meaning to the string

$$(Qx_0 Qx_1 Qx_2 \cdots).$$

The following interpretation is due to Aczel [1975] and is motivated by the observation that, since Q has the monotonicity property,

$$QxP(x) \quad \text{iff} \quad (\exists X \in Q)(\forall x \in X)P(x),$$

so that intuitively we should have the equivalence

$$\begin{aligned}&(Qx_0 Qx_1 Qx_2 \cdots)R(x_0, x_1, x_2, \ldots)\\ &\quad \text{iff} \quad (\exists X_0 \in Q)(\forall x_0 \in X_0)(\exists X_1 \in Q)(\forall x_1 \in X_1) \cdots R(x_0, x_1, \ldots).\end{aligned}$$

This suggests associating with Q as well as with a relation $R \subseteq A^\omega$ the following two-person infinite game $G(Q, R)$ of perfect information:

A *round of the game* $G(Q, R)$ is played by Players I and II who make alternate moves in such a way that I picks a set $X_i \in Q$ and II responds by picking an element $x_i \in X_i$, $i = 0, 1, 2, \ldots$

$$\begin{array}{c|cccccccc} \text{I} & X_0 & & X_1 & & X_2 & & \cdots & (X_i \in Q, \text{ all } i \in I) \\ \hline \text{II} & & x_0 & & x_1 & & x_2 & \cdots & (x_i \in X_i, \text{ all } i \in I) \end{array}.$$

Player I wins the above round if $(x_0, x_1, x_2, \ldots) \in R$; otherwise, Player II wins We say that *Player* I *wins the game* $G(Q, R)$ if I has a systematic way to win every round of the game. This can be made precise by requiring that Player I have a *winning strategy for* $G(Q, R)$; that is, that there be a function $\sigma\colon \bigcup_{n<\omega} (Q \times A)^n \to Q$ with the property that $(x_0, x_1, x_2, \ldots) \in R$ for any round $(X_0, x_0, X_1, x_1, X_2, x_2, \ldots)$ of $G(Q, R)$ in which $X_0 = \sigma((\,))$ and $X_{i+1} = \sigma(X_0, x_0, \ldots, X_i, x_i)$, for every $i \in \omega$. Similarly, we say that *Player* II *wins the game* $G(Q, R)$ if II has a winning strategy $\tau\colon \bigcup_{n<\omega} (Q \times A)^n \times Q \to A$ with which he can win every round of $G(Q, R)$. Finally, we put

$$\begin{aligned}&(Qx_0 Qx_1 Qx_2 \cdots)R(x_0, x_1, x_2, \ldots)\\ &\quad \text{iff} \quad \text{Player I wins the game } G(Q, R).\end{aligned}$$

The following proposition is a simple, but useful tool in manipulating infinite strings of quantifiers. Its proof follows easily from the definitions and the axiom of choice.

4.1.2 Proposition. *Let Q be a monotone quantifier on A and let $R \subseteq A^{\omega}$. Then we have,*

$$Qx\{Qx_0Qx_1Qx_2\cdots\}R(x, x_0, x_1, x_2, \ldots)$$
$$\text{iff} \quad (QxQx_0Qx_1Qx_2\cdots)R(x, x_0, x_1, x_2, \ldots). \quad \square$$

The next theorem provides the basic connection between winning strategies for Player I in the game $G(Q, R)$ and winning strategies for Player II in the dual game $G(\breve{Q}, \neg R)$ associated with the statement

$$(\breve{Q}x_0\breve{Q}x_1\breve{Q}x_2\cdots)\neg R(x_0, x_1, x_2, \ldots), \text{ where of course } \neg R = A^{\omega} - R.$$

4.1.3 Theorem. *Let Q be a monotone quantifier on a set A and let $R \subseteq A^{\omega}$. Then the following are equivalent:*

(i) $(Qx_0Qx_1Qx_2\cdots)R(x_0, x_1, x_2, \ldots)$; *that is to say, Player* I *wins the game* $G(Q, R)$

(ii) *Player* II *wins the game* $G(\breve{Q}, \neg R)$.

Proof. Let σ be a winning strategy for Player I in the game $G(Q, R)$. We will informally describe a winning strategy for Player II in the dual game $G(\breve{Q}, \neg R)$. The argument uses the axiom of choice and the fact that if $X \in Q$ and $Y \in \breve{Q}$, then $X \cap Y \neq \varnothing$. Assume then that Player I starts a round of $G(\breve{Q}, \neg R)$ by playing a set $Y_0 \in \breve{Q}$. If $X_0 = \sigma((\ ))$, then $X_0 \in Q$, and hence $X_0 \cap Y_0 \neq \varnothing$. Now,

> Player II answers Player I in $G(\breve{Q}, \neg R)$ by picking an element $x_0 \in X_0 \cap Y_0$.

If I plays $Y_1 \in \breve{Q}$, then II responds by playing some element x_1 of the non-empty set $X_1 \cap Y_1$, where $X_1 = \sigma(X_0, x_0) \in Q$. If Player II continues in this way, then at the end of time he has produced a round $(Y_0, x_0, Y_1, x_1, \ldots)$ of the game $G(\breve{Q}, \neg R)$ for which there is a round $(X_0, x_0, X_1, x_1, \ldots)$ of $G(Q, R)$ played according to the winning strategy σ for Player I in that game, hence $(x_0, x_1, \ldots) \in R$.

As to the other direction, we will assume that Player II wins the game $G(\breve{Q}, \neg R)$. We will indicate how to define a winning strategy for I in the game $G(Q, R)$. The idea is similar to the one presented earlier; namely, I plays in such a way that he forces his opponent to produce a sequence $(x_0, x_1, x_2, \ldots)$ which corresponds to moves of II in $G(\breve{Q}, \neg R)$ played according to his winning strategy. More precisely, I starts by playing the set

$$X_0 = \{x\colon \text{there is a round of } G(\breve{Q}, \neg R) \text{ of the form } (Y, x, \ldots) \text{ in which Player II follows his winning strategy}\}.$$

Notice that $X_0 \in Q$, since otherwise its complement $(A - X_0) \in \breve{Q}$ and it is thus a legitimate move for I in $G(\breve{Q}, \neg R)$. But then the winning strategy of II in this game produces an element of $X_0 \cap (A - X_0)$. This is a contradiction.

Suppose now that Player II responds with an element $x_0 \in X_0$. Then there is a round of $G(\breve{Q}, \neg R)$ of the form $(Y_0, x_0, \ldots)$ in which II follows his winning strategy. The next move of I in $G(Q, R)$ is the set

$$X_1 = \{x\colon \text{there is a round of } G(\breve{Q}, \neg R) \text{ of the form } (Y_0, x_0, Y, x, \ldots) \\ \text{in which Player II follows his winning strategy}\}.$$

It is easy to see that $X_1 \in Q$. Moreover, if II responds with an element $x_1 \in X_1$, then there is a round of $G(\breve{Q}, \neg R)$ of the form $(Y_0, x_0, Y_1, x_1, \ldots)$ in which II plays according to his winning strategy. In this way, at the end of time the two players in $G(Q, R)$ have produced a sequence $(X_0, x_0, X_1, x_1, X_2, x_2, \ldots)$ such that there is a round $(Y_0, x_0, Y_1, x_1, Y_2, x_2, \ldots)$ of $G(\breve{Q}, \neg R)$ in which II follows his winning strategy. ☐

The proof of the Gale–Stewart theorem (1.2.4) can be easily modified to yield the determinacy of open or closed games associated with the infinite string $(Qx_0 Qx_1 Qx_2 \cdots)$. Thus, if Q is a monotone quantifier and R is a relation which is either open or closed, then Player I or Player II wins the game $G(Q, R)$. By combining this fact with Theorem 4.1.3 we immediately obtain the following

4.1.4 Corollary. *Let Q be a monotone quantifier on A and let $R \subseteq A^\omega$ be a relation which is either open or closed. Then*

$$\textit{Player I does not win } G(Q, R) \quad \textit{iff} \quad \textit{Player I wins } G(\breve{Q}, \neg R)$$

and hence

$$\neg(Qx_0 Qx_1 Qx_2 \cdots)R(x_0, x_1, x_2, \ldots) \Leftrightarrow \\ (\breve{Q}x_0 \breve{Q}x_1 \breve{Q}x_2 \cdots) \neg R(x_0, x_1, x_2, \ldots). \quad ☐$$

4.1.5. Thus far we have considered infinite strings obtained by iterating only one monotone quantifier infinitely often. We might also consider a sequence $\bar{Q} = \{Q_n\}_{n \in \omega}$ of arbitrary monotone quantifiers Q_n, $n \in \omega$, on a set A and the corresponding infinite string $(Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots)$. If $R \subseteq A^\omega$ is a collection of infinite sequences from A, then the statement

$$(Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots)R(x_0, x_1, \ldots, x_n, \ldots)$$

is interpreted via a game $G(\bar{Q}, R)$ which is suggested by the intuitive equivalence

$$\begin{aligned}
&(Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots)R(x_0, x_1, \ldots, x_n, \ldots) \quad \text{iff} \\
&(\exists X_0 \in Q_0)(\forall x_0 \in X_0)(\exists X_1 \in Q_1)(\forall x_1 \in X_1) \\
&\qquad \cdots (\exists X_n \in Q_n)(\forall x_n \in X_n) \cdots R(x_0, x_1, \ldots, x_n).
\end{aligned}$$

The preceding results extend naturally to such arbitrary strings with only minor modifications in the definitions and the proofs. In particular, if $R \subseteq A^\omega$ is either open or closed, then we can push the negation inside, so that we have

$$\begin{aligned}
&\neg(Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots)R(x_0, x_1, \ldots, x_n, \ldots) \\
&\Leftrightarrow (\breve{Q}_0 x_0 \breve{Q}_1 x_1 \cdots \breve{Q}_n x_n \cdots) \neg R(x_0, x_1, \ldots, x_n, \ldots)
\end{aligned}$$

We should point out here that for the infinite string $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)$, the interpretation of the statement $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)R(x_0, y_0, x_1, y_1, \ldots)$ given above is equivalent to the one given in Section 1 of this chapter. Notice, however, that a strategy for I in the sense of this section essentially coincides with a quasi-strategy for I in the sense of Section 1, rather than with a strategy. This is because we have identified the existential quantifier $\exists$ on A with the collection $\{X \subseteq A : X \neq \varnothing\}$.

4.1.6. The infinite string $(Qx_0 Qx_1 Qx_2 \cdots)$ can be viewed as defining a new monotone quantifier Q^* on the set A^ω of infinite sequences from A. More specifically, the *quantifier Q^* on A^ω* is the collection

$$Q^* = \{X \subseteq A^\omega : (Qx_0 Qx_1 Qx_2 \cdots)X(x_0, x_1, x_2, \ldots)\}.$$

If the infinite string $(Qx_0 Qx_1 Qx_2 \cdots)$ is applied to relations R on A^ω which are open or closed, then it gives rise to two monotone quantifiers $Q^\vee$ and $Q^\wedge$ on the set $A^{<\omega}$ of finite sequences from A.

The *quantifier $Q^\vee$ on $A^{<\omega}$* is the collection

$$Q^\vee = \left\{X \subseteq A^{<\omega} : (Qx_0 Qx_1 Qx_2 \cdots) \bigvee_n X(x_0, x_1, \ldots, x_{n-1})\right\},$$

while *the quantifier $Q^\wedge$ on $A^{<\omega}$* is defined by

$$Q^\wedge = \left\{X \subseteq A^{<\omega} : (Qx_0 Qx_1 Qx_2 \cdots) \bigwedge_n X(x_0, x_1, \ldots, x_{n-1})\right\}.$$

The quantifiers $Q^\vee$ and $Q^\wedge$ can be expressed using the quantifier Q^* on A^ω and infinitary connectives. Indeed, if $R \subseteq A^{<\omega}$ is a relation on the set of finite sequences from A, then we first introduce the relations $\bigvee R$ and $\bigwedge R$ on the set of infinite sequences, where

$$\bigvee R = \left\{\alpha \in A^\omega : \bigvee_n R(\alpha \upharpoonright n)\right\} \quad \text{and} \quad \bigwedge R = \left\{\alpha \in A^\omega : \bigwedge_n R(\alpha \upharpoonright n)\right\}.$$

It is now clear that

$$Q^{\vee} sR(s) \Leftrightarrow \text{Player I wins } G(Q, \bigvee R) \Leftrightarrow Q^*\alpha \bigvee R(\alpha)$$

and

$$Q^{\wedge} sR(s) \Leftrightarrow \text{Player I wins } G(Q, \bigwedge R) \Leftrightarrow Q^*\alpha \bigwedge R(\alpha).$$

Since the quantifiers $Q^{\vee}$ and $Q^{\wedge}$ give rise to games which are open or closed, we can use Corollary 4.1.4 to find their dual quantifiers.

4.1.7 Corollary. *Let Q be a monotone quantifier on A. Then*:

(i) *the dual of the quantifier $Q^{\vee}$ is the quantifier $\breve{Q}^{\wedge}$; that is, $(Q^{\vee})^{\smile} = \breve{Q}^{\wedge}$;*
(ii) *the dual of the quantifier $Q^{\wedge}$ is the quantifier $\breve{Q}^{\vee}$; that is $(Q^{\wedge})^{\smile} = \breve{Q}^{\vee}$.* □

4.1.8. The Suslin and the classical $\mathscr{A}$ quantifier are special cases of the quantifiers $Q^{\vee}$ and $Q^{\wedge}$. Indeed, it is obvious that $\forall^{\vee}$ is the Suslin quantifier on the set $A^{<\omega}$, while $\exists^{\wedge}$ is the classical quantifier $\mathscr{A}$ on $A^{<\omega}$. Notice also that $\forall^{\wedge}$ and $\exists^{\vee}$ are respectively the universal and the existential quantifier on the set $A^{<\omega}$ of finite sequences from A.

We now consider the quantifiers $\exists\forall$ and $\forall\exists$ on the set $A^2 = A \times A$, where

$$\exists\forall = \{X \subseteq A^2: (\exists x\, \forall y)((x, y) \in X)\}$$

and

$$\forall\exists = \{X \subseteq A^2: (\forall x\, \exists y)((x, y) \in X)\}.$$

Of course, the quantifier $\forall\exists$ is the dual of $\exists\forall$. Moreover,

$$(\exists\forall)^{\vee} \text{ is the open game quantifier } \mathscr{G} \text{ on } A^{<\omega},$$

and

$$(\forall\exists)^{\wedge} \text{ is the closed game quantifier } \breve{\mathscr{G}} \text{ on } A^{<\omega}.$$

Observe that here we have tacitly identified the sequence $((x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots)$ in $(A \times A)^{\omega}$ with the sequence $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ in A^{ω}.

If Q is a monotone quantifier on A, *then the next quantifier Q^+ of Q* is the quantifier

$$Q^+ = (Q\breve{Q}\exists\forall)^{\vee},$$

where $Q\breve{Q}\,\exists\forall = \{X \subseteq A^4: (Qx\breve{Q}y\, \exists z\, \forall w)((x, y, z, w) \in X)\}$. Therefore, if $R \subseteq A^{<\omega}$, then we have

$$Q^+ sR(s) \Leftrightarrow (Qx_0 \breve{Q}y_0\, \exists z_0\, \forall w_0 Qx_1 \breve{Q}y_1\, \exists z_1\, \forall w_1 \cdots)$$
$$\bigvee_n R(x_0, y_0, z_0, w_0, \ldots, x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}).$$

It follows from the above that the dual quantifier of $Q^+ = (Q\breve{Q}\,\exists\forall)^\vee$ is the quantifier $(\breve{Q}Q\,\forall\exists)^\wedge$. Notice that the open game quantifier $\mathscr{G}$ is the next quantifier of $(\exists\forall)$. As we will see in the sequel, the next quantifier plays an important role in the theory of inductive definability.

4.2. *Game Quantification and Positive Elementary Induction in a Quantifier*

4.2.1. Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be a structure and let Q be a monotone quantifier on the universe A of the structure. The *first-order logic* $\mathscr{L}^{\mathfrak{A}}(Q)$ of the structure $\mathfrak{A}$ has both first-order variables $x, y, z, \ldots$ and second-order variables $S, T, U, \ldots$, but the quantifiers $\forall, \exists, Q, \breve{Q}$ range only over the first-order variables. The "*boldface*" *first-order logic* $\boldsymbol{\mathscr{L}}^{\mathfrak{A}}(Q)$ of the structure $\mathfrak{A}$ is obtained from $\mathscr{L}^{\mathfrak{A}}(Q)$ by adding to the vocabulary a new constant symbol $\mathbf{a}$ for each element $a \in A$. If we do not consider an additional quantifier Q, then we have the logics $\mathscr{L}^{\mathfrak{A}}$ and $\boldsymbol{\mathscr{L}}^{\mathfrak{A}}$ respectively.

If $\varphi(x_1, \ldots, x_n, S)$ is a formula of $\mathscr{L}^{\mathfrak{A}}(Q)$ in which S is a n-ary relation symbol with only positive occurrences, then $\varphi(\bar{x}, S)$ gives rise to a transfinite sequence $\{I_\varphi^\xi\}_{\xi \in \mathrm{Ord}}$ of n-ary relations on A, where

$$I_\varphi^\xi = \left\{\bar{x} \in A^n \colon \varphi\left(\bar{x}, \bigcup_{\eta<\xi} \varphi^\eta\right)\right\}.$$

We put

$$I_\varphi = \bigcup_{\xi \in \mathrm{Ord}} I_\varphi^\xi$$

and call I_φ the set *inductively defined by* φ. It is easy to see that

$$\bar{x} \in I_\varphi \Leftrightarrow \varphi(\bar{x}, I_\varphi)$$

and

$$I_\varphi = \bigcap \{S \colon (\forall \bar{x})(\varphi(\bar{x}, S) \leftrightarrow \bar{x} \in S)\},$$

so that I_φ is the smallest fixed point of φ.

If R is an m-ary relation on A, we say that R is Q-(*positive*) *inductive* in case there is a formula $\varphi(\bar{u}, \bar{v}, S)$ of $\mathscr{L}^{\mathfrak{A}}(Q)$ with S occurring positively and a finite sequence $\bar{a}$ of elements of A such that

$$R(\bar{y}) \Leftrightarrow (\bar{a}, \bar{y}) \in I_\varphi.$$

We say that a relation $R \subseteq A^m$ is *Q-(positive) hyperelementary* if both R and $A^m - R$ are Q-inductive relations. We write

$$\mathbf{IND}(\mathfrak{A}, Q) = \text{the collection of all } Q\text{-(positive) inductive relations on } \mathfrak{A},$$

and

$$\mathbf{HYP}(\mathfrak{A}, Q) = \text{the collection of all } Q\text{-(positive) hyperelementary relations on } \mathfrak{A}.$$

If we do not consider an additional quantifier Q on A, then we have the notions of the (positive) inductive and the (positive) hyperelementary relations on $\mathfrak{A}$. In this case we put

$$\mathbf{IND}(\mathfrak{A}) = \text{all (positive) inductive relations on } \mathfrak{A},$$

and

$$\mathbf{HYP}(\mathfrak{A}) = \text{all (positive) hyperelementary relations on } \mathfrak{A}.$$

The theory of the inductive and the hyperelementary relations has been developed in the monograph Moschovakis [1974a]. Here we will purposely restrict ourselves to stating the results which are directly related to game quantification.

4.2.2. Henceforth, we will confine our attention to structures possessing a first-order coding machinery of finite sequences. We say that a structure $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ is *acceptable* if ω, $\leq_\omega$ are first-order on $\mathfrak{A}$ and there is a total, one-to-one coding function $\langle\ \rangle : A^{<\omega} \to A$ such that the relation seq and the functions lh and q are first-order on $\mathfrak{A}$, where

$$\text{seq}(x) \Leftrightarrow \text{there are } x_1, x_2, \ldots, x_n \text{ such that } x = \langle x_1, x_2, \ldots, x_n \rangle;$$

$$lh(x) = \begin{cases} 0, & \text{if } \neg\ \text{seq}(x) \\ n, & \text{if seq}(x) \text{ and } x = \langle x_1, x_2, \ldots, x_n \rangle; \end{cases}$$

and

$$q(x, i) = (x)_i = \begin{cases} x_i, & \text{if } x = \langle x_1, x_2, \ldots, x_n \rangle \text{ and } 1 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Typical examples of acceptable structures are the structure of arithmetic $\mathbb{N} = \langle \omega, +, \cdot \rangle$, the rationals $\mathbb{Q} = \langle Q, +, \cdot \rangle$, the structure of analysis $\mathbb{R} = \langle \omega \cup \omega^\omega, \omega. +, \cdot, Ap \rangle$ (where $Ap(\alpha, n) = \alpha(n)$, with $\alpha \in \omega^\omega$ and $n \in \omega$), and the structures $\mathbb{V}_\lambda = \langle V_\lambda, \varepsilon \rangle$, for each ordinal $\lambda \geq \omega$, where V_λ is the collection of sets of rank less than λ.

Many of the results in this section are true under a much weaker hypothesis, namely that the structure $\mathfrak{A}$ under consideration has an *inductive pairing function.* Such a function is, of course, a total, one-to-one function $\langle\ \rangle: A \times A \to A$ with an inductive graph. Examples of such structures include the structures $\boldsymbol{\lambda} = \langle \lambda, < \rangle$ and $\mathbb{L}_\lambda = \langle L_\lambda, < \rangle$ for any infinite ordinal λ, all models of Peano arithmetic, and any structure of the form $\mathfrak{A} = \langle A, \in \rangle$ where A is a transitive set closed under pairs.

Every acceptable structure has the property that the weak second-order logic $\mathscr{L}_{\mathrm{wII}}$ on $\mathfrak{A}$ can be subsumed by the first-order logic $\mathscr{L}^{\mathfrak{A}}$ of the structure $\mathfrak{A}$.

If we want to avoid the assumption of acceptability, then we must consider a larger class of inductive definitions, namely the inductive* and the Q-inductive* relations of Barwise [1975, 1978b], or pass from an arbitrary structure $\mathfrak{A} = \langle A, R_1, \ldots, R_m, c_1, \ldots, c_k \rangle$ to the expanded structure $\mathfrak{A}^* = \langle A \cup A^{<\omega} \cup \omega, A, \omega, R_1, \ldots, R_m, \leq_\omega, Ap, c_1, \ldots, c_k \rangle$, where $Ap((a_1, \ldots, a_n), i) = a_i$.

If $\mathfrak{A}$ is an acceptable structure and T is a quantifier on the set $A^{<\omega}$ of finite sequences from A, then T can be identified with a quantifier on A, which we also denote by T and which is defined as follows:

$$T = \{X \subseteq A: \{(x_1, \ldots, x_n) \in A^{<\omega}: \langle x_1, \ldots, x_n \rangle \in X\} \in T\},$$

with $\langle\ \rangle: A^{<\omega} \to A$ a fixed coding function as in the definition of acceptability.

In particular, the quantifiers $Q^\vee$, $Q^\wedge$, Q^+ and $(Q^+)^\cup$ can all be viewed, and indeed will so be viewed from here on, as quantifiers on the universe A of the structure $\mathfrak{A}$. Thus, for example, the open game quantifier on $A^{<\omega}$ is identified with the quantifier

$$\mathscr{G} = \left\{X \subseteq A: (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_n (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle \in X)\right\}$$

on A, while the closed game quantifier $\breve{\mathscr{G}}$ on $A^{<\omega}$ becomes the quantifier

$$\breve{\mathscr{G}} = \left\{X \subseteq A: (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_n (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle \in X)\right\}$$

on A. For the remainder of this section, if $\mathfrak{A}$ is an acceptable structure, then $\langle\ \rangle: A^{<\omega} \to A$ will always denote a total, one-to-one function such that the associated coding and decoding relations and functions seq, lh, q are first-order on $\mathfrak{A}$.

The next theorem provides the basic connection between inductive definability and game quantification. We credit this result to Moschovakis [1974a], [1972] for the inductive relations and to Aczel [1975] for the Q-inductive relations.

4.2.3 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_m, c_1, \ldots, c_k\rangle$ be an acceptable structure and let Q be a monotone quantifier on A. Then,*

(i) *a relation R on A is Q-(positive) inductive if and only if there is a formula $\varphi(u, \bar{z})$ of the "boldface" logic $\mathscr{L}^{\mathfrak{A}}(Q)$ of the structure $\mathfrak{A}$ such that $R(\bar{z}) \Leftrightarrow Q^+ u\varphi(u, \bar{z})$; that is,*

$$R(\bar{z}) \Leftrightarrow (Qv_0 \breve{Q}w_0\, \exists x_0\, \forall y_0\, Qv_1 \breve{Q}w_1\, \exists x_1\, \forall y_1 \cdots) \bigvee_n \varphi(\langle v_0, w_0, x_0, y_0, \ldots, v_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}\rangle, \bar{z});$$

(ii) *in particular, a relation R on A is (positive) inductive if and only if there is a formula φ of the "boldface" logic $\mathscr{L}^{\mathfrak{A}}$ of the structure $\mathfrak{A}$ such that*

$$R(\bar{z}) \Leftrightarrow \mathscr{G}u\varphi(u, \bar{z}) \Leftrightarrow (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_n \varphi(\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle, \bar{z}).$$

Hint of Proof. The inductive analysis of open games given in the proof of the Gale–Stewart theorem (1.2.4) can be used to show that if $(R(\bar{z}) \Leftrightarrow Q^+ u\varphi(u, \bar{z}))$, then the relation R is Q-inductive. For the other direction, one has to show first that if $\psi(\bar{z}, S)$ is a formula of $\mathscr{L}^{\mathfrak{A}}(Q)$ in which S occurs positively, then there is a quantifier-free formula $\theta(\bar{v}, \bar{w}, \bar{x}, \bar{y}, \bar{z})$ such that

$$\psi(\bar{z}, S) \Leftrightarrow (Qv_0)(\breve{Q}w_0)(\exists x_0)(\forall y_0) \cdots (Qv_m)(\breve{Q}w_m)(\exists x_m)(\forall y_m)(\forall \bar{t})\, [\theta(\bar{v}, \bar{w}, \bar{x}, \bar{y}, \bar{z}) \vee S(\bar{t})].$$

Using the equivalence above and the coding machinery on $\mathfrak{A}$, it is not hard to verify that the smallest fixed point I_ψ of the formula $\psi(\bar{z}, S)$ is explicitly definable by the next quantifier Q^+ applied to a formula $\varphi(u, \bar{z})$ of $\mathscr{L}^{\mathfrak{A}}(Q)$. □

4.2.4. The above identification of the inductive relations with the ones definable by open game formulas is an absolute version of Svenonius' theorem (2.1.5), and has many applications in either direction. In particular, results from inductive definability have implications for game quantification and vice-versa. For example, we can use the proof of Theorem 4.2.3 to discover the main properties of the approximations of the open game formulas. Indeed, if $\Phi(\bar{z})$ is an open game formula and $\varphi(\bar{z}, S)$ is a positive in S formula of $\mathscr{L}^{\mathfrak{A}}$ such that $\mathfrak{A} \models (\forall \bar{z})(\Phi(\bar{z}) \leftrightarrow I_\varphi(\bar{z}))$, then the approximations ε_α^Φ of Φ are equivalent on $\mathfrak{A}$ to the stages I_φ^α of φ. In the other direction, Moschovakis [1974a] used Theorem 4.2.3 to show the existence of universal inductive relations on acceptable structures. As a consequence, on every acceptable structure there are inductive relations which are not hyperelementary. Moreover, on such structures the relation of satisfaction "$\mathfrak{A} \models \varphi$", where φ is a sentence of $\mathscr{L}^{\mathfrak{A}}$, is hyperelementary; but it is not, of course, first-order.

The tools of inductive definability can be used to obtain local versions of such global results as Vaught's covering theorem (See Section 2.2.10), the separation and reduction principles and others. One of the main tools is the stage comparison theorem of Moschovakis [1974a] which asserts, intuitively, that we can compare the stages of an inductive definition in an inductive way. Its consequences include the following theorem, a theorem which is true for an arbitrary structure $\mathfrak{A}$.

4.2.5 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be a structure and let Q be a monotone quantifier on A. Then the class* $\mathbf{IND}(\mathfrak{A}, Q)$ *of the Q-inductive relations has the pre-well-ordering property. That is, if $P \subseteq A^n$ is a Q-inductive relation, then there is a map $\sigma\colon P \xrightarrow{\text{onto}} \lambda$, where λ some ordinal, such that the relations $\leq_\sigma^*$ and $<_\sigma^*$ are Q-inductive, where*

$$\bar{x} \leq_\sigma^* \bar{y} \Leftrightarrow (\bar{x} \in P) \,\&\, (\bar{y} \notin P \vee \sigma(\bar{x}) \leq \sigma(\bar{y}))$$

and

$$\bar{x} <_\sigma^* \bar{y} \Leftrightarrow (\bar{x} \in P) \,\&\, (\bar{y} \notin P \vee \sigma(\bar{x}) < \sigma(\bar{y})). \quad \square$$

If P is a Q-inductive relation and $\sigma\colon P \xrightarrow{\text{onto}} \lambda$ is a map such that the relations $\leq_\sigma^*$ and $<_\sigma^*$ are Q-inductive, then we say that σ is a *Q-inductive norm* on P. The existence of Q-inductive norms easily implies the reduction principle for the Q-inductive relations and the separation principle for the complements of the Q-inductive relations.

With any structure $\mathfrak{A}$ we associate the ordinal $\kappa^{\mathfrak{A}}$, where

$$\kappa^{\mathfrak{A}} = \sup\{\text{rank}(<)\colon < \text{ is a hyperelementary pre-well-ordering on } A\}.$$

If Q is a monotone quantifier on the universe A of the structure $\mathfrak{A}$, then we consider also the ordinal

$$\begin{aligned}\kappa^{\mathfrak{A}(Q)} = \sup\{&\text{rank}(<)\colon \\ &< \text{ is a } Q\text{-hyperelementary pre-well-ordering on } A\}.\end{aligned}$$

The stage comparison theorem also yields the following useful boundedness principle.

4.2.6 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be a structure and let Q be a monotone quantifier on A. Assume further that P is a Q-inductive relation and $\sigma\colon P \xrightarrow{\text{onto}} \lambda$ is a Q-inductive norm. Then*

(i) $\lambda \leq \kappa^{\mathfrak{A}(Q)}$;
(ii) *for each $\xi < \lambda$ the set $P^\xi = \{\bar{x} \in P\colon \sigma(\bar{x}) \leq \xi\}$ is Q-hyperelementary;*
(iii) *P is Q-hyperelementary if and only if $\lambda < \kappa^{\mathfrak{A}(Q)}$.* $\square$

The above result can be thought of as a local version of the approximation theorem (2.2.7) and the undefinability of well-order. Actually, Moschovakis [1974a] showed that it implies a covering theorem for the Q-inductive relations on any structure.

4.2.7 The Covering Theorem. *Let P be a Q-inductive relation on a structure $\mathfrak{A}$ and let $\sigma: P \xrightarrow{\text{onto}} \lambda$ be a Q-inductive norm. If R is the complement of a Q-inductive relation and $R \subseteq P$, then there is an ordinal $\xi < \kappa^{\mathfrak{A}(Q)}$ such that*

$$R \subseteq P^{\xi} = \{\bar{x} \in P: \sigma(\bar{x}) \leq \xi\}.$$

In particular, R is contained in a Q-hyperelementary subset of P. $\square$

In order to gain more insight into the relations definable by the game quantifiers on an acceptable structure, we next state various characterizations of the Q-inductive relations in terms of Spector classes, functional recursion, representability in stronger logics, and, finally, admissible sets with quantifiers.

4.2.8. Let Γ be a class of relations on A and let Q be a monotone quantifier on A. We say that Γ is *closed under Q* if, whenever a relation $P \subseteq A^{n+1}$ is in Γ, then the relation $R \subseteq A^n$ is also in Γ, where $R(\bar{x}) \Leftrightarrow (Qy)P(y, \bar{x})$.

The class Γ has the *pre-well-ordering property* if, for each relation P in Γ, there is a mapping $\sigma: P \xrightarrow{\text{onto}} \lambda$, where λ an ordinal, such that the relations $\leq_{\sigma}^{*}$ and $<_{\sigma}^{*}$ are in Γ.

Assume that $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ is an acceptable structure and Γ is a collection of relations on A. We call Γ a *Spector class on* $\mathfrak{A}$ if:

(i) Γ contains all first-order relations on A with parameters from A and is closed under $\cup$, $\cap$, $\forall$, $\exists$;
(ii) Γ has the pre-well-ordering property; and
(iii) Γ is *A-parametrized.* That is to say, for each $n \in \omega$, there is a $(n + 1)$-ary relation U^n in Γ with the property that a relation $R \subseteq A^n$ is in Γ if and only if there is some $a \in A$ such that $R = \{\bar{x} \in A^n: (a, \bar{x}) \in U^n\}$.

It actually turns out that the collections $\mathbf{IND}(\mathfrak{A})$ and $\mathbf{IND}(\mathfrak{A}, Q)$ of the inductive and the Q-inductive relations are both Spector classes. The notion of a Spector class was introduced by Moschovakis [1974a] and provides a framework for developing abstract recursion theory. The following is a theorem of Moschovakis [1974a] and Aczel [1975]. On the one hand, it summarizes the main closure and structural properties of the inductive and the Q-inductive relations while, on the other, it yields a minimality characterization for these classes of relations.

4.2.9 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be an acceptable structure and let Q be a monotone quantifier on A.*

(i) *The collection $\mathbf{IND}(\mathfrak{A}, Q)$ of the Q-inductive relations on A is the smallest Spector class on $\mathfrak{A}$ closed under Q and $\breve{Q}$.*
(ii) *In particular, the collection $\mathbf{IND}(\mathfrak{A})$ of the inductive relations on A is the smallest Spector class on $\mathfrak{A}$.*

We state one further result about Spector classes, a result which shows that these classes possess interesting closure properties and are related to game quantification.

4.2.10 Theorem. *Let $\mathfrak{A}$ be an acceptable structure, Q a monotone quantifier on A and Γ a Spector class on $\mathfrak{A}$. Then*

(i) *Γ is closed under the quantifier Q if and only if Γ is closed under the quantifier $Q^{\vee}$. In particular,*

(ii) *Γ is closed under Q and $\breve{Q}$ if and only if Γ is closed under the next quantifier Q^{+}.*

(iii) *Every Spector class is closed under the open game quantifier $\mathscr{G}$.* □

4.2.11. Let A be a set such that $\omega \subseteq A$ and let $\mathscr{PF}_k$ be the collection of all k-ary partial functions from A to ω. A *functional on A* is a partial mapping

$$\Phi\colon A^l \times \mathscr{PF}_{k_1} \times \cdots \times \mathscr{PF}_{k_m} \to \omega,$$

which is *monotone*. That is, if $\Phi(\bar{x}, g_1, \ldots, g_m) = w$ and $g_1 \subseteq h_1, \ldots, g_m \subseteq h_m$, then $\Phi(\bar{x}, h_1, \ldots, h_m) = w$.

If $\bar{\Phi} = (\Phi_1, \ldots, \Phi_s)$ is a finite sequence of functionals on the universe of a structure $\mathfrak{A}$, then we can define the notion of a *recursive in Φ m-ary* partial function from A to ω. This is done by first associating with $\bar{\Phi}$ the smallest class of functionals having certain closure properties and containing $\bar{\Phi}$, and then iterating the operative functionals in that class. The detailed definitions of functional recursion can be found in Kechris–Moschovakis [1977].

A relation P on A is *semi-recursive in $\bar{\Phi}$* if it is the domain of a recursive in $\bar{\Phi}$ partial function. We say that P is *recursive in $\bar{\Phi}$* if its characteristic function χ_P is recursive in $\bar{\Phi}$. We put

$$\mathbf{ENV}[\bar{\Phi}] = \text{the collection of all semirecursive in } \bar{\Phi} \text{ relations}$$

and

$$\mathbf{SEC}[\bar{\Phi}] = \text{the collection of all recursive in } \bar{\Phi} \text{ relations.}$$

These classes of relations are called, respectively, the *envelope of $\bar{\Phi}$* and the *section of $\bar{\Phi}$*.

Any monotone quantifier Q on A gives rise to a functional $\mathbf{F}_Q^{\#}$ which embodies existential quantification with respect to Q and $\breve{Q}$. This functional is defined by

$$\mathbf{F}_Q^{\#}(p) = \begin{cases} 0, & \text{if } (Qx)(p(x) = 0), \\ 1, & \text{if } (\breve{Q}x)(p(x)\downarrow \neq 0), \\ \uparrow, & \text{otherwise,} \end{cases}$$

where p varies over the partial functions from A to ω. Here $\downarrow$ abbreviates "is defined", while $\uparrow$ stands for "is undefined". If Q is the existential quantifier $\exists$, then we write $\mathbf{E}^{\#}$ for $\mathbf{F}^{\#}_{\exists}$ so that

$$\mathbf{E}^{\#}(p) = \begin{cases} 0, & \text{if } (\exists x)(p(x) = 0), \\ 1, & \text{if } (\forall x)(p(x)\downarrow \neq 0), \\ \uparrow, & \text{otherwise.} \end{cases}$$

It is not hard to show that positive elementary induction in the quantifier Q coincides with recursion in the functionals $\mathbf{E}^{\#}$, $\mathbf{F}^{\#}_{Q}$.

4.2.12 Theorem. *Let $\mathfrak{A}$ be an acceptable structure and Q a monotone quantifier on A, then,*

(i) *A relation is Q-inductive if and only if it is semirecursive in* $\mathbf{E}^{\#}$, $\mathbf{F}^{\#}_{Q}$ *and hence*

$$\mathbf{IND}(\mathfrak{A}, Q) = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{Q}].$$

(ii) *A relation is Q-hyperelementary if and only if it is recursive in* $\mathbf{E}^{\#}$, $\mathbf{F}^{\#}_{Q}$ *and hence*

$$\mathbf{HYP}(\mathfrak{A}, Q) = \mathbf{SEC}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{Q}].$$

In particular, we have

$$\mathbf{IND}(\mathfrak{A}) = \mathbf{ENV}[\mathbf{E}^{\#}] \quad \textit{and} \quad \mathbf{HYP}(\mathfrak{A}) = \mathbf{SEC}[\mathbf{E}^{\#}]. \quad \square$$

4.2.13. Assume that $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ is a structure and T is a system of axioms and rules of inference in a logic $\mathscr{L}$ which has a constant $\mathbf{a}$ for each element $a \in A$. We say that a relation P on A is *weakly representable in T* if there is a formula φ of $\mathscr{L}$ such that

$$P(a_1, \ldots, a_n) \Leftrightarrow T \vdash \varphi(\mathbf{a}_1, \ldots, \mathbf{a}_n).$$

We say that P is *strongly representable* in T if both P and $\neg P$ are weakly representable. Aczel [1970, 1977] characterized the inductive and the Q-inductive relations on an acceptable structure in terms of representability in certain systems. If $\mathfrak{A}$ is a given structure, then the *infinitary system* $T^{\infty}(\mathfrak{A})$ consists of the following axioms and rules of inference:

(i) All standard first-order axioms and rules of inference for the "boldface" first-order logic $\mathscr{L}^{\mathfrak{A}}$.
(ii) All atomic and negated atomic sentences of $\mathscr{L}^{\mathfrak{A}}$ which are true in $\mathfrak{A}$.
(iii) *A-rule*: From $\varphi(\mathbf{a})$ for all $a \in A$, infer $(\forall x)\varphi(x)$.

If Q is a monotone quantifier on A, then the *infinitary system* $T^\infty(\mathfrak{A}, Q)$ *has*, in addition to (i), (ii), and (iii), the following rules:

(iv) *Q-rule*: From $\varphi(\mathbf{a})$ for all $a \in X$, with $X \in Q$, infer $(Qx)\varphi(x)$.
(v) *$\breve{Q}$-rule*: From $\varphi(\mathbf{a})$ for all $a \in X$, with $X \in \breve{Q}$, infer $(\breve{Q}x)\varphi(x)$.

Notice that the $\forall$-rule is the same as the A-rule, while the $\exists$-rule is an axiom of first-order logic, namely from $\varphi(\mathbf{a})$, for some a, infer $\exists x\varphi(x)$.

4.2.14 Theorem. *Let $\mathfrak{A}$ be an acceptable structure and Q a monotone quantifier on A.*

(i) *A relation P on A is weakly representable in $T^\infty(\mathfrak{A}, Q)$ if and only if it is Q-inductive.*
(ii) *A relation P on A is strongly representable in $T^\infty(\mathfrak{A}, Q)$ if and only if it is Q-hyperelementary.*

In particular, the inductive relations are exactly the weakly representable ones in $T^\infty(\mathfrak{A})$ and the hyperelementary relations are the strongly representable ones in $T^\infty(\mathfrak{A})$. $\square$

Notice that if $\mathfrak{A}$ is a countable, acceptable structure, then Svenonius theorem (2.1.5), when combined with Theorems 4.2.3 and 4.2.14, yields a completeness result about the infinitary system $T^\infty(\mathfrak{A})$, namely that if a formula $\varphi(X_1, \ldots, X_n)$ of $\mathscr{L}^{\mathfrak{A}}$ is universally valid, then $T^\infty(\mathfrak{A}) \vdash \varphi(X_1, \ldots, X_n)$. This completeness theorem also has a direct proof which uses the omitting types theorem. In this case, Theorems 4.2.3 and 4.2.14 can be used to give an alternative proof of Svenonius' theorem. On the structure of arithmetic $\mathbb{N} = \langle \omega, +, \cdot \rangle$ these results become the classical representability characterization of the Π_1^1 relations in ω-logic.

Finally, we mention the characterizations of the Q-inductive relations in terms of admissible sets with quantifiers. For simplicity, we restrict our attention to acceptable structures of the form $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ where A is a transitive set.

If A and M are transitive sets, $A \in M$, and Q is a quantifier on A, then we can define what it means for M to be a *$Q^\#$, $\breve{Q}^\#$-admissible set*. The crucial additional axioms are the schemata of Q and $\breve{Q}$-collection, where

$$\textit{Q-collection}: \ (Qx \in A)(\exists y)\varphi \rightarrow (\exists w)(Qx \in A)(\exists y \in w)\varphi,$$

$$\breve{Q}\textit{-collection}: \ (\breve{Q}x \in A)(\exists y)\varphi \rightarrow (\exists w)(\breve{Q}x \in A)(\exists y \in w)\varphi,$$

with φ a $\Delta_0(Q, \breve{Q})$ formula. The detailed definitions are given in Moschovakis [1974a] and Barwise [1978b], while the next theorem comes from Barwise–Gandy–Moschovakis [1971] and Moschovakis [1974a].

4.2.15 Theorem. *Let $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be an acceptable structure such that A is a transitive set and let Q be a quantifier on A. Put*

$$\mathfrak{A}^\#(Q) = \bigcap\{M : \mathfrak{A} \in M \textit{ and } M \textit{ is a } Q^\#, \breve{Q}^\#\textit{-admissible set}\}.$$

Then $\mathfrak{A}^{\#}(Q)$ is a $Q^{\#}$, $\breve{Q}^{\#}$-admissible set, $o(\mathfrak{A}^{\#}(Q)) = \kappa^{\mathfrak{A}(Q)}$ and moreover, for any relation P on A

(i) *P is Q-inductive if and only if P is $\Sigma_1(Q, \breve{Q})$ on $\mathfrak{A}^{\#}(Q)$*
(ii) *P is Q-hyperelementary if and only if $P \in \mathfrak{A}^{\#}(Q)$.* □

At this point, we will collect all the characterizations of the Q-inductive relations into one result which we now present

4.2.16 Theorem. *Let $\mathfrak{A}$ be an acceptable structure and Q a monotone quantifier on A. If $P \subseteq A^n$ is a relation on A, then the following are equivalent:*

(i) *P is explicitly definable by the next quantifier Q^+; that is, there is a formula*

$$\varphi(u, \bar{z}) \text{ of } \mathscr{L}^{\mathfrak{A}}(Q) \quad \text{such that} \quad (\forall \bar{z})(P(\bar{z}) \Leftrightarrow Q^+ u\varphi(u, \bar{z})).$$

(ii) *P is Q-inductive.*
(iii) *P is in the smallest Spector class on $\mathfrak{A}$ closed under Q and $\breve{Q}$.*
(iv) *P is semi-recursive in $\mathbf{E}^{\#}$, $\mathbf{F}_Q^{\#}$.*
(v) *P is weakly representable in $T^{\infty}(\mathfrak{A}, Q)$.*
(vi) *P is $\Sigma_1(Q, \breve{Q})$ on the smallest $Q^{\#}$, $\breve{Q}^{\#}$-admissible set having $\mathfrak{A}$ as element, provided that the universe A of the structure $\mathfrak{A}$ is transitive and $\in \restriction A$ is among the relations of $\mathfrak{A}$.* □

The local results given above suggest certain generalizations of the global results in Section 2. The approximation theory extends to formulas involving the next quantifier; that is to say, it extends to expressions of the form $Q^+ u\varphi(u, \bar{z})$ and $(Q^+)^{\smile} u\varphi(u, \bar{z})$, where Q is an arbitrary monotone quantifier. However, in general, Svenonius' theorem does not hold for an arbitrary quantifier Q—in fact, it is actually false if Q is the open game quantifier $\mathscr{G}$. An interesting problem is to find natural monotone quantifiers Q for which Theorem 2.1.5 goes through. This, of course, is equivalent to the completeness theorem for the infinitary system $T^{\infty}(\mathfrak{A}, Q)$.

4.3. Non-monotone Induction and Recursion in the Game Quantifiers

4.3.1. A *second-order relation on a set A* is a relation $\varphi(x_1, \ldots, x_n, S)$ with arguments elements $x_1, \ldots, x_n$ of A and subsets S of a cartesian product A^m for some $m < \omega$. If $\varphi(x_1, \ldots, x_n, S)$ is a second-order relation on A and $S \subseteq A^n$, then we iterate φ and, by induction on the ordinals, define a sequence of n-ary relations $\{\varphi^{\xi}\}_{\xi}$ on A, where

$$\bar{x} \in \varphi^{\xi} \Leftrightarrow \left(\bar{x} \in \bigcup_{\eta < \xi} \varphi^{\eta}\right) \vee \varphi\left(\bar{x}, \bigcup_{\eta < \xi} \varphi^{\eta}\right).$$

We put

$$\varphi^{\infty} = \bigcup_{\xi} \varphi^{\xi}$$

and call φ^{∞} the set *inductively defined by* φ.

Notice that if φ is a monotone relation, then $(\bar{x} \in \varphi^{\xi} \Leftrightarrow \varphi(\bar{x}, \bigcup_{\eta<\xi} \varphi^{\eta}))$. This was indeed the case for the second-order relations determined by positive formulas in Section 4.1. Here we consider second-order relations which in general are non-monotone.

If $\mathfrak{A}$ is a structure and $\mathscr{F}$ is a collection of second-order relations on A, then we call a (first-order) relation P on A $\mathscr{F}$*-(non-monotone) inductive* in case there is a second-order relation $\varphi(\bar{u}, \bar{v}, S)$ in $\mathscr{F}$ and a sequence $\bar{a}$ of elements of A such that

$$P(\bar{y}) \Leftrightarrow (\bar{a}, \bar{y}) \in \varphi^{\infty}.$$

Let $\mathfrak{A}$ be an acceptable structure, let $\mathscr{G}$ be the open game quantifier on A

$$\mathscr{G} = \left\{X \subseteq A\colon (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_{n} (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle \in X)\right\},$$

and let $P(\bar{x}, S)$ be a second-order relation on A. We say that $P(\bar{x}, S)$ is $\mathscr{G}_1$ *on* $\mathfrak{A}$ if there is a formula $\varphi(u, \bar{x}, S)$ of $\mathscr{L}^{\mathfrak{A}}$ such that

$$P(\bar{x}, S) \Leftrightarrow \mathscr{G}u\varphi(u, \bar{x}, S).$$

We write

$$\mathscr{G}_1 = \text{the collection of all } \mathscr{G}_1 \text{ second-order relations on } \mathfrak{A}.$$

Theorem 4.2.3 has a relativized second-order version which shows that the $\mathscr{G}_1$ relations are exactly the second-order (positive) inductive relations on $\mathfrak{A}$. We will state now a characterization of the $\mathscr{G}_1$-(non-monotone) inductive relations on $\mathfrak{A}$. To do this, however, we need some notions from admissible set theory.

Let M and N be two admissible sets such that $M \subseteq N$. We say that M is N-*stable* if M is a Σ_1-elementary submodel of N, i.e. if for every Σ_1 formula $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in M$

$$\langle M, \in\rangle \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \langle N, \in\rangle \models \varphi(a_1, \ldots, a_n).$$

We say that an admissible set M is $\mathscr{G}_1$*-reflecting* if, for any formula $\varphi(u, \bar{z})$ of set theory and any sequence $\bar{a} = (a_1, \ldots, a_n)$ of parameters from M, we have

$$\langle M, \in\rangle \models \mathscr{G}u\varphi(u, \bar{a}) \Rightarrow \text{there is some admissible set } w \in M \text{ such that } \langle w, \in\rangle \models \mathscr{G}u\varphi(u, \bar{a}).$$

Observe that by Svenonius' theorem (2.1.5) we have that a countable admissible set is $\mathscr{G}_1$-reflecting if and only if it is Π^1_1-reflecting.

4.3.2 Theorem. *An admissible set M is $\mathscr{G}_1$-reflecting if and only if M is M^+ stable, where M^+ is the smallest admissible set having M as element.* ☐

This result is credited to Richter–Aczel [1974] for countable admissible sets. Richter–Aczel [1974] and Moschovakis [1974b] characterized the non-monotone inductions in the open game quantifier using $\mathscr{G}_1$-reflecting admissible sets.

4.3.3 Theorem. *Let $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k\rangle$ be an acceptable structure such that A is a transitive set. A relation P on A is $\mathscr{G}_1$-(non-monotone) inductive if and only if P is $\mathbf{\Sigma}_1$ on the smallest admissible set which is $\mathscr{G}_1$-reflecting and contains $\mathfrak{A}$ as an element.* ☐

This theorem is an absolute version of the following:

4.3.4 Corollary. *Let Π^1_1 be the class of Π^1_1 second-order relations on the structure of arithmetic $\mathbb{N} = \langle \omega, +, \cdot\rangle$. Then a relation P on ω is Π^1_1-(non-monotone) inductive if and only if P is $\mathbf{\Sigma}_1$ on the smallest Π^1_1-reflecting admissible set.* ☐

We next examine the non-monotone inductions in the closed game quantifier

$$\breve{\mathscr{G}} = \left\{X \subseteq A : (\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \right.$$

$$\left. \bigwedge_n (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle \in X)\right\}$$

on an acceptable structure $\mathfrak{A}$.

We say that a second-order relation $P(\bar{x}, S)$ is $\breve{\mathscr{G}}_1$ *on* $\mathfrak{A}$ if there is a formula $\varphi(u, \bar{x}, S)$ of $\mathscr{L}^{\mathfrak{A}}$ such that

$$P(\bar{x}, S) \Leftrightarrow \breve{\mathscr{G}} u \varphi(u, \bar{x}, S).$$

We put

$$\breve{\mathscr{G}}_1 = \text{all } \breve{\mathscr{G}}_1 \text{ second-order relations on } \mathfrak{A}.$$

Harrington–Moschovakis [1974] obtained the following characterization of the non-monotone inductive relations in the quantifier $\breve{\mathscr{G}}$.

4.3.5 Theorem. *Let $\mathfrak{A}$ be an acceptable structure. Then a relation P on A is $\breve{\mathscr{G}}_1$-(non-monotone) inductive if and only if it is $\mathscr{G}$-(positive) inductive, and hence*

$$\breve{\mathscr{G}}_1\text{-IND} = \mathbf{IND}(\mathfrak{A}, \mathscr{G}) = \mathbf{ENV}[\mathbf{E}^\#, \mathbf{F}^\#_{\mathscr{G}}]. \quad ☐$$

4.3.6 Corollary. *Let* $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ *be an acceptable structure such that* A *is a transitive set. A relation* P *on* A *is* $\breve{\mathscr{G}}_1$*-inductive if and only if* P *is* Σ_1 *on the smallest* $\mathscr{G}^{\#}, \breve{\mathscr{G}}^{\#}$*-admissible set with* $\mathfrak{A}$ *as an element.* $\square$

4.3.7. In the light of the preceding theorems, it is natural to ask how do the classes $\mathscr{G}_1$-IND and $\breve{\mathscr{G}}_1$-IND compare. The main theorem of Aanderaa [1974] and the pre-well-ordering property for the second-order (positive) inductive relations (which is the relativized version of Theorem 4.2.5) immediately imply that

$$\mathscr{G}_1\text{-IND} \subsetneqq \breve{\mathscr{G}}_1\text{-IND}.$$

In other words, every $\mathscr{G}_1$-inductive relation is $\breve{\mathscr{G}}_1$-inductive, but the converse is not true. Moreover, the closure ordinals of the $\breve{\mathscr{G}}_1$-inductive relations is much bigger than the closure ordinal of the $\mathscr{G}_1$-inductive relations.

These results show that inductive definability provides ways to distinguish between the open game quantifier and the closed game quantifier. Such distinctions usually do not occur in model theory where a quantifier and its dual are treated on an equal basis, and the properties of the dual are obtained from the ones of the quantifier by involution.

Notice that the functionals $\mathbf{F}^{\#}_{\mathscr{G}}$ and $\mathbf{F}^{\#}_{\breve{\mathscr{G}}}$ do not differentiate the open game quantifier from the closed game quantifier, since it is easy to see that on any acceptable structure

$$\mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{\mathscr{G}}] = \mathbf{IND}(\mathfrak{A}, \mathscr{G}) = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{\breve{\mathscr{G}}}].$$

The recursion-theoretic difference between the quantifiers $\mathscr{G}$ and $\breve{\mathscr{G}}$ is captured by the functional $\mathbf{F}^{\wedge}_{Q}$, which was introduced by Kolaitis [1980] and which, in general, distinguishes the quantifier Q from its dual $\breve{Q}$. The functional $\mathbf{F}^{\wedge}_{Q}$ is defined by

$$\mathbf{F}^{\wedge}_{Q}(p) = \begin{cases} 0, & \text{if } (Qx)(p(x) = 0) \\ 1, & \text{if } p \text{ is total } \& \ (\breve{Q}x)(p(x)\downarrow \neq 0), \\ \uparrow, & \text{otherwise} \end{cases}$$

where p varies over the partial functions from A to ω.

4.3.8 Theorem. *Let* $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ *be an acceptable structure. Then*

$$\mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\mathscr{G}}] \subsetneqq \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\breve{\mathscr{G}}}].$$

Moreover

$$\mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\mathscr{G}}] \subsetneqq \mathscr{G}_1\text{-IND} \subsetneqq \breve{\mathscr{G}}_1\text{-IND} = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\breve{\mathscr{G}}}]. \quad \square$$

4.4. Game Quantification and Descriptive Set Theory

4.4.1. As mentioned in Section 4.1.6, the infinite string $(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots)$ gives rise to a monotone quantifier $(\exists\forall)^*$ on the set A^ω of infinite sequences from A, where

$$(\exists\forall)^* = \{X \subseteq A^\omega \colon (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) X(x_0, y_0, x_1, y_1, \ldots)\}.$$

If $A = \omega$, then the quantifier $(\exists\forall)^*$ is usually denoted by $\mathfrak{D}^1$ or simply by $\mathfrak{D}$ and is called the *game quantifier on* ω^ω, while if $A = R = \omega^\omega$, then $(\exists\forall)^*$ is the *game quantifier* $\mathfrak{D}^2$ *on the set* R^ω of infinite sequences of reals. The properties of the quantifier $\mathfrak{D}$ have been studied in depth by descriptive set theorists. We refer the reader to the book Moschovakis [1980] for a systematic treatment of $\mathfrak{D}$ and its uses in definability theory. Here we will restrict ourselves to stating a sample of the results on the game quantifiers $\mathfrak{D}$ and $\mathfrak{D}^2$, results which are related to topics covered earlier in this chapter.

Assume that Γ is a collection of relations on integers and reals; that is, if $P \in \Gamma$, then P is a relation of the form $P(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)$, where $x_i \in \omega$ for $1 \le i \le n$ and $\alpha_j \in \omega^\omega$ for $1 \le j \le m$. If we quantify every relation in Γ by $\mathfrak{D}$, we then obtain the class

$$\mathfrak{D}\Gamma = \{\mathfrak{D}\alpha P(\bar{x}, \alpha, \bar{\beta}) \colon P(x_1, \ldots, x_n, \alpha, \beta_1, \ldots, \beta_m) \text{ is a relation in } \Gamma\}.$$

In a similar way, we can define the class $\mathfrak{D}^2\Gamma$ for a collection Γ of relations on integers, reals and infinite sequences of reals.

Some of the deeper results in descriptive set theory depend on *transfer theorems* which, in effect, assert that, *under certain assumptions*, properties of a class Γ transfer to the class $\mathfrak{D}\Gamma$ or to the class $\mathfrak{D}^2\Gamma$. In proving such transfer theorems, we usually need certain determinacy theorems or hypotheses about the class Γ.

We say that a relation P on A^ω is *determined* if Player I or Player II wins the game $G(\exists\forall, P)$ associated with P. Of course, for such relations P we have that

$$\begin{aligned}&\neg(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) P(x_0, y_0, x_1, y_1, \ldots)\\ &\quad \Leftrightarrow (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \neg P(x_0, y_0, x_1, y_1, \ldots).\end{aligned}$$

We say that *determinacy holds for a class* Γ of relations on A^ω, and we write $\mathrm{Det}(\Gamma)$, if every relation in Γ is determined.

Martin [1975] established that every Borel set of reals is determined, or equivalently $\mathrm{Det}(\Delta^1_1)$. This is an optimal result in ZFC, since it is well known that $\mathrm{Det}(\Sigma^1_1)$ is not provable in ZFC. Much of the current research in descriptive set theory is carried on under the assumption that certain definable sets of reals are determined. The *hypothesis of projective determinacy* (PD) asserts that every projective set of reals is determined. The *projective sets* are the subsets of the reals which are definable by first-order formulas with parameters over the structure $\mathbb{R} = \langle \omega^\omega \cup \omega, \omega, +, \cdot, Ap\rangle$ of analysis. They are further classified as $\boldsymbol{\Sigma}^1_n$ or $\boldsymbol{\Pi}^1_n$ sets depending on the number of alternations of quantifiers in the defining formula starting respectively with an existential or a universal quantifier. If no parameters are allowed, then we have the "lightface" classes of Σ_n^1 and Π_n^1 sets of reals.

We next state a transfer theorem for the pre-well-ordering property, a result that is due to Moschovakis, and then discuss some of its applications in descriptive set theory.

4.4.2 Theorem. *Let Γ be a class of relations on integers and reals which contains all recursive relations and is closed under finite unions, finite intersections, and substitutions by recursive functions. If Γ has the pre-well-ordering property and* Det(Γ) *holds, then the class $\mathfrak{D}\Gamma$ also has the pre-well-ordering property.* ☐

In order to give concrete applications of this transfer theorem, we first need the following definition. We say that a relation $P(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)$ on integers and reals is Σ_k^0 if there is a recursive relation R such that

$$\begin{aligned} &P(\bar{x}, \alpha_1, \ldots, \alpha_m) \\ &\qquad \Leftrightarrow (\exists l_1)(\forall l_2) \cdots (?l_k) R(\bar{x}, l_1, \ldots, l_k, \bar{\alpha}_1(l_k), \ldots, \bar{\alpha}_m(l_k)), \end{aligned}$$

where all the quantifiers vary over the integers, and if $\alpha \in \omega^\omega$ and $k \in \omega$, then $\bar{\alpha}(k) = \langle \alpha(0), \ldots, \alpha(k-1) \rangle$.

It is quite easy to verify that for each $k \geq 1$ the class of all Σ_k^0 relations is closed under finite unions, finite intersections, recursive substitutions, and has the pre-well-ordering property. Martin's Borel determinacy and the transfer theorem of this section (4.4.2) now immediately imply the following:

4.4.3 Corollary. *The class $\mathfrak{D}\Sigma_k^0$ has the pre-well-ordering property, where $k \geq 1$. Moreover, each $\mathfrak{D}\Sigma_k^0$ is a Spector class.* ☐

The classical normal form for the Π_1^1 relations on the integers and Theorem 2.1.5, in effect, state that

$$\mathfrak{D}\Sigma_1^0 = \Pi_1^1.$$

Solovay has obtained the characterization of the class $\mathfrak{D}\Sigma_2^0$ in terms of non-monotone inductive definitions and this we present in

4.4.4 Theorem. *Let $\mathbb{N} = \langle \omega, +, \cdot \rangle$ be the structure of arithmetic and let Σ_1^1 be the collection of all Σ_1^1 second-order relations on ω. Then a relation P of integers and reals in $\mathfrak{D}\Sigma_2^0$ if and only if it is Σ_1^1-(non-monotone) inductive; that is to say,*

$$\mathfrak{D}\Sigma_2^0 = \Sigma_1^1\text{-IND}. \quad ☐$$

In another direction, we first notice that

$$\mathfrak{D}\Pi_{2n+1}^1 = \Sigma_{2n+2}^1$$

for any $n = 0, 1, 2, \ldots$. Moreover, using the hypothesis of projective determinacy (PD), it is easy to see that

$$\mathfrak{D}\Sigma^1_{2n} = \Pi^1_{2n+1}$$

for any $n = 1, 2, \ldots$.

The computations given above when combined with the transfer theorem (4.4.2) give the next result, a result which was first proved directly by Martin and Moschovakis.

4.4.5 Theorem. *Assuming projective determinacy* (PD), *the classes* Π^1_{2n+1} *and* Σ^1_{2n+2} *have the pre-well-ordering property for all* $n = 0, 1, 2, \ldots$. *In fact,* Π^1_{2n+1} *and* Σ^1_{2n+2} *are Spector classes for all* $n = 0, 1, 2, \ldots$. $\square$

This result is part of the periodicity picture for the projective sets, assuming projective determinacy. For more on the periodicity phenomena as well as on transfer theorems involving much stronger properties, we again refer the reader to Moschovakis [1980].

Recently work has been done on the game quantifier $\mathfrak{D}^2$ on the set R^ω of infinite sequences of reals. This includes transfer theorems of the type we have described here as well as a very useful characterization of the Σ^2_1 in $L(\mathbb{R})$ sets of reals.

The *inner model* $L(\mathbb{R})$ is the smallest model of ZF which contains the structure $\mathbb{R} = \langle \omega^\omega \cup \omega, \omega, +, \cdot, Ap \rangle$ of analysis and all the ordinals as elements. If P is a relation on integers and reals, we say that P is Σ^2_1 *in* $L(\mathbb{R})$ if there is a formula $\varphi(\bar{x}, \bar{a}, X)$ of the first-order language $\mathscr{L}^{\mathbb{R}}$ of the structure $\mathbb{R}$ such that

$$P(\bar{x}, \bar{a}) \Leftrightarrow (\text{in } L(\mathbb{R}) \text{ we have that } \mathbb{R} \models (\exists X)\varphi(\bar{x}, \bar{a}, X))$$

where, of course, the existential quantifier $(\exists X)$ ranges over subsets of reals.

In the terminology of Sections 1 and 2 of this chapter, the Σ^2_1 in $L(\mathbb{R})$ sets of reals are exactly the sets of reals definable in the sense of $L(\mathbb{R})$ by Σ^1_1 second-order formulas of the structure $\mathbb{R}$ of analysis.

We will end this chapter with a theorem of Martin and Steel. This result can be found in Martin–Moschovakis–Steel [1982].

4.4.6 Theorem. *A relation P on integers and reals is* Σ^2_1 *in* $L(\mathbb{R})$ *if and only if it is* $\mathfrak{D}^2\Pi^1_1$; *that is to say, if and only if there is a* Π^1_1 *relation S such that*

$$P(\bar{x}, \bar{\alpha}) \Leftrightarrow (\exists \beta_0\, \forall \gamma_0\, \exists \beta_1\, \forall \gamma_1 \cdots) S(\bar{x}, \bar{\alpha}, \langle \beta_0, \gamma_0, \beta_1, \gamma_1, \ldots \rangle),$$

where the quantifiers in the infinite string range over the reals. $\square$

The above result provides a representation of the Σ^2_1 in $L(\mathbb{R})$ sets of reals in terms of the game quantifier $\mathfrak{D}^2$ applied to a very simple matrix. This representation, together with appropriate transfer theorems and determinacy hypotheses, makes it possible to obtain important structural properties for the class Σ^2_1 in $L(\mathbb{R})$.

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Chapter XI
Applications to Algebra

by P. C. EKLOF

In contrast to the situation in first-order finitary logic, the applications of infinitary logic to algebra are so scattered throughout the literature that it is extremely difficult to discern any coherent pattern. Nevertheless, there are some interesting applications; and, in this chapter, we will survey a few of them. This survey will primarily be for the benefit of the non-specialist. That being so, proofs will not always be given in detail, since our aim is simply to present enough background to state a result, indicate its significance, and explain how infinitary logic enters into the statement of the result and/or its proof.

The separate sections are organized by algebraic subject matter and are essentially independent of each other. The first four sections involve $\mathscr{L}_{\infty\omega}$, while the fifth and sixth make use of $\mathscr{L}_{\infty\kappa}$ for arbitrary κ. The last section is simply a collection of references to other relevant literature.

The first two sections of our survey deal with applications of logic to algebra in the purest sense that results expressible in algebraic terms are proved by logical means. The first section's concern—arguably the most important application to date of infinitary model theory to algebra—is the construction by Macintyre and Shelah of non-isomorphic universal locally finite groups of the same cardinality. In the second section we examine the use by Baldwin of some profound results in the model theory of $\mathscr{L}_{\omega_1\omega}$ to count the number of subdirectly irreducible algebras in a variety. The remaining sections involve applications in which logical notions are employed in the expression as well as in the proof of a result so as to provide new insight into an algebraic notion or problem.

Sections 3, 4 and 5 make use of the notion of infinitary equivalence. In Section 3, the back-and-forth characterization of $\mathscr{L}_{\infty\omega}$ equivalence is used to formulate precisely and prove the heuristic principle in algebraic geometry known as Lefschetz's principle. Classification theorems in abelian group theory are studied in Section 4 to see what information can be gained from their proofs about the $\mathscr{L}_{\lambda\omega}$-equivalence of abelian groups. Section 5 gives a characterization of the algebras in a variety which are $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra, and the question of the existence of non-free such algebras is studied, in general, and specifically in the variety of abelian groups. Finally, Section 6 presents both Hodges' formalization of the notion of a concrete (or effective) construction and an examination of his use of it in proving that certain algebraic constructions are not concrete.

I am deeply appreciative to the authors of the other chapters of this volume for their help. And, to Wilfred Hodges, Carol Jacoby, David Kueker, and Alan Mekler, I extend my special thanks.

1. Universal Locally Finite Groups

Using the model-theory of $\mathscr{L}_{\omega_1\omega}$ Macintyre and Shelah [1976] (to be denoted hereafter simply as [M–S]) answered questions raised by Kegel and Wehrfritz [1973] about the groups in the title.

Recall that a group is *locally finite* if every finitely-generated subgroup is finite. The following class of groups is precisely the class of existentially closed locally finite groups and was first studied by Hall [1959].

1.1 Definition. A group G is *universal locally finite* (or $G \in$ ULF) if it is locally finite and:

(i) every finite group G can be embedded into G; and
(ii) any isomorphism between finite subgroups of G is induced by an inner automorphism of G (see Kegel–Wehrfritz [1973, pp. 177 f]).

Hall has shown that any infinite locally finite group can be embedded in a ULF group of the same cardinality, and that any two countable ULF groups are isomorphic. In fact, the latter result is easily proved by a back-and-forth argument which will show that any two ULF groups are $\mathscr{L}_{\infty\omega}$-equivalent and that if $G \subseteq H$ belong to ULF, then $G \prec_{\infty\omega} H$.

Kegel and Wehrfitz [1973, Chapter 6] posed the following questions:

1.2 Questions. (a) *Are any two* ULF *groups of the same cardinality isomorphic? If not, are there* 2^κ ULF *groups of cardinality* κ?

(b) *Does every* ULF *group of cardinality* $\kappa \geq \aleph_1$ *contain an isomorphic copy of every locally finite group of cardinality* $\leq \kappa$?

The key to the results of Macintyre and Shelah is the observation that ULF is elementary in $\mathscr{L}_{\omega_1\omega}$. Indeed, for each $m \geq 1$ let $\{\varphi_{m,n}(v_1, \ldots, v_m): n \in \omega\}$ be an enumeration of all formulas of $\mathscr{L}_{\omega_1\omega}$ which describe the multiplication table of a set of m generators of a finite group. Then a group G belongs to ULF iff $G \models \sigma$ where σ is the conjunction of the following sentences:

$$(0) \qquad \bigwedge_m \forall v_1 \ldots v_m \bigvee_n \varphi_{m,n}(v_1 \ldots v_m)$$

[that is, G is locally finite];

$$(1) \qquad \bigwedge_m \bigwedge_n \exists v_1 \ldots v_m \varphi_{m,n}(v_1 \ldots v_m)$$

[that is, G satisfies Definition 1.1(i)]; and

$$(2) \qquad \bigwedge_m \bigwedge_n \forall v_1 \ldots v_m \forall u_1 \ldots u_m \left[(\varphi_{m,n}(v_1 \ldots v_m) \wedge \varphi_{m,n}(u_1 \ldots u_m)) \rightarrow \left(\exists x \bigwedge_{i=1}^{m} x^{-1} v_i x = u_i \right) \right]$$

[that is, G satisfies Definition 1.1(ii)].

Since σ has models of all infinite cardinalities, the method of indiscernibles (see Keisler [1971a, Section 13]) implies the following:

1.3 Theorem. *For every $\kappa \geq \aleph_1$ there is a model G_κ of σ of cardinality κ such that for every countable $A \subseteq G_\kappa$, G_κ has only countably many A-types.* ▯

We will now use the following simple group-theoretic observation.

1.4 Lemma. *For every infinite cardinal κ, there is a locally finite group H_κ of cardinality κ^+ and a subset A_κ of H_κ of cardinality κ such that H_κ realizes at least κ^+ quantifier-free A_κ types.*

Proof. Let G be a finite group with two elements α and β such that $\alpha\beta \neq \beta\alpha$. Then G^κ, the direct product of κ copies of G, is locally finite (the reader is referred to [M–S, Lemma 1(b)]). Let H_κ be the subgroup of G generated by $A_\kappa \cup Y$, where A_κ consists of all functions $f_\nu \in G^\kappa$ ($\nu \in \kappa$), where

$$f_\nu(\mu) = \begin{cases} e, & \text{if } \mu \neq \nu, \\ \alpha, & \text{if } \mu = \nu, \end{cases}$$

and Y is a subset of $\{e, \beta\}^\kappa$ of cardinality κ^+. Then any two distinct elements of Y have different quantifier-free types over A_κ; in fact, if $g, h \in Y$, $g(\nu) = e$, and $h(\nu) = \beta$, then g satisfies $xf_\nu = f_\nu x$, but h does not. ▯

We can now proceed to prove (Refer to Questions 1.2)

1.5 Theorem (Macintyre and Shelah [1976]). (i) *For any $\kappa \geq \aleph_1$ there are 2^κ groups in* ULF *of cardinality κ.*

(ii) *For any $\kappa \geq \aleph_1$ there is a locally finite group H_κ of cardinality κ and a group $G \in$* ULF *of cardinality κ such that H_κ is not embeddable in G_κ.*

Proof. The H_κ of Lemma 1.4 is clearly not embeddable in the G_κ of Theorem 1.3, so (ii) holds. Since H_κ is embeddable in some ULF group of cardinality κ, there are clearly at least two non-isomorphic ULF groups of cardinality κ. In order to obtain 2^κ different ULF groups, we appeal to a theorem of Shelah [1972a, Theorem 2.6] which says that if a sentence σ of $\mathscr{L}_{\lambda^+\omega}$ has for every cardinal κ a model $\mathfrak{B}$ with a subset A of cardinality κ such that $\mathfrak{B}$ realizes more than κ quantifier-free A-types, then for all $\kappa > \lambda$ σ has 2^κ models of cardinality κ. (The reader is also referred to Hodges [1984] for a proof of (i) in a more general context). ▯

These results give rise to other questions which have been posed in [M–S].

1.2. Questions (continued) (c) *Which locally finite groups H can be embedded in all* ULF *groups of cardinality $\geq |H|$?* (Such groups are called *inevitable*).

(d) *For $\kappa \geq \aleph_1$ is there a universal* ULF *group of cardinality κ? That is, is there one into which can be embedded every locally finite group of cardinality $\leq \kappa$?*

Hickin [1978] proved that no locally finite group of cardinality $\aleph_1$ is inevitable.

In fact, he constructed a family of $2^{\aleph_1}$ ULF groups of cardinality $\aleph_1$ such that no uncountable subgroup is embeddable in any two of them. Giorgetta and Shelah [1984] obtained the same result with $\aleph_1$ replaced by any κ such that $\aleph_0 < \kappa \leq 2^{\aleph_0}$. Question (d) was answered in the negative (by Grossberg-Shelah [1983]) for $\kappa = 2^{\aleph_0}$; and, assuming GCH, for all κ of uncountable cofinality. The proofs of the results mentioned above do not, however, use infinitary logic.

Problems similar to Questions 1.2(a) and (b) have been studied for algebraically closed groups and for skew fields. Here the statements of some of the results use the notion of $\mathscr{L}_{\infty\omega}$-equivalence, although the proofs themselves use specific algebraic constructions. For example, we have

1.6 Theorem (Shelah-Ziegler [1979]). *Let A be a countable algebraically closed group. Let κ be an uncountable cardinal.*

(i) *There are 2^κ algebraically closed groups of cardinality κ which are $\mathscr{L}_{\infty\omega}$-equivalent to A.*
(ii) *There is an algebraically closed group of cardinality κ which is $\mathscr{L}_{\infty\omega}$-equivalent to A and which contains no uncountable commutative subgroup.* ∎

See also Macintyre [1976], Ziegler [1980], and Giorgetta-Shelah [1984].

2. Subdirectly Irreducible Algebras

Baldwin [1980] observed that some general theorems of the model theory of $\mathscr{L}_{\omega_1\omega}$ have applications to counting the number of subdirectly irreducible algebras in a residually small variety.

Recall that a variety is a class V of algebras (all structures for the same vocabulary τ, consisting only of function symbols) which is closed under the formation of products, subalgebras and homomorphic images. A fundamental theorem of Birkhoff says that V is a variety if and only if it is the class of models of a set of equations, Σ. In the following discussion we will assume that the vocabulary τ of V is countable.

2.1 Definition. An algebra $\mathfrak{A}$ is called *subdirectly irreducible* if whenever $\mathfrak{A}$ is embeddable in a product of algebras, it is also embeddable in one of the factors. This, of course, is equivalent to requiring that every family $\mathscr{F}$ of homomorphisms on $\mathfrak{A}$ which separates points of $\mathfrak{A}$—that is, for all $a \neq b$ in $\mathfrak{A}\ \exists f \in \mathscr{F}$ such that $f(a) \neq f(b)$—contains a one-one homomorphism. A variety V is *residually small* if the class of subdirectly irreducible algebras in V forms a set, or, equivalently, if there is an upper bound to the size of subdirectly irreducible algebras in V. V is *residually countable* if every subdirectly irreducible algebra in V is countable.

Taylor [1972] has shown that if a variety V is residually small then every subdirectly irreducible algebra in V has cardinality $< (2^{\aleph_0})^+$.

2.2 Definition. A *congruence* on $\mathfrak{A}$ is a subset $\theta \subseteq A \times A$ such that there is a homomorphism f on $\mathfrak{A}$ such that $\theta = \{(a, b) \in A \times A : f(a) = f(b)\}$. θ is non-trivial, if $\theta \neq$ the diagonal on A. If $(c, d) \in A \times A$, the *principal congruence generated by* (c, d), which we denote $\theta(c, d)$, is the smallest congruence containing (c, d).

Note that (a, b) belongs to $\theta(c, d)$ if and only if for every homomorphism f on $\mathfrak{A}$ such that $f(c) = f(d)$ we have $f(a) = f(b)$. Thus, by the compactness theorem of finitary logic, we have:

2.3 Lemma. *For any $a, b, c, d \in \mathfrak{A}$, $(a, b) \in \theta(c, d)$ iff there is a positive (existential) formula $\varphi(x, y, z, u) \in \mathscr{L}_{\omega\omega}$ such that*

$$(*) \qquad \models \forall x, z, u[\varphi(x, x, z, u) \rightarrow z = u]$$

and $\mathfrak{A} \models \varphi[c, d, a, b]$. $\square$

Moreover, as an immediate consequence of the definitions we have:

2.4 Lemma. *An algebra $\mathfrak{A}$ is subdirectly irreducible iff there exists $a \neq b$ in A such that for every non-trivial congruence θ on $\mathfrak{A}$, $(a, b) \in \theta$ iff there exists $a \neq b$ in A such that for every $c \neq d$ in A, $(a, b) \in \theta(c, d)$.* $\square$

Using these results, we can now establish

2.5 Proposition. *For any variety V, there is a sentence $\sigma \in \mathscr{L}_{\omega_1\omega}$ such that $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}$ is a subdirectly irreducible algebra in V.*

Proof. Let Φ be the set of all positive existential formulas of $\mathscr{L}_{\omega\omega}$ satisfying $(*)$ in Lemma 2.3. Let σ be the conjunction of the defining equations of V and the following sentence:

$$\exists z, u \, \forall x, y \left[z \neq u \wedge \left(x \neq y \rightarrow \bigvee_{\varphi \in \Phi} \varphi(x, y, z, u) \right) \right].$$

By Lemmas 2.3 and 2.4, σ has the desired property. $\square$

We can now apply the model-theory of $\mathscr{L}_{\omega_1\omega}$.

2.6 Theorem (Harnik–Makkai [1977]). *If σ is a sentence of $\mathscr{L}_{\omega_1\omega}$ and σ has at least $\aleph_1$ and fewer than $2^{\aleph_0}$ countable models, then σ has a model of power $\aleph_1$.* $\square$

2.7 Corollary (Baldwin [1980]). *If V is residually countable, then V has either $\leq \aleph_0$ or exactly $2^{\aleph_0}$ subdirectly irreducible algebras.*

Proof. This result follows immediately from Proposition 2.5 and Theorem 2.6. Baldwin [1980] has noted that all the possibilities for the number of subdirectly irreducible varieties do occur. $\square$

The following theorem was proven by Shelah [1975c] under the assumption that $V = L$ and more recently (Shelah [1983a, b]) assuming only GCH.

2.8 Theorem (G.C.H.). *If σ is a sentence of $\mathscr{L}_{\omega_1\omega}$ which has at least one but fewer than $2^{\aleph_1}$ models of power $\aleph_1$ then it has a model of power $\aleph_2$.* $\square$

2.9 Corollary (Baldwin [1980]) (G.C.H.). *If V is residually small, and it has a subdirectly irreducible algebra of power $\aleph_1$ then it has $2^{\aleph_1}$ subdirectly irreducible algebras of power $\aleph_1$.*

Proof. As we remarked after the statement of Definition 2.1, Taylor has shown that a residually small variety has no subdirectly irreducible algebra of power $(2^{\aleph_0})^+ = \aleph_2$. $\square$

Remarks. (i) Theorems 2.6 and 2.8 can also be used in an analogous way to count the number of simple algebras in certain varieties, because the simple algebras are axiomatized by the following sentence of $\mathscr{L}_{\omega_1\omega}$ (where Φ is as in the proof of Proposition 2.5):

$$\forall x, y\, \forall z, u\left[(x \neq y) \rightarrow \bigvee_{\varphi \in \Phi} \varphi(x, y, z, u)\right].$$

(ii) Mekler [1980b] uses the idea of Lemma 2.3 to prove that the class, $\mathscr{R}$, of residually finite groups is axiomatizable in $\mathscr{L}_{\omega_1\omega}$. It follows immediately from the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega_1\omega}$ that $\mathscr{R}$ is of countable character. That is, a group belongs to $\mathscr{R}$ iff every countable subgroup does. (This result was first proved by B. H. Neumann.)

3. *Lefschetz's Principle*

Using notions from category theory and the model theory of $\mathscr{L}_{\infty\omega}$, Eklof [1973] gave a simple and yet comprehensive formalization of Lefschetz's principle from algebraic geometry. The key idea was inspired by the work of Feferman [1972] and basically asserts that certain simply characterized functors preserve $\mathscr{L}_{\infty\omega}$-equivalence.

Following Weil [1962], we will call K a *universal domain* if K is an algebraically closed field of infinite transcendence degree over its prime field. We recall that the prime field of K is the smallest field contained in K and that it is isomorphic to Q (respectively the field with p elements) if char K, the characteristic of K, is 0 (respectively the prime p).

In his foundational work, Weil [1962, p. 306] gave the following explanation of the heuristic principle attributed to S. Lefschetz:

> "For a given value of the characteristic p[= zero or a prime], every result involving only a finite number of points and varieties, which has been proved for some choice of the universal domain remains valid without restriction; there is but one algebraic geometry of characteristic p for each value of p, not one algebraic geometry for each universal domain."

Seidenberg [1958] has rightly pointed out that Lefschetz had in mind a stronger principle: That algebraic geometry is the same for any two algebraically closed ground fields—not necessarily of infinite transcendence degree—having the same characteristic. We will not deal with this stronger principle at all. The reader should consult Barwise–Eklof [1969, Section 3] for historical remarks on formalizations of Lefschefz's principle.

Notice that two universal domains are $\mathscr{L}_{\infty\omega}$-equivalent if and only if they have the same characteristic. Let $\mathscr{U}$ be the category of universal domains. The nature of the formalization of Lefschetz's principle will be that certain functors on $\mathscr{U}$ into a category $\mathscr{C}$ of algebras preserve $\mathscr{L}_{\infty\omega}$-equivalence; any particular instance of Lefschetz's principle will then follow by checking that the algebraic-geometric result in question is a statement in $\mathscr{L}_{\infty\omega}$ about structures constructed by an appropriate functor.

We shall fix a vocabulary τ consisting of a countable set of function symbols but no relation symbols. Let Alg[τ] be the category of all τ-structures and all τ-homomorphisms.

3.1 Definition. A subcategory $\mathscr{C}$ of Alg[τ] will be called a *quasivariety* if it is a full subcategory (that is, if it contains all τ-homomorphisms between objects in $\mathscr{C}$) and the class of objects of $\mathscr{C}$ is axiomatizable by a set of strict universal Horn sentences, that is, a set of sentences of the form

$$\forall x_1 \ldots \forall x_n[\theta_0 \wedge \cdots \wedge \theta_{n-1} \to \theta_m],$$

where each θ_i is atomic.

Thus defined, the class of objects of $\mathscr{C}$ is closed under products and under substructures. Clearly any variety is a quasivariety. In order to characterize the quasivarieties, we recall an important notion from category theory.

3.2 Definition. Let $\kappa \geq \omega$. $D = (I, \geq)$ is a *κ-directed set* if it is a partially ordered set such that for every subset $X \subseteq D$ of cardinality $< \kappa$, there exists $j \in I$ such that $i \leq j$ for all $i \in X$. A *diagram* $\mathfrak{D}$ over D (in Alg[τ]) is a family of τ-algebras $\mathfrak{A}_i$ for each $i \in I$ and τ-homomorphisms φ_{ij}: $\mathfrak{A}_i \to \mathfrak{A}_j$ for each $i \leq j$ in I such that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ if $i \leq j \leq k$. The *κ-direct limit* of a diagram $\mathfrak{D}$ over a κ-directed set D is a structure $\mathfrak{A}$ together with morphisms ψ_i: $\mathfrak{A}_i \to \mathfrak{A}$ for each $i \in I$ such that given any $\mathfrak{B}$ in Alg[τ] and any family of morphisms θ_i: $\mathfrak{A}_i \to \mathfrak{B}(i \in I)$ such that for all $i \leq j$, $\theta_j \circ \varphi_{ij} = \theta_i$, there is exactly one morphism θ: $\mathfrak{A} \to \mathfrak{B}$ such that for all $i \in I$, $\theta \circ \psi_i = \theta_i$.

If κ is ω, we omit the reference to it, and simply say direct limit instead of ω-direct limit. It is a standard result of category theory that the direct limit is unique up to isomorphism and that in Alg[τ] it may be constructed as the disjoint union of the $\mathfrak{A}_i$ modulo the equivalence relation generated by all identities of the form $y = \varphi_{ij}(x)$. Observe that for the latter result, it is necessary that D be directed. We shall always use the term direct limit in this sense of "colimit over a directed set" (see Mitchell [1965, pp. 44–49]).

Mal'cev [1973, Section 11] characterized the quasivarieties $\mathscr{C}$ in Alg[τ] as the full subcategories which are closed under isomorphism, substructure, and direct limits.

We will be interested in functors which preserve direct limits. The following result gives a large class of such functors (see Feferman [1972, Lemma 4]).

3.3 Lemma. *Let $F: \mathscr{C}_0 \to \mathscr{C}_1$ be a functor, where $\mathscr{C}_0$ and $\mathscr{C}_1$ are quasivarieties. Suppose that:*

(i) *F preserves monomorphisms; and*
(ii) *for every $\mathfrak{A} \in \mathscr{C}_0$ and every finite subset $X \subseteq F(\mathfrak{A})$ there exists a finitely generated substructure $\mathfrak{A}_1$ of $\mathfrak{A}$ such that $X \subseteq F(e)[\mathfrak{A}_1]$, where $e: \mathfrak{A}_1 \to \mathfrak{A}$ is the inclusion morphism.*

Then F preserves direct limits. $\square$

Feferman proved that functors satisfying properties (i) and (ii) of Lemma 3.3—he called them *ω-local functors*—preserve $\mathscr{L}_{\infty\omega}$-equivalence and noted that this (and its generalizations for cardinals $\kappa > \omega$) imply various preservation results for algebraic constructions (see Chapter IX, Sections 4.5.2 and 4.5.3). G. Sabbagh suggested means for obtaining some other preservation results by weakening the hypotheses given in (i) and (ii) above (see Eklof [1975a, Section 3]).

We can now state

3.4 Lefschetz's Principle (Formalized). *Let $\mathscr{C}$ be a quasivariety and $F: \mathscr{U} \to \mathscr{C}$ a functor which preserves direct limits. For any universal domains K_1 and K_2, if* char $K_1 =$ char K_2, *then $F(K_1) \equiv_{\infty\omega} F(K_2)$.*

Proof. K_l is the direct limit of the family $\mathscr{S}_l$ of all of its algebraically closed subfields of finite transcendence degree (the morphisms are inclusions between subfields). Thus, $F(K_l)$ is the direct limit of the $F(k)$, $k \in \mathscr{S}_l$, relative to certain morphisms $\psi_k: F(k) \to F(K_l)$. Let $\tilde{F}(k)$ denote the image of $F(k)$ under ψ_k. It is a subalgebra of $F(K_l)$. If $f: k_1 \to k_2$ is an isomorphism between $k_1 \in \mathscr{S}_1$ and $k_2 \in \mathscr{S}_2$, we will show that the isomorphism $F(f): F(k_1) \to F(k_2)$ induces an isomorphism $\tilde{f}$ between $\tilde{F}(k_1)$ and $\tilde{F}(k_2)$ by means of the rule $\tilde{f}(\psi_{k_1}(x)) = \psi_{k_2}(F(f)(x))$, for $x \in F(k_1)$. It suffices to verify that if $\psi_{k_1}(x) = 0$, then $\psi_{k_2}(F(f)(x)) = 0$. But $\psi_{k_1}(x) = 0$ iff there is a $k_1' \supseteq k_1$ in $\mathscr{S}_1$ such that if $e_1: k_1 \to k_1'$ is the inclusion map, $F(e_1)(x) = 0$. In that case, there is a $k_2' \supseteq k_2$ and an isomorphism $f': k_1' \to k_2'$ extending f such that if $e_2: k_2 \to k_2'$ is inclusion, $f' \circ e_1 = e_2 \circ f$. Hence, $0 = F(f')F(e_1)(x) = F(e_2)F(f)(x)$, and so $\psi_{k_2}(F(f)(x)) = 0$.

Now we appeal to the back-and-forth criterion for $\mathscr{L}_{\infty\omega}$-equivalence (Refer to Chapter IX, Theorem 4.3.1 or to Chapter XIII, Theorem 2.1.1). Indeed, the family, $\mathbf{I}$, of all $\tilde{f}$, as f ranges over all isomorphisms from an element of $\mathscr{S}_1$ to an element of $\mathscr{S}_2$, is a family of partial isomorphisms such that $\mathbf{I}: F(K_1) \simeq_{\omega}^{p} F(K_2)$. ☐

Let us now consider as an example of a use of Lefschetz's principle, the paper of Murthy–Swan [1976]. In this study, Lefschetz's principle is used to carry over a result on uncountable universal domains to the case of countable universal domains. It is striking that the methods used in this paper to justify the appeal to Lefschetz's principle closely mirror the considerations of our general theorem. (In fact, the authors specifically noted this; see pp. 141 f). Murthy and Swan proved that the key constructions they were studying are functors on $\mathscr{U}$ (into Ab, the category of abelian groups, or into Sets, the category of sets) which preserve direct limits (Murthy–Swan [1976, Lemma 5.8]). They then used this result to show that certain properties of the objects constructed by these functors are independent of the choice of universal domain (the reader is referred to Murthy–Swan [1976, pp. 142–143]). For example, one of the properties that concerned them is that a certain abelian group $SA_0(X\ )$—the value at $K \in \mathscr{U}$ of a functor on $\mathscr{U}$ which preserves direct limits—is a divisible group of infinite rank. They make an *ad hoc* argument, using the limit preserving property of the functor, to show that if $SA_0(X_{K_1})$ has this property for some (uncountable) K_1 in $\mathscr{U}$, then $SA_0(X_{K_2})$ has the property for all (including countable) K_2 on $\mathscr{U}$ of the same characteristic. From our point of view, the property of being a divisible abelian group of infinite rank is expressible in $\mathscr{L}_{\infty\omega}$, so by Theorem 3.4, char K_1 = char K_2 implies that $SA_0(X_{K_1}) \equiv_{\infty\omega} SA_0(X_{K_2})$. And hence it follows that $SA_0(X_{K_1})$ is divisible of infinite rank iff $SA_0(X_{K_2})$ is.

Another example of Lefschetz's principle, given by Weil [1962], is worked out in detail in Eklof [1973].

4. Abelian Groups

Classification theorems in abelian group theory, due to Ulm and Warfield, were generalized by Barwise–Eklof [1970] and Jacoby [1980], respectively, to classify a larger class of groups up to $\mathscr{L}_{\infty\omega}$-equivalence. This suggests that the notion of potential isomorphism, which has an algebraic formulation in terms of partial isomorphisms, is a natural one to employ in the study of abelian groups.

For simplicity of exposition—especially in the case of mixed groups—we will restrict attention to the local case. That is, we will fix a prime p and consider abelian groups A which are $\mathbb{Z}_p$-modules, where $\mathbb{Z}_p$ is the ring of rationals with denominators prime to p. This means that every element of A is uniquely divisible by every prime different from p. From now on, we will use the word "module" to mean $\mathbb{Z}_p$-module. A torsion module is then a p-group (i.e., an abelian group A such that for all $a \in A$, there exists $n \in \omega$ such that $p^n a = 0$).

For any module A and ordinal α, define $p^\alpha A$ by induction as follows: $p^0 A = A$; $p^{\alpha+1} = p(p^\alpha A) = \{px : x \in p^\alpha A\}$; $p^\sigma A = \bigcap_{\alpha<\sigma} p^\alpha A$, if σ is a limit ordinal. For any $a \in A$, the *height*, $h(a)$, of a is the unique α such that $a \in p^\alpha A - p^{\alpha+1}A$, if it exists, or $h(a) = \infty$, otherwise. It is easy to see that there exists $\sigma < |A|^+$ such that $p^\sigma A = p^{\sigma+1}A = p^\tau A$, for all $\tau > \sigma$. Then $p^\sigma A$, denoted A_d, is a divisible module and a direct summand of A. The structure of a divisible module is easily explicated: it is a direct sum of copies of $\mathbb{Q}$, the rationals, and of $Z(p^\infty)$, the p-torsion component of $\mathbb{Q}/Z$. Thus, the classification problem easily reduces to the problem of classifying *reduced* modules; that is, modules A such that $A_d = \{0\}$.

Define $p^\alpha A[p] = \{x \in p^\alpha A : px = 0\}$; this is a vector space over $GF(p)$, the field of order p. The dimension of the quotient space $p^\alpha A[p]/p^{\alpha+1}A[p]$ is called the α^{th} Ulm *invariant of* A and is denoted by $f(\alpha, A)$. Let $\hat{f}(\alpha, A) = f(\alpha, A)$ if $f(\alpha, A)$ is finite, and $\hat{f}(\alpha, A) = \infty$ otherwise.

Ulm's theorem asserts that two countable reduced torsion modules A and B are isomorphic iff $f(\alpha, A) = f(\alpha, B)$ for all $\alpha < \omega_1$. This result is not true for arbitrary uncountable torsion modules, although the largest class of torsion modules for which the theorem holds—the class of *totally projective* modules—has been given a number of interesting characterizations (see for example, Fuchs [1973, Chapter XII]). However, the back-and-forth method of proof (see Chapter IX, Theorem 4.3.3) does yield a classification of arbitrary torsion modules up to $\mathscr{L}_{\infty\omega}$-equivalence. More precisely, we have

4.1 Theorem (Barwise–Eklof [1970]). *For any cardinal κ and any reduced torsion modules A and B, A is $\mathscr{L}_{\kappa\omega}$-equivalent to B iff $\hat{f}(\alpha, A) = \hat{f}(\alpha, B)$ for all $\alpha < \kappa$.* □

For an exposition of the proof of Theorem 4.1 the reader should see Barwise [1973b]. The proof shows that every torsion module is $\mathscr{L}_{\infty\omega}$-equivalent to a totally projective module. Barwise–Eklof [1970] also uses the back-and-forth method to classify equivalence with respect to certain subclasses of sentences of $\mathscr{L}_{\kappa\omega}$. For instance, if we let $\hat{r}(B)$ = the rank of B, if finite and $\hat{r}(B) = \infty$, otherwise, we then have

4.2 Theorem (Barwise–Eklof). *If A and B are reduced torsion modules then every existential sentence of $\mathscr{L}_{\kappa\omega}$ true in A is true in B iff for all $\alpha < \kappa$, $\hat{r}(p^\alpha A) \leq \hat{r}(p^\alpha B)$.* □

For countable groups (and, even more generally, by a simple argument, for direct sums of countable groups) this yielded the following result—a result which was apparently not previously known.

4.3 Corollary. *If A and B are countable torsion modules then A is embeddable in B iff* $\operatorname{rank}(p^\alpha A) \leq \operatorname{rank}(p^\alpha B)$ *for all $\alpha < \omega_1$.* □

This result was later extended, by different means, to all totally-projective modules by May–Toubassi [1977].

Remark. The Barwise–Eklof method is employed in Eklof [1977c, Theorem 1.6] to give an $\mathscr{L}_{\infty\omega}$-extension of a theorem of Kaplansky characterizing fully invariant subgroups of a countable p-group; the extended theorem characterizes definable subgroups of arbitrary p-groups.

Warfield [1981] defined a class of modules whose torsion members were precisely the totally projective modules and which included many non-trivial mixed modules, these latter being modules that are not a direct sum of a torsion and a torsion-free module. The modules in this class have come to be called *Warfield modules* and are characterized by the property of being summands of simply presented modules, where a *simply presented* module is a module that can be generated by a set of elements subject only to defining relations of the form $p^n x = 0$ or $p^m x = y$.

A Warfield module M has a *decomposition basis*, such a basis being a linearly independent subset X such that, if $[X]$ denotes the submodule generated by X, $M/[X]$ is torsion, and for all

$$x_0, \ldots, x_n \in X, r_0, \ldots, r_n \in Z_p, \qquad h\left(\sum_{i=0}^{n} r_i x_i\right) = \min\{h(r_i x_i) : i \le n\}.$$

In fact, a countable module is a Warfield module if and only if it has a decomposition basis. For uncountable modules, this is not the case, although the Warfield modules can be characterized as those which have a certain kind of decomposition basis X called *nice*, such that $M/[X]$ is a totally projective torsion module (the reader is referred to Hunter–Richman–Walker [1977]).

Warfield classified the Warfield modules by use of new invariants $g(e, M)$ defined as follow. If $x \in M$, the *Ulm sequence* of x, denoted $U(x)$ is the sequence $(h(p^i x))_{i\in\omega}$. Two Ulm sequences $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ are called *equivalent* if there are positive integers n and m such that for all $i \in \omega$, $\alpha_{i+n} = \beta_{i+m}$. Thus, $U(x)$ and $U(y)$ are equivalent if there exists $r, s \in \mathbb{Z}_p$ such that $rx = sy$. If e is an equivalence class of Ulm sequences, and M is a module with a decomposition basis X, define $g(e, M) =$ cardinality of $\{x \in X : U(x) \in e\}$. Warfield showed that this is an invariant of M and that two reduced Warfield modules M and N are isomorphic iff for all ordinals α and all classes $e, f(\alpha, M) = f(\alpha, N)$ and $g(e, M) = g(e, N)$.

Jacoby [1980] extended Warfield's methods to give a classification result for $\mathscr{L}_{\infty\omega}$-equivalence. Let $\hat{g}(e, M) = g(e, M)$ if finite, and equal to ∞, otherwise.

4.4 Theorem (Jacoby). *If M and N are reduced modules with decomposition bases, then $M \equiv_{\infty\omega} N$ iff for all α and all $e, \hat{f}(\alpha, M) = \hat{f}(\alpha, N)$ and $\hat{g}(e, M) = \hat{g}(e, N)$.* □

Now the class of (non-reduced as well as reduced) modules classified (up to $\mathscr{L}_{\infty\omega}$-equivalence) using Theorem 4.1 is an elementary class in $\mathscr{L}_{\infty\omega}$: It is precisely the class of all torsion modules. But the class of all modules with decomposition bases is not even closed under $\mathscr{L}_{\infty\omega}$-equivalence. Jacoby [1980] defined in a natural algebraic way a larger class of modules closed under $\mathscr{L}_{\infty\omega}$-equivalence (but not $EC_{\infty\omega}$) which can be classified up to $\mathscr{L}_{\infty\omega}$-equivalence using Theorem 4.4. But, surprisingly enough, she was able to show that no class of modules that generalizes the class of modules with decomposition bases in any reasonable way is an elementary class in $\mathscr{L}_{\infty\omega}$. The proof uses her classification theorem for modules with decomposition bases. (Jacoby [1980] contains the proof in the global case).

4.5 Theorem. *Let $\mathscr{C}$ be a class of modules satisfying:*

(i) *every Warfield module is in $\mathscr{C}$; and*
(ii) *if $A \in \mathscr{C}$, then every pair of elements of A is contained in a submodule of A which has a decomposition basis.*

Then $\mathscr{C}$ is not an elementary class in $\mathscr{L}_{\infty\omega}$. □

4.6 Corollary. *The class, $\mathscr{C}$, of all modules which are $\mathscr{L}_{\infty\omega}$-equivalent to a module with a decomposition basis is not an elementary class in $\mathscr{L}_{\infty\omega}$.*

Proof. Clearly $\mathscr{C}$ satisfies (i) of Theorem 4.5. Moreover, since every module with a decomposition basis obviously satisfies (ii) of the Theorem 4.5, we can use the back-and-forth method to show that every module in $\mathscr{C}$ satisfies (ii) of Theorem 4.5. □

5. *Almost-Free Algebras*

Algebras which are $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra in an arbitrary variety have been studied by Kueker, Shelah, Mekler, and Eklof among others.

Fix a variety V in a countable vocabulary (see Section 2). We will say that $A \in V$ is *V-free* (on X) if there is a subset $X \subseteq A$ such that for any $B \in V$ and any set map $f: X \to B$, there is one and only one homomorphism $\hat{f}: A \to B$ such that $\hat{f} \upharpoonright X = f$. X is said to be a *set of free generators for A*. Since V will be fixed, we will simply say *free* instead of V-free.

If B is a subalgebra of C, we say B is a *free factor* of C (written $B|C$) if B and C have sets of free generators, X and Y, respectively, such that $X \subseteq Y$. In this case, *every* set of free generators of B extends to a set of free generators of C. If $B|C$, we say C has *infinite rank over B*, if there are X, Y as above such that in addition $Y - X$ is finite.

It follows easily from the back-and-forth criterion (see Chapter IX, Theorem 4.3.3) that if $\kappa \geq \omega_1$, any two free algebras of cardinality $\geq \kappa$ are $\mathscr{L}_{\infty\kappa}$-equivalent. Define A to be *$\mathscr{L}_{\infty\kappa}$-free*, if A is $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra. The back-and-forth criterion implies that A is $\mathscr{L}_{\infty\kappa^+}$-free iff A is the κ^+-direct limit of a set S of free subalgebras of cardinality κ, where the maps are inclusions, such that S is ω-directed under | (see Definition 3.2). The latter condition means that if $G_0, \ldots, G_n \in S$, then there is an $H \in S$ such that for all $i \leq n$, $G_i | H$.

Surprisingly enough, Kueker [1973] has shown that $\mathscr{L}_{\infty\kappa^+}$-free algebras satisfy the following stronger condition. The proof uses game-theoretic methods (see Kueker [1981]).

5.1 Theorem (Kueker). *A is $\mathscr{L}_{\infty\kappa^+}$-free iff A is the κ^+-direct limit of a set S of free subalgebras of cardinality κ such that S is κ^+-directed under |. In particular, if*

$|A| = \kappa^+$, A is $\mathscr{L}_{\infty\kappa^+}$-free iff $A = \cup_{\nu<\kappa^+} A_\nu$ where each A_ν is a free subalgebra of cardinality κ and for all $\mu < \nu < \kappa^+$, $A_\mu | A_\nu$.

Proof. If Y is a subset of A of cardinality κ, the *Y-Shelah game* on A is the game of length ω, where player I (respectively II) chooses X_n, a subalgebra of A of cardinality κ, when n is even (respectively odd), and II wins if for all k, $X_{2k} \subset X_{2k+1}$ and $Y|X_{2k+1}|X_{2k+3}$. Let $S(A) = \{Y\colon$ player II has a winning strategy in the Y-Shelah game on $A\}$. Observe that if F is the free algebra on κ^+ generators, $Y \in S(A)$ iff for some $B|F$, $(A, Y) \equiv_{\infty\kappa^+} (F, B)$. Hence, $S(A)$ is $\mathscr{L}_{\infty\kappa^+}$-definable (see Chang [1968c, Proposition 7]). Now $S(F)$ is clearly κ^+-directed under $|$ and F is the κ^+-direct limit of $S(F)$. Thus, since these facts are expressible in $L_{\infty\kappa^+}$, the same holds when F is replaced by A. $\square$

5.2 Corollary (Kueker). *If A is $\mathscr{L}_{\infty\kappa^+}$-free then there is a free algebra F on a set of free generators of cardinality κ^+ such that $F \prec_{\infty\kappa} A$.* $\square$

It follows from the back-and-forth criterion that for any uncountable λ, A is $\mathscr{L}_{\infty\lambda}$-free iff A is $\mathscr{L}_{\infty\kappa^+}$-free, for every $\kappa < \lambda$ (see Shelah [1975a, Theorem 2.6(c)]).

A natural question is whether or not there are non-free $\mathscr{L}_{\infty\kappa}$-free algebras. The following profoundly interesting result is due to Shelah (Shelah [1975a, Theorem 2.6(d)]).

5.3 Theorem (Shelah). *If λ is singular and A is $\mathscr{L}_{\infty\lambda}$-free and of cardinality λ then A is free.* $\square$

Remarks. (i) Hodges [1981] gives a very clear exposition of Shelah's "singular compactness theorem" in a general form. For those familiar with Hodges [1981], we now indicate how to derive Theorem 5.3 from the results in that paper. It suffices to prove that for every $\kappa < \lambda$, player II has a winning strategy in the κ^+-Shelah game on A (see Hodges [1981, p. 207]). If S is a set of free subalgebras of cardinality κ such that A is the κ^+-direct limit of S and S is ω-directed under $|$, then player II can win by always choosing his subalgebra B_i (i is odd) to be an element of S such that $B_{i-2}|B_i$.

(ii) Mekler [1980a, Theorem 1.6] proved that if κ is a regular cardinal and A is a κ^+-free group (that is, every subgroup of A of cardinality $<\kappa^+$ is free), then A is $\mathscr{L}_{\infty\kappa}$-free. For varieties in which it is not the case that a subalgebra of a free algebra is always free, a different definition of κ^+-free is needed; one (weak) notion of κ^+-free is that A is to satisfy

$$\neg(\forall x_{2i}\, \exists x_{2i+1})_{i<\kappa} \text{“}\langle x_i\colon i < \kappa\rangle \text{ is not free”}$$

(that is, it is not the case that almost every subalgebra of cardinality κ is non-free. The reader should consult Kueker [1977]). It follows from an argument similar to that in Lemma 3.1 of Hodges [1981] that (for regular κ) if A is a κ^+-free algebra in this sense, then A is $\mathscr{L}_{\infty\kappa}$-free (the reader should compare this result to that in Shelah [1975a, Theorem 2.6(b)]). If A is not E_κ^κ-non-free, then A is κ^+-free (in the above sense). Whether or not an $\mathscr{L}_{\infty\kappa^+}$-free algebra is always κ^+-free remains an open question. (It is true under certain hypotheses on V.)

A general theorem of Shelah (see Chapter IX, Theorem 4.3.7) implies that, assuming $V = L$, if κ is regular and not weakly-compact, then there are either 1 or 2^κ $\mathscr{L}_{\infty\kappa}$-free algebras of cardinality κ. Eklof–Mekler [1982] recently proved the following general result about the existence of non-free $\mathscr{L}_{\infty\kappa}$-free algebras.

5.4 Theorem (Eklof–Mekler). (1) ($V = L$). *If there is a non-free $\mathscr{L}_{\infty\omega_1}$-free algebra of cardinality ω_1, then for every regular non-weakly-compact κ there is a non-free $\mathscr{L}_{\infty\kappa}$-free algebra of cardinality κ.*

(2) *If every $\mathscr{L}_{\infty\omega_1}$-free algebra of cardinality ω_1 is free, then for every κ, every $\mathscr{L}_{\infty\kappa}$-free algebra of cardinality κ is free.* $\square$

Moreover, under certain general conditions on the variety V, the hypothesis given in (2) holds if and only if the class of free algebras is definable in $\mathscr{L}_{\omega_1\omega}$ (see also Kueker [1980]).

Much work has been done on the problem of constructing non-free $\mathscr{L}_{\infty\kappa}$-free algebras for V = the variety of groups or abelian groups. Kueker proved that a group (or abelian group) is $\mathscr{L}_{\infty\omega}$-free iff it is ω_1-free. Higman constructed a non-free ω_1-free group. Mekler, as well as Kueker, constructed a non-free $\mathscr{L}_{\infty\omega_1}$-free group of cardinality ω_1. See Mekler [1980a] for more results on groups. Pope [1982] deals with other varieties of groups and rings.

The $\mathscr{L}_{\infty\kappa}$-free abelian groups (for uncountable κ) are characterized by the property that A is κ-free and every subset of A of cardinality $< \kappa$ is contained in a subgroup B of cardinality $< \kappa$, such that A/B is κ-free (see Eklof [1974]). These groups had arisen naturally in the study of Whitehead's problem: by Chase [1963], CH implies that every Whitehead group is $\mathscr{L}_{\infty\omega_1}$-free. But by Shelah [1979b] MA + $\neg$CH implies that there are Whitehead groups which are not $\mathscr{L}_{\infty\omega_1}$-free. The following theorem sums up the main results about the existence of $\mathscr{L}_{\infty\kappa}$-free abelian groups. (See Eklof [1977c] for more details and references).

5.5 Theorem. (i) (Eklof [1975b]) *For all $n \in \omega$ there is a non-free $\mathscr{L}_{\infty\omega_{n+1}}$-free abelian group of cardinality ω_{n+1}.*

(ii) (Shelah [1979a]). GCH $\Rightarrow$ *for all $\kappa < \aleph_{\omega^2}$ there is a non-free $\mathscr{L}_{\infty\kappa^+}$-free abelian group of cardinality κ^+.*

(iii) (Magidor–Shelah [1983]). (*Assuming the consistency of the existence of many supercompact cardinals*). *It is consistent with* GCH *that if $\kappa = \aleph_{\omega^2}$ every κ^+-free abelian group of cardinality κ^+ is free.*

(iv) *If κ is weakly compact, every κ-free abelian group of cardinality κ is free.*

(v) *If κ is strongly compact, every κ-free abelian group is free.*

(vi) (Gregory). $V = L \Rightarrow$ *for every non-weakly-compact regular κ there exists a non-free $\mathscr{L}_{\infty\kappa}$-free abelian group of cardinality κ.*

(vii) *If there is no inner model with a measurable cardinal, then there exist arbitrarily large κ such that there is a non-free $\mathscr{L}_{\infty\kappa^+}$-free abelian group of cardinality κ^+.* $\square$

5.6 Corollary. *If there is a strongly compact cardinal, then the class of free abelian groups is definable in $\mathscr{L}_{\infty\infty}$. If the class of free abelian groups is definable in $\mathscr{L}_{\infty\infty}$ then there is an inner model with a measurable cardinal.*

Proof. If κ is strongly compact, then by (v) the class of free abelian groups is defined by the sentence of $\mathscr{L}_{\kappa\kappa}$ which says that the group is κ-free. Conversely, we prove a little more: if there is no inner model with a measurable cardinal, then the class of free abelian groups is not definable in any $\mathscr{L}$ which has the following downward Löwenheim–Skolem property: there is a cardinal λ such that for every sentence θ of $\mathscr{L}$, if $\mathfrak{A} \models \theta$, then $\mathfrak{B} \models \theta$, for some substructure $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq \lambda$. Suppose there is a sentence θ of $\mathscr{L}$ which is true in a group G iff G is a free abelian group, and let λ be as above. By (vii), there is a non-free $\mathscr{L}_{\infty\kappa^+}$-free abelian group A of cardinality κ^+ for some $\kappa \geq \lambda$. So $A \models \neg\theta$. But every subgroup of A of cardinality $\leq \lambda$ is free and hence satisfies θ—a contradiction. $\square$

For Theorem 5.5(vii) and Corollary 5.6, the crucial fact from the theory of the core model is that if there is a largest κ such that $\square_{\kappa^+}$ holds, then there is an inner model with many measurable cardinals.

6. *Concrete Algebraic Constructions*

Using notions from $\mathscr{L}_{\infty\kappa}$, Hodges gave a formalization of the intuitive idea of an effective algebraic construction and used it in conjunction with set-theoretic methods to give a negative answer to Taylor's question (Taylor [1971]) as to whether or not there is a concrete construction of the pure injective hull of every abelian group.

6.1 Notation. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be vocabularies consisting of function symbols (possibly of infinite arity), and let Alg[$\boldsymbol{\sigma}$] (respectively Alg[$\boldsymbol{\tau}$]) be the category of σ-structures (respectively $\boldsymbol{\tau}$-structures). Let $\mathscr{B}$ (respectively $\mathscr{C}$) be a quasivariety in Alg[$\boldsymbol{\tau}$] (respectively in Alg[$\boldsymbol{\sigma}$]) (see Definition 3.1). The following definition is a formalization of the intuitive idea of a construction which is uniformly definable by generators and relations (see Hodges [1975]).

6.2 Definition. Let $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, $\mathscr{B}$, $\mathscr{C}$ be as in Notation 6.1, and let κ be a regular cardinal. A function F from objects of $\mathscr{B}$ to objects of $\mathscr{C}$ is a *κ-word-construction* if there is a vocabulary $\boldsymbol{\sigma}'$ extending $\boldsymbol{\sigma}$, a set T of terms of $\mathscr{L}[\boldsymbol{\sigma}']$, a set A of atomic formulas of $\mathscr{L}[\sigma']$, and a function $\Gamma\colon T \cup A \to \mathscr{L}_{\infty\kappa}[\boldsymbol{\tau}]$ such that, for all $\alpha \in T \cup A$, $\Gamma(\alpha)$ has free variables among those in α; and, for all $\mathfrak{B}$ in $\mathscr{B}$, $F(\mathfrak{B})$ is isomorphic to $df\langle X, \Phi\rangle$, the structure given by the presentation $\langle X, \Phi\rangle$ where $X = X^{\mathfrak{B}}$, the set of generators, is $\{t(\vec{b})\colon t \in T,\ \mathfrak{B} \models \Gamma(t)[\vec{b}]\}$ and $\Phi = \Phi^{\mathfrak{B}}$, the set of relations, is $\{\varphi(\vec{b})\colon \varphi \in A,\ \mathfrak{B} \models \Gamma(\varphi)[\vec{b}]\}$. (Let $\vec{b}$ run over all sequences of elments of $\mathfrak{B}$ of length $<\kappa$). More precisely, $df\langle X, \Phi\rangle$ is the structure whose universe is the closure $\bar{X}$ of X under the function symbols of σ, modulo the equivalence relation $\sim$ on $\bar{X}$ defined by

$$x_1 \sim x_2 \quad \text{iff} \quad \Phi \models x_1 = x_2;$$

if $\tilde{x}$ denotes the equivalence class of $x \in \bar{X}$, the operations on $df\langle X, \Phi\rangle$ are given by: if f is a n-ary function symbol of σ,

$$f(\tilde{x}_1, \ldots, \tilde{x}_n) = f(x_1, \ldots, x_n)^{\sim}$$

F is a *word-construction* if it is a κ-word-construction for some κ.

6.3 Examples. (1) Let $\mathscr{B} = \mathscr{C} =$ the variety of rings in the vocabulary $\boldsymbol{\sigma} = \boldsymbol{\tau} = \{+, \cdot\}$. We will now show that the function F which takes a ring B to the formal power series ring $B[[Y]]$ is an ω_1-word construction. Let $\boldsymbol{\sigma}'$ add to $\boldsymbol{\sigma}$ the extra ω-ary function symbol p. Let $T = \{p(v_0, v_1, v_2, \ldots)\}$ and let $\Gamma(p(v_0, v_1, v_2, \ldots))$ be $\forall x(x = x)$. Hence, $X = \{p(\vec{b}): \vec{b} \in B^\omega\}$. Let $A = \{\varphi_1, \varphi_2\}$, where φ_1 is $p(v_0, v_1, v_2, \ldots) + p(u_0, u_1, u_2, \ldots) = p(w_0, w_1, w_2, \ldots)$, and φ_2 is $p(v_0, v_1, v_2, \ldots) \cdot p(u_0, u_1, u_2, \ldots) = p(w_0, w_1, w_2, \ldots)$. Let $\Gamma(\varphi_1)$ be $\bigwedge_{i\in\omega}(v_i + u_i = w_i)$, and let $\Gamma(\varphi_2)$ be $\bigwedge_{j\in\omega}(w_j = \sum_{l+k=j} v_l + u_k)$. Observe that these are formulas of $\mathscr{L}_{\infty\omega_1}$ but not of $\mathscr{L}_{\infty\omega}$, since they have infinitely many free variables. Now it is easy to check that $df\langle X^B, \Phi^B\rangle \cong B[[Y]]$.

(2) Let $\mathscr{B} =$ the variety of sets; that is, $\mathscr{B} = \mathrm{Alg}[\boldsymbol{\tau}]$, where $\boldsymbol{\tau} = \varnothing$ and $\mathscr{C} =$ the variety of groups ($\subseteq \mathrm{Alg}[\boldsymbol{\sigma}]$, where $\boldsymbol{\sigma} = \{\cdot\}$). We shall show that the function F which takes a set B to the free group on B is an ω-word construction. Let $\boldsymbol{\sigma}'$ be obtained by adding to $\boldsymbol{\sigma}$ a unary function symbol i, and a 0-ary function symbol (constant) e. Let T be the set of all terms in $\mathscr{L}[\boldsymbol{\sigma}']$ and let $\Gamma(t)$ be $\forall x(x = x)$, for all $t \in T$. Let $A = \{v \cdot i(v) = e,\ i(v) \cdot v = e,\ e \cdot v = v,\ v \cdot e = v,\ v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3\}$; and, for each $\varphi \in A$, let $\Gamma(\varphi)$ be $\forall x(x = x)$. Then $df(X^B, \Phi^B)$ is the free group on B.

Other examples of word-constructions are the following—the first three being ω-word-constructions, and the last an ω_1-word-construction.

(3) An integral domain to its quotient field (the example is worked out in Hodges [1975, Example 6]).

(4) An ordered field to its real closure (see Hodges [1976, Theorem 2.1]).

(5) A valued field to its Henselization (see Hodges [1976, Theorem 2.4]).

(6) A rank 1 valued field to its completion (see Hodges [1976, Theorem 2.6]).

By using several sorts, the word construction can be defined so that (for example, in Example (4)) it gives the embedding $F \to \tilde{\bar{F}}$ of F in its real closure.

Hodges [1975] advances the thesis that every effective—or, synonymously, concrete—construction occurring naturally in algebra can be put into the form of a word-construction. That word-constructions are effective is given by the following result. Let $\mathscr{P}_{<\kappa}$ denote the function given by $\mathscr{P}_{<\kappa}(X) = \{Y: Y \subseteq X$ and $\mathrm{Card}(Y) < \kappa\}$.

6.4 Theorem (Hodges [1975]). *If $F: \mathscr{B} \to \mathscr{C}$ is a κ-word-construction, then F is provably $\Sigma_1(\mathscr{P}_{<\kappa})$. That is, there is a formula $\theta(x, y)$ in the language of set theory including the symbol $\mathscr{P}_{<\kappa}$ (possibly with parameters) which has all universal quantifiers bounded and which satisfies*

$$\mathrm{ZF} + \textit{definition of } \mathscr{P}_{<\kappa} \vdash \forall x \exists ! y\theta(x, y)$$

and for all $\mathfrak{B} \in \mathscr{B}$,

$$\mathrm{ZF} + \textit{definition of } \mathscr{P}_{<\kappa} \vdash \theta(\mathfrak{B}, F(\mathscr{L})). \quad \square$$

Hodges [1975] also proves that κ-word-constructions preserve $\mathscr{L}_{\infty\kappa}$-equivalence, and discusses connections with Feferman [1972] Eklof [1973, 1975a] and Gaifman [1974].

The following result provides a useful algebraic method of proving that certain constructions are word-constructions (see, Hodges [1980a, Lemma]).

6.5 Lemma. *Let $\mathscr{B}$, $\mathscr{C}$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ be as in Notation 6.1 and let κ be a regular cardinal. If $F: \mathscr{B} \to \mathscr{C}$ is a functor which preserves κ-direct limits (see Definition 3.2) then F is a κ-word-construction.*

Proof. Any structure $\mathfrak{B}$ is the κ-direct limit of $\mathscr{D}(\mathfrak{B})$, the κ-directed diagram of the κ-generated—that is, is generated by fewer than κ elements—substructures of $\mathfrak{B}$, where the maps between substructures are inclusions. So it suffices to define a word construction which sends every $\mathfrak{B}$ to the κ-direct limit of $F(\mathscr{D}(\mathfrak{B}))$. To do this, let $\{\mathfrak{B}_\nu: \nu \in \lambda\}$ be the set of all κ-generated substructures of $\mathfrak{B}$, and for each $\mathfrak{B}_\nu$ let $f_\nu: \rho_\nu \to \mathfrak{B}_\nu$ be a function ($\rho_\nu < \kappa$) whose image is a set of generators of $\mathfrak{B}_\nu$. Then we extend $\boldsymbol{\sigma}$ to $\boldsymbol{\sigma}'$ by adding a set T of function symbols $\zeta_{\nu,c}$ where c ranges over all elements of $F(\mathfrak{B}_\nu)$ and the arity of $\zeta_{\nu,c}$ is ρ_ν. Let A be the set of all atomic formulas of $\mathscr{L}[\boldsymbol{\sigma}']$. We claim that, for all $\mathfrak{B}$ in $\mathscr{B}$, the κ-direct limit of $F(\mathfrak{D}(\mathfrak{B}))$ is $df\langle X, \Phi\rangle$, where X is the set of all $\zeta_{\nu,c}(\vec{b})$ and where the map $f_\nu(i) \mapsto b_i$, for $i < \rho$, induces an isomorphism of $\mathfrak{B}_\nu$ to the substructure $\langle\vec{b}\rangle$ of $\mathfrak{B}$ generated by $\vec{b}$; and where, furthermore, Φ consists of atomic formulas which are of the form $\varphi(\zeta_{\nu,c_1}(\vec{b}), \ldots, \zeta_{\nu,c_n}(\vec{b}))$, where the $\zeta_{\nu,c_i}(\vec{b})$ are in X and $F(\mathfrak{B}_\nu) \models \varphi[c_1, \ldots, c_n]$, or are of the form $\zeta_{\nu,c}(\vec{b}) = \zeta_{\mu,e}(\vec{d})$ (both terms in X), where there is an inclusion $\iota: \langle\vec{b}\rangle \to \langle\vec{d}\rangle$ and $F(\iota)c = e$; or are logical consequences in the quasivariety $\mathscr{C}$ of formulas of these forms. We leave it to the reader to verify that there is a function $\Gamma: T \cup A \to \mathscr{L}_{\infty\kappa}[\tau]$ which determines X and Φ as in Definition 6.2 (see Hodges [1975, pp. 457 ff] for details). $\square$

Note that, in fact, we need only that F preserve κ-direct limits over diagrams whose map are monomorphisms. For example, for any right R-module M, the functor which takes a left R-module N to the abelian group $M \otimes_R N$ preserves ω-direct limits and is thus an ω-word construction. Observe that this functor is not ω-local, but does preserve $\mathscr{L}_{\infty\omega}$-equivalence (see Section 3).

Let Div be the functor which takes an abelian group A to the push-out diagram illustrated below

$$\begin{tikzcd} \bigoplus_{a \in A} \mathbb{Q}c_a \arrow[r] & \mathrm{Div}'(A) \\ \bigoplus_{a \in A} \mathbb{Z}c_a \arrow[u, "i"] \arrow[r, "g"'] & A \arrow[u] \end{tikzcd}$$

where i is inclusion and g takes c_a to a. It is not hard to check that Div preserves ω-direct-limits. Hence, by Lemma 6.5 there is a concrete construction which takes every abelian group A to an embedding of A into a divisible group, $\mathrm{Div}'(A)$, containing A. On the other hand, we have

6.6 Theorem (Hodges [1980a, Corollary 5]). *There is no word-construction F on the variety of abelian groups such that for all A, $F(A)$ is an embedding of A in a divisible hull of A.*

Sketch of Proof. Suppose to the contrary, that there is such an F. Then, using the definition of a word-construction, it is easy to see that F induces an embedding of the automorphism group of A into the automorphism group of $F(A)$. One obtains a contradiction by taking $A = \mathbb{Z}_5 \oplus \mathbb{Z}_5$, the direct sum of two copies of the cyclic group of order 5, and by showing—via a direct computation—that the automorphism of A, given by the matrix,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has order 5 but has no extension to the divisible hull—or even to $\mathbb{Z}_{25} \oplus \mathbb{Z}_{25}$—which has order 5. (This argument—though different from the one in Hodges [1980a]—is also due to Hodges). $\square$

As a corollary, Hodges [1980a, Theorem 6] also gives a negative answer to Taylor's question: if there were a word-construction sending an abelian group A to an embedding of A in a pure-injective hull of A, then, using some constructions satisfying the hypothesis of Lemma 6.5 one could define a word-construction of divisible hulls.

Other negative results are given in Hodges [1976], such as, for example, that there is no word-construction sending a field to its algebraic closure, or a formally real field to its real closure (the reader should compare this to Example 6.3(4)).

7. *Miscellany*

Here we mention a few other examples of the interaction of infinitary logic and algebra.

One important application is Shelah's construction of arbitrarily large rigid real closed fields. This result uses infinitary logic in a general construction that has a variety of other uses (see Shelah [1983d]).

The model theory of $\mathscr{L}_{\omega_1\omega}$ has also been applied to group theory by Kopperman–Mathias [1968]. There use was made of the downward Löwenheim–Skolem notions for $\mathscr{L}_{\omega_1\omega}$ to give new proofs of results of Hall, results showing that certain classes of groups are bountiful, where a class $\mathscr{C}$ of groups is called *bountiful* if whenever $G \subseteq H$ and $H \in \mathscr{C}$, then there exists $H' \in \mathscr{C}$ such that $G \subseteq H'$ and $|H'| = |G| + \aleph_0$.

Dickman has analyzed the Erdös–Gillman–Henriksen isomorphism theorem for real closed fields from the point of view of the back-and-forth method (see Chapter IX, Theorems 4.5.8 and 4.5.9).

Using the model theory of $\mathscr{L}_{\infty\omega}$, Eklof [1977b] contains a new proof of a result of Hill characterizing the classes of abelian groups closed under substructures and direct limits.

Eklof and Sabbagh [1971] discuss $\mathscr{L}_{\infty\omega}$-equivalence and $\mathscr{L}_{\infty\omega}$-definability for various classes of modules and rings. For example, it is proved that the class of Noetherian rings is not definable in $\mathscr{L}_{\infty\omega}$. But it is definable in $\mathscr{L}_{\omega_1\omega_1}$ (see Kopperman [1969]). An algebraic result of Gordon and Robson [1973, Theorem 9.8] implies that the class $\mathscr{C}$ of *commutative* Noetherian rings is not definable in $\mathscr{L}_{\infty\omega}$. (The argument in Eklof–Sabbagh [1971, p. 644] immediately implies that $\mathscr{C}$ is not definable in $\mathscr{L}_{\omega_1\omega}$, but an argument found by Hodges—an argument which uses the ordinal rank of prime ideals—yields the stronger conclusion).

Part D

Second-Order Logic

This part of the book is devoted to the study of second-order logic and some of its applications. We discuss the two chapters in the opposite order from that in which they appear.

Chapter XIII is about monadic second-order logic, logic that allows quantification over arbitrary subsets of the domain, but not over arbitrary relations or functions. While this does not make any difference on structures like the natural numbers with plus and times, where sequences can be coded by numbers, it turns out to make an enormous difference in more algebraic settings. In these cases, monadic second-order logic is a good source of theories that are both highly expressive yet manageable. Section 2 illustrates the uses of finite automata and games in the proof of decidability results. It begins with a simple case, the monadic theory of finite chains, which it works out in complete detail, and shows how the method generalizes to a number of results, including one of the most famous, Rabin's theorem on the decidability of the monadic second-order theory of two successor functions. In Section 3 more model-theoretic methods, generalized products, are used to prove some of the same and related results. Some undecidability results are also presented. Proofs of these have to be novel, since we are dealing with theories where one cannot interpret first-order arithmetic.

If we think of monadic second-order logic as the part of second-order logic obtained restricting the quantification in a simple definable manner, we can ask whether there are any other natural sublogics that can be obtained by restricting the second-order quantifiers in some other first-order definable manner. There is one other. Namely, one might quantify not over arbitrary functions, but over permutations of the domain. This is called permutational logic. It arose in Shelah's study of symmetric groups. However, as it turns out, that's all! Up to a strong form of equivalence, the only sublogics of second-order logic given by first-order restricted second-order quantifiers are first-order logic, monadic second-order logic, permutational logic, and full second-order logic. This result, first proved in Shelah [1973c], is established by some new methods in Chapter XII. In addition, a number of newer, related results are presented.

Chapter XII
Definable Second-Order Quantifiers

by J. BALDWIN

In this chapter we investigate the class of second-order quantifiers which are definable in a sense which will shortly be made precise. This subject arose from investigations of the following sort. Let κ be an infinite cardinal and let S_κ denote the symmetric group on κ elements. What can we say about the first-order theories T_κ of the groups S_κ? Isbell showed that there is a sentence in the language of group theory that is true of S_κ just in case $\kappa = \omega$. McKenzie [1971] showed that $T_{\aleph_\alpha} = T_{\aleph_\beta}$ implies α and β are elementarily equivalent as ordered sets. We can describe the Isbell result as asserting that ω is characterized by a sentence of group theory. McKenzie asked whether or not the set of cardinals characterized in this way was the same as the set of second-order definable cardinals. Shelah [1973a] showed that this was not so. McKenzie had also reformulated the notion of characterization so as to make the question more natural. Instead of discussing the first-order theory of the group S_κ, we can discuss the theory of the set κ in a logic allowing quantification over permutations. Shelah [1973a, b] showed that the Hanf number of this logic is $\aleph_{\Omega^\omega}$, where $\Omega = (2^\omega)^+$. This answers McKenzie's questions, since there certainly are larger cardinals that are definable in second-order logic. In his proof, Shelah discussed a similar quantifier: quantification over permutations of order two. The first quantifier is certainly stronger than the second; moreover, it is easy to describe the first quantifier in terms of the second. To see this, we simply replace an arbitrary permutation f by three permutations g, h, j of order two such that on each orbit of f, g fixes "every other" element while, at the same time, h and j are a product of two-cycles. These cycles agree with f on the elements fixed by g and on the elements moved by g, respectively.

Prompted by questions raised by Stavi, Shelah [1973c] addressed the problem of determining which quantifiers the discussion was about and how many of them there were. The main aim of this chapter is to report his answer to this question. That is, that there are four second-order quantifiers (which are definable in the sense of Section 1.2 below): First-order (Q_{I}), monadic second-order (Q_{mon}), permutational (Q_{1-1}), and full second-order (Q_{II}). These quantifiers range over, respectively, elements, subsets, 1–1 functions, and arbitrary relations. In Section 1, we will formulate the entire question more precisely as well as provide some further examples of this class of quantifier. In Section 2 we will prove Shelah's theorem that there are only four second-order quantifiers. The proofs in Sections 1 and 2 focus attention on two ideas. The argument that Q_{II} is not interpretable in Q_{1-1} depends on the computation of the Hanf number of Q_{1-1}. On the other hand, the argument that any quantifier weaker than Q_{II} is interpretable in Q_{1-1} depends on a decomposition theorem. This kind of decomposition or Feferman–Vaught theorem is discussed in Section 3 (see also Chapter XIII) and is applied in Sections 4 and 5. In Section 4, we will explore the requirements on the notion of interpretation that are necessary to give a proof of non-interpretability via the computation of Hanf numbers. Section 5 surveys the classification of first-order theories by the interpretability of second-order quantifiers. This classification naturally falls into the unstable case (discussed in Section 4) and the stable case (discussed in Section 5). Section 6 contains a brief survey of some other generalizations that were found by Shelah.

1. Definable Second-Order Quantifiers

1.1. Logics, Theories, and Quantifiers

In Chapter II a logic L is defined as a function L which assigns to each vocabulary τ a set of sentences $L(\tau)$ and a semantics $\models_\tau$. In discussing higher-order quantifiers it is natural to examine theories rather than logics. For, the properties of a specific logic—say, monadic logic—vary tremendously depending on the vocabulary involved. In a vocabulary with only unary predicates the Hanf and Löwenheim numbers of monadic logic are $\aleph_0$ and the Feferman–Vaught theorem holds. On the other hand, if the vocabulary contains a binary function symbol f, then, by specifying f to be a pairing function, we extend from monadic logic to full second-order logic and all these pleasant properties are thus destroyed. Notice, however, that we must not only make a binary function symbol available but we must, in addition, specify that it defines a pairing function in order to induce the tragedy. The major results in this chapter concern the relative interpretability of theories in logics with second-order quantifiers.

Following are some notations and conventions which are perhaps peculiar to this chapter. Small Roman letters x, y, z etc. will represent individual variables while small Roman letters r, s, t etc. will represent predicate variables. Similarly, capital Roman letters R, S, T etc. represent relations, and small Roman letters a, b, c etc. individuals. We will use $\bar{a}$ to denote a finite sequence of individuals and $\bar{R}$ for a finite sequence of relations. We will also write $\bar{a} \in A$, and $\bar{R} \in A$, without writing the appropriate exponent on A. If $\phi(\bar{x}, \bar{y}, \bar{s})$ is a formula and A is a structure with $\bar{b} \in A$ and $\bar{S}$ a relation on A, then $\phi(A, \bar{b}, \bar{S}) = \{\bar{a} \in A : A \models \phi(\bar{a}, \bar{b}, \bar{S})\}$. We will regard the ordinary equality sign as a logical symbol. For any formula $\phi(x, \bar{y}, \bar{r})$, $(\exists^{<k}x)\phi(x, \bar{y}, \bar{r})$ abbreviates:

$$(x_0), \ldots, (x_k)\left(\bigwedge_{i<k+1} \varphi(x_i, \bar{y}, \bar{r}) \rightarrow \bigvee_{i<j<k+1} x_i = x_j\right)$$

1.2. Definable Second-order Quantifiers

For any structure M, let M_n denote the power set of M^n. Now full second-order logic allows quantification over $\bigcup_{n<\omega} M_n$. We could consider restricting our quantification to n-ary relations for a fixed n. More restrictively, we could allow $\exists X^n$ to range only over a specified subset of M_n. If we require that subset to be definable by a formula in pure equality theory, quantifying only over elements of M, we arrive at the class of definable second-order quantifiers. More formally, we have:

1.2.1 Notation. If $\psi(r)$ is a formula whose only non-logical symbol is the n-ary relation r, then for each infinite set A, $\mathscr{R}_\psi(A)$ is the collection of n-ary relations R on A such that $A \models \psi(R)$. We will use the same notation even if ψ contains a finite sequence $\bar{r}$ of relation variables.

1.2.2 Definition. Let $\psi(r)$ be a formula whose only symbols are r, $=$, first-order quantifiers, and propositional connectives. Then $Q_{\psi(r)}$ is the *second-order quantifier* whose semantics are given by:

$$M \models Q_{\psi(r)}\phi(r) \quad \text{iff} \quad (\exists R) \in \mathscr{R}_\psi(M), \quad M \models \phi(R).$$

There is a first-order theory naturally associated with each quantifier Q_ψ, namely the theory, T_ψ, whose only non-logical symbol is R and whose only non-logical axiom is $\psi(R)$. Note, however, that this theory does not contain all the information that the quantifier does. For, expressions in the language with the generalized quantifier can contain more than one instance of R.

Naturally, first-order quantification (Q_{I}) and full second-order quantification (Q_{II}) are definable second-order quantifiers. As we will see in Section 2, the only other examples are:

Monadic Quantification. Let r be unary and let $\phi(r)$ be any valid formula. Then $Q_{\phi(r)}$ is merely another name for the monadic second-order quantifier.

Permutational Quantification. Let r be binary and let $\phi(r)$ assert that r is an equivalence relation such that every class has two elements. We call $Q_{\phi(r)}$ the permutational quantifier. The name "permutational" will be justified shortly.

Note that quantification over L-automorphisms of M is not a definable second-order quantifier, since the assertion that f is an automorphism cannot be given in pure predicate calculus.

1.2.3 Definition. For T a first-order theory and Q_ψ a definable second-order quantifier we write (T, Q_ψ) for *the collection of all Q_ψ sentences in $L(T)$ valid on the models of T.*

Convention. We write Q_ψ for $(\mathrm{Th}(=), Q_\psi)$ where $\mathrm{Th}(=)$ is the theory of equality. We write $(\mathrm{Th}(<), Q_\psi)$ for the Q_ψ theory of order.

1.2.4 Definition. Let r be k-ary. We say $Q_{\psi(r)}$ is *(first-order) interpretable* in $(T, Q_{\psi(s)})$ if the following conditions hold. There exist first-order formulas $\theta_0(x_0, \bar{y}, \bar{s})$, $\theta(x_0, \ldots, x_{k-1}, \bar{y}, \bar{s})$ and $\chi(\bar{x}, \bar{s})$ such that:

(i) If $A \models \chi(\bar{a}_0, \bar{S}_0)$ then $\theta_0(A, \bar{a}_0, \bar{S}_0)$ is infinite, $\bar{S}_0 \in R_\psi(A)$, and

$$\begin{aligned}(\theta_0(A, \bar{a}_0, \bar{S}_0), \{\theta(A, \bar{a}, \bar{S})\colon \bar{a} \in A, \bar{S} \in \mathscr{R}_\psi(A)\})\\ = (\theta_0(A, \bar{a}, \bar{S}_0), \mathscr{R}_\phi(\theta_0(A, \bar{a}, \bar{S}_0))).\end{aligned}$$

(ii) For every infinite B, there exist A, $\bar{a}_0$, and $\bar{S}_0$ such that $A \models \chi(\bar{a}_0, \bar{S}_0)$ and $\theta_0(A, \bar{a}_0, \bar{S}_0) \approx B$.

Even though ϕ may contain only a single relation symbol r, the interpreting formulas may contain a sequence $\langle s_0, \ldots, s_n\rangle$. Note that by modifying θ_0 we can require, without loss of generality, that each structure $(\theta_0(A), \theta(A))$ satisfies ψ.

In accordance with our convention we will write $Q_{\phi(r)} \leq Q_{\psi(s)}$ whenever T is the theory of equality.

In this definition the theory which is interpreted is in the language with only the equality symbol. No other notion is needed for Section 2. For the discussion in Sections 4 and 5, we will extend the definition to $(T_1, Q_\psi) \leq (T_2, Q_\phi)$ by requiring that, for each relation symbol in the language of T_1, there be an interpreting formula in the language of (T_2, Q_ϕ). We actually employ this more general notion only when T_1 is the theory of order or $T_1 = T_2$.

The major results of this paper deal with the classification of the theories (T, Q_ψ), where T is a first-order theory. Section 2 concerns the case in which T is the theory of equality. It is easy to see that, for any theory T, we have

$$Q_\psi \leq Q_\phi \quad \text{implies} \quad (T, Q_\psi) \leq (T, Q_\phi).$$

Another formulation of this remark is that if $Q_\psi \leq Q_\phi$, then, for every vocabulary L, $L_{\omega,\omega}(Q_\psi) \leq L_{\omega,\omega}(Q_\phi)$, where '$\leq$' is taken in the sense of Chapter II. That is to say, the finitary logic associated with Q_ψ is weaker than that associated with Q_ϕ. Moreover, this result obviously extends to infinitary logics. Thus, the work described in this chapter provides a refinement of the notions in Chapter II.

We will now use this observation to show that the four quantifiers we have discussed are distinct. However, these quantifiers may coalesce on some T. For example, in the presence of a pairing function, Q_{mon} is equivalent to Q_{II}. This phenomena is discussed in detail in Section 5.

One way to show that quantifiers are distinct is to observe that interpretations as defined in Definition 1.2.4 preserve Hanf number. The Hanf number of a theory (T, Q_ψ) is the least cardinal such that any (Q_ψ)-sentence which has a model of at least that cardinality has arbitrarily large models. A number of variants on this notion are discussed in Baldwin–Shelah [1982], and we discuss it in somewhat more detail in Section 4. For the present, however, a quick application of this observation shows the following.

1.2.5 Theorem. *The four quantifiers are distinct*: $Q_{\mathrm{mon}} \not\leq Q_{\mathrm{I}}$, $Q_{1-1} \not\leq Q_{\mathrm{mon}}$, $Q_{\mathrm{II}} \not\leq Q_{1-1}$.

Proof. The class of well-orders is definable in $(\mathrm{Th}(<), Q_{\mathrm{mon}})$ but not in $(\mathrm{Th}(<), Q_{\mathrm{I}})$. Thus, $Q_{\mathrm{mon}} \not\leq Q_{\mathrm{I}}$.

Every sentence in $(\mathrm{Th}(=), Q_{\mathrm{mon}})$, is either true on all infinite sets or is false on infinite sets. Thus, the Hanf number of $(\mathrm{Th}(=), Q_{\mathrm{mon}})$ is $\aleph_0$. As remarked in the introduction, there are Q_{1-1} sentences of equality theory with only uncountable models. Thus, $Q_{1-1} \not\leq Q_{\mathrm{mon}}$.

Shelah [1973a, b] showed the Hanf number of $(\mathrm{Th}(=), Q_{1-1})$ is $\aleph_{\Omega^\omega}$ and thus that $Q_{\mathrm{II}} \not\leq Q_{1-1}$. $\square$

In the introduction we showed that quantification over arbitrary permutations is bi-interpretable in the sense of Definition 1.2.4 with quantification over permutations of order 2. It is clear that quantification over permutations of order 2 is bi-interpretable with the permutational quantification introduced above.

We will now give a few easy examples to show that a definable second-order quantifier which can define certain kinds of relations must be stronger than our standard examples, monadic and permutational quantification. The key to our argument will be to deal with very simple Q_ψ formulas, namely those of the form $\phi(\bar{x}, \bar{R})$ with $\bar{R} \in \mathscr{R}_\psi(A)$ and ϕ a first-order formula.

1.2.6 Definition. If the relation S on A is defined by $\phi(\bar{x}, \bar{b}, \bar{R})$ with $\bar{R} \in \mathscr{R}_\psi(A)$, where ϕ is of the first-order, then we say S is *simply definable* by Q_ψ.

It is easy to show from the definitions that Q_{II} is maximal among all the definable quantifiers

1.2.7 Proposition. *For any* ϕ, $Q_\phi \leq Q_{\mathrm{II}}$. $\square$

1.2.8 Lemma. *If* Q_ψ *simply defines an infinite, coinfinite set, then* $Q_{\mathrm{mon}} \leq Q_\psi$.

Proof. Consider a definable second-order quantifier Q_ψ, and a structure A. Suppose that for some first-order formula $\phi(x, \bar{a}, \bar{R})$, with $\bar{R} \in \mathscr{R}_\psi(A)$ and $\bar{a} \in A$, both $\phi(A, \bar{a}, \bar{R})$ and $\neg\phi(A, \bar{a}, \bar{R})$ are infinite. We will show that each subset of A is definable by a formula $\theta(x, \bar{a}, \bar{R})$, with $\bar{R} \in \mathscr{R}_\psi(A)$. Call X a regular subset of A if $|X| = |A - X| = |A|$. Since ψ contains no non-logical symbols, the assumption that one regular subset of A is definable by a first-order formula $\phi(\bar{x}, \bar{b}, \bar{R})$ implies that any other regular subset is also. But any subset of A is a boolean combination of regular subsets so that all subsets of A are $Q_{\psi(r)}$ definable. Thus, $Q_{\mathrm{mon}} \leq Q_{\psi(r)}$. $\square$

We can view these remarks from another perspective, one that makes discussion of their consequences more concise. If $\bar{R} \in \mathscr{R}_\psi(A)$, then $(A, \bar{R})$ can be thought of as a model of a first-order theory in a language with non-logical symbols $\bar{R}$ and whose only axiom is $\psi(\bar{R})$. Then our last observation is simply the assertion that every infinite model of this theory is strongly minimal in the sense of Baldwin–Lachlan [1971] that T is strongly minimal if every definable (with parameters) subset is finite or cofinite. Moreover, standard compactness arguments show that this implies that if for each $\bar{b}$, $\phi(A, \bar{b}, \bar{R})$ is finite, then there is a uniform bound on the cardinalities of these sets.

1.2.9 Lemma. *If Q_ψ simply defines an equivalence relation with infinitely many infinite classes and $Q_{\text{mon}} \leq Q_\psi$, then $Q_{1-1} \leq Q_\psi$.*

Proof. Suppose there is a formula $\theta(x, y, \bar{a}, \bar{R})$, an $\bar{R} \in \mathscr{R}_\psi(A)$, and an $\bar{a} \in A$ such that $\theta(x, y, \bar{a}, \bar{R})$ defines on some infinite subset B of A an equivalence relation having infinitely many classes with more than two elements. By shrinking B, we may assume that each class has exactly two elements and that $A - B$ is infinite. By the compactness and Löwenhiem-Skolem theorems, we may assume that every infinite set C contains a regular subset B_C with such a definable equivalence relation. Using again the fact that ψ contains no nonlogical symbols, we see that a similar equivalence relation can be defined on $C - B_C$. But then, since B is simply definable (as every subset is simply definable), we can easily define an equivalence relation on all of C such that each class has exactly two elements. Thus we have defined Q_{1-1}. ☐

The main result asserts that the four quantifiers we have discussed are (up to bi-interpretability) the only definable second-order quantifiers and that, in fact, they are linearly ordered by interpretation. In fact, the argument shows that we would gain no additional cases by considering definable second-order quantifiers with finite strings of variables (that is, by replacing $Q_{\phi(r)}$ by $Q_{\phi(\bar{r})}$).

Most of the definitions in this section have described definable second-order quantifiers in pure logic. We can, of course, consider the more general situation in which we add definable second-order quantifiers to a non-trivial first-order theory. We will consider this situation in some detail in Section 5.

1.3. Some Conditions for Interpretability

In this section we will describe a few conditions which suffice for interpreting second-order logic into another logic.

We remarked in Section 1.1 that the introduction of a pairing function transforms monadic logic into full second-order logic. We now want to discuss a slightly weaker condition which has the same effect.

1.3.1 Definition. The theory T is *codable* if, for some n and some model M of T, there are infinite sets $\langle B_i : i < n \rangle$ and C contained in M and a first-order formula (possibly with parameters), $\phi(x, y_0, \ldots, y_{n-1})$, which defines a 1–1 map from $B_0 \times \cdots \times B_{n-1}$ onto C.

If T is codable, then, for any cardinal κ, we have a pairing function from two sets of power κ onto a third. We can thus easily code any binary relation on κ in terms of the pairing function and a subset of the third set. This argument is carried out in detail in Section II.2.4 of Baldwin–Shelah [1982]. Formally, we have

1.3.2 Theorem. *If T is codable, then $Q_{\text{II}} \leq (T, Q_{\text{mon}})$.* ☐

Arguments like those for 1.3.2 show:

1.3.3 Lemma. *If there is a first-order formula $\phi(x, y)$ which defines on some model M of a first-order theory T and on some infinite subset A of M an equivalence relation with infinitely many infinite classes, then $Q_{\mathrm{II}} \leq (T, Q_{1-1})$.* □

This is Chapter II, Section 2.6 of Baldwin–Shelah [1982].

2. Only Four Second-Order Quantifiers

In this section we will prove the main result of Shelah [1973c]: that up to interpretation (in the sense defined in Definition 1.2.4) there are only four (definable) second-order quantifiers. In Section 2.1 we will begin by deriving some consequences of Ramsey's theorem and the Δ-system lemma which will be used several times in the proof of the main theorem. That done, we will then show successively in Section 2.2 that if $Q_{\mathrm{mon}} \not\leq Q_\psi$, then $Q_\psi \leq Q_{\mathrm{I}}$; in Section 2.3 that if $Q_{1-1} \leq Q_\psi$, then $Q_\psi \leq Q_{\mathrm{mon}}$; and finally in Section 2.4 that if $Q_{\mathrm{II}} \not\leq Q_\psi$, then $Q_\psi \leq Q_{1-1}$. These three assertions and Proposition 1.2.7 yield the following theorem.

2.0 Theorem. *If Q_ψ is a definable second-order quantifier, then Q_ψ is bi-interpretable with one of Q_{I}, Q_{mon}, Q_{1-1}, or Q_{II}.*

The proof of the first two of the three assertions constituting this theorem is just a reworking of the argument given in Shelah [1973c]. We give the main idea of the proof for the third in Section 2.4. In Sections 2.5 and 2.6 we give alternate arguments for the crucial Theorem 2.4.6. The argument in Section 2.5 is derived from Baldwin–Shelah [1982], while that in Section 2.6 is a modification of the argument given in Shelah [1973c].

The argument for each of the three cases follows the same general line. To show that $Q_\phi \leq Q_\psi$, we first define an appropriate notion of "$\bar{a}$ and $\bar{b}$ are Q_ψ-similar over" respectively a finite set of elements in Section 2.2, a finite set of elements and a finite set of subsets in Section 2.3, and a finite set of elements, a finite set of subsets, and a finite set of 1–1 functions in Section 2.4. We say $\bar{S}$ determines θ if $\bar{a}$ and $\bar{b}$ are Q_ψ similar over some sequence $\bar{S}$ satisfying ψ implies $\bar{a}$ and $\bar{b}$ satisfy the same formulas $\theta(\bar{x}; \bar{R})$, for $\bar{R} \in \mathscr{R}_\phi(A)$. It is easy to see that if $\bar{S}$ determines each θ then $Q_\phi \leq Q_\psi$. The bulk of the argument which differs from case to case consists in showing by induction on $\lg(\bar{x})$ that each $\theta(\bar{x}; \bar{R})$ is so determined.

2.1 Consequences of Some Combinatorial Lemmas. Our first result is an application of Ramsey's theorem to the problem of interpretation.

2.1.1 Lemma. *Let $\theta(z, y, \bar{x}, \bar{R})$ be a first-order formula. Suppose that for every A and every $\bar{R} \in \mathscr{R}_\psi(A)$ and for some $m < \omega$ we have*

$$A \models (\bar{x})(y)(\exists^m z)\theta(y, z, \bar{x}, \bar{R})$$

and

$$A \models (z)(y)\theta(y, z, \bar{x}, \bar{r}) \to z \neq y.$$

Then either

(1) *for some $n < \omega$, we have $A \models (\bar{x})(\exists^n)[(\exists y)\theta(y, z, \bar{x}, \bar{R})]$; or*
2(a) $Q_{\text{mon}} \leq Q_\psi$ *and*
2(b) $Q_\psi \leq Q_{1-1}$.

Proof. Assuming that (1) fails, we first show

(*) There are $C = \langle c_i : i < \omega\rangle$ and $B = \langle b_j : j < \omega\rangle$ and $\bar{d}$ such that $B \cap C = \varnothing$; $c_i = c_j$ iff $i = j$; $\models \theta(b_i, c_j, \bar{d}, \bar{R})$ iff $i = j$; and $\models \neg\theta(c_i, c_j, \bar{d}, \bar{R})$ if $i \neq j$.

If, for each $\bar{d}$, there are only finitely many c such that $\models (\exists z)\theta(z, c, \bar{d}, \bar{R})$, then (1) holds by an easy compactness argument and we are finished. If not, then we can certainly find disjoint sets B and C such that $A \models \theta(b_i, c_j, \bar{d}, \bar{R})$ but $A \models \neg\theta(b_i, c_j, \bar{d}, \bar{R})$, for $i < j$. By applying Ramsey's theorem to the partition of pairs $\{i, j\}$ for $i < j < \omega$ induced by whether or not $\theta(b_i, c_j, \bar{d}, \bar{R})$ holds, we can pass to subsets of B and C so that the truth of $\theta(b_i, c_j, \bar{d}, \bar{R})$ depends only on the order of i and j. We know that $\theta(b_i, c_j, \bar{d}, \bar{R})$ fails if $i < j$ and since $(\exists^{<m} z)\theta(z, c_j, \bar{d}, \bar{R})$ and some c_j has more than m predecessors, we must also have $\neg\theta(b_i, c_j, \bar{d}, \bar{R})$ if $i > j$. A similar use of Ramsey's theorem allows us to assume that $\neg\theta(c_i, c_j)$ also if $i \neq j$. This establishes (*).

We will now define a formula $\chi(y, \bar{d}, \bar{R}, \bar{R}')$ such that $A \models \chi(c_i, \bar{d}, \bar{R}, \bar{R}')$ iff $i \equiv 0 \bmod 3$. Since we will have thus defined an infinite and coinfinite set, it will follow by Lemma 1.2.8 that $Q_{\text{mon}} \leq Q_\psi$. We can assume that none of the b_i's or c_j's occur in $\bar{d}$. Let f be the permutation of A which interchanges c_{3i+2} and c_{3i+1} and leaves all other elements of A fixed. Let $\bar{R}'$ be the image of R under f. That is, f is an isomorphism between $(A, \bar{R})$ and $(A, \bar{R}')$. Then

$$A \models (x)[\theta(x, c_j, \bar{d}, \bar{R}) \leftrightarrow \theta(x, c_j, \bar{d}, \bar{R}')]$$

if and only if $j \equiv 0 \bmod 3$. Thus, letting $\chi(y, \bar{d}, \bar{R}, \bar{R}')$ be

$$(x)[\theta(x, y, \bar{d}, \bar{R}) \leftrightarrow \theta(x, y, \bar{d}, \bar{R}')],$$

we have (2a).

To obtain (2b), we note that $\theta(x, y, \bar{d}, \bar{R})$ defines on a subset of $B \cup C$ an equivalence relation having infinitely many classes with two elements. In the light of (2a) and Lemma 1.2.9 we have (2b).

Our next step is an application of a weak version of the Δ-system lemma. The remainder of this section is applied in Lemma 2.3.6 and 2.5.8.

2.1.2 Definition. A *Δ-system with heart H* is a family of sets $\{C_i: i < \kappa\}$ such that if $i \neq j$, then $C_i \cap C_j = H$. We will frequently fix an enumeration $\bar{h}$ of H. Then $\bar{h}$ will be taken to mean either the sequence $\bar{h}$ or the range of that sequence (that is, H), whichever is appropriate.

An easy combinatorial argument establishes

2.1.3 Lemma (The Weak Δ-System Lemma). *If $\langle C_i: i < \omega\rangle$ is a sequence of distinct sets with the same finite cardinality n, then there is a subsequence of the C_i which is a Δ-system with some heart H, and $|H| < n$.*

For our application we want to distinguish the following families of formulas.

2.1.4 Definition. A family of formulas $\{\theta_n(z_0, \ldots, z_{n-1}, \bar{y}, \bar{r}): n < \omega\}$ is *malleable* if

(i) θ_n is predicate of the set $\{z_0, \ldots, z_{n-1}\}$, not the sequence $\bar{z}$.
(ii) If $\{\bar{c}_i: i < \omega\}$ is a Δ-system with heart $H(|\bar{c}_i| = n$ and $|H| = m < n)$ and $A \models \theta_n(\bar{c}_i, \bar{b}, \bar{R})$ for i, then $A \models \theta_m(\bar{h}, \bar{b}, \bar{R})$.

2.1.5 Example. If $\theta_n(z_0, \ldots, z_{n-1}, \bar{y}, \bar{r})$ is

$$(\bar{x})\left[\phi(x_0, \ldots, x_{m-1}, \bar{y}, \bar{r}) \rightarrow \bigwedge_{i<m} \bigvee_{j<n} x_i = z_j\right],$$

then $\{\theta_n: n < \omega\}$ is a malleable family. To see this, we let $\bar{d}$ be a solution of $\phi(\bar{x}, \bar{b}, \bar{R})$ and let $\bar{c}_i$ be a Δ-system of n-tuples such that $\bar{d} \subseteq \bar{c}_i$ for each i. Then $\bar{d}$ is clearly contained in $\bar{h}$.

For $\{\theta_n: n < \omega\}$ a malleable family, we introduce the following notation: θ_n^* denotes $(\exists z_1) \cdots (\exists z_{n-1})\theta_n(z_0, z_1, \ldots, z_{n-1}, \bar{y}, \bar{R})$. $\theta_n'(\bar{z}, \bar{y}, \bar{R})$ denotes the conjunction of $\theta_n(\bar{z}, \bar{y}, \bar{R})$ with the formulas $(z_0), \ldots, (z_{m-1}) \neg\, \theta_m(\bar{z}, \bar{y}, \bar{R})$, for $m < n$.
This definition is designed to yield the following lemma.

2.1.6 Lemma. *Suppose θ_n is a malleable family of formulas such that for every A, every $R \in \mathscr{R}_\psi(A)$, and every $\bar{b}$ in A, there is a finite sequence $\bar{c}$ with $|\bar{c}| < M$ (for some integer M) such that $A \models \theta_{|c|}(\bar{c}, \bar{b}, \bar{R})$. Then*

(i) *There is an integer $n(\bar{b})$ such that: $\theta_{n(\bar{b})}^*(A, \bar{b}, \bar{R})$ is finite and*

$$A \models (\exists \bar{z})\theta_{n(\bar{b})}'(\bar{z}, \bar{b}, \bar{R}) \wedge \bigwedge_{i<n(\bar{b})} \theta_{n(\bar{b})}^*(z_i, \bar{b}, \bar{R}).$$

(ii) *If, in addition,* $Q_{\text{mon}} \leq Q_\psi$ *then there is an integer* k *and a formula* $\theta^*(z, \bar{y}, \bar{R})$ *such that*:

(a) $A \models (\bar{y})(\exists^{<k} z)\theta^*(z, \bar{y}, \bar{R})$;
(b) $A \models (\bar{y})(\bar{z})[\theta'_m(\bar{z}, \bar{y}, \bar{R}) \rightarrow \wedge \{\theta^*(z_i, \bar{y}, \bar{R}): i < m\}]$, *for all* $m < M$.

Proof. (i) Fix $\bar{b}$ and some $\bar{c}$ of smallest cardinality such that $A \models \theta_{|c|}(\bar{c}, \bar{b}, \bar{R})$ and suppose $|\bar{c}| = n$. Suppose $\theta_n^*(A, \bar{b}, \bar{R})$ is infinite. Then there is an infinite family of n-element sets C_i such that if $\bar{c}_i$ is any enumeration of C_i, $A \models \theta(\bar{c}_i, \bar{b}, \bar{R})$. By the Δ-system lemma, we can find a heart H $(\bar{h})$ for the C_i's with $|H| = m < n$. Moreover, by the very definition of malleable family, $A \models \theta_m(\bar{h}, \bar{b}, \bar{R})$. But this contradicts the minimality of n and so yields (i).

(ii) Since $Q_{\text{mon}} \leq Q_\psi$, we know by Lemma 1.2.8 that any $(A, \bar{R})$ with each $R \in \mathscr{R}_\psi(A)$ is strongly minimal. In particular, there is an integer k such that all the sets $\theta_{n(\bar{b})}^*(A, \bar{b}, \bar{R})$ have cardinality $< k$. Recall that by hypothesis all the $n(\bar{b}) < M$; and, furthermore, let θ^* be the formula:

$$\bigvee_{j<M} (\exists u_0) \cdots (\exists u_{j-1})\theta_j^*(\bar{u}, \bar{y}, \bar{R}) \rightarrow \theta_j^*(z, \bar{y}, \bar{R}).$$

This formula clearly meets conditions (a) and (b). □

2.2 Lemma. *If* $Q_{\text{mon}} \not\leq Q_\psi$, *then* $Q_\psi \leq Q_I$.

This subsection is devoted to the proof of Lemma 2.2. We will proceed by induction to show that the hypothesis implies that every formula with parameters $\bar{R}$ in $\mathscr{R}_\psi(A)$ and k free variables is expressible in first-order logic. When k reaches the arity of R we must then have the lemma (see Lemma 2.2.2). In addition to the notions from Sections 1 and 2.1, we will require the following concept.

2.2.1 Definition. (i) Let X be a finite set of relation symbols or formulas. By $\text{tp}_X(\bar{a}; B)$ we mean the *collection of formulas* $\psi(\bar{x}; \bar{b})$ such that $\bar{b} \in B$, $\psi(\bar{x}; \bar{y}) \in X$ and $\models \psi(\bar{a}, \bar{b})$. We will simply write, $t_=(\bar{a}; B)$ for $t_{\{=\}}(\bar{a}; B)$.

(ii) Two finite sequences of the same length, $\bar{a}$ and $\bar{b}$, are (first-order) *similar* over B if $\text{tp}_=(\bar{a}; B) = \text{tp}_=(\bar{b}; B)$.

(iii) The set $D \cup \bar{c}$ *determines* $\phi(\bar{x}; \bar{c}, \bar{R})$ if for any sequences $\bar{a}$ and $\bar{b}$ which are similar over $D \cup \bar{c}$: $A \models \phi(\bar{a}, \bar{c}, \bar{R}) \leftrightarrow \phi(\bar{b}, \bar{c}, \bar{R})$

Note. The notion $D \cup \bar{c}$ determines $\phi(\bar{x}; \bar{c}, \bar{R})$ depends not just on the formula $\phi(\bar{x}, \bar{y}, \bar{R})$ but on the partition of the sequence $\bar{x}\bar{y}$.

2.2.2 Lemma. *If, for every formula* $\phi(\bar{x}, \bar{y}, \bar{r})$, *there exists a formula* $\phi^*(z, \bar{y}, \bar{r})$ *and an integer* n *such that for every* A *and every* $\bar{b}$ *in* A *and* $\bar{R}$ *in* $\mathscr{R}_\psi(A)$

(i) $|\phi^*(A, \bar{b}, \bar{R})| \leq n$; *and*
(ii) $\phi^*(A, \bar{b}, \bar{R}) \cup \{\bar{b}\}$ *determines* $\phi(\bar{x}, \bar{b}, \bar{R})$,

then $Q_\psi \leq Q_I$.

Proof. We apply the hypothesis, taking $r(\bar{x})$ as $\phi(\bar{x}; \bar{y}, \bar{r})$. Then $R(\bar{x})$ is determined by the finite set $\phi^*(A, R)$ so that a suitable coding of the equality types over $\phi^*(A, R)$ defines $R(\bar{x})$ as required. □

2.2.3 Definition. The formula $\chi(w)$ is an $=$-diagram (read simply as *equality diagram*) if χ is a maximal consistent conjunction of equalities and inequalities among the w_i.

Shelah [1973] calls χ a complete formula. The following lemma yields Lemma 2.2.

2.2.4 Lemma. *If $Q_{\mathrm{mon}} \not\leq Q_\psi$, then for every formula $\phi(\bar{x}, \bar{y}, \bar{r})$, there is a formula $\phi^*(z, \bar{y}, \bar{r})$ and an integer k such that for every $A, \bar{b}$, and $\bar{R}$ in $\mathscr{R}_\psi(A)$: $|\phi^*(A, \bar{b}, \bar{R})| = k$ and $\phi^*(A, \bar{b}, \bar{R}) \cup \{\bar{b}\}$ determines $\phi(\bar{x}, \bar{b}, \bar{R})$.*

Proof. The proof is by induction on the length of $\bar{x}$ for arbitrary sequences $\bar{y}$ and $\bar{r}$. If $\lg(\bar{x}) = 1$, the result is immediate from the remark following the proof of Lemma 1.2.8.

We now consider a formula $\phi(\bar{x}; \bar{y}, \bar{r})$. Let $\bar{x} = \bar{x}'\, w$ and $\bar{y}' = w\, \bar{y}$. Now, we have $\phi_0 = \phi(\bar{x}; \bar{y}, \bar{r})$ and $\phi_1 = \phi(\bar{x}'; \bar{y}', \bar{r})$ which differ only in the position of the semicolon. Suppose we have constructed by induction a formula $\phi_0^*(z, \bar{y}', \bar{r})$ and an m such that for each $a, \bar{b}$, and $\bar{R} \in \mathscr{R}_\psi(A)$:

(i) $|\phi_0^*(A, a, \bar{b}, \bar{R})| < m$;
(ii) $\phi_0^*(A, a, \bar{b}, \bar{R}) \cup \{a, \bar{b}\}$ determines $\phi(\bar{x}'; a, \bar{b}, \bar{R})$;
(iii) $\phi_0^*(z, w, \bar{b}, \bar{R}) \rightarrow z \neq w$.

By explicitly listing $\{a, \bar{b}\}$ in (ii), we are left free to assume that (iii) holds. Now, applying Lemma 2.1.1, we see $A \models (\bar{y})(\exists^{<k}z)(\exists w)(\phi_0^*(z, w, \bar{y}, \bar{R}))$. Let $\phi_1^*(z, \bar{y}, \bar{R})$ be $(\exists w)\phi_0^*(z, w, \bar{y}, \bar{R})$. Then, for each a and $\bar{b}$, it is easy to see that $\phi_1^*(A, \bar{b}, \bar{R}) \cup \{a, \bar{b}\}$ determines $\phi(\bar{x}', a, \bar{b}, \bar{R})$. It remains to remove the dependence on a. To do this, however, we must first look more carefully at how the determination occurs.

Let $\bar{c}$ be an enumeration of $\phi_1^*(A, \bar{b}, \bar{R})$. Fix $\lg(\bar{z}) = \lg(\bar{c})$ and let $\chi_i(\bar{x}'; w, \bar{y}, \bar{z})$ for $i < p$ be a complete list of the equality diagrams in the displayed variables. For each $a \in A$ and each i, we must have either

(i) $A \models (\bar{x}')[\chi_i(\bar{x}', a, \bar{b}, \bar{c}) \rightarrow \phi(\bar{x}', a, \bar{b}, \bar{R})]$; or
(ii) $A \models (\bar{x}')[\chi_i(\bar{x}', a, \bar{b}, \bar{c}) \rightarrow \neg\phi(\bar{x}', a, \bar{b}, \bar{R})]$.

Now, for each $S \subseteq p$, let $\chi_S(a, \bar{b}, \bar{R})$ hold just if (i) above holds for exactly those $i \in S$. Now, by strong minimality, there is an L (depending on S and $\bar{b}$) such that if $A \models \chi_S(a, \bar{b}, \bar{R})$ for more than L choices of a, then $A \models \chi_S(a, \bar{b}, \bar{R})$ for all but finitely many a. By compactness and the fact that there are only finitely many choices for S, we can choose a single L with this property for all $\bar{b}$ and S. Now, $\phi(\bar{x}; \bar{b}, \bar{R})$ is clearly determined by $\phi_1^*(A, \bar{b}, \bar{R}) \cup C(\bar{b}) \cup \bar{b}$, where $C(\bar{b})$ denotes the set of those a such that $A \models \chi_S(a, \bar{b}, \bar{R}) \rightarrow (\exists^{<L}x)\chi_S(x, \bar{b}, \bar{R})$. Moreover, we now see that $\phi_1^*(A, \bar{b}, \bar{R}) \cup C(\bar{b})$ has less than $(k + L)$ elements and is uniformly definable from $\bar{b}$. Thus, we have proven the lemma. □

In Shelah's original proof, the $C(\bar{b})$ are defined by an appeal to Lemma 2.1.6 so that the structure of his argument is actually closer to that which follows in the proof of Lemma 2.3.

2.3 Lemma. *If $Q_{1-1} \not\leq Q_\psi$ then $Q_\psi \leq Q_{\text{mon}}$.*

Our proof of this result is parallel to the proof of Lemma 2.2. We will require the following concept—a concept that is analogous to the notion given in Definition 2.2.1.

2.3.1 Definition. (i) Two finite sequences, $\bar{a}$, $\bar{b}$, of the same length, n, are *monadically similar* over $\langle D; C_0, \ldots, C_{m-1}\rangle$ if for any d in D and any $i < n$, $b_i = d$ iff $a_i = d$; and, for $j < m$, $a_i \in C_j$ iff $b_i \in C_j$.

(ii) *A finite equivalence relation* over F is an equivalence relation (on k-tuples, for some k) which is definable with parameters from F and has only finitely many equivalence classes.

(iii) The set D and the finite equivalence relation E *monadically determine* $\phi(\bar{x}, \bar{c}, \bar{R})$ if, for any sequences $\bar{a}$ and $\bar{b}$ of the same length: if $\bar{a}$ and $\bar{b}$ are monadically similar over $\langle D \cup \bar{c}; C_0, \ldots, C_{k-1}\rangle$ where the C_i are the equivalence classes of E, then $\phi(\bar{a}, \bar{c}, \bar{R}) \leftrightarrow \phi(\bar{b}, \bar{c}, \bar{R})$.

2.3.2 Lemma. *If for every formula $\phi(\bar{x}, \bar{y}, \bar{r})$ there exist formulas $\phi^*(x, u, \bar{y}, \bar{r})$ and $\theta(z, \bar{y}, \bar{r})$ such that for every A, $\bar{c}$, and $\bar{R} \in \mathscr{R}_\psi(A)$:*

(i) *$\theta(A, \bar{c}, \bar{R})$ is finite;*

(ii) *$\phi^*(x, u, \bar{c}, \bar{R})$ is a finite equivalence relation;*

(iii) *$\theta(A, \bar{c}, \bar{R})$ and $\phi^*(x, u, \bar{c}, \bar{R})$ monadically determine $\phi(\bar{x}, \bar{c}, \bar{R})$, then $Q_\psi \leq Q_{\text{mon}}$.*

2.3.3 Definition. For any formula $\phi(\bar{x}; \bar{y}, \bar{r})$, any A and $\bar{c}$, any $\bar{R} \in \mathscr{R}_\psi(A)$, and any $C \subseteq A$, define $e(\phi(\bar{x}; \bar{c}, \bar{R}), C, A) = e(\phi, C, A)$ by

$$e(\phi, C, A) = \{\langle a, b\rangle : \text{tp}_{\{=, \phi(\bar{x}, \bar{c}, \bar{r})\}}(a; A - C) = \text{tp}_{\{=, \phi(\bar{x}, \bar{c}, \bar{r})\}}(b; A - C).$$

The formulas in $\text{tp}_{\{=, \phi(\bar{x}, \bar{c}, \bar{r})\}}(a; X)$ are obtained by fixing any entry in $\bar{x}$ for substitution of a and leaving the others for substitutions from X. Note that $e(\phi, C, A)$ is first-order definable (with parameters C, $\bar{c}$ and $\bar{R}$).

2.3.4 Lemma. *If $Q_{1-1} \leq Q_\psi$, then for every A, C, $\bar{b}$, and $\phi(\bar{x}; \bar{y}, \bar{r})$, $e(\phi, C, A)$ has only finitely many equivalence classes.*

Proof. By Lemma 2.2, we can assume that $Q_{\text{mon}} \leq Q_\psi$. We first note that by Lemma 1.2.9, since $e(\phi, C, A)$ is definable, it can have only finitely many equivalence classes with two or more elements. Since replacing C by a smaller set refines the equivalence relation, we can, by proper choice of C, assume that each class of $e(\phi, C, A)$ is a singleton. If $e(\phi, C, A)$ has infinitely many classes, we will define in terms of $\bar{R} \in \mathscr{R}_\psi(A)$, an equivalence relation possessing infinitely many classes with two or more elements. We thereby contradict Lemma 1.2.9. For this, fix a permutation f with order 2 of A whose set of fixed points is $(A - C) \cup \{\bar{b}\}$. Let $\bar{R}_1 = f(\bar{R})$. Let S_0 denote the relation defined by $\phi(\bar{x}, \bar{b}, \bar{R})$ and S_1 the relation defined by $\phi(\bar{x}, \bar{b}, \bar{R}_1)$. Let e_1 be the following equivalence relation (this relation is clearly definable from $\bar{R}$, $\bar{R}_1$, C and $\bar{b}$ and therefore by Q_ψ):

$$\{\langle a, c\rangle: \mathrm{tp}_{\{S_0, =\}}(a: (A - C) \cup \{\bar{b}\}) = \mathrm{tp}_{\{S_1, =\}}(c; (A - C) \cup \{\bar{b}\})$$

and

$$\mathrm{tp}_{\{S_0, =\}}(c; (A - C) \cup \{\bar{b}\}) = \mathrm{tp}_{\{S_1, =\}}(a; (A - C) \cup \{\bar{b}\}).$$

Clearly, if $a, c \in C$ and $f(a) = c$, then $\langle a, c\rangle \in e_1$. Now, if $\langle a, c\rangle \in e_1$, then

$$\mathrm{tp}_{\{S_0, =\}}(c; (A - C) \cup \{\bar{b}\}) = \mathrm{tp}_{\{S_1, =\}}(a; (A - C) \cup \{\bar{b}\}) = q.$$

But since $e(\phi(\bar{x}; \bar{b}, \bar{R}), C, A)$ has only singleton equivalence classes, the unique element realizing q in the S_1 interpretation is $f(c)$. So $a = f(c)$. Since e_1 is clearly symmetric, we see that $e_1(a, c)$ if and only if $a = f(c)$. That is, we can define by $e_1(x, y) \vee x = y$ an equivalence relation with infinitely many two element classes.

Note that by invoking the compactness theorem, we can find a uniform n such that, for all C and $\bar{b}$, $e(\phi(\bar{x}; \bar{b}, \bar{R}), C, A)$ has less than n equivalence classes.

The following technical result asserts that if a definable symmetric, reflexive relation has a bounded number of pairwise incomparable elements then its transitive closure also is definable. We need it for the next lemma.

2.3.5 Proposition. *Suppose $\phi(x, y)$ defines a symmetric reflexive relation such that, for some m and for any set of distinct elements $\{a_i: i < m\}$, there are $i \neq j$ such that $\phi(a_i, a_j)$. Then the equivalence relation E which is obtained by forming the transitive closure of the relation defined by $\phi(x, y)$ is itself defined by:*

$$(\exists z_0), \ldots, (\exists z_{2m-3}) \bigwedge_{i<2m-3} \phi(z_i, z_{i+1}) \wedge z_0 = x \wedge z_{2m-3} = y).$$

Proof. Let $\{a_0, \ldots, a_k\}$ be the shortest path connecting a_0 and a_k, and let $k = 2u$ or $k = 2u + 1$, depending on the parity of u. No pair from $\{a_0, \ldots, a_u\}$ satisfies ϕ. Thus, $u \leq m - 1$ which yields the result. □

2.3.6 Lemma. *If $Q_{1-1} \leq Q_\psi$, then for each formula $\phi(\bar{x}; \bar{c}, \bar{R})$ there are formulas $\phi^*(x, u, \bar{y}, \bar{R})$ and $\theta^*(z, x, u, \bar{y}, \bar{r})$ such that for every A, $\bar{c}$, a and $\bar{R} \in \mathscr{R}_\psi(A)$:*

(i) *$\phi^*(x, y, \bar{c}, \bar{R})$ defines an equivalence relation with finitely many classes.*
(ii) *If $A \models \phi^*(a, b, \bar{c}, \bar{R})$, then $\langle a, b\rangle \in e(\phi(\bar{x}, \bar{c}, \bar{R}), \theta^*(A, \bar{c}, \bar{R}) \cup \{a, b\}, A)$.*
(iii) *$\theta^*(A, \bar{c}, \bar{R})$ is finite.*

Proof. We first use Proposition 2.3.5 to establish (i) and the weakened version of (ii) which is obtained by replacing $e(\phi(\bar{x}; \bar{c}, \bar{R}), \theta(A, \bar{c}, \bar{R}) \cup \{a, b\}, A)$ by the equivalence relation e^* which holds for two elements if and only if for some finite B, $e(\phi, B, A)$ also holds of those elements. The formula $\phi^*(x, y, \bar{b}, \bar{R})$ defines a finite equivalence relation which refines the finite equivalence relation $e^*(\phi(\bar{x}; \bar{b}, \bar{R}), A, \bar{R})$. Then two applications of Lemma 2.1.1 yield the full result.

For the first step, define for each A, $\bar{b}$ and $\bar{R} \in \mathscr{R}_\psi(A)$ the binary relation $e_n = e_n(\phi(\bar{x}; \bar{b}, \bar{R}), A, \bar{R})$ to hold for $\langle a, b\rangle$ just if for some n-element subset B of A, $\langle a, b\rangle \in e(\phi(\bar{x}, \bar{b}, \bar{R}), B, A)$. Note that e_n is reflexive and symmetric but not transitive. Moreover, there is a formula $\phi_n(x, y, \bar{b}, \bar{R})$ which defines e_n. Finally, e_n refines e_{n+1}. Now, let the equivalence relation $e^* = \bigcup \{e_n : n < \omega\}$. For a fixed m, not depending on B, each $e_n(\phi, B, A)$ has at most m classes so there is no set of $m + 1$ elements, each pair of which does not satisfy ϕ_n. Thus e^* has at most m classes. So for some l, the set of sentences

$$\Gamma_l = \{\psi(R)\} \bigcup_{\substack{n<l \\ 1 \le i < j \le m+1}} \{\neg \phi_n(x_i, x_j, \bar{d}, \bar{R})\}$$

is inconsistent. Let p be the least integer such that Γ_p is inconsistent. By Proposition 2.3.5, the transitive closure of $\phi_p(x, y, \bar{d}, \bar{r})$ is definable by a formula ϕ^*, and defines an equivalence relation with at most m classes. ϕ^* clearly satisfies (i) and the weakened form of (ii). Thus, each equivalence class of e^* is a union of ϕ^* equivalence classes.

To establish the full strength of (ii), we define the malleable family of formulas $\theta_n(x, y, \bar{z}, \bar{u}, \bar{r})$ which assert that $\langle x, y\rangle \in e(\phi(\bar{x}; \bar{u}, \bar{r}), \{x, y, z_0, \ldots, z_{n-1}\}, A)$. Taking p for the bound M in the hypothesis of Lemma 2.1.6, we deduce that there is a formula $\theta^*(z, x, y, \bar{u}, \bar{r})$ such that for some k (first a $k(\bar{b})$ but then, by compactness, independent of $\bar{b}$) we have:

(a) $A \models (x)(y)(\exists^{<k} z)\theta^*(z, x, y, \bar{c}, \bar{R})$.
(b) If $A \models \phi^*(a, b, \bar{c}, \bar{R})$ then $\langle a, b\rangle \in e(\phi, \theta^*(A, a, b, \bar{c}, \bar{R}) \cup \{a, b\}, A)$.
(c) If $A \models \neg\phi^*(a, b, \bar{c}, \bar{R})$ then $\theta^*(A, a, b, \bar{c}, \bar{R}) = \varnothing$.

Now, applying Lemma 2.1.1 twice to condition (a) we obtain

$$A \models (\exists^{<k} z)(\exists x)(\exists y)\theta^*(z, x, y, \bar{b}, \bar{R}) \quad \text{so}$$

$$(\exists^{<k} z)(\exists x)(\exists y)\theta^*(z, x, y, \bar{u}, \bar{R}).$$

Now, to complete the proof of Lemma 2.3, we show by induction that every formula is monadically determined.

2.3.7 Lemma. *If $Q_{1-1} \not\leq Q_\psi$, then for any $\phi(\bar{x}; \bar{y}, \bar{r})$, there are formulas $\phi^*(x, u, \bar{y}, \bar{r})$ and $\theta(z, \bar{y}, \bar{r})$ which monadically determine $\phi(\bar{x}; \bar{y}, \bar{r})$.*

Proof. The proof is by induction on $\lg(\bar{x})$. If $\lg(\bar{x}) = 1$, we are merely restating Lemma 2.3.6. Thus, suppose that we have the result if $\lg(\bar{x}) < n$, and consider a formula $\phi(\bar{x}; \bar{y}, \bar{R})$ with $\lg(\bar{x}) = n$. By Lemma 2.3.6 we can find a finite equivalence relation $\phi^*(x, y, \bar{b}, \bar{R})$ and a set $\theta(A, \bar{b}, \bar{R})$ such that if

$$A \models \phi^*(a, c, \bar{b}, \bar{R}), \quad \text{then} \quad \langle a, c\rangle \in e(\phi, \theta^*(A, a, c, \bar{b}, \bar{R}) \cup \{a, c\}, A).$$

This means that the equivalence classes of $e(\phi, \theta^*(A, a, c, \bar{b}, \bar{R}) \cup \{a, c\}, A)$ are finite unions of equivalence classes of ϕ^*. Now, for each element d of $\theta^*(A, \bar{b}, \bar{R}) \cup \{a, c\}$, let $\phi_{d,i}(\bar{x}'; \bar{b}, d, \bar{R})$ be the $(n - 1)$-ary relation obtained by substituting d for x_i in ϕ_i. Then, $\phi(\bar{x}; \bar{b}, \bar{R})$ is first-order definable from the equivalence classes of ψ^*, the elements of $\theta(A, \bar{b}, \bar{R}) \cup \{a, c\}$, and the $\phi_{d,i}(\bar{x}', \bar{b}, \bar{R})$. For, if $\bar{a} \cap \theta(A, \bar{b}, \bar{R}) = \emptyset$, then $\phi(\bar{a}; \bar{b}, \bar{R})$ depends only on the ϕ^* equivalence class of the a_i. If $\bar{a} \cap \theta(A, \bar{b}, \bar{R}) \neq \emptyset$, then $\phi(\bar{a}; \bar{b}, \bar{R})$ depends on one of the $\phi_{d,i}$ which are monadically determined by induction. This completes the proof of Lemma 2.3. ☐

2.4 Lemma. *If $Q_{\mathrm{II}} \not\leq Q_\psi$ then $Q_\psi \leq Q_{1-1}$.*

Once we have established this lemma, we will have completed the proof of the four second-order quantifier theorem. We will first show that a certain decomposition of all structures $(A, \bar{R})$ with $\bar{R} \in \mathscr{R}_\psi(A)$ implies that $Q_\psi \leq Q_{1-1}$. Afterwards, we will show that the hypothesis $Q_{\mathrm{II}} \not\leq Q_\psi$ implies that such a decomposition exists.

An extremely simple example of such a decomposition is the division of models of $\mathrm{Th}(Z, S)$ into connected components. More complicated examples are elaborated in Baldwin–Shelah [1982].

2.4.1 Definition. (i) If E is an equivalence relation then two sequences $\bar{a}$ and $\bar{b}$ are *similar for E* if $\lg(\bar{a}) = \lg(\bar{b}) = k$ and there is a partition of k into, say, n sets $J_0, \ldots, J_{n-1}$ such that for any elements of the sequences a_i, a_j, b_i, b_j we have $a_i E a_j$ if and only if $b_i E b_j$ if and only if i and j are members of the same partition element J_l. We write $\bar{a} = \langle \bar{a}_0, \ldots, \bar{a}_{n-1}\rangle$ where $\bar{a}_j$ is the set of a_i with $i \in J_j$.

(ii) The model M is decomposed over N if there is an equivalence relation E on $M - N$ such that if $\bar{a}$ is similar for E to $\bar{b}$ and, for each $\bar{a}_i, \bar{b}_i$, we have $\mathrm{tp}(\bar{a}_i; N) = \mathrm{tp}(\bar{b}_i; N)$, then for each $\lg(\bar{a})$-ary relation symbol, R, in the vocabulary of the structure $M \models R(\bar{a}) \leftrightarrow R(\bar{b})$. We say E is an *L-congruence*.

(iii) The L-structure M is *strongly decomposed* over N by E if each equivalence class of E has no more than $|L|$ elements.

(iv) The theory T is (*strongly*) *decomposable* if, for each $M \models T$ and each $N \prec M$ with $|N| \leq |T|$, M is (strongly) decomposed over N.

We will show that if $Q_{\mathrm{II}} \not\leq Q_\psi$, then each structure $(A, \bar{R})$ with $\bar{R} \in \mathscr{R}_\psi(A)$ is strongly decomposed by the following natural equivalence relation. (The hypothesis $Q_{\mathrm{II}} \not\leq Q_4$ is required to show the relation is symmetric.)

2.4.2. Definition. (i) For an element a and a set B, we write $a \in \mathrm{cl}(B)$ if, for some formula $\phi(x)$ with parameters from B, $\phi(a)$ holds and ϕ has only finitely many solutions.

(ii) Let $N \prec M$, then for $a, b \in M - N$, $a \sim_N b$ if $a \in \mathrm{cl}(N \cup \{b\})$.

We will show that such a decomposition suffices for the interpretation of Q_ψ in Q_{1-1} and then that the decomposition exists. For the first task we require a few more definitions.

2.4.3 Definition. Let $C_0, \ldots, C_{m-1}$ be a sequence of subsets of A and let $f_0, \ldots, f_{k-1}$ be a sequence of partial 1–1 functions on A. Then

(i) Two finite sequences $\bar{a}$ and $\bar{b}$ of the same length are $1-1$ *similar* over $\langle D; C_0, \ldots, C_{m-1}; f_0, \ldots, f_{k-1}\rangle$ if for any d in D and any $i < n = \mathrm{lg}(\bar{a})$, $b_i = d$ iff $a_i = d$, and for $j < m$, $a_i \in C_j$ iff $b_i \in C_j$ and for $l < k$, $f_l(a_i) = d\ (\in C_j)$ if and only if $f_l(b_i) = d\ (\in C_j)$.

(ii) The sequence $\langle D; C_0, \ldots, C_{m-1}; f_0, \ldots, f_{k-1}\rangle$ 1-1 *determines* $\phi(\bar{x}, \bar{c}, \bar{R})$ if for any sequences $\bar{a}$ and $\bar{b}$ of the same length we have that if $\bar{a}$ and $\bar{b}$ are 1-1 similar over $\langle D; C_0, \ldots, C_{m-1}; f_0, \ldots, f_{k-1}\rangle$, then $\phi(\bar{a}, \bar{c}, \bar{R}) \leftrightarrow \phi(\bar{b}, \bar{c}, \bar{R})$.

2.4.4 Definition. A formula $\phi(x, y, \bar{n})$ is called a *binding-formula* if, for some integer k, $\models (x)(\exists^{<k} y)\phi(x, y, \bar{n}) \wedge (y)(\exists^{<k} x)\phi(x, y, \bar{n})$.

Note that if M is strongly decomposed via $\sim_N$, then for any pair of elements $a, b \in M - N$, if a and b are equivalent, then for some binding formula M $\phi(x, y, \bar{n})$ with the $\bar{n}$ from N: $\models \phi(a, b, \bar{n})$. Moreover, if $\bar{a}$ is a sequence of equivalent elements from $M - N$, $t(\bar{a}; N)$ is implied by the union of the types $t(a_i; N)$, for $i < n$ with the binding formulas which relate the a_i. Finally, if $\bar{a}$ is a sequence from $M - N$ involving elements from different equivalence classes, then $t(\bar{a}; N)$ is implied by the types of the singleton a_i, the binding formulas which tie together the elements from the same classes and the negations of all binding formulas which might relate pairs that are not in the same class. With this in mind, we will establish a final lemma and complete the proof of the theorem.

2.4.5 Theorem. *If for every infinite A and every $\bar{R} \in \mathscr{R}_\psi(A)(A, \bar{R})$ is strongly decomposable by $\sim_N$, for some proper elementary submodel N of $(A, \bar{R})$, then $Q_\psi \leq Q_{1-1}$.*

Proof. Let $M = (A, \bar{R})$ be strongly decomposed over N. Note that for any $M^* \succ N$, M^* is also strongly decomposed over N. Thus, for any model M of $\mathrm{Th}(N)$ and any $\bar{a} \in M - N$, there is a type $q(\bar{a})$ such that each formula in q contains only one a_i, or is a binding formula, or the negation of a binding formula and is such that $q \vdash t(\bar{a}; N)$ and if $M \models R(\bar{a})$, then $t(\bar{a}; N) \vdash R(\bar{x})$. (The existence of this type is guaranteed by the discussion preceding this lemma.) Now, a standard "double compactness" argument shows that $R(\bar{x})$ is equivalent to a disjunction of formulas over a finite set N_0 and that each of these formulas either contains at most one x_i, or is a binding formula, or is the negation of a binding formula. Now, if D is N_0, C_i picks out the solution set of the ith disjunct with only one x_i; and, for each binding formula $\phi_i(x, y, \bar{d})$, the functions f_i^j for $j < k$ (the number of solutions of $\phi_i(a, x, \bar{n})$) are defined so that $\{f_i^j(a): j < k\} = \{b: \phi_i(a, b, \bar{d})\}$). Then R is 1–1 determined by $D, C_0, \ldots, C_p$ and f_i^j for $i < m$ and $j < k$ (for appropriate p, k, m). □

We will complete the proof of Lemma 2.4 by establishing in the rest of Section 2:

2.4.6 Theorem. *If $Q_{\text{II}} \not\leq Q_\psi$, then for every $(A, \bar{R})$ with $\bar{R} \in \mathscr{R}_\psi(A)$ and for some elementary submodel N of $(A, \bar{R})$, A is strongly decomposed by $\sim_N$.* □

We will explain two proofs of the above result. The first is both the most natural and the most useful. We will continue to use its methods later in the paper. However, it requires a minimal knowledge of stability theory (for instance, the first half of Lascar–Poizat [1979]) so for those who might be unfamiliar with those basic facts, we have included in Section 2.6 an *ad hoc* but self-contained proof of Theorem 2.4.6.

2.5 Theorem. *If $Q_{\text{II}} \not\leq Q_\psi$, then for every A and every $\bar{R} \in \mathscr{R}_\psi(A)$, $(A, \bar{R})$ is strongly decomposable.* (1st Proof).

We first observe

2.5.1 Lemma. *$Q_{\text{II}} \not\leq Q_\psi$ implies T is stable.*

We give two arguments for this. Note that T being unstable implies there is a definable linear ordering of n-tuples. In Chapter VIII of Baldwin–Shelah [1982] it is shown that in any theory with a definable linear order on n-tuples one can monadically define a linear order on singletons. From this one constructs an equivalence relation with infinitely many infinite classes and finishes by Lemma 1.3.3. Alternatively, we use more of the machinery set up in Section 2.6 and deduce directly from the definable linear order on n-tuples the existence of a definable equivalence relation on n-tuples with infinitely many non-pseudofinite (see Definition 2.6.3) classes which contradicts Lemma 2.6.4. □

2.5.2 Definition (The Fundamental Equivalence Relation). Let $N \prec M$ and M a model of a stable theory. We define a relation E_N on $M - N$ by aE_Nb just if $t(a; N \cup b)$ forks over N.

Now the standard properties of forking in a stable theory assure us that E is reflexive and symmetric. In general, E is not transitive. However, in our situation we obtain this and more.

2.5.3 Lemma. *If T is stable, $N \prec M$ and E is the fundamental equivalence relation then M is decomposed over N by E.*

Proof. We must show that E is an equivalence relation and that condition (ii) of Definition 2.4.1 is satisfied. We will give a brief outline of the argument.

2.5.4 Lemma. *Suppose that in a model of T, there exists an element a and $B = \langle b_i : i < \omega \rangle$ and $C = \langle c_j : j < \omega \rangle$ such that*

(i) *B is a set of indiscernibles;*
(ii) *C is a set of indiscernibles over B and there is a formula $\phi(x, y, z)$ such that $\models \phi(a, b_i, c_j)$ if and only if $i = j$.*

Then T is codable.

This lemma is an easy reworking of the definition of codable given in Definition 1.3.1. Its proof as well as that of the following lemma are detailed as Sections IV.2.4 and IV.2.6 of Baldwin–Shelah [1982]. The following lemma is a fairly routine calculation using the properties of the forking relation and Lemma 2.5.4.

2.5.5 Lemma. *If T is stable and either:*

(i) *There exists a subset A of a model of T and elements a, b, c such $t(a; A \cup b)$ forks over A and $t(b; A \cup c)$ forks over A, but $t(a; A \cup c)$ does not fork over A, or*
(ii) *There exists a subset A of a model of T and elements $a, b_1, \ldots, b_n$ such that for each i $t(a; A \cup b_i)$ does not fork over A but $t(a; A \cup \{b_1, \ldots, b_n\})$ forks over A.*

Then T is codable.

This result shows that if $Q_{\mathrm{II}} \not\leq Q_{\mathrm{mon}}$ and $\bar{R} \in \mathscr{R}_\psi(A)$, then $\mathrm{Th}(A, \bar{R})$ is decomposable (see Baldwin–Shelah [1982]). In order to show that it is actually strongly decomposable, we will need one further fact from stability theory.

2.5.6 Lemma. *If T is stable and there exist $a, b \in A \models T$ and $B \subseteq A$ such that $t(a; B \cup b)$ forks over B but $t(b; B \cup a)$ is not algebraic, then on some subset of a model of T there is a definable equivalence relation which has infinitely many infinite classes.*

(This result is Lemma VI.1.1 of Baldwin–Shelah [1982].) Now by Definition 1.2.4 we see the conclusion of Lemma 2.5.6 cannot hold unless $Q_{\mathrm{II}} \leq Q_\psi$ (as we have $Q_{1-1} \leq A_\psi$). Thus, we have established Theorem 2.5.

We turn now to the other proof of Theorem 2.5.

2.6 Theorem. *If $Q_{\mathrm{II}} \not\leq Q_\psi$, then for every A and every $\bar{R} \in \mathscr{R}_\psi(A)$, $(A, \bar{R})$ is strongly decomposable.* (2nd Proof)

We first use an argument similar to the ones given in Baldwin–Shelah [1982] to show that for any model N, $\sim_N$ is symmetric and thus is an equivalence relation. For this, we will require a few other concepts. The first is given in

2.6.1 Definition. Let $N \subseteq A$ and $p \in S(A)$, then p is *finitely satisfied in N* if every formula in p has a solution in N.

Using compactness it is easy to see that if $A \subseteq B \subseteq C$ and $p \in S(B)$ is finitely satisfied in A, then p extends to a $p' \in S(C)$ which is also finitely satisfied in A. Next, we consider

2.6.2 Lemma. *If $Q_{\text{II}} \not\leq Q_\psi$ and $\bar{R} \in \mathscr{R}_\psi(A)$, then for any $N \prec (A, \bar{R})$ if neither a nor b is algebraic in N and $t(a, N \cup b)$ is algebraic, then $t(b, N \cup a)$ is also algebraic.*

Proof. Suppose not and choose b_i for $i < \omega$, which are distinct, with $t(b_i; N \cup a) = t(b; N \cup a)$. Let $\bar{c}_0 = \bar{b}a$ and choose $\bar{c}_i$ for $i < \omega$ such that $t(\bar{c}_i; C_i) = t(\bar{c}_{i+1}; C_i)$ and $t(\bar{c}_{i+1}; N \cup C_{i+1})$ is finitely satisfied in N. Here $C_i = N \cup \{\bar{c}_j : j < i\}$. Thus $\bar{c}_i$ has the form $\langle b_{i,j}: \text{for } j < \omega\rangle a_i$. Clearly a_i is algebraic in each $b_{i,j}$ by the same formula ψ. But no a_j is algebraic in $N \cup b_{i,k}$ with $i > j$. For, if it were, we could, by finite satisfiability, find a $b' \in N$ with a_i algebraic in $N \cup b'$ and hence, in N, also, which is impossible. But no a_i can be algebraic in $b_{j,k}$, with $j < i$ since all a_l with $l > i$ realize the same type as a_i over $N \cup b_{j,k}$. Thus, by adding predicates A and B to pick out the a's and b's, we can define an equivalence relation on B with infinitely many infinite classes by $E(x, y) \leftrightarrow (\exists z)\phi(x, z) \wedge \phi(y, z) \wedge A(z)$. This contradicts Lemma 1.3.3 and establishes the lemma. $\square$

We now want to show that if N is chosen appropriately, then $\sim_N$ actually determines a strong decomposition of $(A, \bar{R})$. To accomplish this, we return to the original Shelah argument. We will proceed by extending the properties of strongly minimal sets to finite sequences. We will accordingly arrive at a notion reminiscent of the weakly minimal formulas that are examined in Shelah [1974a].

2.6.3 Definition. The family $F = \{\bar{f}_i : i < \alpha\}$ is *pseudo-finite*, if there is a finite set C such that for every i, $C \cap \bar{f}_i \neq \emptyset$.

The formula $\phi(\bar{x}, \bar{a}, \bar{R})$ is *pseudo-algebraic* in $(A, \bar{R})$ if its solution set is pseudo-finite. The sequence $\bar{a}$ is *pseudo-algebraic over B*, if for some formula $\phi(\bar{x})$ with parameters from B, $\models \phi(\bar{a})$ and ϕ is pseudoalgebraic.

Note that $\bar{a}$ is *not-pseudo-finite* over B means that we can find arbitrarily many disjoint finite sequences which realize $t(\bar{a}; B)$.

2.6.4 Lemma. *If $Q_{\text{II}} \not\leq Q_\psi$, then for any A and any $\bar{R} \in \mathscr{R}_\psi(A)$, there is no formula $\phi(\bar{x}, \bar{y}, \bar{c}, \bar{R})$ which defines an equivalence relation with infinitely many non-pseudo-finite equivalence classes.*

Proof. If $\lg(\bar{x}) = \lg(\bar{y}) = 1$, then this assertion is only Lemma 1.3.3. Using $Q_{1\text{-}1} \leq Q_\psi$, we will reduce the case $n > 1$ to the case $n = 1$ and thus finish the argument. By induction choose sequences $\bar{a}_{i,j}$ such that $\bar{a}_{i,j}$ is equivalent to $\bar{a}_{k,l}$ just if $i = k$ and such that the $\bar{a}_{i,j}$ having distinct indices are pairwise disjoint and all are disjoint from c. Now, define for each $m < n$ a permutation f_m of A which exchanges the first and mth members of each sequence $\bar{a}_{i,j}$ and which fixes all other elements of A. Let B^* consist of the first coordinates of the $\bar{a}_{i,j}$. Now the formula

$$\phi^*(x, y, \bar{c}, \bar{R}, f_1, \ldots, f_n) = \phi(f_1(x), \ldots, f_n(x), f_1(y), \ldots, f_n(y), \bar{c}, \bar{R})$$

defines on B^* an equivalence relation with infinitely many infinite equivalence classes. This is, of course, contrary to Lemma 1.3.3 and we are done.

2.6.5 Definition. The formula $\psi(\bar{x}, \bar{c}, \bar{R})$ is $\phi(\bar{x}, \bar{y}, \bar{r})$-minimal, if ψ is not pseudo-finite but for every $\bar{d}$ either $\psi(\bar{x}) \wedge \phi(\bar{x}, \bar{d}, \bar{R})$ or $\psi(\bar{x}) \wedge \neg\phi(\bar{x}, \bar{d}, \bar{R})$ is pseudo-finite.

The search for a ϕ-minimal formula is similar to the search for a strongly minimal formula in an ω-stable theory. We will show that we cannot build a complete binary tree of instances of ϕ and negations of ϕ such that each path is not pseudo-finite. The main step for this is

2.6.6 Lemma. *If $Q_{\mathrm{II}} \not\leq Q_\psi$ then there are no A and $\bar{R} \in \mathscr{R}_\psi(A)$ such that there exist a $\phi(\bar{x}, \bar{y}, \bar{R})$ and $\bar{a}_n$ for $n < \omega$ so that for each $n < \omega$, the formula*

$$\theta_n = \bigwedge_{m<n} \phi(\bar{x}, \bar{a}_m, \bar{R}) \wedge \neg\phi(\bar{x}, \bar{a}_n, \bar{R})$$

is not pseudo-algebraic.

Proof. Assume that the lemma fails. By the compactness theorem, we can assume that each θ_n is satisfied by more than $2^{\aleph_0}$ disjoint sequences. Let B be the collection of elements which appear in any of the parameter sequences $\bar{a}_n$. Define two sequences $\bar{b}, \bar{c}$ from A to be e equivalent just if for every $\bar{a}$ from B $\phi(\bar{b}, \bar{a}, \bar{R}) \leftrightarrow \phi(\bar{c}, \bar{a}, \bar{R})$. Now, for each n and m, if $n \neq m$ a sequence satisfying θ_m and a sequence satisfying θ_n are not equivalent so that e has infinitely many classes. But each of these classes is not pseudo-finite. For, there are more than $2^{\aleph_0}$ disjoint sequences satisfying θ_n and at most (since B is countable) $2^{\aleph_0}$ classes of e so that some e-class intersects θ_n in uncountably many disjoint sequences and thus that class is not pseudo-finite. Thus, for each n, we find a distinct class of the definable equivalence relation e which is not pseudo-finite. By Lemma 2.6.4, $Q_{\mathrm{II}} \leq Q\psi$. $\square$

2.6.7 Lemma. *If $Q_{\mathrm{II}} \not\leq Q_\psi$, then for any $\phi(\bar{x}, \bar{y}, \bar{r})$ there is an integer $m(\phi)$ and there are formulas $\chi_i(\bar{x}, \bar{z}, \bar{r})$ (depending on ϕ) for $i < m(\phi)$, such that for any A and any $\bar{R} \in \mathscr{R}_\psi(A)$, there is a $\bar{c} \in A$ such that the formulas $\chi_i(\bar{x}, \bar{c}, \bar{R})$ partition A and each $\chi(\bar{x}, \bar{c}, \bar{R})$ is ϕ-minimal.*

Proof. Build a binary tree of instances of $\phi(\bar{x}, \bar{y}, \bar{R})$ and its negation. Either, for some n, each path of length n defines a ϕ-minimal set; or, for arbitrary k, we can find $\bar{a}_i$ for $i < k$ such that taking $\lambda_i(\bar{x}, \bar{y}, \bar{R})$ as $\phi(\bar{x}, \bar{y}, \bar{R})$ or $\neg\phi(\bar{x}, \bar{y}, \bar{R})$ (depending on i) $\wedge$ $\{\lambda_i(\bar{x}, \bar{a}_i, \bar{R}) : i < k\}$ is not pseudo-algebraic. If $k = 2m + 2$, the formula θ_m from Lemma 2.6.6 is not pseudo-algebraic and we violate Lemma 2.6.6.

We will need one more nice property of pseudo-algebraic formulas to complete the proof.

2.6.8 Lemma. If $\bar{a} = \langle a_0, \ldots, a_n \rangle$ *is pseudo-algebraic over B, then some a_i is algebraic over B.*

Proof. Let $\phi(\bar{x}, \bar{b}, \bar{R})$ be a pseudo-algebraic formula satisfied by $\bar{a}$. Let C be a set with minimal cardinality n such that if $A \models \phi(\bar{a}', \bar{b}, \bar{R})$, then $\bar{a}^* \cap C \neq \varnothing$. Recall from Example 2.1.5 that if $\theta_n(z_0, \ldots, z_{n-1}, \bar{y}, \bar{r})$ is

$$(x)\left[\phi(x_0, \ldots, x_{m-1}, \bar{y}, \bar{r}) \rightarrow \bigvee_{\substack{m<n \\ j<n}} x_i = y_j\right],$$

then $\{\theta_n: n < \omega\}$ is a malleable family. Now, by applying Lemma 2.1.6, we see that some component of $\bar{a}$ satisfies the algebraic formula $\theta^*(x, \bar{b}, \bar{R})$ and we are done. $\square$

2.6.9 Theorem. *If $Q_{\mathrm{II}} \not\leq Q_\psi$, then for any A and any $\bar{R} \in \mathscr{R}_\psi(A)$, there is an elementary submodel N of $(A, \bar{R})$ such that $\sim_N$ strongly decomposes $(A, \bar{R})$ over N.*

Proof. For each $\phi(\bar{x}, \bar{y}, \bar{r})$, choose a sequence $\bar{c}$ and formulas χ_i as in Lemma 2.6.7 and let N contain all the $\bar{c}$. By induction on n we will prove that if $\bar{a}$ and $\bar{b}$ with length n are similar for $\sim_N$ and for each $\bar{a}_i, \bar{b}_i$ (see notation in Definition 2.4.1) $t(\bar{a}_i; N) = t(\bar{b}_i; N)$, then $t(\bar{a}; N) = t(\bar{b}; N)$. If $n = 1$, this assertion is tautogical. Suppose that we have proved the claim for n. To prove it for $n + 1$, we consider a formula $\phi(x, \bar{y}, \bar{z}, \bar{r})$ with $\lg(\bar{y}) = n$, and let $\bar{n}$ be in N. If all elements of $\bar{a}$ are in the same $\sim_N$ equivalence class, then there is nothing to prove. Let $\bar{a}_1$ be a maximal pairwise equivalent subsequence of $\bar{a}$—as is indeed implied by our notation. Then, if we let $\bar{a}'$ (respectively $\bar{b}'$) denote $\bar{a}$ without $\bar{a}_1$ (respectively $\bar{b}$ without $\bar{b}_1$), no component of $\bar{a}'$ (respectively $\bar{b}'$) is algebraic in $N \cup \bar{a}_1$ (respectively in $N \cup \bar{b}_1$) and thus $\bar{a}'$ ($\bar{b}'$) is not pseudo-algebraic in $N \cup \bar{a}_1$ (in $N \cup \bar{b}_1$), (by Lemma 2.6.8).

We must prove that for any $\bar{n} \in N$, $A \models \phi(\bar{a}_1, \bar{a}', \bar{n}, \bar{R}) \leftrightarrow \phi(\bar{b}_1, \bar{b}', \bar{n}, \bar{R})$. By Lemma 2.6.7 and the choice of N, we can find a $\bar{d} \in N$ and a ϕ-minimal $\chi(\bar{x}, \bar{d}, \bar{R})$ such that $A \models \chi(\bar{a}', \bar{d}, \bar{R})$. By the definition of ϕ-minimality, one of

$$\chi(\bar{x}, \bar{d}, \bar{R}) \wedge \phi(\bar{a}_1, \bar{x}, \bar{c}, \bar{R}) \quad \text{and} \quad \chi(\bar{x}, \bar{d}, \bar{R}) \wedge \neg\phi(\bar{a}_1, \bar{x}, \bar{c}, \bar{R})$$

is pseudo-algebraic. Without loss of generality, we can take it to be the second one. By a simple application of compactness, this means that for some $m_1(\phi)$, the formula is satisfied by no more than $m_1(\phi)$ pairwise disjoint sequences. As $\bar{a}'$ is not pseudo-algebraic over $\bar{a}_1$, we have $A \models \phi(\bar{a}_1, \bar{a}', \bar{n}, \bar{R}')$. By induction, $\bar{a}'$ and $\bar{b}'$ have the same type over N so $A \vdash \chi(\bar{b}', \bar{d}, \bar{R})$. Since $\bar{a}_1$ and $\bar{b}_1$ have the same type over N, $\chi(\bar{x}, \bar{d}, \bar{R}) \wedge \neg\phi(\bar{b}_1, \bar{x}, \bar{n}, \bar{R})$ is not satisfied by more than $m_1(\phi)$ pairwise disjoint sequences. Since $\bar{b}'$ is not pseudo-algebraic over $N \cup \bar{b}_1$, we thus have $A \models \phi(\bar{b}_1, \bar{b}', \bar{n}, \bar{R})$ as was required. $\square$

3. *Infinitary Monadic Logic and Generalized Products*

Our primary focus so far has been on the classification of theories of equality, Q_ψ. Now we will consider the following question: What are the possibilities for theories of the form (T, Q_ψ), where T is a complete first-order theory and Q_ψ is one of the four second-order quantifiers? The notion of a decomposable model is a key tool in the proof of Lemma 2.4. We will develop a generalization of this idea and use it, for example, to compute the Hanf numbers of some logics (see Sections 4.5 and 5.2). The major device for these computations is a Feferman–Vaught type theorem for monadic logic. As Gurevich pointed out to me, this is a natural development of the original Feferman–Vaught theorem which described the first-order properties of a generalized product of a family $\{M_i: i \in I\}$ in terms of the first-order theory of the factors and the monadic theory of the index set (enriched by unary predicates which pick out the indices whose models have the same theory). The material in this section is largely taken from Shelah [1975e] and Gurevich [1979a].

In many cases, it is artificial to consider the first-order monadic theory of a class of structures, because this theory already encodes a certain amount of information that we would normally think of as "$L_{\omega_1, \omega}$" information. For example, we can monadically define the closure of a subset of a group. Or, consider the class of all structures containing two infinite classes, P_0, P_1, and a binary extensional relation, E, between them. (That is to say, one is the set of subsets of the other). Now, if T is the monadic theory of this class, any model of the monadic sentence

$$(X) \subseteq P_0(\exists y) \in P_1(z) \in P_0(z \in X \leftrightarrow z \in y)$$

has models only of power $\geq \beth_1$ This kind of argument shows that the Hanf number of $L_{\omega, \omega}(Q_{\text{mon}}) \geq$ the Hanf number of $L_{\omega_1, \omega}$; furthermore, it leads us to consider infinitary monadic logic. We are going to prove a Feferman–Vaught type theorem by way of a back-and-forth argument. This requires some means of handling variables. Rather than deal with variables explicitly we will expand the language by adding additional constant symbols. Since this is monadic logic, we must add not only names for individuals but for subsets as well. We want to describe a specific sentence in $L_{\infty, \lambda}(Q_{\text{mon}})$ which contains the information we need in order to make our induction. Individuals are considered to be subsets with only one element. Note that if (A, R) and (B, R) are equivalent for existential first-order sentences, then R is a singleton in A iff it is a singleton in B.

This section repeats the discussion in Section 3 of Chapter XIII in a superficially more general situation. The chief differences here are that Chapter XIII restricts itself to finitary logic and, for expository purposes, merely works out the preservation theorem for ordered sums. Here, however, we will give an abstract notion of product in Section 3.4, a notion which focuses attention on exactly those properties (for example, of the ordered sum construction) which allow the argument for the preservation theorem to go through. In Chapter XIII monadic logic is interpreted into a first-order logic; here, on the other hand, the monadic logic is taken as basic. The following glossary connects the two chapters.

Chapter XIII	Chapter XII
a sequence ξ	an ordinal $\alpha = \lg(\xi)$
an l-tuple of elements	a λ-tuple of elements
$\xi - \mathrm{Th}(M, a_1, \ldots, a_l)$	$t_{\alpha, \lambda}(M, \bar{Q})$
$\xi - l$-Box	$t_{\alpha, \lambda}(L)$

Observe that the correspondence suggested by the tabular arrangement is not exact since a $\xi - l$-Box depends on a (suppressed) theory T.

$$\begin{array}{ll} X_1, \ldots, X_l & \langle Q_t : t \in \lambda \rangle, \\ P(\xi, X, t) & Q_t(I), \\ P(\xi, X) & \langle Q_t(I) : t \in t_{\alpha,\lambda}(L) \rangle. \end{array}$$

Another difference in the presentation of results arises from the fact that one chapter emphasizes decidability results, while the other stresses preservation results. In Chapter XIII, the bounded theories are viewed as objects in their own right and the ξ-theory of the product is computed from the $H(\xi, l)$ theory of the index set. In this chapter, however, the bounded theories are viewed as properties of structures and the theorem has the following form: If *the bounded theories of two index structures are the same, then so are the theories of the product structures.*

3.1 Definition. We define by induction the set of formulas $t_{\alpha,\lambda}(M)$ as follows:

(i) For any L-structure M, let $t_{0,\lambda}(M) = \{\theta : M \models \theta\}$.

Here θ ranges over existential first-order formulas with at most λ variables. Note that $t_{0,\lambda}(M)$ is the same for all infinite λ. We would just say the existential theory of M, but the decidability results require that if λ is finite, then so is $t_{0,\lambda}$. We require existential rather than quantifier-free formulas in $t_{0,\lambda}(M)$ in order that we may know the cardinality (mod $\aleph_0$) of every subset of M defined by a boolean combination of unary predicates.

Now, for any α and λ, we define $t_{\alpha,\lambda}(M)$ as follows: $t_{\alpha+1,\lambda}(M) = \{t_{\alpha,\lambda}(M, \bar{Q}) : \lg(Q) = \lambda\}$

$$t_{\delta,\lambda}(M) = \bigcup \{t_{\alpha,\lambda}(M) : \alpha < \delta\}, \text{ if } \delta \text{ is a limit ordinal.}$$

(ii) For any α and λ, let $t_{\alpha,\lambda}(L)$ denote $\{t_{\alpha,\lambda}(M) : M \text{ is an } L\text{-structure}\}$.

Thus, $t_{\alpha+1,\lambda}(M)$ describes the $L^{\alpha}_{\infty,\lambda}(Q_{\text{mon}})$-theory of the expansion of M by λ unary predicates. Similarly, $t_{\alpha,\lambda}(L)$ denotes the set of all possible $L^{\alpha}_{\infty,\lambda}(Q_{\text{mon}})$-theories.

Observe here that if α, λ, and L are finite, then so is $t_{\alpha,\lambda}(L)$. Moreover, for each L-structure M, $t_{\alpha,\lambda}(M)$ is equivalent (that is, it holds of the same structures) to a sentence in $L_{\beth(\alpha,\lambda+|L|),\lambda}$. The following lemma illustrates the expressive power of the $t_{\alpha,\lambda}(M)$. And, interestingly enough, it also provides the key technical step for our Feferman–Vaught like theorem.

3.2 Lemma. *Let λ, λ' and κ be cardinals with $\lambda + \lambda' \leq \kappa$. Let $\mathscr{I}$ and $\mathscr{J}$ be structures (having, for the sake of simplicity, a finite language) and universes I and J respectively. Suppose the sets I and J are partitioned by the sequences $Q_t(I)$, $Q_t(J)$, respectively, for $t \in \lambda$, and suppose further that*

$$t_{\alpha+1,\kappa}(\langle \mathscr{I}, Q_t(I) : t \in \lambda \rangle) = t_{\alpha+1,\kappa}(\langle \mathscr{J}, Q_t(J) : t \in \lambda \rangle).$$

If $\langle X_i : i \in \lambda' \rangle$ *is a partition of* I *refining the partition* $\langle Q_t(I) : t \in \lambda \rangle$, *then there exists* $\langle Y_i : i \in \lambda' \rangle$, *a partition of* J, *such that*:

$$t_{\alpha,\kappa}(\langle \mathcal{I}, Q_t(I), X_i : t \in \lambda, i \in \lambda' \rangle) = t_{\alpha,\kappa}(\langle \mathcal{J}, Q_t(J), Y_i : t \in \lambda, i \in \lambda' \rangle). \quad \square$$

3.3 Generalized Products. We begin our treatment of the Feferman–Vaught theorem by giving a rather "soft" definition of a generalized product. This notion differs from that in Feferman–Vaught in several respects. Perhaps the most basic is that it is designed to describe only operations taking a set of L-structures to an L-structure. Thus, the definition focuses on the relation between the truth of basic relations in the language L (as opposed to arbitrary definable relations) in the factor structure and the product structure. The intent of this definition is to emphasize those properties of the definition of the basic relations in the product structure which allow the assertion, "truth of basic relations depends on truth in the factors" to propagate to, "truth of all sentences in first-order logic (in infinitary monadic logic) depends on their truth in the factors". This definition is abstracted from the accounts of the monadic preservation theorem in Shelah [1975e] and Gurevich [1979b]. The emphasis here differs from that in Feferman [1972] where the role of functors from one similarity type to another is of central importance.

Examples of the notion of generalized product defined here—not of minor modifications of it—include direct product, disjoint union, ordinal sum of linear orderings, ultraproduct, and reduced product. Observe that in the last two, the language for the index set contains symbols binding subsets. Note also that the notion we are here examining does not include the concept of a sheaf over a boolean space.

Following is the key idea of the definition. Since we are going to give a proof by induction on quantifiers, we must describe how the product operation behaves with respect to structures obtained by naming elements and—since we will work in monadic logic—subsets. In fact, the notion of projection which we formulate below would be harder to explicate if we were to deal with elements rather than with sets since (for example, in disjoint unions) we frequently want to project to the empty set.

3.4 Definition. A *generalized product* is a function (or a family of functions) which, to each language L and each sequence $\langle A_i : i \in I \rangle$ of L-structures, assigns an L-structure $F(\langle A_i : i \in I \rangle) = A^*$ satisfying the following conditions:

(i) For each i there exists a function $\rho_i : \mathcal{P}(A^*) \to \mathcal{P}(A_i)$ such that if $\bar{P}$ is a sequence of subsets of A^*, then

$$F(\langle (A_i, \rho_i(\bar{P})) : i \in I \rangle) = \langle F(A_i : i \in I), \bar{P} \rangle.$$

(ii) For any sequence $\bar{a}$ (of arbitrary length $< |A^*|$) and for each L-symbol R, letting $K_R(\bar{a}) = \{i : A_i \models R(\rho_i(\bar{a}))\}$ and analogously in B^*, if $t_{0,\lambda}(\langle \mathcal{I}, K_R(\bar{a}) \rangle) = t_{0,\lambda}(\langle \mathcal{J}, K_R(\bar{b}) \rangle)$, then $A^* \models R(\bar{a})$ if and only if $B^* \models R(\bar{b})$.

Here and below, when $\bar{a}$ is a sequence of individuals, we will simply write $\rho_i(\bar{a})$ for $\langle\rho_i(\{a_0\}), \ldots, \rho_i(\{a_{k-1}\})\rangle$. Each $\rho(\{a_i\})$ has cardinality at most 1. Now we can state our version of the Feferman–Vaught theorem. The proof is similar to that of Theorem 2 of Chapter XIII.

3.5 Theorem (Preservation Theorem). *Suppose F is a generalized product operation and suppose also that $\langle A_i: i \in I\rangle$ and $\langle B_j: j \in J\rangle$ are families of L structures. For $t \in t_{\alpha,\lambda}(L)$, let $Q_t(I) = \{i: t_{\alpha,\lambda}(A_i) = t\}$ and let $Q_t(J) = \{j: t_{\alpha,\lambda}(B_j) = t\}$. Moreover, let $W = \{t_{\alpha,\lambda}(A_i): i \in I\} \cup \{t_{\alpha,\lambda}(B_j): j \in J\}$. There exists a $\kappa = \kappa(\alpha, |W|)$ such that if*

$$t_{\alpha,\kappa}(\langle\mathscr{I}, Q_t(I): t \in W\rangle) = t_{\alpha,\kappa}(\langle\mathscr{J}, Q_t(J): t \in W\rangle)$$

then

$$t_{\alpha,\lambda}(A^*) = t_{\alpha,\lambda}(B^*). \quad \square$$

As a corollary, we get a result mentioned in Chapter IX.

3.6 Corollary. *If κ is strongly inaccessible, then $L_{\kappa,\lambda}$-equivalence is preserved by generalized product.*

Proof. If $\phi \in L_{\kappa,\lambda}(Q_{\text{mon}})$ then for some $\mu < \kappa$, $\phi \in L_\mu$. But then $\phi \in L^\alpha_{\mu,\lambda}(Q_{\text{mon}})$ where $\alpha < \mu^+$ (this is a straightforward computation). Thus, the truth of ϕ in M is determined by $t_{\alpha,\lambda}(M)$ which is equivalent to a formula in $L_{\kappa,\lambda}$ since κ is strongly inaccessible. $\square$

This argument also yields the results in 3.3.4, Corollary 2.3.5, and 3.3.6 of Chapter IX.

We will now describe a generalization of disjoint union which is the example of generalized product that is of most use in the study of second-order quantifiers. This is a generalization of the notion of decomposition that was employed in Lemma 2.4. If we form a disjoint union, no relation holds between sequences $\bar{a}$, $\bar{b}$ from different constituents of the union. We want to allow such relations to hold but we also want to require that whether $R(\bar{a}, \bar{b})$ holds shall depend only the separate properties of $\bar{a}$ and $\bar{b}$. To make this notion precise, we require several preliminary definitions.

3.7 Definition. (1) If $\langle M_i: i \in I\rangle$ is a sequence of L-structures with $M_i \cap M_j - N$, we call the M_i a *sequence with heart N*.

(2) Let $\langle M_i: i \in I\rangle$ be a sequence with heart N. To define *the free union (with respect to σ) over N of the M_i*, we first need the following auxilliary notions:

(i) An n-condition τ is a pair $\langle P, \langle\phi_0, \ldots, \phi_{k-1}\rangle\rangle$ consisting of a partition, P, of n into sets $P_0, \ldots, P_{k-1}$ and a k-tuple of first order formulas such that ϕ_i has $|P_i|$ free variables.

(ii) σ is a map which assigns to each m-ary relation symbol R of L a finite set of m-conditions.

(iii) Let M be $\bigcup\{M_i: i \in I\}$. If $\bar{a} \in M$, then $\bar{a}$ satisfies the n-condition $\langle P, \langle \phi_0, \ldots, \phi_{k-1} \rangle\rangle$ if for some $M_{i_0}, \ldots, M_{i_{k-1}}$, we have $P_j = \{m: a_m \in M_{i_j}\}$; and, letting $\bar{a}_j = \{a_m: m \in P_j\}$, taken in increasing order of subscript, $M_{i_j} \models \phi_j(\bar{a}_j)$.

Now the free union of the M_i over N is the structure whose universe is $\cup \{M_i: i \in I\}$, where $R^M = \{\bar{a}: \bar{a}$ satisfies an m-condition in $\sigma(R)\}$.

It is easy to see that such a free union satisfies the definition of generalized product. Technically, we note that one must make allowance for the amalgamation, but this is straightforward. The details of Theorem 3.3.5 are, in this special case, carried out in III.1.13 in Baldwin–Shelah [1982]. In that paper, the free union is defined in terms of $t(\bar{a}; N)$. An easy application of compactness shows that when every model of T containing N can be decomposed in the sense of Definition 2.4.1, then each such model can, in fact, be written as free union over N in the sense of 3.7.3.

4. *The Comparison of Theories*

This section discusses a nuance in Shelah's argument, reported in Theorem 1.2.5, that $Q_{\mathrm{II}} \not\leq Q_{1-1}$. Namely, we consider the exact rôle of the assertion that interpretations preserve Hanf number. We show that a similar in form but technically easier argument shows $Q_{\mathrm{II}} \not\leq (\mathrm{Th}(<, Q_{\mathrm{mon}})$, the monadic theory of order. This last remark is apparently paradoxical in the light of the proof (Gurevich–Shelah [1982]) that it is consistent to interpret Q_{II} into $(\mathrm{Th}(<), Q_{\mathrm{mon}})$. To resolve this paradox we must distinguish the usual notion of interpretation from the stronger notions used in this paper.

4.1 Definition. The theory T_1 is *syntactically-interpretable* in the logic T_2 if there is a map* assigning to each T_1-sentence ϕ a T_2-sentence ϕ^* such that $T_1 \vdash \phi$ iff $T_2 \vdash \phi^*$.

Clearly, if* is recursive the Turing degree of T_1 is less than or equal to the Turing degree of T_2. However, this map need not preserve model-theoretic properties. Thus, using the Feferman–Vaught theorem for monadic logic, we will show that the Hanf number for monadic sentences on linearly ordered models (the Hanf number of the monadic theory of order) is strictly less than the Hanf number of second-order logic. It is easily seen that this implies that there can be no strong interpretation (in the sense of Definition 3.2) of Q_{II} into $\mathrm{Th}(<)$, Q_{mon}, (see Baldwin–Shelah [1982, VIII.2.12]). Nevertheless, Gurevich–Shelah [1982] have shown that it is consistent—indeed, it follows from the GCH—that there be a syntactic interpretation of Q_{II} into the monadic theory of order. The reader should consult Chapter XIII for more details on the monadic theory of order.

Several variants on the notion of interpretation and their roles are discussed in Baldwin–Shelah [1982]. We will use here only interpretations which satisfy the following condition.

4.2 Definition. The logic T_1 is *semantically interpretable* in the logic T_2 if there exist a pair of maps (both denoted by *) taking T_1-sentences to T_2-sentences and the models of T_2 onto the models of T_2 such that:

(i) $M \subseteq M^*$;
(ii) $M \models \phi$ iff $M^* \models \phi^*$.

If, in addition, we have

(iii) $|M^*|$ can be computed from $|M|$,

then we say T_1 is *strongly semantically interpretable* into T_2.

We will now show how bounds on the Hanf number of a theory can be used to show that there is no strong semantic interpretation of one theory into another. This, however, requires the technical notion given in

4.3 Definition. We say that the Hanf number of T_1 is bounded in terms of the Hanf number of T_2 and write $B(T_1, T_2)$ if there is a second-order definable function $f(x)$ such that $H(T_1) \leq f(H(T_2))$.

Observe that this relation is obviously transitive. Now, if $B(Q_{\text{II}}, T)$, it is fairly easy to see that there can be no strong semantic interpretation of Q_{II} into T. Since our notion of $\leq$ is a strong semantic interpretation, this gives a more general explanation for Theorem 1.2.5. We will now show that that theorem can be extended to the monadic theory of order.

In some respects, Silver [1971] begins this program with his explicit computation of an upper bound for the Hanf number for logic with the well-ordering quantifier (Chapter XVII). This shows that fewer classes of cardinals are characterized as cardinals (that is, as, well-ordered sets) in the monadic theory of order than in second-order logic. This leaves open the possibility that we might be able to characterize the missing classes as sets of cardinals in which a sentence in the monadic theory of order has a model (although not necessarily a well-ordered one).

We use the following notation.

4.4 Notation. We denote the Hanf number of $(\text{Th}(<), Q_{\text{mon}})$, the monadic theory of well-orderings, and Q_{II} respectively by HL, HW, and H_{II}.

We write $H(T)$ for the Hanf number of theory T. If $H(T)$ can be bounded by a cardinal definable in second-order logic (for example, HW), then H_{II} cannot be bounded in terms of $H(T)$. As, we would then have a second-order definable bound on H_{II}, which is clearly impossible. Thus, the assertion $HL < H_{\text{II}}$ follows immediately from the next lemma.

4.5 Lemma. *HL is bounded in terms of HW.*

Proof. Specifically, we will show that $HL \leq \Sigma\,\{2^\lambda\colon \lambda < HW\}$. Let $(M, <)$ be a linear order and suppose that λ can be embedded in M as $\langle a_i\colon i < \lambda\rangle$. Now, M is a free union (in the sense of Theorem 2.6) of the intervals determined by the a_i. For any fixed monadic sentence ϕ, say with quantifier depth m, we can find a k such that $t_{2,k}(\lambda, Q_t(\lambda))$, where t ranges over the finitely many monadic theories of quantifier depth n, determines whether M satisfies ϕ. Since $\lambda > HW$, we can replace λ with an arbitrarily large λ' with $t_{2,k}(\lambda, Q_t(\lambda)) = t_{2,k}(\lambda, A_t)$ for appropriate subsets A_t of λ'. But it is an easy matter to find an M' such that M' is a free union of intervals indexed by λ' and so that $Q_t(\lambda') = A_t$. But then $M' \models \phi$.

Since, for any linear order M, if $|M| > 2^\lambda$, there is an order embedding of either λ or λ^* into M—and since the preceding argument works equally well for λ^*—we see $HL \leq \Sigma\{2^\lambda : \lambda < HW\}$.

Clearly, if Q_{II} could be strongly interpreted in $(\mathrm{Th}(<), Q_{\mathrm{mon}})$, then H_{II} would be bounded in terms of HL. Thus, we have

4.6 Theorem. *There is no strong semantic interpretation of* Q_{II} *into* $(\mathrm{Th}(<), Q_{\mathrm{mon}})$. □

5. *The Classification of Theories by Interpretation of Second-Order Quantifiers*

We will not investigate the partial order of interpretability among theories (T, Q_ψ). That order refines the interpretability order of among first-order theories and so defies model-theoretic analysis. Rather, we will discuss the following question for a given first-order theory T: Do the four second-order quantifiers coalesce when restricted to models of T? The answer to this question can be viewed either as a comment on the quantifiers or as a comment on the theory T. We will adopt the latter viewpoint here. The non-interpretability of second-order logic imposes an extremely strong structure theory on the models of T. This structure theory and some of its consequences are outlined below. In particular, we measure the complexity of (T, Q_ψ) by computing Hanf and Löwenheim numbers.

5.1. *Outline of the Classification*

In making such a classification, we consider those theories for which (T, Q_{mon}) interprets Q_{II} as being beyond analysis. The remainder can then be divided into four classes as follows. Assume $Q_{\mathrm{II}} \not\leq (T, Q_{\mathrm{mon}})$.

	$Q_{\mathrm{II}} \leq (T, Q_{1-1})$	$Q_{\mathrm{II}} \not\leq (T, Q_{1-1})$
$\mathrm{Th}(<, Q_{\mathrm{mon}}) \leq (T, Q_{\mathrm{mon}})$ (unstable)	prototype $(\mathrm{Th}(<, Q_{\mathrm{mon}})$	impossible
$\mathrm{Th}(<, Q_{\mathrm{mon}}) \not\leq (T, Q_{\mathrm{mon}})$ (stable)	tree decomposable prototypes $\lambda^{\leq\omega}, \lambda^{<\omega}$	strongly decomposable

We could discuss the desirable properties of a particular entry of this table in two ways. We could prove a specific theorem (for example, that: the Löwenheim number of a countable theory such that $Q_{\mathrm{II}} \not\leq (T, 1\text{-}1)$ is $\aleph_0$). Even when such precise information cannot be obtained, we may be able to reduce such questions to the computation of, for instance, Löwenheim numbers for a specific theory T_0 by showing, for example, that if $Q_{\mathrm{II}} \not\leq (T, Q_{\mathrm{mon}})$, then (T, Q_{mon}) is bi-interpretable with the models of T_0. In some cases, we will prove a slightly weaker reduction than the second alternative: We will replace the theory T_0 by a class of structures which is not first-order definable. In some respects, of course, such a reduction is actually stronger than proving a particular theorem since it provides a "normal form" for models of T; the strength of the reduction depends on how well we are able to analyze the class to which we reduce.

The first line of the table distills an argument for the importance of studying the monadic theory of order. First, interpretability of the monadic theory of order is related to the important distinction between stable and unstable first-order theories.

5.1.1 Lemma. *If the complete first-order theory T is unstable, then* $(\mathrm{Th}(<), Q_{\mathrm{mon}}) \leq (T, Q_{\mathrm{mon}})$.

This result is proven in detail in Baldwin–Shelah [1982]. In outline, the proof proceeds by noticing (see Shelah [1978a]) that T is unstable iff T admits a definable linear ordering of an infinite set of n-tuples. A fairly complicated analysis of order indiscernibles (see Baldwin–Shelah [1982, VIII.1.3] shows that with additional unary predicates a linear ordering of a definable subset can be specified.

A second reason for the intensive study of the monadic theory of order as opposed to $(\mathrm{Th}(<), Q_\psi)$, for some other ψ, is that no other ψ is really possible. We have already shown in Section 2 that the only possibility for Q_ψ is Q_{1-1}. The next theorem rules out even that. It is fairly easy to deduce from Lemma 1.2.9 that $Q_{\mathrm{II}} \leq (\mathrm{Th}(<), 1\text{-}1)$. Combining this result with Section 5.1, we obtain

5.1.2 Theorem. *If T is unstable, then $Q_{\mathrm{II}} \leq (T, Q_{1-1})$.*

Further expansion of the argument that $\mathrm{Th}(<, Q_{\mathrm{mon}})$ is the prototype for those monadic unstable theories which can be analyzed occurs in Shelah [198?b, 198?d].

We will now discuss the situation characterized by bottom line of this table: The situation in which T is stable. In Section 2.5, we outlined the argument that if $Q_{\mathrm{II}} \not\leq (T, Q_{1-1})$ and T is stable, then T is strongly decomposable. If $Q_{\mathrm{II}} \leq (T, Q_{1-1})$, the argument that the fundamental equivalence relation is the same as algebraic closure and thus that each class is small (see Lemma 2.5.6) does not apply so that the classes may indeed be large. In this case, we iterate the procedure by choosing submodels inside each equivalence class and decomposing the class over this model. Since T is stable, this process cannot be iterated more than $|T|$ times (see Baldwin–Shelah [1982, IV.2.1]). This decomposes each model by a tree of height $\leq |T|$ in the sense of the following definition.

Before we examine the definition in detail, observe that the notation τ^- denotes the result of deleting the last symbol from a sequence.

5.1.3 Definition. The model M is *tree-decomposed* by the tree I of sequences of length at most κ if there exist models $\{\langle M_\eta, N_\eta\rangle : \eta \in I\}$ such that:

(i) $|N_\eta| = |T|$ for every η.
(ii) If $\eta \subseteq \rho$ then $N_\eta \subseteq N_\rho \subseteq M_\rho \subseteq M_\eta$.
(iii) For each $\tau \in I$ there are index sets J and functions σ such that:
  (a) M_τ is the free union of the $\{M_{\tau\frown j}: j \in J\}$ amalgamated over N_τ and taken with respect to σ.
  (b) M is the free union over N_τ (with respect to σ) of $\{M_{\tau\frown j}: j \in J\} \cup \{M_\rho \cup N_\tau: \rho \neq \tau \text{ but } \rho^{--} = \tau^{-}\}$;
(iv) $M_{\langle\rangle} = M$; if $\lg(\eta)$ is a limit ordinal then $N_\eta = \bigcup\{N_\tau: \tau \subseteq \eta\}$, $M_\eta = \bigcap\{M_\tau: \tau \subseteq \eta\}$;
(v) $M = \bigcup\{N_\tau: \tau \in I\}$.

If a theory is κ tree-decomposable (that is, every model of T is tree-decomposed by a tree of height κ), then the models of T are short in the sense that no matter how large a model is, complete information about a finite sequence of elements from the model depends only on the less than κ elements which precede it in the tree.

5.1.4 Definition. The theory T is *shallow* if every model of T can be tree-decomposed by a well-founded tree. Otherwise T is *deep*.

If T is shallow, then we assign a rank to models of T, namely, the ordinal rank of the tree.

Now we can describe our prototypes.

5.1.5 Notation. Let K_0 be the class of all trees $\{\lambda^{<\omega}: \lambda \in \mathrm{Ord}\}$ and K_1 the class of all trees $\{\lambda^{\leq\omega}: \lambda \in \mathrm{Ord}\}$. If $Q_{\mathrm{II}} \leq (T, Q_{\mathrm{mon}})$, then the models of T are very closely tied to the trees which arise as skeletons when the models are tree-decomposed. Specifically, we have

5.1.6 Theorem. (i) *If T is a countable superstable deep theory and $Q_{\mathrm{II}} \leq (T, Q_{\mathrm{mon}})$, then $(T, L_{\omega_1,\omega}(Q_{\mathrm{mon}})$ and $(K_0, L_{\omega_1,\omega})$ are bi-interpretable.*
(ii) *If T is a countable stable but not superstable theory and $Q_{\mathrm{II}} \not\leq (T, Q_{\mathrm{mon}})$, then $(T, L_{\omega,\omega}(Q_{\mathrm{mon}})$ and $(K_1, L_{\omega,\omega})$ are bi-interpretable.*

This is Theorem VII.2.1 of Baldwin–Shelah [1982].

5.2. *Computations of Hanf and Löwenheim Numbers*

In this section we will briefly discuss the results on Hanf and Löwenheim numbers which can be derived from the preceding classification. We will then indicate how such computations are made. For the sake of simplicity, we will discuss only the case of countable languages here. The results extend to uncountable languages and such extensions are considered in Baldwin–Shelah [1982].

5.2.1. Finitary Monadic Logic

	Löwenheim Number	Hanf Number
$\lambda^{\not\leq\omega}$	?	$(\beth_\omega)^+$
$\lambda^{<\omega}$ deep	*	$(\beth_\omega)^+$
$\lambda^{<\omega}$ shallow depth $= \beta$	$\min(\beth_\beta, \beth_\omega)$	$\min(\beth_\beta, \beth_\omega)$
strongly decomposable	$\beth_1$	$\beth_1$

* Shelah [1983b] has shown that there are superstable deep theories such that the Löwenheim number of (T, Q_{mon}) is (assuming $V = L$) the same as that of second-order logic.

This table and the one in 5.2.3 reports the Hanf number for sets of sentences. For a single sentence the '$^+$' can be dropped in some cases. See III.2 of Baldwin–Shelah [1982].

In order to completely determine the Löwenheim number, we must consider one further property. This we do in

5.2.2 Definition. The free union of $\langle M_i : i \in I \rangle$ over N is *nice* if for each i there exist finite subsets H_i of N and U_i of M_i such that for any $m \in M$, $t(m; H_i \cup U_i) \vdash t(m; N)$.

If the decomposition is nice, then the Löwenheim number of a shallow theory is $\aleph_0$; otherwise, it is $2^{\aleph_0}$. Details on this nicety are given in VI.2 of Baldwin–Shelah [1982].

5.2.3 Infinitary Monadic Logic ($L^\alpha_{\infty,\omega}$). For the sake of simplicity, assume that $\alpha \geq \omega_1$, then the following arrangement is possible.

	Löwenheim Number	Hanf Number
$\lambda^{\leq\omega}$	?	$\beth^+_{2+1}$
$\lambda^{<\omega}$ deep	*	$\beth^+_{\alpha+1}$
$\lambda^{<\omega}$ shallow shallow: depth $= \beta$	$(\beth_\beta)^+$	$(\beth_\beta)^+$
strongly decomposable	$(\beth_1)^+$	$(\beth_1)^+$

* Shelah [198?b] has shown that for every superstable deep theory such that the Löwenheim number in infinitary logic of (T, Q_{mon}) is, assuming that $V = L$, the same as that of second-order logic.

5.2.4 Outline of the Argument. These computations depend on (i) the decomposition of the models; (ii) the generalized Feferman–Vaught theorems; and (iii) the computation of the cardinality of $t_{\alpha,\lambda}(L)$. The general program is simply this: to decompose a model as free union of structures N_i for $i \in I$. Suppose we are trying to extend (Hanf number) or restrict (Löwenheim number) M for a sentence with λ

quantifiers (either individual or monadic) and α alternations. Let $W = \bigcup_{i \in I} t_{\alpha,\lambda}(N_i)$. Then, by Theorem 2.5, we can find a κ such that $t_{\alpha,\lambda}(M)$ is determined by $t_{\alpha,\kappa}(\langle \mathscr{I}, Q_t(I): t \in W \rangle)$. Thus, if we can guarantee the cardinality of I to be sufficiently greater than $|W|$, there will be a large number of indices with the "same theory". We can then expand or contract this set at will. The full details are given in Chapters III, VI, and VII of Baldwin–Shelah [1982]. One sample is perhaps instructive. If T is strongly decomposable, then each model is a free union of countable structures. Since there are only $\beth_1$ possible $L_{\omega,\omega}(Q_{\text{mon}})$ theories of a countable structure, this reduces both the Hanf and Löwenheim numbers of (T, Q_{mon}) to $\beth_1$ precisely. In fact, for theories with a nice decomposition these numbers can be reduced to $\aleph_0$.

The situation when T is only tree-decomposable is somewhat more subtle. We can compute the Hanf number for $L_{\infty,\lambda}$ by noting that if $|M| > \beth_{\alpha+1}$ somewhere in the tree, we have a free union with more that $|t_{\alpha,\lambda}(L)|$ factors and then extend M. But this argument yields no information on the Löwenheim number. If T is shallow and β is the sup of the ranks of models of T, then we obtain the bound $\min(\beth_\beta, \beth_\omega)$ for both the Hanf and Löwenheim numbers by induction on this rank.

6. Generalizations

This work can be extended in several directions. In particular, the results in Section 5 can be sharpened, and the notion of quantifier can be extended. With respect to the first direction, Shelah [198?d] confirms the close connection between Hanf number and interpretability by showing

6.1 Theorem. *For any first-order theory T the Hanf number of (T, Q_{mon}) is at most H_{II} iff $Q_{\text{II}} \not\leq (T, Q_{\text{mon}})$.* $\square$

In the other direction, we again return to the definition of a second-order quantifier.

1.2.2 Definition. If $\psi(\bar{r})$ is a formula of pure identity theory, then $Q_\psi(\bar{r})$ is the second-order quantifier whose semantics are given by:

$$M \models Q_\psi(\bar{r})\phi(\bar{r}) \quad \text{iff for some sequence} \quad \bar{R} \in \mathscr{R}_\psi(M),\ M \models \phi(\bar{R}).$$

There are several ways to extend this definition. Perhaps the most obvious one is to replace the requirement that ϕ be a first-order formula by introducing a parameter for the language. Thus, we have been discussing first-order definable second-order quantifiers. One could discuss infinitarily definable second-order quantifiers, or second-order quantifiers defined in stationary logic, or second-order definable second-order quantifiers etc. *ad nauseum.* A second possibility is to partition the variables $\bar{r}$ into a sequence $\bar{s}\bar{t}$. Then, by freezing the $\bar{s}$, we move out of pure logic and are thus able to discuss automorphisms, congruences and other algebraic concepts. Finally, we could remove the restriction that the relations $\bar{r}$ be subsets of A_n, for some n, and allow them, for example, to be families of subsets. Thus, we would obtain definable third-order quantifiers. At this level, we spread our net to include $L(\mathrm{aa})$. Another approach is to relax the definability requirement and allow the class of subsets defining a quantifier to be any class that is closed under isomorphism. This is the line adopted by Shelah [1983a]. Thus, we identify a quantifier with a class K of subsets of $\bigcup A^n$. Naturally, we may also deal with a finite sequence of quantifiers (classes) $\bar{K} = \langle K_0, \ldots, K_n \rangle$.

In discussing this widened class of quantifiers, Shelah weakens the notion of interpretability somewhat.

6.2 Definition. We say $\bar{K}$ is *expressible* in K if for each $R \in K$ there is a formula $\phi(\bar{x}, \bar{r})$ (with quantifiers over the K_i) such that for some $R_0, \ldots, R_n$ each in one of the K_i, $R(\bar{x}) \leftrightarrow \phi(\bar{x}, \bar{R})$. The problem—already hinted at in Shelah [1973c]—was finally addressed in Shelah [1983a], and it asks the following: Is every quantifier (that is, class K) bi-interpretable with a finite sequence $\bar{K}$, where each K_i is an equivalence relation? The main result on this is given in

6.3 Theorem (Expressibility with Equivalence Relations). (i) If $V = L$, then every K is bi-expressible with an equivalence relation (see Shelah [1983c]; p. 53).

(ii) It is consistent that there is a K which is not biexpressible with an equivalence relation. (Shelah [1983c]; pp. 48–57).

There is still another way these methods might be used. In many of the technical successes of stability theory over the last few years—for example, Vaught's conjecture for ω-stable T (Harrington–Makkai–Shelah [1983]) and the solution by Shelah [1982f, 198?c] of Morley's conjecture that (with the obvious exception) the spectrum function is increasing—the part of the proof showing there are many models can be viewed as an interpretation of Q_{II} into the $L_{\omega_1,\omega}(Q_{\mathrm{mon}})$ theory of T.

Chapter XIII
Monadic Second-Order Theories

by Y. Gurevich

In the present chapter we will make a case for the monadic second-order logic (that is to say, for the extension of first-order logic allowing quantification over monadic predicates) as a good source of theories that are both expressive and manageable. We will illustrate two powerful decidability techniques here—the one makes use of automata and games while the other uses generalized products *à la* Feferman–Vaught. The latter is, of course, particularly relevant, since monadic logic definitely appears to be the proper framework for examining generalized products.

Undecidability proofs must be thought out anew in this area; for, whereas true first-order arithmetic is *reducible* to the monadic theory of the real line R, it is nevertheless not *interpretable* in the monadic theory of R. Thus, the examination of a quite unusual undecidability method is another subject that will be explained in this chapter. In the last section we will briefly review the history of the methods thus far developed and give a description of some further results.

1. Monadic Quantification

Monadic (second-order) logic is the extension of the first-order logic that allows quantification over monadic (unary) predicates. Thus, although binary, ternary, and other predicates, as well as functions, may appear in monadic (second-order) languages, they may nevertheless not be quantified over.

1.1. Formal Languages for Mathematical Theories

We are interested less in monadic (second-order) logic itself than in the applications of this logic to mathematical theories. We are interested in the monadic formalization of the language of a mathematical theory and in monadic theories of corresponding mathematical objects. Before we explore this line of thought in more detail, let us argue that formalizing a mathematical language—not necessarily in monadic logic, but rather in first-order logic or in any other formal logic for that matter—can be useful.

We begin by observing that the first-order Zermelo–Fraenkel set theory stands as a very important case in point, since it provides the most popular way to avoid known paradoxes in set theory. Another excellent example is related to the Lefschetz principle in algebraic geometry. This principle asserts that any algebraic statement that is true for the field of complex numbers is also true for any algebraically closed field of characteristic 0. Tarski proved a meaningful exact version of the Lefschetz principle, namely, that all algebraically closed fields of characteristic 0 are elementarily equivalent.

The task of classifying all mathematical structures of a kind up to isomorphism (or homeomorphism, etc.) may be impossible. For example, nobody can classify all abelian groups up to isomorphism. Formalizing (a portion of) the language may allow classification by properties that are expressible in the formal language. Szmielew [1955] did, in fact, classify all abelian groups up to elementary equivalence. The classification of structures up to indistinguishability in a formal language may indeed be a reasonable alternative to the original classification problem provided, of course, that the formal language expresses enough of the relevant mathematics.

Another impossible task is that of learning everything about a specific structure. For example, nobody can learn all about the binary tree of words in a two-letter alphabet. Formalizing (a portion of) the language may enable us to learn all about the structure that is capable of being expressed in the formal language. It is, of course, a reasonable approach if the formal language is sufficiently rich. Indeed, Rabin [1969] has constructed an algorithm which is capable of recognizing the true statements in the very expressive monadic (second-order) language of the binary tree with two successor functions.

The study of mathematical structures in a formal language may, of course, degenerate to a mere logic exercise if the language is not sufficiently expressive. For example, imagine studying first-order properties of dense linear orders. On the other hand, the study itself may become intractable if the language is over-expressive. For instance, imagine studying second-order properties of dense linear orders. A good formal language has to meet two conflicting demands. It must express an interesting portion of the relevant mathematics, and it must also provide a manageable theory. One of the main aims of this chapter is to demonstrate that the monadic (second-order) logic is a good source of expressive and manageable theories.

1.2. Ordered Abelian Groups and Restricted Monadic Quantification

I began to think in terms of monadic logic while I was working on ordered abelian groups. The original problem I faced was the decision problem for the elementary theory of such groups—a question of Tarski. It appeared, however, that monadic logic gives a better formalization of the informal theory of o.a. groups. The story was an important lesson for me and I will briefly relate it to you.

An o.a. group is a group and a chain, the two structures being connected by the law

$$x < y \to x + z < y + z.$$

Here is a particular example: the additive group of complex numbers ordered thus:

$$a + bi < c + di \quad \text{iff} \quad b < d, \text{or } b = d \text{ and } a < c.$$

The subgroups of an ordered abelian group that form intervals are called *convex subgroups*. For example, the real numbers form a convex subgroup in the o.a. group of complex numbers just described. It is easy to verify that the convex subgroups of any o.a. group are linearly ordered by inclusion. Before proceeding, we should point out that throughout this chapter the terms *chain* and *linear ordering* will be used interchangeably.

The elementary first-order theory of o.a. groups was shown to be decidable in Gurevich [1964], there was proven that two o.a. groups are elementarily equivalent iff their chains of definable convex subgroups with some definable weights are elementarily equivalent. Of course, in that study most of the informal theory of o.a. groups was left aside, such theory tending as it does to deal with convex subgroups. In particular, we note that the o.a. group of complex numbers described above is elementarily equivalent to the naturally ordered additive group of real numbers, although only one of these o.a. groups has a non-trivial convex subgroup.

The elementary language of o.a. groups was expanded in Gurevich [1977a] by adding quantifiable variables that range over arbitrary convex subgroups, and the expanded theory of such groups was there proven to be decidable. You might suspect that the expanded theory is decidable because the expansion did not greatly increase the expressive power, and that the restricted monadic quantification can be essentially eliminated. However, this is not at all the case! Not only does the expansion considerably increase the expressive power, but it is also the *elementary* quantification that can be essentially eliminated in the expanded theory. Two o.a. groups are equivalent in the expanded language iff their chains of convex subgroups with some definable weights are elementarily equivalent. Moreover, the decision procedure is clearer and less cumbersome in the case of the expanded theory. Thus, in the case of o.a. groups, the monadic logic really does provide a better formalization.

Not too much work has yet been done on this kind of algebraic application of *restricted* monadic quantification. In this connection, the reader might consult Kokorin–Pinus [1978], an informative, although somewhat biased, survey. The remainder of this chapter is devoted mainly to *unrestricted* monadic quantification, an area in which some very impressive progress has been made. In the original papers, many of the results on unrestricted monadic quantification are accompanied by restricted monadic quantification results. The work on unrestricted monadic quantification seems to be a natural step in the development of ways that are capable of dealing with the presumably more applicable restricted monadic quantification.

1.3. Monadic Languages

The monadic (second-order) logic is the fragment of the full second-order logic allowing quantification only over elements and monadic predicates. One way to define the monadic version of an elementary language L is to augment L by a sequence of quantifiable set variables and by new atomic formulas $t \in X$, where t is an elementary term and X is a set variable. The intended interpretation here is that $\in$ is the membership relation and the set variables range over all subsets of a structure for L. Observe, however, that in the case of restricted monadic quantification the set variables range only over special subsets; that is to say, they only range over subgroups, or normal subgroups, etc.

The following proposition shows that the monadic theory of a structure may easily be intractable.

1.3.1 Proposition. *Let P be a ternary predicate on a non-empty set S. Suppose that, for every $x, y \in S$, there is $z \in S$, with $(x, y, z) \in P$, and for every $z \in S$ there is at most one pair (x, y) with $(x, y, z) \in P$; such P may be called a pairing predicate. Then the true (full) second-order theory of S is interpretable in the monadic theory of (S, P).*

Proof. The proof is quite clear. First, we code ternary, quaternary, etc., predicates by binary ones. That done, we then code a binary predicate B by a monadic predicate $\{z$: there is a pair (x, y) in B with $(x, y, z) \in P\}$. $\square$

We will be interested in the monadic theories that are not able to express pairing such as monadic theories of (linear) orders, monadic theories of trees, etc. In these theories it is useful in many cases for us to rid ourselves entirely of elementary variables by coding the original structure on singleton sets. For example, we consider the monadic language of order as the (formally) first-order language whose vocabulary consists of the binary predicate symbols $\subseteq$ and $\leq$. Every chain (that is, every linearly ordered set) gives a standard model: the variables range over all subsets of the chain, $\subseteq$ is the usual inclusion, and $X < Y$ means that there are elements $x < y$ with $X = \{x\}$, $Y = \{y\}$. The (formally) first-order theory of these standard models is, by the definition, the monadic theory of linear order.

2. The Automata and Games Decidability Technique

The first technique for dealing with nontrivial monadic theories originated in the theory of finite automata. In Section 2.1 we will demonstrate this technique on an easy example of the monadic theory of finite chains. Section 2.2 is devoted to the monadic theory of the chain ω of natural numbers, while Section 2.3 is devoted to the central result proven by the technique which is decidability of the monadic theory of the binary tree.

2.1. Monadic Theory of Finite Chains

We define the *monadic language of one successor* as formally the first-order language with binary predicates $\subseteq$ and SUC. It is convenient here for us to view a finite chain as a model for the monadic language of one successor, that is, the variables range over the subsets of the chain, $\subseteq$ is ordinary inclusion, and $\mathrm{SUC}(X, Y)$ means that there are points x, y such that $X = \{x\}$, $Y = \{y\}$, and y is the successor of x. The linear order (on singleton sets) is then easily definable.

Throughout this section Σ is an alphabet (all of our alphabets are finite and are not empty). A Σ-*automaton* is a quadruple $A = (S, T, s_{\text{in}}, F)$, where S is the finite *set of states*, $T \subseteq S \times \Sigma \times S$ is the *transition table*, $s_{\text{in}} \in S$ is the *initial state*, and $F \subseteq S$ is the *set of final* (or *accepting*) *states*. A is generally a non-deterministic automaton. It is *deterministic* if T is a total function from $S \times \Sigma$ to S.

A *run* of the Σ-automaton A on a word $\sigma_1 \ldots \sigma_l$ in Σ is a sequence $s_1 \ldots s_l$ of states such that $(s_{\text{in}}, \sigma_1, s_1) \in T$ and every $(s_i, \sigma_{i+1}, s_{i+1}) \in T$. The automaton *accepts* $\sigma_1, \ldots, \sigma_l$ if there is a run $s_1 \ldots s_l$ on this word with $s_l \in F$.

2.1.1 Theorem. *There is an algorithm that, given an alphabet Σ and a Σ-automaton A, constructs a deterministic Σ-automaton accepting exactly the words accepted by A.*

Proof. See any standard text in automata theory or, for the original proof Rabin–Scott [1959]. ☐

2.1.2 Theorem. *There is an algorithm that, given an alphabet Σ and a Σ-automaton A, decides whether A accepts at least one non-empty word.*

Proof. Let $A = (S, T, s_{\text{in}}, F)$. First, we construct a singleton alphabet $\Sigma' = \{a\}$ and a Σ'-automaton $A' = (S, T', s_{\text{in}}, F)$ that accepts a non-empty word iff A accepts a non-empty word. Set

$$T' = \{s_1 a s_2 : s_1 \sigma s_2 \in T, \text{ for some } \sigma \in \Sigma\}.$$

Second, we use the algorithm of Theorem 2.1.1 to construct a deterministic Σ'-automaton A'' that accepts exactly the words accepted by A'.

Third, let n be the number of states of A''. Consider now the unique run $s_1 \ldots s_{n+1}$ of A'' on the Σ'-word of length $(n + 1)$. There are $i < j \leq n + 1$ with $s_i = s_j$. Hence, any run of A'' is purely periodic from the ith place on. Thus, A'' accepts a non-empty word iff a final state appears among $s_1, \ldots, s_{j-1}$. ☐

A finite chain C with n subsets $X_1, \ldots, X_n$ can be considered as a word $\mathrm{Word}(C, X_1, \ldots, X_n)$ of length $|C|$, in the alphabet Σ_n that is the Cartesian product of precisely n copies of $\{0, 1\}$. If $n = 0$, then Σ_0 is a singleton. In case $n > 0$, a letter of Σ_n can be viewed as a column of n zeros and ones. For example, if C is the chain Sunday, ..., Saturday and $X_1 = \{\text{Monday, Thursday}\}$ and $X_2 = \{\text{Monday, Tuesday, Wednesday}\}$, then we have

$$\text{Word}(C, X_1, X_2) = \begin{matrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0. \end{matrix}$$

2.1.3 Theorem. *There is an algorithm that, given n and a Σ_n-automaton A, constructs a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of one successor such that for every finite chain C and any subsets $X_1, \ldots, X_n$ of C, we have that*

$$C \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \text{Word}(C, X_1, \ldots, X_n).$$

Proof. Without loss of generality, C can be taken as the chain $1, \ldots, l$ for some l. Let $s_1, \ldots, s_m$ be the states of A. The desired formula says that there are subsets $Y_1, \ldots, Y_m$ describing an accepting run of A on Word($C, X_1, \ldots, X_n$). The intended meaning of Y_k is $\{i$: A is in the state s_k after reading the ith letter$\}$. ☐

2.1.4 Theorem. *There is an algorithm that, given a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of one successor (with free variables as shown), constructs a Σ_n-automaton A such that for every finite chain C and any subsets $X_1, \ldots, X_n$ of C, we have that*

$$C \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \text{Word}(C, X_1, \ldots, X_n).$$

Proof. We will merely sketch the proof. The automaton is built by induction on the formula. The atomic cases and the case of disjunction are quite easy. As to the case in which $\phi = \exists X_{n+1}\psi$, the desired Σ_n-automaton guesses X_{n+1} and mimics the Σ_{n+1}-automaton corresponding to ψ. The case of negation is easy for deterministic automata. We will now use Theorem 2.1.1 and the result will follow. ☐

Theorems 2.1.3 and 2.1.4 together constitute a kind of normal form theorem for the monadic theory of finite chains.

2.1.5 Theorem. *The monadic theory of finite chains is decidable.*

Proof. Given a sentence ϕ, we use the algorithm of Theorem 2.1.4 to find an appropriate automaton. The sentence ϕ is satisfiable iff the automaton accepts at least one non-empty word. Now, using Theorem 2.1.2, the assertion follows immediately. ☐

2.2. Monadic Theory of ω

As usual, ω will denote the chain of natural numbers. We consider it here as a model for the monadic language of one successor: The variables range over the subsets of ω, $\subseteq$ is the usual inclusion, and SUC(X, Y) means that there is a natural number x such that $X = \{x\}$ and $Y = \{x + 1\}$. The monadic theory of ω is known as S1S which is an acronym for second-order (monadic) theory of one successor. Observe that the linear order (on singleton sets) is easily definable in S1S.

A *sequential Σ-automaton* is a quadruple $A = (S, T, s_{\text{in}}, F)$, where S is the set of finite *states*, $T \subseteq S \times \Sigma \times S$ is the *transition table*, s_{in} is the *initial state* and F is the set of *final collections* of states. A is generally a non-deterministic automaton. However, it is *deterministic* if T is a total function from $S \times \Sigma$ to S. A *run* of A on a sequence $\sigma_1\sigma_2 \ldots$ is a sequence $s_1s_2 \ldots$ of states such that $(s_{\text{in}}, \sigma_1, s_1) \in T$, and every $(s_i, \sigma_{i+1}, s_{i+1}) \in T$. It is an *accepting run* if $\{s: s_n = s$ for infinitely many $n\}$ belongs to F. And, finally, A *accepts* a sequence $\sigma_1\sigma_2 \ldots$ if there is an accepting run of A on this sequence.

2.2.1 Theorem. *There is an algorithm that, given an alphabet Σ and a sequential Σ-automaton A, constructs a deterministic sequential Σ-automaton accepting exactly the sequences accepted by A.*

This result is proven in McNaughton [1966]. However, simpler proofs can be found in Rabin [1972], Choueka [1974], Thomas [1981]. □

2.2.2 Theorem. *There is an algorithm that, given an alphabet Σ and a sequential Σ-automaton A, decides whether A accepts at least one sequence.*

Proof. The argument here is simple, since we only need repeat the proof of Theorem 2.1.2, speaking about sequences rather than words and changing the last sentence to: Thus A'' accepts the unique Σ'-sequence iff the collection $\{s_i, \ldots, s_{j-1}\}$ is final. □

Subsets $X_1, \ldots, X_n$ of ω form a sequence SEQ$(X_1, \ldots, X_n)$ in the alphabet Σ_n. The following three theorems and their proofs are similar to the corresponding theorems and proofs in Section 2.1.

2.2.3 Theorem. *There is an algorithm that, given n and a Σ_n-automaton A, constructs a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of one successor such that for any subsets $X_1, \ldots, X_n$ of ω,*

$$\omega \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \mathrm{SEQ}(X_1, \ldots, X_n). \quad \square$$

2.2.4 Theorem. *There is an algorithm that, given a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of one successor (with free variables as shown), constructs a Σ_n-automaton A such that for any subsets $X_1, \ldots, X_n$ of ω,*

$$\omega \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \mathrm{SEQ}(X_1, \ldots, X_n).$$

2.2.5 Theorem. *The monadic theory of ω is decidable.* □

2.3. *Monadic Theory of the Binary Tree*

The *binary tree* is here defined as the set $\{l, r\}^*$ of all words in the alphabet $\{l, r\}$. The empty word e is the root of the tree. The words xl and xr are respectively the *left* and the *right successors* of a word x.

The *monadic language of two successors* is (formally) the first-order language with binary predicates $\subseteq$, Left and Right. We regard the binary tree as a model for the monadic language of two successors: the variables range over the subsets, $\subseteq$ is the usual inclusion, Left(X, Y) means that there is a word x with $X = \{x\}$, $Y = \{xl\}$, and Right(X, Y) means that there is a word x with $X = \{x\}$, $Y = \{xr\}$. The monadic theory of the binary tree is known as S2S which is an acronym for the second-order (monadic) theory of two successors.

S2S is a very expressive theory. The relation "x is the initial segment of y" and "x precedes y lexicographically" are easily expressible (when coded on singleton sets). Rabin [1969] interpreted in S2S the monadic theories of 3, 4, etc. successors, the monadic theory of ω successors, and a good deal more.

A mapping V from the binary tree to an alphabet Σ will be called a Σ-*valuation* or a Σ-*tree*. We say that a *tree* Σ-*automaton* is a quadruple $A = (S, T, T_{\text{in}}, F)$ where S is the finite alphabet of *states*, $T \subseteq S \times \{l, r\} \times \Sigma \times S$ is the *transition table*, $T_{\text{in}} \subseteq \Sigma \times S$ is the *initial state table*, and F is the set of *final collections* of states. In order to describe when the automaton A accepts a Σ-tree V, we introduce a game $\Gamma(A, V)$ between the automaton A and another player called Pathfinder.

A chooses:	Pathfinder chooses:
s_0	
	d_1
s_1	
	d_2
s_2	
	d_3
s_3	
	$\ldots$

Here each $s_n \in S$ and each $d_n \in \{l, r\}$. The choices of A are restricted by the following conditions:

$$(V(e), s_0) \in T_{\text{in}} \quad \text{and} \quad (s_n, d_{n+1}, V(d_1 \ldots d_{n+1}), s_{n+1}) \in T.$$

We would like to avoid the possibility of the automaton not being able to make the next move. One way to do this is to provide our automata with an additional state FAILURE in such a way that a transition into FAILURE is always possible, but a transition from a FAILURE to another state is never possible. Of course, the singleton set {FAILURE} will not be a final collection.

The automaton A *wins* a play $s_0 d_1 s_1 d_2 \ldots$ if $\{s \in S: s_n = s$ for infinitely many $n\}$ belongs to F. Otherwise, Pathfinder *wins*. The automaton A *accepts* V if it has a winning strategy in $\Gamma(A, V)$. Otherwise, it *rejects* V. The notion of strategy is clarified below.

A *position* in $\Gamma(A, V)$ is a word in the alphabet $S \cup \{l, r\}$ that is an initial segment of some play $s_0 d_1 s_1 d_2 \ldots$. The *last appearance record* LAR(p) in a position p is the string of last appearances of states in p. Consider the following example:

A	Pathfinder	Position	LAR
		e	e
a		a	a
	l	al	a
b		alb	ab
	r	$albr$	ab
a		$albra$	ba
	l	$albral$	ba
c		$albralc$	bac
	r	$albralcr$	bac
c		$albralcrc$	bac
	l	$albralcrcl$	bac
a		$albralcrcla$	bca

Here is an inductive definition of the last appearance record LAR(p). If p is the empty word e (that is, the initial position), then LAR(p) is empty. If $p = ql$ or $p = qr$, then LAR(p) = LAR(q). Suppose now that $p = qs$, u = LAR(q) and u' is obtained from u by erasing all appearances of s. Then LAR(p) = $u's$. Every last appearance record is a word in alphabet S, where each state appears at most once.

A (deterministic) *strategy* for the automaton A in the game $\Gamma(A, V)$ is a function assigning a legal state to every position of even length. A (deterministic) *strategy* for Pathfinder is a function assigning a direction l or r to each position of odd length.

Unfortunately, deterministic tree automata are too weak and Theorem 2.1.1 cannot be generalized to them. That theorem played a key role in Section 2.1; and in the case of tree automata the proper form of determinacy will play an analogous role.

2.3.1 Theorem (Forgetful Determinacy Theorem). *One of the players has a winning strategy f in $\Gamma(A, V)$ such that if p, q are two positions, where the winner makes moves and p, q define the same residual game (that is, they have the same continuation) and have the same last appearance records, then $f(p) = f(q)$.*

Proof. See Gurevich and Harrington [1982]. ▯

A strategy f for a player in $\Gamma(A, V)$ will be called *forgetful* if $f(p) = f(q)$, for all positions p, q such that the player makes moves in p, q and p, q define the same residual games, and moreover, the last appearance records in p and in q coincide. The reason for this term is that any value $f(p)$ depends on the residual game and an only limited information about the history. Thus, in brief, we may say that a forgetful strategy "forgets" most of the history.

2.3.2 Theorem. *There is an algorithm that, given an alphabet Σ and a tree Σ-automaton A, decides whether A accepts at least one Σ-tree.*

Proof. As in the proof of Theorem 2.1.2, we first reduce the problem to the case of a singleton alphabet. Thus, suppose that Σ is a singleton and V is the unique Σ-tree. By the forgetful determinacy theorem, one of the players has a forgetful strategy winning $\Gamma(A, V)$. List all forgetful strategies $f_1, \ldots, f_m$ for the automaton A and all forgetful strategies $g_1, \ldots, g_n$ for Pathfinder. It is possible to check each f_i against each g_j because the play eventually becomes periodic. This way we can find the desired winning strategy and determine whether or not A accepts V. ◻

Subsets $X_1, \ldots, X_n$ of the binary tree give a Σ_n-tree that will be called $\mathrm{TREE}(X_1, \ldots, X_n)$, where Σ_n is as in Section 2.1.

2.3.3 Theorem. *There is an algorithm that, given n and a tree Σ_n-automaton A, constructs a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of two successors such that for any n subsets $X_1, \ldots, X_n$ of the binary tree,*

$$\{l, r\}^* \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \mathrm{TREE}(X_1, \ldots, X_n).$$

Proof. A *run* of a tree Σ-automaton A on a Σ-tree V is a function R from the binary tree to the set of states of A such that every sequence

$$R(e)d_1R(d_1)d_2R(d_1d_2)\ldots$$

is a legal play in $\Gamma(A, V)$. If A wins all these plays then the run R is *accepting*.

The desired formula says that there are subsets Y_s, where s ranges over the states of the given tree Σ_n-automaton A, that describe an accepting run R of A on $\mathrm{TREE}(X_1, \ldots, X_n)$. The intended meaning of Y_s is

$$\{x \in \{l, r\}^* : R(x) = s\}. \quad ◻$$

2.3.4 Theorem. *There is an algorithm that, given a formula $\phi(X_1, \ldots, X_n)$ in the monadic language of two successors, constructs a tree Σ_n-automaton A in such a way that for any n subsets $X_1, \ldots, X_n$ of the binary tree,*

$$\{l, r\}^* \models \phi(X_1, \ldots, X_n) \quad \textit{iff} \quad A \textit{ accepts } \mathrm{TREE}(X_1, \ldots, X_n).$$

Proof. The argument here is similar to that given for Theorem 2.1.4, except for the case of negation which is treated in Theorem 2.3.6 below. ◻

2.3.5 Theorem. *The monadic theory of the binary tree is decidable.*

Proof. The argument here is similar to that given for Theorem 2.1.5. □

2.3.6 Theorem (Complementation Theorem). *There is an algorithm that, given an alphabet Σ and a tree Σ-automaton A, constructs a tree Σ-automaton accepting exactly the Σ-trees rejected by A.*

Proof. Let V be a Σ-tree rejected by A. By the forgetful determinacy theorem, Pathfinder has a forgetful strategy f winning $\Gamma(A, V)$. If p is a position in $\Gamma(A, V)$, let Node(p) be the string of even letters in p. For example, if $p = albralcrcla$ then Node(p) $= lrlrl$. If p, q are two positions of odd length, Node(p) = Node(q), and A is in the same state in p, q (that is to say, p, q have the same last letter), then p, q define the same residual game. This allows us to code f by an appropriate valuation of the binary tree.

Let RECORDS be the set of words u in the alphabet of states of A such that every state appears at most once in u. Elements of RECORDS will be called records. Let Σ' be the set of functions assigning a letter l or r to each record. There is a Σ'-tree V' such that for every position p in $\Gamma(A, V)$ we have

$$f(p) = (V'(\text{Node } p))(\text{LAR } p).$$

Since f is winning, every path

$$e, d_1, d_1d_2, d_1d_2d_3, \ldots$$

through the binary tree $\{l, r\}^*$ satisfies the following condition:

(*) There are no sequences $s_0s_1s_2\ldots$ and $u_0u_1u_2\ldots$ such that $s_0d_1s_1d_2\ldots$ is a play with respect to f and $u_0, u_1, u_2, \ldots$ are corresponding last appearance records and $\{s\colon$ for every i there is $j > i$ with $s_j = s\}$ is a final collection of states.

Clearly (*) abbreviates a formula in the monadic language of one successor whose parameters code the path $e, d_1, d_1d_2, d_1d_2d_3, \ldots$ and the corresponding sequences $V(e), V(d_1), V(d_1d_2), \ldots$ and $V'(e), V'(d_1), V'(d_1d_2), \ldots$. By Theorem 2.2.4 there is a sequential automaton $A' = (S', T', s'_{\text{in}}, F')$ over the alphabet $(\Sigma \times \Sigma') \cup (\{l, r\} \times \Sigma \times \Sigma')$ that accepts a sequence

$$V(e)V'(e), d_1V(d_1)V'(d_1), d_2V(d_1d_2)V'(d_1d_2), \ldots$$

iff it satisfies (*).

Let $A'' = (S', T'', T'_{\text{in}}, F')$ be the deterministic tree $\Sigma \times \Sigma'$-automaton with $T''(s, d, \sigma\sigma') = T'(s, d\sigma\sigma')$ and $T''_{\text{in}}(\sigma\sigma') = T'(s'_{\text{in}}, \sigma\sigma')$. A'' mimics A' and accepts the $\Sigma \times \Sigma'$-tree $V \times V'$ given by V and V'. Finally, let $\bar{A}$ be the Σ-automaton that guesses V' and mimics A''. Note that each successor in the row A, A', A'', $\bar{A}$ is computable from the predecessor. Evidently $\bar{A}$ accepts V.

$\bar{A}$ is the desired Σ-automaton complementing A. For, suppose that $\bar{A}$ accepts a Σ-tree V. There is a Σ'-tree V' such that A'' accepts $V \times V'$. Then A' accepts every sequence

$$V(e)V'(e), d_1 V(d_1)V'(d_1), d_2 V(d_1 d_2)V'(d_1 d_2), \ldots .$$

Thus, every path e, d_1, $d_1 d_2, \ldots$ through the binary tree satisfies $(*)$, where f is the strategy for Pathfinder defined by

$$f(p) = (V'(\text{Node } p))(\text{LAR } p).$$

Evidently f is winning. Hence, A rejects V. ▯

3. The Model-Theoretic Decidability Technique

The most important tools for dealing with monadic theories are *composition theorems*. The term "composition" here means generalized products in the sense of Feferman–Vaught [1959]. The Feferman–Vaught theorem reduces the first-order theory of the given composition to the first-order theories of the parts (summands, factors) and the monadic (!) theory of the index structure. Monadic composition theorems reduce the monadic theory of the given composition to the monadic theory of the parts and the monadic theory of the index structure (see, for example, the monadic composition theorem for chains in Section 3.2). Thus, monadic composition theorems appear to be more natural. Moreover, the interplay of monadic theories opens absolutely new and unexpected approaches to the decision problem. One of these approaches is demonstrated in Section 3.3 by a model-theoretic proof of decidability of the monadic theory of ω. Limited by the size of this chapter, we have chosen in the present section to explain only an easy part of the model-theoretic technique for proving decidability of monadic theories and to make this exposition as comprehensible as possible. We hope that this discussion—selective though it may be—will assist the interested reader in examining the more comprehensive exposition to be found in either Shelah [1975e] or in the papers Gurevich [1979a] and Gurevich–Shelah [1979].

3.1. Bounded Theories

Recall that the prefix of a prenex first-order formula is a word in the alphabet $\{\forall, \exists\}$. Blocks of universal quantifiers alternate with blocks of existential quantifiers in a prefix. The *alternation type* of a prefix is the sequence of lengths of the quantifier blocks. For example the alternation type of both $\forall^3\exists^4\forall^5$ and $\exists^3\forall^4\exists^5$ is 3, 4, 5. Clearly, the alternation type of the empty prefix is the empty sequence. Letters ξ and η (without subscripts) will be used to denote alternation types. We use the symbol $^\frown$ to denote concatenation of sequences. Thus, if ξ is 3, 4, 5 then $\xi ^\frown 8$ is 3, 4, 5, 8.

Let L be a first-order language. For every n, indistinguishability by prenex sentences with prefix of length n gives an equivalence relation on structures for L. The n-step Ehrenfeucht game was introduced to provide a convenient sufficient condition for this equivalence relation to hold. Indistinguishability by prenex sentences with prefix of a given alternation type is also an equivalence relation on structures for L. We generalize Ehrenfeucht games to provide convenient sufficient conditions for these new equivalence relations to hold.

Proviso 1. *The vocabulary of L consists of finitely many relation symbols and individual constants.*

Let M and N be structures for L and ξ be an alternation type $\xi_1 \dots \xi_n$. The game $\xi - \Gamma(M, N)$ is played between players I and II in n steps. On the kth step, player I chooses a structure M or N and a tuple of ξ_k elements of the chosen structure; and, in response, player II chooses a tuple of ξ_k elements of the remaining structure. Let $a_1, \dots, a_m$ be the tuple of all $\xi_1 + \cdots + \xi_n$ elements chosen in M; the ξ_1-tuple of the first step concatenated with the ξ_2-tuple of the second step, etc. Let $b_1, \dots, b_m$ be the corresponding tuple of elements chosen in N. Player II wins if the quantifier-free type of $a_1, \dots, a_m$ in M coincides with the quantifier-free type of $b_1, \dots, b_m$ in N, otherwise player I wins.

3.1.1 Theorem. *If player* II *has a winning strategy in $\xi - \Gamma(M, N)$, then M and N are indistinguishable by prenex sentences with prefix of type ξ.*

Proof. Any distinguishing prenex sentence of type ξ gives a winning strategy for player I. ▯

We will say that L-structures M and N are *ξ-equivalent* if player II has a winning strategy in $\xi - \Gamma(M, N)$.

By induction on the length of ξ, we define the ξ-theory of an L-structure M with a tuple of additional elements. $0 - \mathrm{Th}(M, a_1, \dots, a_l)$ is the quantifier-free type of $a_1, \dots, a_l$ in M. If ξ is $\eta ^\frown k$ then $\xi - \mathrm{Th}(M, a_1, \dots, a_l)$ is the set of all $\eta - \mathrm{Th}(M, a_1, \dots, a_l, b_1, \dots, b_k)$ where $b_1, \dots, b_k \in M$.

3.1.2 Theorem. *Two structures for L are ξ-equivalent iff they have the same ξ-theory.*

Proof. The proof is simple and we will omit it here. ▯

The usual n-step Ehrenfeucht game corresponds to the case when ξ is a sequence of n ones. This sequence will be denoted 1^n. 1^n-equivalent structures are called usually n-equivalent. The 1^n-theory of a structure is called usually the n-theory.

It is important for us that our bounded theories—in particular, quantifier-free types—are finite objects. This explains Proviso 1. This proviso is, however, too restrictive for applications. Is there any way to have finite quantifier-free types in a situation when Proviso 1 fails? The answer is Yes. In fact, consider the first-order theory of boolean algebras. There are infinitely many terms in a given finite set of variables, but only finitely many of these terms are in disjunctive normal form and each term is equal to one in disjunctive normal form.

Proviso 2. *L may have function symbols but it has only finitely many relation symbols. T is a theory in L, T allows a definition of normal terms in such a way that:*

(i) *there are only finitely many normal terms for any given finite set of variables; and*
(ii) *every term is equal in T to a normal term (in the same variables).*

An atomic formula $P(\tau_1, \ldots, \tau_k)$ will be called *standard* if the terms $\tau_1, \ldots, \tau_k$ are normal. We identify the quantifier-free type of a tuple $(a_1, \ldots, a_l)$ in a model M of T with the set of standard atomic formulas $\phi(v_1, \ldots, v_l)$ such that $M \models \phi(a_1, \ldots, a_l)$. Now we can simply repeat the definition of ξ-theories. Proviso 2 will suffice for our purposes here. A more liberal proviso can be found in Gurevich [1979a].

3.1.3 Theorem. *T is decidable if there is an algorithm computing $\{\xi - \mathrm{Th}(M): M \models T\}$ from ξ. T is decidable if there is an algorithm computing $\{1^n - \mathrm{Th}(M): M \models T\}$ from n.*

Proof. As in the case of Theorem 3.1.2, the proof of this result is simple and will not be given here. ▯

Even if T is not decidable, there is often an algorithm which computes a box including $\{\xi - \mathrm{Th}(M): M \models T\}$ from ξ. We define these boxes by induction on the length of ξ. The 0-l-Box is

$$\{0 - \mathrm{Th}(M, a_1, \ldots, a_l): M \models T \text{ and } a_1, \ldots, a_l \in M\}.$$

If ξ is $\eta \wedge k$, then the ξ-l-Box is the power-set of the η-$(l + k)$-Box. We now turn our attention to

3.1.4 Proposition. *If $M \models T$ and $a_1, \ldots, a_l \in M$ then*

$$\xi - \mathrm{Th}(M, a_1, \ldots, a_l) \in \xi\text{-}l\text{-Box}.$$

Proof. Again, the argument for this result is obvious and is omitted here. ▯

It will be convenient to view elements of every ξ-l-Box as ordered in a standard manner. For example, the order may be lexicographical.

3.2. Monadic Composition Theorem for Chains

To fit this section into the framework of Section 3.1, we should say what the language L and the theory T are. Let BOOL be the first-order language of boolean algebras containing all the usual boolean operations and the equality predicate. L is the monadic language of order that is obtained from BOOL by adding the predicate $X \leq Y$. Every chain gives a standard model for L in the following way: We consider the boolean algebra of subsets and define $X \leq Y$ iff there are points $x \leq y$ with $X = \{x\}$ and $Y = \{y\}$. T is the monadic theory of order in L. In other words, T is simply the first-order theory of the described standard models for L. L and T satisfy Proviso 2 and we can freely use ξ-theories as well as other notions defined in Section 3.1.

Suppose that M is the lexicographic sum

$$L\Sigma\langle M_i : i \in I\rangle$$

of chains M_i with respect to a chain I. This means that M is itself a chain, the chains M_i are disjoint, the universe of M is the union of the universes of the chains M_i, and a point $x \in M_i$ precedes in M a point $y \in M_j$ iff $i < j$ or $i = j$ and $x < y$ in M_i.

Let X be an l-tuple $X_1, \ldots, X_l$ of subsets of M. For $i \in I$, the l-tuple $X_1 \cap M_i, \ldots, X_l \cap M_i$ will be denoted $X | M_i$. For every alternation type ξ and every $t \in \xi$-l-Box, let

$$P(\xi, X, t) = \{i : \xi - \mathrm{Th}(M_i, X | M_i) = t\}.$$

Furthermore, let $P(\xi, X)$ be the sequence $\langle P(\xi, X, t) : t \in \xi\text{-}l\text{-Box}\rangle$ that partitions I.

3.2.1 Lemma. *There is an algorithm that computes* $0 - \mathrm{Th}(M, X)$ *from* $0 - \mathrm{Th}(I, P(0, X))$ *when* I, M *and* X *are varied.*

Proof. Let $P = P(0, X)$ and $P_t = P(0, X, t)$. If τ is a boolean term in variables $v_1, \ldots, v_l$, then we let $\tau^* = \tau(X_1, \ldots, X_l)$, where the complements are taken in M. It is easy to check that

$$\tau^* \cap M_i = \tau(X_1 \cap M_i, \ldots, X_l \cap M_i),$$

where the complements are taken in M_i.

In order to compute $0 - \mathrm{Th}(M, X)$ it suffices to compute the truth values of statements $\sigma^* = \tau^*$ and $\sigma^* \leq \tau^*$, where σ and τ are in disjunctive normal form.

$\sigma^* = \tau^*$ iff $\sigma^* \cap M_i = \tau^* \cap M_i$, for every $i \in I$, iff for every $t \in 0$-l-Box, we have that either $P_t = 0$ or t implies $\sigma = \tau$. Given $0 - \mathrm{Th}(I, P)$, we can check the last necessary and sufficient condition.

Note that $\tau \le \tau$ means that τ is a singleton set. τ^* is a singleton iff there is $s \in 0$-l-Box such that P_s is a singleton, s implies $\tau \le \tau$ and for every other $t \in 0$-l-Box, we have that either $P_t = 0$ or t implies $\tau = 0$. Given $0 - \mathrm{Th}(I, P)$, we can check the necessary and sufficient condition.

Finally $\sigma^* \le \tau^*$ iff both σ^* and τ^* are singleton and either

(i) there are distinct $s, t \in 0$-l-Box such that $P_s \le P_t$ and s implies $\sigma \ne 0$, t implies $\tau \ne 0$; or
(ii) there is $t \in 0$-l-Box such that $P_t \ne 0$, and t implies $\sigma \le \tau$.

Given $0 - \mathrm{Th}(I, P)$, we can check the necessary and sufficient condition. □

3.2.2 Definition. If ξ is empty, then for every k, $H(\xi, k)$ is the empty alternation type. If ξ is $\eta \frown j$, then $H(\xi, k) = H(\eta, k + j) \frown p$, where p is the cardinality of η-$(k + j)$-Box.

3.2.3 Theorem. *There is an algorithm* COMP *that computes* $\xi - \mathrm{Th}(M, X)$ *from* $H(\xi, l)$-$\mathrm{Th}(I, P(\xi, X))$, *when* I, M, X *and* ξ *are varied.*

Proof. By induction on n, we construct algorithms COMP_n such that every COMP_n computes $\xi - \mathrm{Th}(M, X)$ from $H(\xi, l) - \mathrm{Th}(I, P(\xi, X))$, for every ξ of length n. The construction is uniform in n and results in the desired algorithm COMP.

The case $n = 0$ was treated in Lemma 3.2.1. Suppose that COMP_n is already constructed. Instead of defining COMP_{n+1} formally, we will simply explain how it works.

Let ξ be an alternation type of length n. $\xi \frown k - \mathrm{Th}(M, X)$ is the set

$$\mathrm{S1} = \{\xi - \mathrm{Th}(M, X \frown Y) \colon lh(Y) = k\},$$

where Y ranges over tuples of k subsets of M. COMP_n will compute S1 from

$$\mathrm{S2} = \{\eta - \mathrm{Th}(I, P(\xi, X \frown Y)) \colon lh(Y) = k\},$$

where $\eta = H(\xi, l + k)$. S2 is computable from

$$\mathrm{S3} = \{\eta - \mathrm{Th}(I, P(\xi \frown k, X), P(\xi, X \frown Y)) \colon lh(Y) = k\}.$$

From the other side, $H(\xi \frown k, l) - \mathrm{Th}(I, P(\xi \frown k, X))$ is the set

$$\mathrm{S4} = \{\eta - \mathrm{Th}(I, P(\xi \frown k, X) \frown Q) \colon lh(Q) = |\xi\text{-}(l + k)\text{-Box}|\},$$

where η is again $H(\xi, l + k)$. Evidently, S3 is included into S4. We give a verifiable necessary and sufficient condition for an element $u = \eta - \mathrm{Th}(I, P(\xi \frown k, X) \frown Q)$ of S4 to belong to S3:

The sequence

$$Q = \langle Q_t \colon t \in \xi\text{-}(l + k)\text{-Box}\rangle$$

partitions I, and $t \in s$ whenever Q_t meets $P(\xi \frown k, X, s)$.

The argument for necessity is obvious. To prove the sufficiency, suppose that u satisfies the condition. We need to find a tuple Y of k subsets of M such that $P(\xi, X \frown Y) = Q$. For every $i \in I$, there are $s \in \xi \frown k$-l-Box and $t \in \xi$-$(l + k)$-Box such that $i \in P(\xi \frown k, X, s) \cap Q_t$. Then $t \in s$; that is to say, $t \in \xi \frown k - \mathrm{Th}(M_i, X | M_i)$. Hence, $t = \xi - \mathrm{Th}(M_i, (X | M_i) \frown Y^i)$, for some tuple Y^i of k subsets of M_i. Now choose Y such that $Y | M_i = Y^i$, for $i \in I$. ▯

3.3. Monadic Theory of Countable Ordinals

3.3.1 Theorem. *There is an algorithm* PLUS *such that if M is the lexicographic sum $M_1 + M_2$ of chains M_1 and M_2 and if X is a tuple of subsets of M, then for every alternation type ξ,*

$$\xi - \mathrm{Th}(M, X) = \mathrm{PLUS}(\xi - \mathrm{Th}(M_1, X | M_1), \xi - \mathrm{Th}(M_2, X | M_2)).$$

Proof. Simply take $I = \langle 1, 2 \rangle$ in the composition theorem and the result follows. ▯

We write $t = t_1 + t_2$ if $t = \mathrm{PLUS}(t_1, t_2)$. The induced addition of bounded theories is obviously associative.

3.3.2 Theorem. *The monadic theory of finite chains is decidable.*

Proof. By Section 3.1, it suffices to show that $\{1^n - \mathrm{Th}(M)\colon M$ is a finite chain$\}$ is computable from n. Given n, we compute the 1^n-theory t_1 of singleton chains. We thus compute $t_2 = t_1 + t_1$, $t_3 = t_2 + t_1$, etc., stopping when we find $i < j$ with $t_i = t_j$. The set $\{t_1, \ldots, t_{j-1}\}$ is equal to $\{1^n - \mathrm{Th}(M)\colon M$ is finite$\}$. ▯

3.3.3 Theorem. *There is an algorithm* MULT *satisfying the following condition. Let M be the lexicographical sum of chains M_i with respect to a chain I, and let X be a tuple of l subsets of M. If $\xi - \mathrm{Th}(M_i, X | M_i) = s$ for every i and $\eta = H(\xi, l)$, then*

$$\xi - \mathrm{Th}(M, X) = \mathrm{MULT}(\eta - \mathrm{Th}(I), s).$$

Proof. The algorithm COMP computes $\xi - \mathrm{Th}(M, X)$ from $\eta - \mathrm{Th}(I, P(\xi, X))$ which is itself computable from $\eta - \mathrm{Th}(I)$ and s, because $P(\xi, X, s) = I$ and any other $P(\xi, X, t) = 0$. ▯

We write $s' = t \cdot s$ if $s' = \mathrm{MULT}(t, s)$.

3.3.4 Theorem. *The monadic theory of ω is decidable.*

Proof. By induction on n, we construct an algorithm f_n such that, given an alternation type ξ of length n and a natural number l, f_n computes $\{\xi - \mathrm{Th}(\omega, X)\colon X$ is an l-tuple of subsets of $\omega\}$. The construction is uniform in n and provides an algorithm which will subsume every f_n. By Section 3.1, we know that this is enough for decidability.

Case $n = 0$ is easy. Suppose that $n > 0$ and f_{n-1} is already constructed. Given ξ and l, we compute $\eta = H(\xi, l)$ which is equal to $\tilde{\eta} \frown k$, for some alternation type $\tilde{\eta}$ of length $n - 1$ and some k. Also, we compute

$$\begin{aligned} t = \eta - \mathrm{Th}(\omega) &= \{\tilde{\eta} - \mathrm{Th}(\omega, Y)\colon Y \text{ is a } k\text{-tuple of subsets of } \omega\} \\ &= f_{n-1}(\tilde{\eta}, k). \end{aligned}$$

Using the decision procedure for the monadic theory of finite chains, we compute $A = \{\xi - \mathrm{Th}(M, X)\colon M$ is a finite chain and X is an l-tuple of subsets of $M\}$. And, finally, using the algorithms PLUS and MULT, we compute $B = \{s_0 + t \cdot s\colon s_0, s \in A\}$.

Evidently, $B \subseteq C = \{\xi - \mathrm{Th}(\omega, X)\colon X$ is an l-tuple of subsets of $\omega\}$. We prove that $B = C$, which fact allows us to compute C.

Given an l-tuple X of subsets of ω color every non-empty interval $[i, j)$ of natural numbers by the "color" $\xi - \mathrm{Th}([i, j), X \mid [i, j))$. By the Ramsey theorem, there is an infinite sequence $0 < n_1 < n_2 < \cdots$ such that all intervals $[n_i, n_{i+1})$ have the same color s. If s_0 is the color of $[0, n_1)$, then $\xi - \mathrm{Th}(\omega, X) = s_0 + t \cdot s \in B$. ☐

3.3.5 Theorem. *The monadic theory of countable ordinals is decidable.*

Proof. We explain how to compute $\{1^n - \mathrm{Th}(\alpha)\colon \alpha$ is a countable ordinal$\}$ from a given number n. First, we use the algorithm of Theorem 3.3.4 to compute $t = \eta - \mathrm{Th}(\omega)$, where $\eta = H(1^n, 0)$. By Theorem 3.3.3 $1^n - \mathrm{Th}(\alpha \cdot \omega) = t \cdot (1^n - \mathrm{Th}(\alpha))$, for any α. Second, compute the minimal set S of 1^n-theories which contains the 1^n-theory of singleton chains and which is also closed under addition and under multiplication by t. It is easy to see that S is the desired $\{1^n - \mathrm{Th}(\alpha)\colon \alpha$ is a countable ordinal$\}$. ☐

4. *The Undecidability Technique*

The *monadic topology* of a topological space U is the first-order theory of the structure $\langle \mathrm{PS}(U), \subseteq, \mathrm{OPEN}\rangle$, where $\mathrm{PS}(U)$ is the power-set of U, $\subseteq$ is the usual inclusion and OPEN is the unary predicate "X is open." In this section, we will describe a proof of undecidability of the monadic topology of the Cantor discontinuum CD. The monadic topology of CD is easily interpretable in the monadic theory of the real line R. In this way, we get undecidability of the monadic theory of R. We could, of course, deal directly with the monadic theory of R—it would be practically the same proof. Undecidability of the monadic topology of CD seems to be even more mysterious and more difficult to prove.

In Section 4.1 we will give a rough idea how one can talk about natural numbers in the monadic topology of CD—explaining the details would require more space. However, the details can be found in Gurevich–Shelah [1982]. There is a serious restriction on how much we can say about natural numbers in the monadic topology of CD: true first-order arithmetic is not interpretable (in the usual sense of this word, for example Monk [1976]) in the monadic theory of R, see Gurevich–Shelah [1981a]. In Section 4.2, we show that whatever we can say about natural numbers in the monadic topology of CD is enough to reduce true first-order arithmetic to the monadic topology of CD. Actually, a much stronger result is proven in Section 4.2.

4.1. *How Can One Speak About Natural Numbers in the Monadic Topology of the Cantor Discontinuum?*

The idea is to slice a countable everywhere dense set D into everywhere dense slices $S_0, S_1, \ldots$ and to code this decomposition by parameters. First, we choose an everywhere subset D^0 of D such that $D - D^0$ is everywhere dense also. Then, we slice D in such a way that the sets $A_0 = S_0 \cap D^0$, $A_1 = S_1 \cap D^0$, $A_2 = S_2 \cap D^0, \ldots$ are disjoint as well as everywhere dense. We then prove that there is a parameter W such that a certain monadic formula $\phi(X)$ with parameters D, D^0, W defines the slices locally: that is, every S_n satisfies ϕ and if some X satisfies ϕ, then every non-empty open set G has a non-empty open subset H where X coincides with one of the slices S_n. We have not said anything about sets $S_0 - A_0$, $S_1 - A_1, \ldots$. They can be used to code additional information. In particular, a pairing function can be coded.

The coding described is best explained in Gurevich–Shelah [1982]. Here we can only summarize results of the coding in a convenient form. There are monadic topological formulas Premise($\bar{u}$), Share($\bar{u}$, v_0) and Pairing($\bar{u}$, v_0, v_1, v_2, v_3) which satisfy the following conditions. Both $\bar{u}$ and (v_0, v_1, v_2, v_3) are sequences of (set) variables. The formulas Premise, Share, and Pairing do not have any free variables except those shown. Premise($\bar{u}$) is satisfiable in CD. If t is a sequence of point sets and Premise(t) holds in CD then there is a sequence $\langle A_i : i < \omega \rangle$ of disjoint subsets of CD which satisfy the conditions C0–C2 below:

- C0. Each A_n is everywhere dense and each intersection $A_i \cap A_j$, with $i \neq j$, is empty.
- C1. Share(t, X) holds iff every non-empty open set G has a non-empty open subset H such that $X \cap H$ is equal to some $A_n \cap H$.

We will say that X is a *t-share* if Share(t, X) holds. We order the ordered pairs of natural numbers first by the maximum and then lexicographically:

$$(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (1, 2), (2, 1), \ldots.$$

Let P be the set of triples (i, j, k) of natural numbers such that (i, j) is the kth pair (when $(0, 0)$ is pair number 0).

- C2. Suppose that X, Y, Z are t-shares and G is a non-empty open set. Then, Pairing(t, X, Y, Z, G) holds iff, for every non-empty open $G_1 \subseteq G$, there is a triple $(i, j, k) \in P$ and a nonempty open $H \subseteq G_1$ with $X \cap H = A_i \cap H$, $Y \cap H = A_j \cap H$, $Z \cap H = A_k \cap H$.

Before we go on to discuss reduction, let us recall that an open subset G of a topological space is called *regular* if the interior of the closure of G coincides with G. The following propositions is well known.

4.1.1 Proposition. *The regular open subsets of any topological space U form a complete boolean algebra with:*

(i) $G \cdot H = G \cap H$;
(ii) $G + H = \text{Interior}(\text{Closure}(G \cup H))$;
(iii) $-G = \text{Interior}(U - G)$; *and*
(iv) $1 = U$, and $0 = \varnothing$.

4.2. Reduction

Models of ZFC, the Zermelo–Fraenkel set theory with the axiom of choice, will be called *worlds*. In this discussion we will work in a world V. By sets is meant elements of V. For every complete boolean algebra B (in the world V) a standard construction provides a B-valued world V^B (see Jech [1978]). If ϕ is a formula in the language of ZFC with possible parameters from V^B, then the boolean value of ϕ will be denoted as usual $\|\phi\|$. Some simple but useful facts about V^B are summarized in the following

4.2.1 Proposition. (a) *Suppose that $\{b_i: i \in I\}$ is an antichain in B (which means that $b_i \cdot b_j = 0$ for $i \neq j$). For every $\{\sigma_i \in V^B: i \in I\}$ there is $\sigma \in V^B$ such that $b_i \leq \|\sigma_i = \sigma\|$ for $i \in I$.*

(b) *Let $\psi(v)$ be a formula in the language of* ZFC *with exactly one free variable and perhaps some parameters from V^B, then there is $\sigma \in V^B$ such that $\|\psi(\sigma)\| = \|\exists v \psi(v)\|$.*

(c) *Let $\psi(v)$ be as above and $\tau \in V^B$. Suppose $\|\exists v(v \in \tau)\| = 1$, then there is $\sigma \in V^B$ such that $\|\sigma \in \tau\| = 1$, and $\|\psi(\sigma)\| = \|(\exists v \in \tau)\psi(v)\|$.*

Proof. For the proof of (a), see Lemma 18.5 in Jech [1978]. As to part (b), see Lemma 18.6 in Jech [1978]. Turning now to part (c), we let $b = \|(\exists v \in \tau)\psi(v)\|$. By part (b), there are σ_0 and σ_1 such that $\|\sigma_0 \in \tau\| = 1$ and $\|\sigma_1 \in \tau \text{ and } \psi(\sigma_1)\| = b$. Moreover, by part (a), there is σ such that $(-b) \leq \|\sigma = \sigma_0\| \leq \|\sigma \in \tau\|$, and then $b \leq \|\sigma = \sigma_1\| \leq \|\sigma \in \tau\| \cdot \|\psi(\sigma)\|$. σ is the desired element of V^B. □

In the remainder of this subsection B is the boolean algebra of regular open subsets of the Cantor discontinuum CD (in V). An element $\sigma \in V^B$ will be called a *quasi-element* (of ω) if $\|\sigma \in \omega\| = 1$. It will be called a *quasi-set* (of natural numbers) if $\|\sigma \subseteq \omega\| = 1$. Hereafter, we ignore the difference between an element of V and the canonical name for it in V^B.

Let t be a sequence of subsets of CD satisfying Premise(t). We will say that a t-share X *represents* a quasi-element σ if

$$\Sigma\{b \in B: X \cap b = A_n \cap b\} = \|\sigma = n\| \quad \text{for} \quad n < \omega.$$

Subsets of CD will be called *point sets*, and we will say that a point-set Y *represents* a quasi-set τ if

$$\Sigma\{b \in B: A_n \cap b \subseteq Y\} = \|n \in \tau\| \quad \text{for} \quad n < \omega.$$

4.2.2 Proposition. (a) *Every t-share represents some quasi-element, and every quasi-element is represented by some t-share.*

(b) *Suppose that t-shares X_0, X_1, X_2 represent quasi-elements $\sigma_0, \sigma_1, \sigma_2$. For every $b \in B$, Pairing (t, X_0, X_1, X_2, b) holds in CD iff $b \leq \|(\sigma_0, \sigma_1, \sigma_2) \in P\|$.*

(c) *Every point set represents some quasi-set, and every quasi-set is represented by some point set.*

(d) *Suppose that a t-share X represents a quasi-element σ, and a point set Y represents a quasi-set τ. Then*

$$\|\sigma \in \tau\| = \Sigma\{b \in B: X \cap b \subseteq Y\}.$$

Proof. (a) Given a t-share X let

$$b_n = \Sigma\{b \in B: X \cap b = A_n \cap b\} \quad \text{for} \quad n < \omega.$$

By condition C0, distinct regular open sets b_n are disjoint. Moreover, by condition C1, they partition CD. By Proposition 4.2.1, there is σ with $\|\sigma = n\| \geq b_n$, for all n. σ is the desired quasi-element. Conversely, if σ is a quasi-element, then the desired t-share is

$$X = \bigcup \{A_n \cap \|\sigma = n\|: n < \omega\}.$$

For the proof of part (b) we use condition C2.

Turning now to part (c), we see that if Y is a point set, then the desired quasi-set τ is a function from ω to B with

$$\tau(n) = \Sigma\{b \in B: A_n \cap b \subseteq Y\} \quad \text{for all} \quad n.$$

Conversely, if τ is a quasi-set, then the desired point set is

$$Y \doteq \bigcup \{A_n \cap \|n \in \tau\|: n < \omega\}.$$

We now consider part (d). To prove $\subseteq$, we will suppose that $0 < a \leq \|\sigma \in \tau\|$. It then suffices to show that there is $0 < b \leq a$ with $X \cap b \subseteq Y$. Since σ is a quasi-element and τ is a quasi-set, there are n and $0 < a_1 \leq a$ such that $a_1 \leq \|\sigma = n\|$ and $a_1 \leq \|n \in \tau\|$. Since X represents σ, there is $0 < a_2 \leq a_1$ such that $X \cap a_2 = A_n \cap a_2$. Since Y represents τ, there is $0 < b \leq a_2$ such that $A_n \cap b \subseteq Y$. Thus, $X \cap b \subseteq Y$.

To prove $\supseteq$, we will suppose that $a > 0$ and $X \cap a \subseteq Y$. It then suffices to show that there is $0 < b \leq a$ with $b \leq \|\sigma \in \tau\|$. Since σ is a quasi-element, there are n and $0 < a_1 \leq a$ with $a_1 \leq \|\sigma = n\|$. Since X represents σ, there is $0 < b \leq a_1$

such that $X \cap b = A_n \cap b$ and, therefore, $A_n \cap b \subseteq Y$. Since Y represents τ, we have $b \leq \|n \in \tau\|$. Thus, $b \leq \|\sigma \in \tau\|$. □

4.2.3 Theorem. *The full second-order theory of $\aleph_0$ in the world V^B is reducible to the monadic topology (in the world V) of the Cantor discontinuum. In other words, there is an algorithm (not depending on the choice of the ground world V) that assigns a sentence ϕ^* in the language of monadic topology to every second-order sentence ϕ in such a way that* CD $\models \phi^*$ *iff* $\|\omega \models \phi\| = 1$. □

This theorem tells us that the monadic topology of CD is very complicated. In particular, true first-order arithmetic is reducible to the monadic topology of CD. For, V and V^B share the same true first-order arithmetic. Moreover, there is an algorithm interpreting true first-order arithmetic in (and therefore reducing it to) the full second-order theory of $\aleph_0$ in any world. This algorithm, in conjunction with the algorithm of Theorem 4.2.3, reduces true first-order arithmetic to the monadic topology of CD.

Proof of Theorem 4.2.3. The algorithm of Proposition 1.3.1 interprets the full second-order V^B-theory of ω in the monadic V^B-theory of the structure (ω, P), where P is the pairing predicate defined in Section 4.1. Let L be the monadic language of (ω, P). We will view individual variables (respectively set variables) of L as variables ranging over quasi-elements (respectively quasi-sets). Thus, we view L as a sublanguage of the language of ZFC. If ϕ is a sentence that is an L-formula with parameters, we will write $\|\phi\|$ instead of $\|\omega \models \phi\|$.

Let t be a tuple of point sets such that Premise(t) holds in CD. By induction on L-formulas $\phi(u_1, \ldots, u_m, V_1, \ldots, V_n)$, we define a formula

$$(w \leq \|\phi(u_1, \ldots, u_m, V_1, \ldots, V_n)\|)_t$$

in the language of monadic topology in such a way that if t-shares $X_1, \ldots, X_m$ represent quasi-elements $\sigma_1, \ldots, \sigma_m$, and point sets $Y_1, \ldots, Y_n$ represent quasi-sets $\tau_1, \ldots, \tau_n$ and $b \in B$, then

$$(*) \qquad \text{CD} \models (b \leq \|\phi(X_1, \ldots, X_m, Y_1, \ldots, Y_n)\|)_t \quad \text{iff} \quad b \leq \|\phi(\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n)\|.$$

In the case $m = n = 0$, $b = 1$ we will have the desired:

$$\text{Premise}(t) \rightarrow (1 \leq \|\phi\|)_t \text{ holds in CD} \quad \text{iff} \quad \|\phi\| = 1.$$

Case 1. ϕ is $(u_0, u_1, u_2) \in P$. Let $(w \leq \|\phi\|)_t$ be Pairing(t, u_0, u_1, u_2, w), and use Proposition 4.2.2(b).

Case 2. ϕ is $u \in V$. Let $(w \leq \|\phi\|)_t$ be a formula saying that $u \cap w - V$ is nowhere dense, and use the result of Proposition 4.2.2(d).

Case 3. ϕ is ϕ_1 & ϕ_2. Set

$$(w \leq \|\phi\|)_t = (w \leq \|\phi_1\|)_t \,\&\, (w \leq \|\phi_2\|)_t.$$

Case 4. ϕ is $\sim\psi$. Let $(w \leq \|\phi\|)_t$ be a formula saying that there is no $0 < w' \leq w$ satisfying $(w' \leq \|\psi\|)_t$. To check $(*)$, we suppose for simplicity that ϕ is a sentence. Then $(b \leq \|\phi\|)_t$ holds iff there is no $0 < a \leq b$ with $a \leq \|\psi\|$ iff $b \leq \|\phi\|$.

Case 5. ϕ is $\exists u\psi(u)$. Let $(w \leq \|\phi\|)_t$ be a formula saying that there is a t-share u satisfying $(w \leq \|\psi(u)\|)_t$. To check $(*)$ assume for simplicity that ϕ is a sentence. We first suppose that $b \leq \|\phi\|$. By the results of Proposition 4.1.1(c), there is a quasi-element σ with $\|\psi(\sigma)\| = \|\phi\| \geq b$. If a t-share X represents σ, then by the induction hypothesis $(b \leq \|\psi(X)\|)_t$ holds. Hence, $(b \leq \|\phi\|)_t$ holds. Next, we suppose that some t-share X satisfies $(b \leq \|\psi(X)\|)_t$. It represents some quasi-element σ. By the induction hypothesis, $b \leq \|\psi(\sigma)\|$. Hence, we have $b \leq \|\phi\|$.

Case 6. ϕ is $\exists V\psi(V)$. Let $(w \leq \|\phi\|)_t$ be a formula asserting that there is a point set V which satisfies $(w \leq \|\psi(V)\|)_t$. To check $(*)$ in this situation is similar to the task of checking in Case 5. $\square$

5. *Historical Remarks and Further Results*

We will first review very briefly the history of the method of automata and games. We will also mention delimiting undecidability results and some other closely related results obtained by model-theoretic methods. In Section 5.2 we will, very briefly review the history of the model-theoretic methods used to deal with monadic theories. Some later results use model-theoretic methods as well as the method of automata and games. It seems to make no real sense to divide the two approaches too sharply, however.

5.1. *Emphasizing the Method of Automata and Games*

Church [1963] gave "a summary of recent work in the application of mathematical logic to finite automata." Exploring connections between logic and finite automata proved fruitful indeed; but the most interesting applications appeared to be applications of finite automata to the decision problems for monadic second-order theories. Decidability of the monadic theory of finite chains could have been the first, the most natural and the easiest example—but it was not. I only just made up this particular application and inserted it into Section 2 for expository purposes. Arithmetic was too much on the minds of those who first explored the connections between logic and finite automata. The first results were related to the weak monadic theory of ω with the successor relation. This theory was called weak second-order arithmetic. (Let us recall that the weak monadic theory of a structure is the theory of that structure in the monadic second-order language when the set variables range over finite sets of elements.) We will not speak about weak monadic theories here. A survey of the results in this area can be found in Thatcher–Wright [1968]. Let us note merely that the game technique given in Section 2 can be used to give an alternative (and relatively simple) proof of decidability of the weak monadic theory of the binary tree. We should also note that the decidability schema of Section 2, a schema that is based on correspondence between monadic formulas and automata, had already taken shape in the work on weak monadic theories.

Decidability of the monadic theory S1S of ω with the successor relation was proved by Büchi [1962]. He established a correspondence between S1S formulas and Büchi automata. These machines are ordinary finite automata $A = (S, T, s_{\text{in}}, F)$ with $F \subseteq S$ that work on sequences. A is said to accept a sequence $\sigma_1\sigma_2 \ldots$ in the input alphabet of A if there is a run $s_1 s_2 \ldots$ of A on the given sequence (which means, of course, that $(s_{\text{in}}, \sigma_1, s_1) \in T$ and every $(s_i, \sigma_{i+1}, s_{i+1}) \in T$) such that for every i there is $j > i$ with $s_j \in F$. Büchi also solved the emptiness problem for Büchi automata. Unfortunately, a non-deterministic Büchi automaton may be not equivalent to any deterministic Büchi automaton, and Büchi used the Ramsey theorem to solve the complementation problem for Büchi automata. Our sequential automata were introduced by Muller [1963] in order to prove Theorem 2.2.1. However, the first correct proof of that theorem was published by McNaughton [1966]. Simplifications of McNaughton's proof can be found in Rabin [1970], Choueka [1974], Thomas [1981].

Decidability of the monadic theory S2S of the binary tree with two successor relations was proven by Rabin [1969]. He established a correspondence between S2S formulas and Rabin automata that are somewhat different from our tree automata, and his proof of the complementation theorem is an extremely difficult induction on countable ordinals. He used the same technique to solve the emptiness problem for Rabin automata, although Rackoff [1972] found a simple reduction of the emptiness problem for Rabin automata to the emptiness problem for automata on finite binary trees. Our simple proof of the decidability of S2S follows Gurevich and Harrington [1982].

The idea of using games had been exploited earlier however. Büchi–Landweber [1969] used a strong determinacy of more special games to prove the following: Suppose that a sentence $\forall X\ \exists Y \phi(X, Y)$ holds in S1S where X, Y are tuples of variables. Then there is a deterministic sequential automaton which outputs an appropriate output Y when reading X. In particular, there is an S1S formula $\phi^*(X, Y)$ uniformizing ϕ; that is, ϕ^* implies ϕ in S1S and, for every X, there is a unique Y such that $\phi^*(X, Y)$ holds in S1S. Büchi [1977] sketched a reduction of the complementation problem for Rabin automata to a strong determinancy for boolean-F_σ games. This determinacy result was proven independently in Gurevich–Harrington [1982] and in the manuscript Büchi [1981]. The latter solution, however, is much more complicated (and it still uses an induction on countable ordinals).

Let me add a few words about Rabin's uniformization problem for S2S. Suppose that a sentence $\forall X\ \exists Y \phi(X, Y)$ holds in S2S (where for the sake of simplicity, X, Y are just single variables). Is there an S2S formula $\phi^*(X, Y)$ such that ϕ^* implies ϕ in S2S and, for every X, there is a unique Y such that $\phi^*(X, Y)$ holds in S2S? Using model-theoretic methods and forcing Gurevich–Shelah [1983b] solved this problem negatively. Their counterexample $\phi(X, Y)$ asserts that if X is not empty, then Y is a singleton subset of X.

Rabin [1969] proved the decidability of many interesting theories by interpreting them in S2S. Among those theories we find the monadic theory of countable chains and the theory of the real line with quantification over countable sets. More direct model-theoretic proofs of these two results as well as delimiting undecidability results can be found in Gurevich–Shelah [1979]. For more on this the reader may also see Section 5.2. Finally, we note that Rabin [1969] also proved that S2S allows us to quantify over F_σ subsets of (infinite) branches of the binary tree. (Basic open sets of the topology in question are sets of branches through a given node.)

Open Question. *If we augment the language of* S2S *by allowing quantification of arbitrary Borel sets over branches, is the resulting theory of the binary tree in the augmented language decidable?*

Shelah [1975e] states the reducibility of the monadic theory of a tree of height ω with a given structure S on the successors of each node to the monadic theory of S. The details appear in Stupp [1975]. Their proof uses Rabin's technique. The game technique of Gurevich–Harrington [1982] gives the generalized result fairly easily.

Büchi [1973] used automata to prove decidability of the monadic theory of ω_1 (with the order). See also Litman [1972], Büchi–Siefkes [1973], Büchi–Zaiontz [1983] for additional results about monadic theories of ordinals of cardinality at most $\aleph_1$. There is a good reason why these results cannot be generalized to ω_2. Using model-theoretic methods and assuming the existence of a weakly compact cardinal, Gurevich, Magidor, and Shelah [1983] prove:

(i) for any given $S \subseteq \omega$, there is a forcing extension of the given set-theoretic world, where the monadic theory of ω_2 has the Turing degree of S; and
(ii) there is a forcing extension of the given set-theoretic world, where the monadic theory of ω_2 and the full second-order theory of ω_2 are reducible each to the other.

5.2. *Model-Theoretic Methods*

The paper Shelah [1975e] represented a breakthrough in the study of monadic theories of chains. Shelah developed the model-theoretic decidability method, which we illustrated in Section 3, and proved all known decidability results about monadic theories of chains in a uniform way. Assuming the continuum hypothesis, he reduced true first-order arithmetic to the monadic theory of the real line. This was the first undecidability result in the area.

Shelah's decidability method was rooted in achievements of his predecessors. In this connection, let me mention Feferman–Vaught [1959], Ehrenfeucht [1961], and Läuchli [1968]. Working on well-orderings, Shelah used ideas of Büchi and

Rabin. For more on this, see the references in Shelah [1975e]. A detailed version of the model-theoretic decidability method, a version which prepared the ground for stronger results, is given in Gurevich [1979a]. Shelah's undecidability method was absolutely new. Actually, he wanted to prove decidability of the monadic theory of the real line. He was developing and sharpening the decidability method to achieve this goal when he discovered the undecidability. Later, he reduced true first-order arithmetic to the monadic theory of the real line just in ZFC, without making any additional set-theoretic assumptions. See Gurevich–Shelah [1982] in this connection.

Sometimes model-theoretic analysis is less informative than is the automaton-theoretic. For example, the decision procedure in Section 2 for the monadic theory of ω gives more than the corresponding decision procedure in Section 3: It establishes the correspondence between monadic formulas and deterministic sequential automata. In many other cases, however, the model-theoretic analysis is more informative. For example, Shelah answered negatively a question posed by Rabin, a question asking whether or not countable orders can be characterized in the monadic theory of chains.

Let us examine the monadic theory of countable chains a bit further. Shelah [1975e] conjectured that the monadic theory of countable chains can be finitely axiomatizable in the monadic theory of chains. However, Gurevich [1977b] refuted this conjecture. He provided a certain axiomatization of the monadic theory of countable chains. A chain is *short* if it embeds neither ω_1 nor ω_1^*, where ω_1^* is the dual of ω_1. A chain without jumps (that is, a densely ordered chain) is *perfunctorily n-modest* if for all everywhere dense subsets $X_1, \ldots, X_n$, there is a perfect subset Y without jumps such that $Y \subseteq X_1 \cup \cdots \cup X_n$ and every $X_i \cap Y$ is dense in Y. A chain is *n-modest* if all its subchains without jumps are perfunctorily n-modest. A chain is *modest* if it is n-modest, for every n. It appears that a chain is monadically equivalent to a countable chain iff it is short and modest. Rabin [1969] proved decidability of the monadic theory of countable chains. Thus, the monadic theory of short modest chains is decidable. Gurevich–Shelah [1979] proved directly decidability of short modest chains.

The situation is very different for non-modest chains. Assuming the continuum hypothesis, Gurevich–Shelah [1979] reduced true first-order arithmetic to the monadic theory of any nonmodest chain. The use of the continuum hypothesis was removed in Gurevich–Shelah [1982]. The reader may also consult Gurevich–Shelah [1979] for a model-theoretic analysis of the theory of the real line with quantification over countable subsets.

In order to discuss undecidability results, we need to clarify the terminology. A *reduction* of a theory T to a theory T^* is an algorithm associating a sentence ϕ^* in the language of T^* with each sentence ϕ in the language of T in such a way that ϕ^* holds in T^* iff ϕ holds in T. An *interpretation* of one theory in another is a special case of reduction when models of T are defined inside models of T^*. An exact definition of interpretation can be found in Monk [1976] for example.

As we mentioned above, Shelah [1975e] reduced true first-order arithmetic to the monadic theory of the real line. In Section 4 we did not say much about the undecidability method of Shelah [1975e]. This method was augmented in Gurevich

[1977b] by a technique of towers, a technique that has been exploited extensively in subsequent papers. Confirming Shelah's conjecture, Gurevich [1979b] reduced true third-order arithmetic to the monadic theory of the real line (in fact, to the monadic theory of any short non-modest chain) in Gödel's constructive universe. The converse reduction is obvious. Only during the Jerusalem Logic Year 1980–81 we—Saharon Shelah and I—realized that our reductions are really a kind of interpretation of (in terms of Section 4) theories in the "next world" V^B in theories in "this world" V. Subsuming all mentioned undecidability results, Gurevich-Shelah [1981a] managed:

(i) reduce true second-order arithmetic in V^B to the monadic V-theory of any short non-modest chain; and also
(ii) to reduce true third-order arithmetic in V^B to the monadic V-theory of any short non-modest chain if the continuum hypothesis holds in V.

In contrast to this, Gurevich-Shelah [1981a] proved that true first-order arithmetic is not interpretable in the monadic theory of the real line.

Gurevich-Shelah [1983a] reduce true second-order logic to the monadic theory of (linear) order under very weak set-theoretical assumptions. This gives the complexity of the monadic theory of order. It does not mean, however, that the monadic theory of order is as un-manageable as second-order logic. From a model-theoretical point of view, there is an enormous difference between these two theories (reflected somewhat in different Löwenheim and Hanf numbers). This topic is, however, beyond the scope of this chapter and the reader may see Chapter 12 in this connection.

A few words about topology. Grzegorczyk [1951] introduced the monadic topology (see Section 4) and interpreted (in a simple and natural way) true first-order arithmetic in the monadic topology of the Euclidean plane. It does not take much more sophistication to verify that the monadic topology of the Euclidean plane and true third-order arithmetic are interpretable, each in the other. For more on this, the reader may see Gurevich [1980]. Grzegorczyk's question about the decision problem for the monadic topology of the real line was, however, long open. Reading the paper Shelah [1975e], I noted that Shelah had solved negatively the question of Grzegorczyk under the continuum hypothesis. Several papers—especially Gurevich-Shelah [1981c]—give undecidability results about the monadic topology. In particular, all mentioned above undecidability results about the monadic theory of the real line apply to the monadic topology of the Cantor discontinuum. For a positive result on monadic topology see Gurevich [1982].

Gurevich-Shelah [1981b] use both model-theoretic methods and the method of automata and games to construct a decision procedure for the theory of trees (all trees, not necessarily well-founded) with quantification over maximal branches.

Finally, let us mention some results that are not directly related to decision problems. Gurevich [1977b] proved (thus refuting Shelah's conjecture) that the predicate "X is countable" is expressible in the monadic theory of the real line if the continuum hypothesis holds. Gurevich [1979b] also proved (and thus partly refuted and partly confirmed Shelah's conjectures) that the monadic theory of the real line can be finitely axiomatizable (in the monadic theory of chains) and categorical under natural set-theoretic assumptions. By "Shelah's conjectures" here, we mean the collection of conjectures that are given in Shelah [1975e]. Almost all of these conjectures have been decided by now, and a majority of those decided are true. Thus, the program sketched in Shelah [1975e] is essentially fulfilled. Moreover, I have an impression that an important and natural phase in the study of monadic second-order theories is now completed.

Part E

Logics of Topology and Analysis

This part of the book is devoted to logics which presuppose different kinds of structures than those underlying first-order logic and its extensions so far dealt with in Parts B, C and D.

Chapter XIV is about logics where the underlying structure is a probability space, a structure with a countably additive probability measure. In addition to the usual propositional operations, the basic form of quantification is given by allowing formulas

$$(Px \geq r)\phi(x),$$

which means that the probability of the set $\{x: \phi(x)\}$ is at least r. Structures take the form of probability spaces with countably additive measures. To have a successful theory here a number of changes in perspective must be made. In the first place, one must arrange things so that all definable sets are measurable. As a result, the logics considered here are not closed under the usual quantifiers $\forall$ and $\exists$. Consequently, these logics do not contain first-order logic, nor do they satisfy all the assumptions on logics given in the general definition. They also have model-theoretic properties that have no first-order analogue, like the Law of Large Numbers.

While the lack of ordinary quantifiers entail a loss in expressive power, we can compensate for that, in part, by the use of countable conjunctions and disjunctions, as in $\mathscr{L}_{\omega_1\omega}$, since such operations preserve measurability (due to countable additivity of probability measures). Expressed in terms of admissible sets, one finds the appropriate forms of completeness and compactness results. Interestingly, there is also an analogue of the Robinson consistency property, which fails for $\mathscr{L}_{\omega_1\omega}$. This chapter should be read after reading the relevant sections of Chapter VIII.

In his retiring address as president of the Association for Symbolic Logic in 1972, Abraham Robinson (Robinson [1973]) asked what logic for topological structures was the analogue of first-order logic for algebraic structures. Chapter XV presents the work that has gone into this problem. Obviously the structures to be considered are of the form $(\mathfrak{M}, \tau)$, where τ is a topology on the domain of the first-order structure $\mathfrak{M}$. Examples include topological space, topological groups, and topological fields. It has taken a lot of effort to arrive at what appears to be the right answer to Robinson's question. The chapter begins by describing three of the logics for such structures that have been studied: $\mathscr{L}(I^n)$, logic with the interior operator, $\mathscr{L}^{\text{mon}}$, a version of monadic second-order logic but where the set quantification is taken to be only over open sets, and a sublogic of this, $\mathscr{L}^t$, where such second-order quantifiers are restricted in a certain way. The logic $\mathscr{L}^t$ is stronger than $\mathscr{L}(I^n)$ but weaker than $\mathscr{L}^{\text{mon}}$. Chapter XV presents results and arguments to support the claim that $\mathscr{L}^t$ is the solution to Robinson's problem by being the "right" analogue of first-order logic for topological logic. Unlike $\mathscr{L}^{\text{mon}}$, $\mathscr{L}^t$ (and *a fortiori*, its sublogic $\mathscr{L}(I^n)$) is compact, has the Löwenheim–Skolem property and has a completeness theorem. However, unlike $\mathscr{L}(I^n)$, $\mathscr{L}^t$ allows one to express continuity, surely a desirable property for the logic of topology.

The logic $\mathscr{L}^t$ also satisfies the interpolation property, a result which leads to a persuasive analogue of Lindstrom's theorem: *$\mathscr{L}^t$ is the strongest logic for topological structures which is compact and has the Löwenheim–Skolem property.* The chapter concludes with some applications of the theory to specific topological theories, including the theory of abelian Hausdorff groups, the theory of the complex numbers as a topological field and topological vector spaces. (The reader may find this chapter is rather dense, but it repays study.)

Chapter XVI presents some previously unpublished work on the logic of Borel structures, due largely to Harvey Friedman. Friedman's basic idea is that while there are some very pathological sets and relations of real numbers, the collection of Borel sets and relations is much better behaved. Why not restrict attention to structures on the reals that are Borel and study the resulting logic? A *Borel structure* is one whose domain is a Borel subset of R and whose relations and functions are all Borel. Given a logic $\mathscr{L}$, a structure is *totally Borel* for $\mathscr{L}$ if all relations definable using $\mathscr{L}$-formulas are Borel.

Thus, whereas an essential feature of the other logics discussed in this part is the structures they consider are richer, the logics studied here are richer in that their structures are constrained to be totally Borel. The chapter applies the notion to two different logics, $\mathscr{L}(Q, Q_m)$ and $\mathscr{L}(Q, Q_c)$ where Q is "there exist uncountably many," Q_m is "there exist a set not of measure 0" and Q_c is "there is a set which is not meager." For example,

$$Q_m x Q_m y \phi(x, y) \leftrightarrow Q_m y Q_m x \phi(x, y)$$

expresses a version of the Fubini theorem, which is true of all totally Borel structures for $\mathscr{L}(Q_m)$. The main results of Chapter XVI are abstract and concrete completeness theorems for the logics $\mathscr{L}(Q, Q_m)$ and $\mathscr{L}(Q, Q_c)$ relative to the collection of totally Borel structures. These logics are less well known but seem very interesting in their potential applications and because they represent a really different direction in the study of extended logics.

Chapter XIV
Probability Quantifiers

by H. J. KEISLER

In this chapter we develop logics appropriate for probability structures, these being first-order structures endowed with a probability measure on the universe. We consider logics having the property that in every probability structure, every definable set is measurable. The price for this is high: The logics do not have the ordinary quantifiers $\forall x$ and $\exists x$. Instead, they have probability quantifiers and countable conjuctions. The main probability logic $L_{\mathbb{A}P}$ satisfies the Barwise completeness and compactness theorems, but does not satisfy finitary compactness. In spite of this, however, this logic does possess the Robinson consistency property. And it also has model-theoretic properties with no first-order analog, such as the law of large numbers, a principle that is presented in Section 3. In Section 4 we will study logics for richer structures with conditional expectations. This development will lead to a model theory which is closely tied to current research in stochastic processes and which has applications to stochastic differential equations.

1. Logic with Probability Quantifiers

In this section we will introduce the logic $L_{\mathbb{A}P}$, which is quite similar to the infinitary logic $L_{\mathbb{A}}$ except that instead of the ordinary quantifiers $(\forall x)$ and $(\exists x)$, the logic $L_{\mathbb{A}P}$ possesses the probability quantifiers $(Px \geq r)$. A structure for this logic is a first-order structure with a (countably additive) probability measure on the universe, such that each relation is measurable. The formula

$$(Px \geq r)\varphi(x)$$

means that the set $\{x \mid \varphi(x)\}$ has probability at least r. Axioms and rules of inference appropriate to our investigation will be presented in this section. The following sections will then examine the subject in more detail.

1.1. Syntax

1.1.1 Convention. We will assume throughout this chapter that $\mathbb{A}$ is an admissible set (possibly with urelements) such that $\omega \in \mathbb{A}$, and each $a \in \mathbb{A}$ is countable (that is, $\mathbb{A} \subseteq \mathrm{HC}$, where HC is the set of hereditarily countable sets).

We refer the reader to Chapter VIII of this volume for a detailed treatment of admissible sets and the infinitary logic $L_{\mathbb{A}}$. Briefly, however, we note that the set of formulas of $L_{\mathbb{A}}$ is the set of all expressions in $\mathbb{A}$ that are built from atomic formulas using negation $\neg$, finite or infinite conjuction, and the quantifier $(\forall x)$.

1.1.2 Definition. We will assume throughout our exposition that L is a countable $\mathbb{A}$-recursive set of finitary relation and constant symbols (no function symbols). The logic $L_{\mathbb{A}P}$ has the following logical symbols:

(a) A countable list of individual variables v_n, for $n \in \mathbb{N}$.
(b) The connectives $\neg$ and $\bigwedge$.
(c) The quantifiers $(P\bar{x} \geq r)$, where $\bar{x} = \langle x_1, \ldots, x_n \rangle$ is a tuple of distinct variables and $r \in \mathbb{A} \cap [0, 1]$.
(d) The equality symbol $=$ (optional).

1.1.3 Definition. The set of *formulas* of $L_{\mathbb{A}P}$ is the least set such that:

(a) Each atomic formula of first-order logic is a formula of $L_{\mathbb{A}P}$.
(b) If φ is a formula of $L_{\mathbb{A}P}$, then $\neg\varphi$ is a formula of $L_{\mathbb{A}P}$.
(c) If $\Phi \in \mathbb{A}$ is a set of formulas of $L_{\mathbb{A}P}$ with only finitely many free variables, then $\Lambda\Phi$ is a formula of $L_{\mathbb{A}P}$;
(d) If φ is a formula of $L_{\mathbb{A}P}$ and $(P\bar{x} \geq r)$ is a quantifier of $L_{\mathbb{A}P}$, then $(P\bar{x} \geq r)\varphi$ is a formula of $L_{\mathbb{A}P}$.

It is understood that the formulas are constructed set theoretically so that $L_{\mathbb{A}P} \subseteq \mathbb{A}$. We denote $L_{\mathbb{A}P}$ where $\mathbb{A} = \mathrm{HC}$ by $L_{\omega_1 P}$. Thus,

$$L_{\mathbb{A}P} = \mathbb{A} \cap L_{\omega_1 P}.$$

The notions of free and bound variables are defined as usual, with the quantifier $(P\bar{x} \geq r)$ binding all the variables in the tuple $\bar{x}$.

The equality relation plays only a minor role in the logic $L_{\mathbb{A}P}$, a fact which stems from the absence of the universal quantifier and of function symbols.

1.1.4 Definition. It is convenient ot use the following abbreviations in $L_{\mathbb{A}P}$:

(i) $(P\bar{x} < r)\varphi$ for $\neg (P\bar{x} \geq r)\varphi$.
(ii) $(P\bar{x} \leq r)\varphi$ for $(P\bar{x} \geq 1 - r) \neg \varphi$.
(iii) $(P\bar{x} > r)\varphi$ for $\neg (P\bar{x} \geq 1 - r) \neg \varphi$.
(iv) $\bigvee_{\varphi \in \Phi} \varphi$ for $\neg \bigwedge_{\varphi \in \Phi} \neg \varphi$.
(v) The finitary connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ are defined as usual.

The quantifier $(P\bar{x} \geq 1)$ is a weak analog of $(\forall\bar{x})$, while $(P\bar{x} > 0)$ is a strong analog of $(\exists\bar{x})$. In principle, it would be possible to make do with the one-variable probability quantifier $(Px \geq r)$ alone and introduce the n-variable quantifier $(P\bar{x} \geq r)$ as an abbreviation. However, this abbreviation would be quite complicated, and it is simpler to include $(P\bar{x} \geq r)$ explicitly in the language.

1.2. Probability Models

We will begin with some very basic notions from probability theory. First, a *finitely additive probability space* is a triple $\langle M, S, \mu \rangle$ where S is a field of subsets of M, $\mu: S \to [0, 1]$, $\mu(M) = 1$, and for $X, Y \in S$,

$$\mu(X \cup Y) = \mu(X - Y) + \mu(Y - X) + \mu(X \cap Y).$$

The sets $X \in S$ are *μ-measurable*, and μ is called a *finitely additive probability measure* on M. Next, we say that $\langle M, S, \mu \rangle$ is a *probability space* if, in addition, S is a σ-field and μ is countably additive; that is, whenever $X_0 \subseteq X_1 \subseteq \cdots$ in S, then

$$\mu\left(\bigcup_n X_n\right) = \lim_n \mu(X_n).$$

In this case, μ is said to be a *probability measure* on M. We emphasize that "probability measure" without an adjective will mean "countably additive probability measure."

A set X is said to be a *null set* of μ if there is a $Y \supseteq X$ with $\mu(Y) = 0$. The *product* of two probability spaces $\langle M, S, \mu \rangle$ and $\langle N, T, \nu \rangle$ is the probability space

$$\langle M \times N, S \otimes T, \mu \otimes \nu \rangle,$$

where $S \otimes T$ is the σ-algebra generated by the set of measurable rectangles $X \times Y$, with $X \in S$, $Y \in T$ and where

$$(\mu \otimes \nu)(X \times Y) = \mu(X) \cdot \nu(Y).$$

The n-fold product space is denoted by (M^n, S^n, μ^n).

In general, the diagonal set

$$\{\langle x, x \rangle : x \in M\}$$

is not μ^2-measurable. However, if each singleton is measurable, then there is a canonical way to extend the product measure to the diagonal. In the case that every singleton has measure zero, the diagonal is given measure zero. In general, however, only countably many singletons have positive measure, and the measure of the diagonal is the sum of the squares of the measures of the singletons.

1.2.1 Definition. Let $\langle M, S, \mu \rangle$ be a probability space such that each singleton is measurable. Then, for each $n \in \mathbb{N}$, we have that $\langle M^n, S^{(n)}, \mu^{(n)} \rangle$ is the *probability space* such that $S^{(n)}$ is the σ-algebra generated by the measurable rectangles and the diagonal sets

$$D_{ij} = \{\bar{x} \in M^n : x_i = x_j\},$$

and $\mu^{(n)}$ is the unique extension of μ^n to $S^{(n)}$ such that

$$\mu^{(n)}(D_{ij}) = \sum_{x \in M} \mu(\{x\})^2.$$

In the sequel, we will use $A \Delta B$ for the symmetric difference of the sets A and B. The above ideas clear, we will now consider

1.2.2 Proposition. *If $\langle M, S, \mu \rangle$ is a probability space such that each singleton is measurable, then the measure $\mu^{(n)}$ on $S^{(n)}$ given in Definition* 1.2.1 *exists and is unique. Moreover, for each set $X \in S^{(n)}$ there is a μ^n-measurable set U such that $\mu^{(n)}(X \Delta U) = 0$.*

Proof. We will give the proof in the case $n = 2$. Here, $S^{(2)}$ is the set of all sets $X \subseteq M^2$ of the form

$$X = (Y \cap D_{12}) \cup (Z - D_{12}), \qquad Y, Z \in S^2.$$

Let $\nu: S^{(2)} \to [0, 1]$ be defined by

$$\nu(X) = \sum_{\langle x, x\rangle \in Y} \mu(\{x\})^2 + \mu^2(Z) - \sum_{\langle x, x\rangle \in Z} \mu(\{x\})^2.$$

Then $\nu = \mu^{(2)}$ is the unique countably additive probability measure on $S^{(2)}$ which extends μ^2 and satisfies

$$\text{(1)} \qquad \nu(D_{12}) = \sum_{x \in M} \mu(\{x\})^2.$$

Finally, let $E = \{\langle x, x\rangle \in D_{12}: \mu(\{x\}) > 0\}$ and let $U = (Y \cap E) \cup (Z - E)$. Since E is countable and each singleton is measurable, E and hence U are μ^2-measurable. Also, $\nu(D_{12} - E) = 0$, and $X \Delta U \subseteq D_{12} - E$, whence, we have that $\nu(X \Delta U) = 0$. □

We are now ready to define a probability structure for L, where L is a set of n_i-placed relation symbols R_i, for $i \in I$, and constant symbols c_j, for $j \in J$.

1.2.3 Definition. A *probability structure for L* is a structure

$$\mathscr{M} = \langle M, R_i^{\mathscr{M}}, c_j^{\mathscr{M}}, \mu\rangle_{i \in I, j \in J},$$

where μ is a (countably additive) probability measure on M such that each singleton is measurable, each $R_i^{\mathscr{M}}$ is $\mu^{(n_i)}$-measurable, and each $c_j^{\mathscr{M}} \in M$.

1.2.4 Theorem. Let $\mathscr{M}$ be a probability structure for L. The *satisfaction relation* $\mathscr{M} \models \varphi[\bar{a}]$, for $\varphi(\bar{x}) \in L_{\mathbb{A}P}$ and $\bar{a}$ in M, is defined recursively exactly as for $L_{\mathbb{A}}$ except for the following quantifier clause:

$$\mathscr{M} \models (P\bar{y} \geq r)\varphi(\bar{x}, \bar{y})[\bar{a}] \quad \text{iff} \quad \{\bar{b} \in M^n: \mathscr{M} \models \varphi[\bar{a}, \bar{b}]\}$$
$$\text{is } \mu^{(n)}\text{-measurable and has measure at least } r.$$

Moreover, $\mathscr{M}$ is a *model* of a sentence φ if $\mathscr{M} \models \varphi$.

1.2.5 Theorem. *For each probability structure $\mathscr{M}$, formula $\varphi(\bar{x}, \bar{y}) \in L_{\mathbb{A}P}$, and tuple $\bar{a}$ in M, the set $\{\bar{b} \in M^n : \mathscr{M} \models \varphi[\bar{a}, \bar{b}]\}$ is $\mu^{(n)}$-measurable.* $\square$

This theorem is needed to show that the satisfaction relation has the intended meaning for $L_{\mathbb{A}P}$, and its proof follows easily by induction from a "diagonal" form of the Fubini theorem. A function $f: M \to \mathbb{R}$ is *μ-measurable* if $f^{-1}(-\infty, r]$ is μ-measurable for each $r \in \mathbb{R}$.

1.2.6 Fubini Theorem. *Let μ be a probability measure such that every singleton is measurable, and let $B \subseteq M^{m+n}$ be $\mu^{(m+n)}$-measurable. Then*

(i) *Each section $B_{\bar{x}} = \{\bar{y} \in M^n : \bar{x}\bar{y} \in B\}$ is $\mu^{(n)}$-measurable.*
(ii) *The function $f(\bar{x}) = \mu^{(n)}(B_{\bar{x}})$ is $\mu^{(m)}$-measurable.*
(iii) *We have $\mu^{(m+n)}(B) = \int f(\bar{x})\, d\mu^{(m)}$.* $\square$

The proof here is exactly like the proof of the usual Fubini theorem for product measures. Theorem 1.2.5 would fail if we were to include both the universal quantifier and the probability quantifiers in the language, because projections of measurable sets need not themselves be measurable.

The model-theoretic notions of isomorphism, $L_{\mathbb{A}P}$-equivalence, and $L_{\mathbb{A}P}$-elementary substructure are defined as one would expect, and are respectively written as $\mathscr{M} \cong \mathscr{N}$, $\mathscr{M} \equiv_{\mathbb{A}P} \mathscr{N}$, and $\mathscr{M} \prec_{\mathbb{A}P} \mathscr{N}$.

1.3. Examples

The following examples of sentences of $L_{\mathbb{A}P}$ indicate the expressive power of the language.

(1) "There is a countable set of measure one" is expressed by:

$$(Px \geq 1)(Py > 0)x = y.$$

(2) "There are no point masses" (that is, there are no singletons of positive measure) is expressed by:

$$(Px \geq 1)(Py \geq 1)x \neq y.$$

Every model of this last sentence is uncountable. In the class of structures with no point masses, every sentence of $L_{\mathbb{A}P}$ with equality is equivalent to the sentence without equality that is obtained by replacing $v_n = v_n$ by "true," and $v_m = v_n$ by "false" if $m \neq n$.

(3) The reader can check that no two of the sentences

$$(Px \geq \tfrac{1}{2})(Py \geq \tfrac{1}{2})R(x, y),$$

$$(Py \geq \tfrac{1}{2})(Px \geq \tfrac{1}{2})R(x, y);$$

and

$$P(xy \geq \tfrac{1}{4})R(x, y),$$

are equivalent. (Consider structures with three elements of measure $\frac{1}{3}$.)

A measurable function $X: M \to \mathbb{R}$ is sometimes called a *random variable*. By the Fubini theorem, each binary relation $R(x, y)$ in a probability structure $\mathscr{M}$ induces the random variable $X(u) = \mu\{v \mid R(u, v)\}$. In the following examples, let the language L have binary relation symbols R, R_n, $n \in \mathbb{N}$, and denote the corresponding random variables by X, X_n, $n \in \mathbb{N}$.

(4) The condition $X(u) \geq r$ is expressed by:

$$(Pv \geq r)R(u, v).$$

(5) $|X_1(u) - X_2(u)| \leq r$ is expressed by:

$$\bigwedge_{q \in \mathbb{Q}} (X_1(u) \geq q \to X_2(u) \geq q - r) \wedge (X_2(u) \geq q \to X_1(u) \geq q - r).$$

(6) $X_n \to X$ almost surely (a.s.) is expressed by:

$$(Pu \geq 1) \bigwedge_n \bigvee_m \bigwedge_{k \geq m} |X_k(u) - X(u)| \leq \frac{1}{n}.$$

(7) $X_n \to X$ in probability is expressed by:

$$\bigwedge_n \bigvee_m \bigwedge_{k \geq m} \left(Pu \geq 1 - \frac{1}{n}\right) |X_k(u) - X(u)| \leq \frac{1}{n}.$$

(8) X_1 and X_2 have the same distribution is expressed by

$$\bigwedge_{q \in \mathbb{Q}} \bigwedge_{r \in \mathbb{Q}} (Pu \geq r)(X_1(u) \geq q) \leftrightarrow (Pu \geq r)(X_2(u) \geq q)$$

(9) X_1 and X_2 are independent is expressed by

$$\bigwedge_{q, r \in \mathbb{Q}} \bigwedge_{a, b \in \mathbb{Q}} (Pu \geq a)X_1(u) \geq q \wedge (Pu \geq b)X_2(u) \geq r \to (Pu \geq ab)(X_1(u) \geq q \wedge X_2(u) \geq r),$$

and similarly with $(Pu \leq)$ in place of $(Pu \geq)$.

(10) $1/X(u)$ is integrable is expressed by

$$\neg \bigwedge_{m\in\mathbb{N}} \bigvee_{\substack{s_1+\cdots+s_n\geq m \\ s_i\in\mathbb{Q}}} \bigwedge_{k=1}^{n} (Pu \geq s_k)|X(u)| \leq \frac{1}{k}.$$

1.4. Proof Theory

$L_{\mathbb{A}P}$ has the following set of axioms, where $\varphi \in L_{\mathbb{A}P}$ and $r, s \in \mathbb{A} \cap [0, 1]$. All but the last axiom B4 are in Hoover [1978a, b].

1.4.1 Definition. The Axioms for weak $L_{\mathbb{A}P}$ are as follows:

A1. All axioms of $L_{\mathbb{A}}$ without quantifiers.
A2. Monotonicity:

$$(P\bar{x} \geq r)\varphi \to (P\bar{x} \geq s)\varphi, \quad \text{where} \quad r \geq s.$$

A3. $(P\bar{x} \geq r)\varphi(\bar{x}) \to (P\bar{y} \geq r)\varphi(\bar{y})$.
A4. $(P\bar{x} \geq 0)\varphi$.
A5. Finite additivity:
  (i) $(P\bar{x} \leq r)\varphi \wedge (P\bar{x} \leq s)\psi \to ((P\bar{x} \leq r + s)(\varphi \vee \psi))$;
  (ii) $(P\bar{x} \geq r)\varphi \wedge (P\bar{x} \geq s)\psi \wedge (P\bar{x} \leq 0)(\varphi \wedge \psi) \to (Px \geq r + s)(\varphi \vee \psi)$.
A6. The Archimedean property:

$$(P\bar{x} > r)\varphi \leftrightarrow \bigvee_{n\in\mathbb{N}} \left(P\bar{x} \geq r + \frac{1}{n}\right)\varphi$$

1.4.2 Definition. The axioms for (full) $L_{\mathbb{A}P}$ consist of the axioms for weak $L_{\mathbb{A}P}$ plus:

B1. Countable additivity:

$$\bigwedge_{\Psi\subseteq\Phi} (P\bar{x} \geq r) \bigwedge \Psi \to (P\bar{x} \geq r) \bigwedge \Phi,$$

where Ψ ranges over the finite subsets of Φ.
B2. Symmetry:

$$(Px_1 \cdots x_n \geq r)\varphi \leftrightarrow (Px_{\pi 1} \cdots x_{\pi n} \geq r)\varphi,$$

where π is a permutation of $\{1, \ldots, n\}$.
B3. Product independence:

$$(P\bar{x} \geq r)(P\bar{y} \geq s)\varphi \to (P\bar{x}\bar{y} \geq rs)\varphi,$$

provided all variables in $\bar{x}, \bar{y}$ are distinct.

B4. Product measurability: For each $r < 1$,

$$(P\bar{x} \geq 1)(P\bar{y} > 0)(P\bar{z} \geq r)(\varphi(\bar{x}\bar{z}) \leftrightarrow \varphi(\bar{y}\bar{z})),$$

provided all variables in $\bar{x}$, $\bar{y}$, $\bar{z}$ are distinct.

The central purpose of Axiom B4 is to guarantee that $\varphi(\bar{x}, \bar{y})$ can be approximated by a finite union of measurable rectangles. It is obviously valid if $\varphi(\bar{x}, \bar{z})$ is a "rectangle" $\psi(\bar{x}) \wedge \theta(\bar{z})$. We will see later on that it is valid in general (the Soundness Theorem).

1.4.3 Definition. The Rules of Inference for $L_{\mathbb{A}P}$ are as follows:

R1. *Modus Ponens*:

$$\varphi, \varphi \to \psi \vdash \psi.$$

R2. Conjunction:

$$\{\varphi \to \psi \mid \psi \in \Psi\} \vdash \varphi \to \bigwedge \Psi.$$

R3. Generalization:

$$\varphi \to \psi(\bar{x}) \vdash \varphi \to (P\bar{x} \geq 1)\psi(\bar{x}),$$

provided $\bar{x}$ is not free in φ.

1.4.4 Definition. The notion of a *deduction* of a formula ψ from a set of sentences Φ, and the expressions

$$\Phi \vdash \psi, \quad \vdash \psi, \quad \Phi \models \psi, \quad \models \psi$$

are defined in the usual way. A *theorem* of $L_{\mathbb{A}P}$ is a sentence ψ such that $\vdash \psi$.

1.4.5 Deduction Theorem. *In either $L_{\mathbb{A}P}$ or weak $L_{\mathbb{A}P}$, if ψ is a sentence and $\Phi \cup \{\psi\} \vdash \theta$, then $\Phi \vdash \psi \to \theta$.*

1.4.6 Proposition. *The following are theorems of $L_{\mathbb{A}P}$, and their proofs do not require use of Axiom* B4:

(i) $(P\bar{x} \leq 1)\varphi$.
(ii) $(P\bar{x} > r) \bigvee \Phi \leftrightarrow \bigvee_{\Psi \subseteq \Phi} (P\bar{x} > r) \bigvee \Psi$, *where Ψ is finite.*
(iii) $(P\bar{x} \geq r)\varphi \leftrightarrow \bigwedge_n (Px \geq r - 1/n)\varphi$.
(iv) $(P\bar{x} \leq a)\varphi(\bar{x}) \wedge (P\bar{y} \leq b)\psi(\bar{y}) \to (P\bar{x}\bar{y} \leq ab)(\varphi(\bar{x}) \wedge \psi(\bar{y}))$.
(v) $(P\bar{x} \geq r)\varphi(\bar{x}) \to (P\bar{x}\bar{y} \geq r)\varphi(\bar{x})$.
(vi) $(P\bar{x}\bar{y} \geq a + b - ab)\varphi \to (P\bar{x} \geq a)(P\bar{y} \geq b)\varphi$.

Taking $a = b = 1$:

(vii) $(P\bar{x} \geq 1)(P\bar{y} \geq 1)\varphi \leftrightarrow (P\bar{x}\bar{y} \geq 1)\varphi$ □

1.4.7 Theorem (Soundness Theorem). *Any set Φ of sentences of $L_{\mathbb{A}P}$ which has a model is consistent.*

Outline of Proof. As usual, to prove the soundness theorem it suffices to show that each axiom is valid and the rules of inference preserve validity. The only difficulty lies in checking the validity of the product measurability axiom, (Axiom B4). In view of part (iii) of Proposition 1.4.6, it suffices to show that for each $q, r < 1$,

$$(P\bar{x} \geq q)(P\bar{y} > 0)(P\bar{z} \geq r)(\varphi(\bar{x}\bar{z}) \leftrightarrow \varphi(\bar{y}\bar{z}))$$

is valid. This can be proven by use of the Fubini theorem, Proposition 1.2.2, and the direction (ii) implies (i) of the following lemma. $\square$

1.4.8 Lemma. *Let μ, ν, and λ be probability measures on M, N, and $M \times N$ such that $\mu \otimes \nu \subseteq \lambda$. Let U be λ-measurable. The following are equivalent:*

(i) *For every $\varepsilon > 0$, there is a finite union B of $\mu \otimes \nu$-measurable rectangles such that $\lambda(U \Delta B) < \varepsilon$.*
(ii) *There is a $\mu \otimes \nu$-measurable set C with $\lambda(U \Delta C) = 0$.*

Idea of Proof. From (i) to (ii), we may take C to be a limit of the B's. We then use the monotone class theorem to show that for each $\mu \otimes \nu$-measurable U, (i) holds. It then follows at once that (ii) implies (i). $\square$

Remark. D. Hoover has pointed out the curious fact that the logic $L_{\mathbb{A}P}$ is equivalent to the richer logic on L with ord($\mathbb{A}$) variables, which allows formulas with $\mathbb{A}$-finitely many free variables and quantifiers $(P\bar{x} \geq r)$ over $\mathbb{A}$-finite sequences $\bar{x}$. The axioms and rules are as before with the additional scheme

$$(Px_1x_2 \ldots \geq r)\varphi(x_{i_1} \ldots x_{i_n}) \leftrightarrow (Px_{j_1} \ldots x_{j_n} \geq r)\varphi(x_{j_1} \ldots x_{j_n}),$$

where none of the other x_j's are free in φ. It can be shown by the logical monotone class theorem (Keisler [1977c]) that every sentence of the richer logic is equivalent to a sentence of $L_{\mathbb{A}P}$. The situation is radically different, however, when universal quantifiers are present, since well-ordering is definable in $L_{\omega_1\omega_1}$.

1.5. Weak Models

We will now begin working toward the completeness theorem for $L_{\mathbb{A}P}$. To this end, we first examine

1.5.1 Definition. A *weak structure for $L_{\mathbb{A}P}$* is a structure

$$\mathscr{M} = \langle M, R_i^{\mathscr{M}}, c_j^{\mathscr{M}}, \mu_n \rangle_{i \in I, j \in J, n \in \mathbb{N}}$$

such that each μ_n is a finitely additive probability measure on M^n with each singleton measurable, and (with the natural definition of satisfaction) the set

$$\{\bar{b} \in M^n \colon \mathscr{M} \models \varphi[\bar{a}, \bar{b}]\}$$

is μ_n-measurable for each $\varphi(\bar{x}, \bar{y}) \in L_{\mathbb{A}P}$ and $\bar{a}$ in M.

By Theorem 1.2.5, every probability structure $\mathscr{M}$ induces a weak structure for $L_{\omega_1 P}$ with $\mu_n = \mu^{(n)}$.

1.5.2 Weak Soundness Theorem. *Let Φ be a set of sentences of $L_{\mathbb{A}P}$. If Φ has a weak model, then Φ is consistent in weak $L_{\mathbb{A}P}$.*

1.5.3 Weak Completeness Theorem (Hoover [1978b]). *Let $\mathbb{A}$ be countable. If Φ is consistent in weak $L_{\mathbb{A}P}$, then Φ has a weak model.*

Sketch of Proof. Let C be a countable set of new constants, and let $K = L \cup C$. By a Henkin construction, Φ can be extended to a maximal weak $K_{\mathbb{A}P}$-consistent set Γ of sentences with the following witness properties:

(1) If $\Phi \subseteq \Gamma$ and $\bigwedge\Phi \in K_{\mathbb{A}P}$, then $\bigwedge\Phi \in \Gamma$;
(2) If $\varphi(\bar{c}) \in \Gamma$ for all $\bar{c}$ in C, then $(P\bar{x} \geq 1)\varphi(\bar{x}) \in \Gamma$.

Let C' be the set of constants of K. Γ induces a first-order structure

$$\mathscr{M}_0 = \langle M, R_j^{\mathscr{M}}, c^{\mathscr{M}}\rangle_{i \in I,\ c \in C'},$$

for K in the usual way, and $M = \{c^{\mathscr{M}} \mid c \in C'\}$. Define μ_n by

$$\mu_n\{\bar{c}^{\mathscr{M}} \mid \varphi(\bar{c}, \bar{d}) \in \Gamma\} = \sup\{r \mid (P\bar{x} \geq r)\varphi(\bar{x}, \bar{d}) \in \Gamma\}$$

for each $\varphi(\bar{x}, \bar{y})$ and $\bar{d}$. Axioms A1 through A5 insure that μ_n is well defined and is a finitely additive probability measure. This gives us a weak structure $\mathscr{M}$. Axiom A6 (in the dual form of Proposition 1.4.6(iii)) insures that the above supremum is always attained, and it follows by induction that $\mathscr{M} \models \Gamma$; and, hence, $\mathscr{M} \models \Phi$. ☐

1.6. *Atomic and Countable Models*

In this section we will dispose of the degenerate case in which there is a countable set of measure one; that is, we will consider the situation in which

$$(*) \qquad (Px \geq 1)(Py > 0)x = y$$

holds. Notice that the last axiom of $L_{\mathbb{A}P}$, namely Axiom B4, is provable from $(*)$ and the other axioms.

Assume first that L has no constant symbols.

1.6.1 Definition. Let $\mathscr{M}$ be a probability structure. An element $a \in M$ is an *atom* if $\{a\}$ has positive measure. $\mathscr{M}$ is *atomic* if every element is an atom.

We list some easy facts in the next proposition.

1.6.2 Proposition. (i) *If $\mathscr{M}$ is atomic, then $\mathscr{M}$ is countable.*

(ii) *If $\mathscr{M}$ is countable, then $\mathscr{M}$ satisfies* $(*)$.

(iii) *If $\mathscr{M}$ satisfies $(*)$, then there is a unique atomic model $\mathscr{N}$ such that $\mathscr{N} \prec_{\mathbb{A}P} \mathscr{M}$.*

(iv) *If $\mathscr{M}$ and $\mathscr{N}$ are atomic and $L_{\mathbb{A}P}$-equivalent, then they are isomorphic.*

(v) *If $\mathscr{M}$ is atomic, then for every formula $\varphi(x\bar{y})$ of $L_{\mathbb{A}P}$ and $\bar{b}$ in M, we have*

$$\mathscr{M} \models (\forall x)\varphi(x\bar{b}) \leftrightarrow (Px \geq 1)\varphi(x\bar{b});$$

and

$$\mathscr{M} \models (\exists x)\varphi(x\bar{b}) \leftrightarrow (Px > 0)\varphi(x\bar{b}). \quad \square$$

Part (v) of the proposition shows that in atomic structures the ordinary quantifiers can be defined in terms of probability quantifiers.

1.6.3 Theorem (Completeness Theorem for Atomic Models). *A countable set of sentences Φ of $L_{\mathbb{A}P}$ has an atomic model if and only if $\Phi \cup \{(*)\}$ is consistent in $L_{\mathbb{A}P}$.*

Sketch of Proof. We may take $\mathbb{A}$ to be countable. Suppose $\Phi \cup \{(*)\}$ is consistent in $L_{\mathbb{A}P}$. Then it has a weak model $\mathscr{M}_0$ in which all theorems of $L_{\mathbb{A}P}$ hold. From Section 1.4, for each m, the following are deducible from $(*)$ in $L_{\mathbb{A}P}$:

$$(Px \geq 1) \bigvee_n \left(P_1 \geq \frac{1}{n}\right) x = y_y;$$

$$\left(Px > 1 + \frac{1}{m}\right) \bigvee_n \left(Py \geq \frac{1}{n}\right) x = y;$$

and

$$\bigvee_n \left(Px > 1 - \frac{1}{m}\right)\left(Py \geq \frac{1}{n}\right) x = y.$$

It follows that M_0 has a finite subset of μ_1-measure greater than $1 - 1/m$. Thus μ_1 can be extended to a probability measure μ defined on all subsets of M_0 by

$$\mu(X) = \sup\{\mu_1(Y) \colon Y \subseteq X,\ Y \text{ finite}\},$$

forming a probability structure $\mathscr{M} \equiv_{\mathbb{A}P} \mathscr{M}_0$. The atomic model $\mathscr{N} \prec_{\mathbb{A}P} \mathscr{M}$ is the required model of Φ. $\square$

We now consider the general case where L has constant symbols. Define $\mathscr{M}$ to be *atomic* if every element of $\mathscr{M}$ is either of positive measure or equal to a constant symbol. With this definition, all the results remain true except for part (v) of Proposition 1.6.2. If the set of constant symbols is A-finite, we can still define the ordinary quantifiers in terms of probability quantifiers in an atomic structure $\mathscr{M}$ by

$$\mathscr{M} \models \forall x\varphi(x\bar{b}) \leftrightarrow (Px \geq 1)\varphi(x\bar{b}) \wedge \bigwedge_{j\in J} \varphi(c_j\bar{b});$$

and

$$\mathscr{M} \models \exists x\varphi(x\bar{b}) \leftrightarrow (Px > 0)\varphi(x\bar{b}) \vee \bigvee_{j\in J} \varphi(c_j\bar{b}).$$

2. Completeness Theorems

The main result of this section, to be given in Section 2.3, states that the set of axioms given in Section 1 is complete. As a preliminary result, in Section 2.2 we prove a completeness theorem for $L_{\mathbb{A}P}$ without using the axiom B4. However, this is done for a wider class of models. The chief difficulty is the construction of a countably additive probability structure from a finitely additive one. The key to getting past this difficulty is the Loeb measure construction from non-standard analysis, and this we examine in the following discussion.

2.1. The Loeb Measure

We assume once and for all that we have an ω_1-saturated non-standard universe

$$*: \langle V_\omega(U), \in\rangle \to \langle V_\omega(*U), \in\rangle,$$

where U is a set of urelements large enough for our purposes (see Keisler [1976] or Loeb [1979a] for the details). For $r \in *\mathbb{R}$, ${}^\circ r$ denotes the standard part of r. We will briefly state the definition and main facts about the Loeb measure. They are due to Loeb [1975].

2.1.1 Definition. Let M be an internal set in $V_\omega(*U)$ and let $\langle M, S, \mu\rangle$ be an internal $*$-finitely additive probability space. (Thus, μ and S are internal and $\mu: S \to *[0, 1]$.) The *Loeb measure* of μ is the unique (countably additive) probability space $\langle M, \sigma(S), \hat{\mu}\rangle$ such that:

(i) $\sigma(S)$ is the σ-algebra generated by S.
(ii) $\hat{\mu}(X) = {}^\circ\mu(X)$ for all $X \in S$.

2.1.2 Theorem. *The Loeb measure exists and is unique.*

Proof. Use ω_1-saturation and the Caratheodory extension theorem. □

2.1.3 Theorem. *Let $X \in \sigma(S)$. Then,*

(i) *for each $n \in \mathbb{N}$, there exist $Y, Z \in S$ such that $Y \subseteq X \subseteq Z$ and $\mu(Z - Y) < 1/n$;*
(ii) *there exists $Y \in S$ such that $\hat{\mu}(X \,\Delta\, Y) = 0$.*

Proof. Part (i) of the theorem uses the monotone class theorem, and part (ii) follows from part (i) by ω_1-saturation. □

Intuitively, part (i) says that every Loeb measurable set can be approximated above and below by internal measurable sets.

2.2. Graded Probability Models

A graded probability structure is a generalization of a probability structure in which the diagonal product $\mu^{(n)}$ is replaced by any probability measure on M^n which satisfies the Fubini theorem. We will show that the set of axioms for $L_{\mathbb{A}P}$ without axiom B4 is sound and complete for these structures.

2.2.1 Definition. A *graded probability structure* for L is a structure

$$\mathcal{M} = \langle M, R_j^{\mathcal{M}}, c_j^{\mathcal{M}}, \mu_n \rangle_{i \in I, j \in J, n \in \mathbb{N}}$$

such that:

(a) Each μ_n is a (countably additive) probability measure on M^n.
(b) Each n-placed relation $R_i^{\mathcal{M}}$ is μ_n-measurable, and the identity relation is μ_2-measurable.
(c) If B is μ_m-measurable, then $B \times M^n$ is μ_{m+n}-measurable.
(d) The symmetry property holds; that is, each μ_n is preserved under permutations of $\{1, \ldots, n\}$.
(e) $\langle \mu_n | n \in \mathbb{N} \rangle$ has the *Fubini property*: If B is μ_{m+n}-measurable, then
 (1) For each $\bar{x} \in M^n$, the section $B_{\bar{x}} = \{\bar{y} | B(\bar{x}, \bar{y})\}$ *is* μ_n-measurable.
 (2) The function $f(\bar{x}) = \mu_n(B_{\bar{x}})$ is μ_m-measurable.
 (3) $\int f(\bar{x}) d\mu_m = \mu_{m+n}(B)$.

2.2.2 Proposition. (i) *If $\mathcal{M}$ is a probability structure, then*

$$\langle M, R_i^{\mathcal{M}}, c_j^{\mathcal{M}}, \mu^{(n)} \rangle_{n \in \mathbb{N}}$$

is a graded probability structure.
(ii) *Every graded probability structure is a weak structure for $L_{\omega_1 P}$.* □

2.2.3 Proposition. *In a graded probability structure $\mathscr{M}$, μ_n is an extension of $\mu_1^{(n)}$.* □

An important example of a graded probability structure arises from the Loeb measure construction.

2.2.4 Theorem (Keisler [1977b]). *Let M be a $*$-finite set. For each n, let ν_n be the internal probability measure on M^n giving each element the same weight (the counting measure), and let $\mu_n = \hat{\nu}_n$ be the Loeb measure of ν_n. Then $\langle \mu_n \mid n \in \mathbb{N} \rangle$ has the Fubini property. Hence, if each n-ary relation of $\mathscr{M}$ is μ_n-measurable, $\mathscr{M}$ is a graded probability structure.* □

The following example of Hoover provides a graded probability structure which is not $L_{\mathbb{A}P}$-equivalent to any ordinary probability structure.

2.2.5 Example (White Noise). Let H be an infinite $*$-finite set, let $M = {}^*\mathscr{P}(H)$ be the set of all internal subsets of H, and take μ_n as in Theorem 2.2.4. Let $f\colon M \to H$ be an internal function partitioning M into H equal parts. Let $R(x, y)$ iff $f(x) \in y$. Then R is internal and hence μ_2-measurable.

If $f(a) \neq f(b)$, then the sets $R(a, v)$ and $R(b, v)$ are independent; similarly, for $f(a_1), \ldots, f(a_n)$ distinct. This suggests the name "white noise." Thus,

$$\mathscr{M} \models (Px \geq 1)(Py \geq 1)(Pz \leq \tfrac{1}{2})R(xz) \leftrightarrow R(yz).$$

But then,

$$\mathscr{M} \models (Px \leq 0)(Py > 0)(Pz > \tfrac{1}{2})R(xz) \leftrightarrow R(yz).$$

Thus axiom B4 fails in $\mathscr{M}$. In fact, R is not measurable in the completion of $\mu_1^{(2)}$.

2.2.6 Definition. By *graded $L_{\mathbb{A}P}$* we mean $L_{\mathbb{A}P}$ with all the axioms except for the product measurability axiom B4.

One may check that all axioms except axiom B4 hold in all graded probability structures.

2.2.7 Theorem (Graded Soundness Theorem). *Every set of sentences of $L_{\mathbb{A}P}$ which has a graded model is consistent in graded $L_{\mathbb{A}P}$.* □

2.2.8 Theorem (Graded Completeness Theorem by Hoover [1978b]). *Every countable set Φ of sentences which is consistent in graded $L_{\mathbb{A}P}$ has a graded model.*

Sketch of Proof. Let $\mathbb{A}$ be countable, and assume L has countably many constants not appearing in Φ. From the proof of the weak completeness theorem, Φ has a weak model

$$\mathscr{M} = \langle M, R_i, c_j, \mu_n \rangle_{i \in I, j \in J, n \in \mathbb{N}}$$

such that $\mathscr{M}$ satisfies each theorem of graded $L_{\mathbb{A}P}$, $M = \{c_j | j \in J\}$, and the domain of each μ_n is the set of $L_{\mathbb{A}P}$-definable subsets of M^n. Form the internal structure

$$*\mathscr{M} = \langle *M, *R_j, *c_j, *\mu_n \rangle_{i \in *I, j \in *J, n \in *\mathbb{N}}.$$

Let

$$\hat{\mathscr{M}} = \langle *M, *R_j, *c_j, \hat{\mu}_n \rangle_{i \in I, j \in J, n \in \mathbb{N}},$$

where $\hat{\mu}_n$ is the Loeb measure of μ_n. By Theorem 2.1.3, every $\hat{\mu}_n$- measurable set can be approximated above and below by $*$-definable sets in n variables. Using this fact and axioms B2 and B3 in $\mathscr{M}$, it can be shown that $\hat{\mathscr{M}}$ is a graded probability structure. An induction on formulas will show that $\hat{\mathscr{M}}$ is $L_{\mathbb{A}P}$-equivalent to $\mathscr{M}$. Axiom B1 is used in the $\wedge$ step, and axioms B2 and B3 in the quantifier step. Therefore, $\hat{\mathscr{M}} \models \Phi$. $\square$

Remark. The graded soundness and completeness theorems hold with little change if L has function symbols, and graded probability structures are defined so that the interpretation of every atomic formula in n variables is μ_n-measurable. This is done in Hoover [1978b].

2.3. The Main Completeness Result

We are now ready to prove the completeness theorem for $L_{\mathbb{A}P}$. The results of this section, including the completeness theorem, are new. We make use of axiom **B4** by way of the following lemma.

2.3.1 Lemma (Rectangle Approximation Lemma). *Let $\mathscr{M}$ be a graded probability structure satisfying every theorem of $L_{\mathbb{A}P}$. Then for each $\varepsilon > 0$ and formula $\varphi(\bar{y})$ of $L_{\mathbb{A}P}$ there are finitely many formulas $\psi_{ij}(\bar{x}y_j)$, where $i = 1, \ldots, m$, and $j = 1, \ldots, n$, such that*

$$\mathscr{M} \models (P\bar{x} > 0)(P\bar{y} > 1 - \varepsilon)(\varphi(\bar{y}) \leftrightarrow \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} \psi_{ij}(\bar{x}y_j)). \ \square$$

The lemma asserts that any definable set $\varphi(\bar{y})$ in $\mathscr{M}$ can be approximated within ε by a finite union of definable rectangles, uniformly in parameters $\bar{x}$ from a set of positive measure. The proof is rather technical, and axiom B4 is used n times.

2.3.2 Definition. Let $\mathscr{M}$ and $\mathscr{N}$ be graded probability structures. We say that $\mathscr{M} = \mathscr{N}$ *almost surely*, (in symbols, $\mathscr{M} = \mathscr{N}$ a.s.) if $\mathscr{M}$ and $\mathscr{N}$ have the same universe, constants, and measures, and if for each atomic formula $\varphi(\bar{x})$ of $L_{\mathbb{A}P}$,

$$\mathscr{M} \models \varphi[\bar{a}] \quad \text{iff} \quad \mathscr{N} \models \varphi[\bar{a}]$$

for μ_n-almost all $\bar{a}$.

2.3.3 Lemma. *If $\mathscr{M} = \mathscr{N}$ a.s. then $\mathscr{M}$ and $\mathscr{N}$ are $L_{\mathbb{A}P}$-equivalent. Also for each formula $\varphi(\bar{x})$ of $L_{\mathbb{A}P}$,*

$$\mathscr{M} \models \varphi[\bar{a}] \quad \textit{iff} \quad \mathscr{N} \models \varphi[\bar{a}]$$

for μ_n-almost all $\bar{a}$.

Proof. The proof here is by induction on φ. □

The following theorem is the last step needed for the completeness theorem. The proof of this result would break down if we were to allow function symbols in L.

2.3.4 Theorem. *Let $\mathscr{M}$ be a graded probability structure satisfying every theorem of $L_{\mathbb{A}P}$, and let $\mu = \mu_1$. Then there is a graded probability structure $\mathscr{N}$ such that $\mathscr{M} = \mathscr{N}$ a.s., and each relation $R_i^{\mathscr{N}}$ is $\mu^{(n)}$-measurable. Thus, $\mathscr{N}$ induces an ordinary probability structure.*

Sketch of Proof. By the Rectangle Approximation Lemma, for each $\varepsilon > 0$ and $L_{\mathbb{A}P}$-definable set $U \subseteq M^n$ in $\mathscr{M}$, there is a finite union B of μ^n-measurable rectangles such that $\mu_n(B \,\Delta\, U) < \varepsilon$. Then, by Lemma 1.4.8, there is a μ^n-measurable set C such that $\mu_n(C \,\Delta\, U) = 0$. By patching diagonals together, we find that for each $i \in I$, there is a $\mu^{(n)}$-measurable relation $R_i^{\mathscr{N}}$ such that $\mathscr{M} = \mathscr{N}$ a.s. □

2.3.5. Theorem (Completeness Theorem for $L_{\mathbb{A}P}$). *Every countable consistent set Φ of sentences of $L_{\mathbb{A}P}$ has a model.*

Proof. The proof of this result is by the Graded Completeness Theorem, Theorem 2.3.4, and Lemma 2.3.3. □

By the usual $L_{\mathbb{A}}$ arguments (as given in Chapter IX), we obtain Barwise-type results. Similar results for graded $L_{\mathbb{A}P}$ are given in Keisler [1977b].

2.3.6 Theorem (Barwise Completeness Theorem). *The set of valid sentences of* $L_{\mathbb{A}P}$ is Σ *on* $\mathbb{A}$.

2.3.7 Theorem (Barwise Compactness Theorem). *Let $\mathbb{A}$ be countable and let Φ be a set of sentences of $L_{\mathbb{A}P}$. If Φ is Σ on $\mathbb{A}$ and every $\mathbb{A}$-finite $\Psi \subseteq \Phi$ has a model, then Φ has a model.*

2.4. *Hanf and Löwenheim Numbers*

We have seen in Section 1 that the sentence stating that $\mathscr{M}$ is atomless has no countable models. Thus, the Löwenheim number of $L_{\mathbb{A}P}$ is at least ω_1. On the other hand, given any probability structure $\mathscr{M}$, we can obtain $L_{\omega_1 P}$-equivalent structures of arbitrarily large cardinality by adding a set of new elements of measure zero. Thus, the Hanf number is ω but for a trivial reason. When considering cardinalities, we should restrict our attention to *reasonable* structures.

2.4.1 Definition. A probability structure $\mathscr{M}$ is *reasonable* if every set of measure one has the same cardinal as M. The *reasonable* Löwenheim or Hanf number of $L_{\mathbb{A}P}$ is obtained by restricting to reasonable probability structures.

2.4.2 Proposition. *A reasonable structure is countable if and only if the set of atoms has measure one.* $\square$

Let $\mathscr{M}$ and $\mathscr{N}$ be probability structures for L. Notice that if $\mathscr{M} \prec_{\mathbb{A}P} \mathscr{N}$, then the first-order part of $\mathscr{M}$ is a substructure of the first-order part of $\mathscr{N}$ but is not necessarily an elementary substructure in the sense of $L_{\omega\omega}$.

2.4.3 Proposition. *Every probability structure $\mathscr{M}$ has a reasonable $L_{\mathbb{A}P}$-elementary substructure $\mathscr{N}$ such that $\mu(N) = 1$ and v is the restriction of μ to N. The cardinal of N is unique.* $\square$

The following theorem and corollary are new.

2.4.4 Theorem (Downward Löwenheim–Skolem Theorem). *Let $\mathscr{M}$ be a reasonable probability structure of power at least λ, where $\lambda = \lambda^{\omega}$. Then, for every set $X \subseteq M$ of power $\leq \lambda$, $\mathscr{M}$ has a reasonable $L_{\omega_1 P}$-elementary substructure $\mathscr{N}$ of power λ with $X \subseteq N$.*

Proof. Let $X \subseteq X_0 \subseteq M$ where X_0 has power λ and contains all constants. Form a chain X_α, $\alpha < \lambda$ of subsets of M of power λ such that for every Borel combination B of sets $L_{\omega_1 P}$-definable in $\mathscr{M}$ with parameters in X_α,

(1) if $B \neq \varnothing$ then $B \cap X_{\alpha+1} \neq \varnothing$;
(2) if $\mu(B) = 1$ then $|B \cap X_{\alpha+1}| = \lambda$.

Take unions at limit α. Form the structure $\mathscr{N}$, with $N = \bigcup_\alpha X_\alpha$ and $v(B \cap N) = \mu(B)$, for each Borel combination B of sets $L_{\omega_1 P}$-definable with parameters in N. Then $\mathscr{N}$ is as required. $\square$

2.4.5 Corollary. *Let λ be the reasonable Löwenheim number for $L_{\mathbb{A}P}$. Then,*

(i) $\omega_1 \leq \lambda \leq 2^{\omega}$;
(ii) *Martin's axiom implies* $\lambda = 2^{\omega}$.

Proof. As to the argument for (i), we note that Theorem 2.4.4 shows that $\lambda \leq 2^{\omega}$. In order to prove (ii), we let φ say that $R_n(x)$, $n \in \mathbb{N}$ are independent sets of probability $\frac{1}{2}$. By Martin's axiom, every subset of $2^{\mathbb{N}}$ of power $< 2^{\omega}$ has Lebesgue measure zero, and it thus follows that φ has no reasonable model of power $< 2^{\omega}$. $\square$

2.4.6 Theorem (Hoover [1978b]). *Every uncountable reasonable probability structure $\mathscr{M}$ has reasonable $L_{\omega_1 P}$-elementary extensions of arbitrarily large cardinality.*

Sketch of Proof. Working in a κ^+-saturated universe, form $^*\mathscr{M}$ and use the Loeb process to get a graded structure $\hat{\mathscr{M}} \succ_{\omega_1 P} \mathscr{M}$ and probability structure $\mathscr{N} \succ_{\omega_1 P} \mathscr{M}$.

By Proposition 2.4.2, $\mathscr{M}$ has an atomless set of positive measure ε. By κ^+-saturation, every internal set in $^*M$ of measure $> 1 - \varepsilon/2$ has power at least κ^+, so every Loeb measurable set of measure one has power $\geq \kappa^+$. $\square$

2.4.7 Corollary. *$L_{\mathbb{A}P}$ has reasonable Hanf number ω_1.* $\square$

2.5. Random Variables

In this section we will consider structures with random variables instead of relations. From the examples of Section 3.1 we saw that structures with random variables are of interest in probability theory. In general, one could consider random variables with values in a Polish space. We will restrict our discussion here to random variables with values in $\mathbb{R}$ and will use a language $L = \{X_i, c_j | i \in I, j \in J\}$.

2.5.1 Definition. An *n-fold random variable* on a probability space $\langle M, S, \mu\rangle$ is a $\mu^{(n)}$-measurable function $X: M^n \to \mathbb{R}$. A *random variable structure for L* is a structure

$$\mathscr{M} = \langle M, X_i^{\mathscr{M}}, c_j^{\mathscr{M}}, \mu\rangle_{i \in I, j \in J},$$

where μ is a probability measure on M, $X_i^{\mathscr{M}}$ is an n_i-fold random variable, and $c_j \in M$, and each

2.5.2 Definition. The *auxiliary language* of L is the language L' which has the same constant symbols c_j of L but has new relation symbols $[X_i(\bar{u}) \geq r]$, and $[X_i(\bar{u}) \leq r]$, for each $i \in I$ and $r \in \mathbb{Q}$.

Each random variable structure $\mathscr{M}$ for L induces a probability structure $\mathscr{M}'$ for L', where $[X_j(\bar{u}) \geq r]$ is interpreted in the natural way.

2.5.3 Definition. We will use the following abbreviations:

$$[X(\bar{u}) > r] \quad \text{for} \quad \neg[X(\bar{u}) \geq r],$$

$$[X(\bar{u}) < r] \quad \text{for} \quad \neg[X(\bar{u}) \geq r].$$

2.5.4 Definition. The *language* $L_{\mathbb{A}P}(\mathbb{R})$ has the same set of formulas as $L'_{\mathbb{A}P}$. It has all the axioms and rules of inference of $L'_{\mathbb{A}P}$, plus four new axioms, where $r, s \in \mathbb{Q}$:

C1. $[X(\bar{u}) \geq r] \to [X(\bar{u}) \geq s]$, where $r \geq s$;
C2. $[X_i(\bar{u}) > r] \leftrightarrow \bigvee_n [X_i(\bar{u}) \geq r + 1/n]$;
C3. $[X_i(\bar{u}) \geq r] \leftrightarrow \bigwedge_n [X_i(\bar{u}) \geq r - 1/n]$;
C4. $\bigvee_n ([X_i(\bar{u}) \geq -n] \wedge [X_i(\bar{u}) \leq n])$, and each singleton is measurable.

2.5.5 Theorem (Soundness and Completeness Theorem for $L_{\mathbb{A}P}(\mathbb{R})$). *A countable set Φ of sentences of $L_{\mathbb{A}P}(\mathbb{R})$ has a random variable model if and only if it is consistent in $L_{\mathbb{A}P}(\mathbb{R})$.*

Proof. The soundness is easy. Suppose Φ is consistent. Let Ψ be the set of sentences of the form $(P\bar{v} \geq 1)\psi$, where ψ is one of the axioms C_1 through C_4. Then, $\Phi \cup \Psi$ is consistent and has a probability model $\mathscr{M}'$. Make $\mathscr{M}'$ into a random variable model $\mathscr{M}$ by defining

$$X^{\mathscr{M}}(\bar{a}) = \sup\{r \in \mathbb{Q} \mid \mathscr{M}' = [\dot{X}(\bar{a}) \geq r]\}.$$

Use Ψ to show that $X^{\mathscr{M}}$ is almost surely finite and uniquely defined. $\square$

2.6. Finitary Probability Logic

We will now discuss the situation when ω is not an element of $\mathbb{A}$, so that each formula of $L_{\mathbb{A}P}$ is finite. We will assume that the rationals are defined in such a way that $\mathbb{Q} \subseteq \mathbb{A}$, so $L_{\mathbb{A}P}$ has at least the quantifiers $(P\bar{x} \geq r)$, $r \in \mathbb{Q} \cap [0, 1]$. By throwing additional reals into $\mathbb{A}$ as urelements, we can obtain more probability quantifiers. When $\omega \notin \mathbb{A}$, the infinitary axiom B1 and the infinite conjunction rule R2 become finite. However, the other infinitary axiom A6 is outside the language $L_{\mathbb{A}P}$ and must be replaced by a new infinite rule of inference, a rule which is due to Hoover [1978a].

2.6.1 Definition. The rule of inference for finitary $L_{\mathbb{A}P}$ is given by

$$\{\psi \rightarrow (P\bar{x} \geq r)(P\bar{y} \geq s - 1/n)\varphi \mid n \in \mathbb{N}\} \vdash \psi \rightarrow (P\bar{x} \geq r)(P\bar{y} \geq s)\varphi.$$

With this new rule of inference, the weak, graded, and full completeness theorems hold for the finitary case $\omega \notin \mathbb{A}$. Hoover [1978b] has shown that when $\mathbb{A} = \mathbb{HF}$, the set of valid sentences of $L_{\mathbb{A}P}$ is complete Π_1^1 and thus not recursively enumerable. This was done by interpreting the standard model of number theory in a finite theory of $L_{\mathbb{A}P}$. The compactness theorem fails for $L_{\mathbb{A}P}$, so that some infinitary rule of inference is needed.

2.6.2 Example. Let Φ be the set of sentences containing $(Px > 0)R(x)$, and $(Px \leq 1/n)R(x)$, for $n = 1, 2, \ldots$. Then every finite subset of Φ has a model, but Φ itself has no model.

However, there is a compactness theorem for special sentences, which we will state for $L_{\omega_1 P}$.

2.6.3 Definition. The set of *universal conjunctive formulas* of $L_{\omega_1 P}$ is the least set containing all quantifier-free formulas and closed under arbitrary $\bigwedge$, finite $\vee$, and the quantifiers $(P\bar{x} \geq r)$.

2.6.4 Theorem (Finite Compactness Theorem (see Hoover [1978b])). *Let Φ be a set of universal conjunctive sentences of $L_{\mathbb{A}P}$. If every finite subset of Φ has a graded model, then Φ has a graded model.*

Proof. Suppose each finite subset $\Psi \subseteq \Phi$ has a model $\mathscr{M}_\Psi$. Take an ultraproduct $^*\mathscr{M}$ of the $\mathscr{M}_\Psi$'s such that, for each $\varphi \in \Phi$, almost every $\mathscr{M}_\Psi$ satisfies φ. Form a graded probability structure $\hat{\mathscr{M}}$ from $^*\mathscr{M}$ by the Loeb construction. Then, by induction show that every universal conjunctive formula holding in almost all $\mathscr{M}_\Psi$ holds in $\hat{\mathscr{M}}$ also. $\square$

The above proof, of course, does not work for probability models, because axiom B4 is not universal conjunctive.

2.6.5 Example. Let Φ be the set of universal conjunctive sentences

$$(Px \geq 1)\left(Py \geq 1 - \frac{1}{n}\right)(Pz \geq \tfrac{1}{2})\neg(R(xz) \leftrightarrow R(yz)),$$

where $n \in \mathbb{N}$. Each finite subset of Φ has a (finite) probability model. However, Φ implies the white noise sentence

$$(Px \geq 1)(Py \geq 1)(Pz \leq \tfrac{1}{2})(R(xz) \leftrightarrow R(yz))$$

of Example 2.2.3. Thus, Φ has no probability model.

However, if in Theorem 2.6.4 every instance of axiom B4 is deducible from Φ in graded $L_{\mathbb{A}P}$, then Φ does have a probability model.

2.7. *Probabilities on Sentences of $L_{\omega_1\omega}$*

We can easily generalize our treatment of $L_{\mathbb{A}P}$ to two-sorted logic. It is more interesting that there is a mixed two-sorted logic which has probability quantifiers on one sort and ordinary quantifiers on the other sort. In this mixed two-sorted logic, we can study models which assign probabilities to sentences of $L_{\mathbb{A}}$. We will use $x, y, \ldots$ for the first sort of variables, and $s, t, \ldots$ for the second.

2.7.1 Definition. $L_{\mathbb{A}}(P, \forall)$ is the two-sorted logic which has probability quantifiers $(P\bar{x} \geq r)$ on the first sort and the universal quantifier $(\forall t)$ on the second sort. *Probability structures for* $L(P, \forall)$ have the form

$$\mathscr{M} = \langle M, T, R_i, c_j, \mu\rangle_{i \in I, j \in J},$$

where μ is a probability measure on M, and $R_i(\bar{x}; \bar{t})$ is $\mu^{(n)}$-measurable for each $\bar{t}$ in T.

If T is countable, there is no difficulty in defining the satisfaction relation in $\mathscr{M}$, with the usual clause for $(\forall t)$. This is the case which is needed for the completeness theorem.

There is also a definition of satisfaction which applies to any probability structure for $L(P, \forall)$, as introduced by Gaifman [1964] and extended by Krauss–Scott [1966]. The idea underlying this development is to assign, for each $\varphi(\bar{x}; \bar{t})$ and $\bar{b}$ in T, an element $\varphi(\bar{x}; \bar{b})^{\mathscr{M}}$ of the measure algebra of $\mu^{(n)}$ modulo the null sets. The $\forall$ clause is

$$(\forall t)\varphi(\bar{x}; \bar{b})^{\mathscr{M}} = \inf\{\varphi(\bar{x}; \bar{b}, c)^{\mathscr{M}} \mid c \in T\},$$

taking inf in the measure algebra. This coincides with the natural definition of satisfaction when T is countable, but not when T is uncountable.

2.7.2 Definition. The axioms for $L_{\mathbb{A}}(P, \forall)$ consist of all axiom schemes for $L_{\mathbb{A}P}$ and $L_{\mathbb{A}}$, with quantifiers on the appropriate sort, plus the new axiom

$$(P\bar{x} \geq r)(\forall t)\varphi(\bar{x}; t) \leftrightarrow \bigwedge_{n} (\forall t_1) \ldots (\forall t_n)(P\bar{x} \geq r) \bigwedge_{k=1}^{n} \varphi(\bar{x}; t_k).$$

The rules of inference for $L_{\mathbb{A}}(P, \forall)$ are the natural combination of rules for $L_{\mathbb{A}P}$ and $L_{\mathbb{A}}$.

2.7.3 Theorem (Soundness Theorem). *Every set Φ of sentences of $L_{\mathbb{A}}(P, \forall)$ which has a model is consistent.* $\square$

2.7.4 Theorem (Completeness Theorem). *Every countable consistent set of sentences of $L_{\mathbb{A}}(P, \forall)$ has a model $\mathscr{M}$ with T countable.*

Proof. Form a countable weak model. Then keep the second sort fixed while using the method of Sections 2.2 and 2.3 to extend the first sort to a probability model. $\square$

The following simpler logic is of particular interest.

2.7.5 Definition. Let L be a first-order language with variables $t_0, t_1, \ldots$ and relation symbols $R(\bar{t})$. By *L-probability logic* we mean the two-sorted logic $L'_{\mathbb{HC}}(P, \forall)$ which has only one variable x of the new sort and where L' is formed by replacing each relation $R(\bar{t})$ of L by $R(x; \bar{t})$.

L-probability logic is a logic which assigns probabilities to sentences of $L_{\omega_1\omega} = L_{\mathbb{HC}}$. Its model theory was studied by Krauss–Scott [1966]. A probability structure

$$\mathscr{M} = \langle M, T, R_i, \mu\rangle$$

for L-probability logic may be regarded as an indexed family $\langle \mathscr{M}_x \mid x \in M\rangle$ of first-order structures $\mathscr{M}_x$ for L each with universe T, together with a probability measure μ on M such that each $\{x \mid R_i(x; \bar{t})\}$ is measurable.

2.7.6 Definition. A *probability on* $L_{\omega_1\omega}$ is a function μ from sentences of $L_{\omega_1\omega}$ to $[0, 1]$ which is countably additive with respect to $\bigvee$, $\neg$ and such that each valid sentence has measure one.

Each structure $\mathscr{M}$ for L-probability logic induces the probability $\mu^{\mathscr{M}}$ on $L_{\omega_1\omega}$ given by

$$\mu^{\mathscr{M}}(\varphi) \geq r \quad \text{iff} \quad \mathscr{M} \models (Px \geq r)\varphi.$$

The axioms and rules for L- probability logic are like those for $L_{\mathbb{HC}}(P, \forall)$ except that axioms A3, B2, B3, and B4 disappear. The soundness and completeness theorems still hold and have easier proofs which avoid graded structures.

The following completeness theorem was proved by Krauss–Scott [1966], extending results of Gaifman [1964] and Łoś [1963]. Although it does not follow from Theorem 2.7.4, it can be proven by a similar argument.

2.7.7 Theorem. *Let μ be a probability on $L_{\omega_1\omega}$ which assigns* 0 *or* 1 *to pure equality sentences. For every countable set $\Psi \subseteq L_{\omega_1\omega}$, there is a structure $\mathscr{M}$ for L-probability logic such that $\mu^{\mathscr{M}}$ agrees with μ on Ψ.* ☐

Other work on probabilities of sentences can be found in Havranek [1975], Fenstad [1967], Fagin [1976], Compton [1980], Lynch [1980], Gaifman–Snir [1982], and Krauss [1969].

The logic $L_{\mathbb{A}P}$ should be compared with the logic $L(Q_m)$ of H. Friedman, which is discussed in Chapter XVI. This logic also has models with measures as well as both the classical quantifier $(\forall x)$ and the measure quantifier $(Q_m x)$ which has the same interpretation as our $(Px > 0)$. In order to have both quantifiers, one must pay the price of restricting attention to those structures in which every definable set is Borel (the absolutely Borel structures). A similar treatment of logic with both quantifiers $\forall x$ and $(P\bar{x} \geq r)$ for absolutely Borel structures should be possible and interesting.

3. *Model Theory*

In this section, we will develop the model theory of the logic $L_{\mathbb{A}P}$. In Section 3.1 we state a model-theoretic form of the law of large numbers, showing that every probability structure is "approximated" by almost every sequence of finite substructures. This result is used in Section 3.2 to prove the existence and uniqueness of hyperfinite models, which play the role for $L_{\mathbb{A}P}$ that saturated models play in classical model theory. These models are used in Section 3.3 to prove the Robinson consistency and Craig interpolation theorems for $L_{\mathbb{A}P}$. The section concludes with the development of integrals, which eliminate quantifiers from $L_{\mathbb{A}P}$ in a manner analogous to Skolem functions in classical logic.

3.1. Laws of Large Numbers

The results in this section hold for all graded probability structures. First, we have

3.1.1 Definition. A *finite universal formula of* $L_{\mathbb{A}P}$ is a formula of the form

$$(P\bar{x}_1 \geq r_1) \ldots (P\bar{x}_n \geq r_n)\psi,$$

where ψ is a finite quantifier-free formula of L. A *finite existential formula* of $L_{\mathbb{A}P}$ is a formula of the form

$$(P\bar{x}_1 > r_1) \ldots (P\bar{x}_n > r_n)\psi,$$

where ψ is as before.

Note that since $\neg(P\bar{x} > r)\psi$ is equivalent to $(P\bar{x} \geq 1 - r) \neg \psi$, the negation of a finite existential formula is equivalent to a finite universal formula, and vice versa. We shall see that the laws of large numbers for $L_{\mathbb{A}P}$ deal with finite existential sentences. To state them, however, we need the notion of a finite sample of $\mathscr{M}$.

3.1.2 Definition. Let $\mathscr{M}$ be a graded probability structure for L, and let $\bar{a}_k = \langle a_1, \ldots, a_k \rangle \in M^k$ be a k-tuple of elements of M. The *finite sample* $\mathscr{M}(\bar{a}_k)$ is the probability structure whose universe is the union of $\{a_1, \ldots, a_k\}$ and the constants (if any) of $\mathscr{M}$, whose first-order part is a substructure of $\mathscr{M}$, and whose measure v is given by

$$v(S) = |\{m \leq k \mid a_m \in S\}|/k.$$

Thus, the finite set $\{a_1, \ldots, a_k\}$ has measure one in $\mathscr{M}(\bar{a}_k)$, and the measure of a singleton $\{a\}$ is $1/k$ times the number of occurrences of a in the sequence $\bar{a}_k$.

3.1.3 Theorem. *Let $\mathscr{M}$ be a graded probability structure for L with measures μ_n, and let φ be a finite existential sentence of $L_{\mathbb{A}P}$ such that $\mathscr{M} \models \varphi$.*

(i) *Weak Law of Large Numbers for $L_{\mathbb{A}P}$:*

$$\lim_{k \to \infty} \mu_k\{\bar{a}_k \in M^k \mid \mathscr{M}(\bar{a}_k) \models \varphi\} = 1.$$

(ii) *Strong Law of Large Numbers for $L_{\mathbb{A}P}$. Let $\mu_{\mathbb{N}}$ be the completion of the measure on $M^{\mathbb{N}}$ determined by the μ_n. Then, for $\mu_{\mathbb{N}}$ almost all sequences $\bar{a} \in M^{\mathbb{N}}$, $\mathscr{M}(\bar{a}_k) \models \varphi$ for all but finitely many $k \in N$.* □

The above theorem is a reformulation of Lemma 6.13 in Keisler [1977b]. In the special case in which φ has the form $(Px > r)\psi(x)$, the result follows directly from the weak and strong laws of large numbers in probability theory. Hoover has pointed out that the case in which φ has the form $(P\bar{x} > r)\psi(\bar{x})$ can be proved by the same argument as the proof of the strong law in probability theory, using the martingale convergence theorem. The general case uses an induction on the number of quantifiers.

3.1.4 Theorem (Normal Form Theorem (Hoover [1982])). *Every formula $\varphi(\bar{x})$ of graded $L_{\omega_1 P}$ is equivalent to a countable boolean combination of formulas of the form $(P\bar{y} \geq r)\psi(\bar{x}\bar{y})$, where $\psi(\bar{x}\bar{y})$ is a finite conjunction of atomic formulas of L.*

Proof. By a prenex normal form argument, it can be shown that every formula of graded $L_{\omega_1 P}$ is equivalent to a countable boolean combination of finite universal formulas (with the same free variables). By the Weak Law of Large Numbers, each statement below implies the next, where ψ is a finite quantifier-free formula.

(1) $$\mathcal{M} \models (P\bar{x} \geq r)(P\bar{y} \geq s)\psi$$

(2) $$\bigwedge_n \mathcal{M} \models \left(P\bar{x} > r - \frac{1}{n}\right)\left(P\bar{y} > s - \frac{1}{n}\right)\psi$$

(3) $$\bigwedge_n \lim_{k\to\infty} \mu_k\left\{\bar{a}_k \mid \mathcal{M}(\bar{a}_k) \models \left(P\bar{x} > r - \frac{1}{n}\right)\left(P\bar{y} > s - \frac{1}{n}\right)\psi\right\} = 1$$

(4) $$\bigwedge_n \lim_{k\to\infty} \mu_k\left\{\bar{a}_k \mid \mathcal{M}(\bar{a}_k) \models \left(P\bar{x} \geq r - \frac{1}{n}\right)\left(P\bar{y} \geq s - \frac{1}{n}\right)\psi\right\} > 0$$

(5) $$\bigwedge_n \mathcal{M} \models \left(P\bar{x} \geq r - \frac{1}{n}\right)\left(P\bar{y} \geq s - \frac{1}{n}\right)\psi$$

(6) $$\mathcal{M} \models (P\bar{x} \geq r)(P\bar{y} \geq s)\psi$$

Hence, these statements are equivalent. Each property

$$\mathcal{M}(\bar{z}_k) \models \left(P\bar{x} \geq r - \frac{1}{n}\right)\left(P\bar{y} \geq s - \frac{1}{n}\right)\psi$$

is expressible in $\mathcal{M}$ by a finite quantifier-free formula $\theta(\bar{z}_k)$ of L. It follows, then, that each formula is equivalent to a countable Boolean combination of formulas of the form $(P\bar{z} \geq t)\theta$, where θ is finite and quantifier-free. Finally, we reduce to the case in which θ is a conjunction of atomic formulas using the probability rules

$$P(\neg\varphi) = 1 - P(\varphi),$$

and

$$P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi). \quad \square$$

3.1.5 Corollary. *Let $\mathscr{M}$ and $\mathscr{N}$ be graded probability structures for L. The following are equivalent.*

(a) $\mathscr{M} \equiv_{\omega_1 P} \mathscr{N}$.
(b) $\mathscr{M} \equiv_{\mathbb{A}P} \mathscr{N}$.
(c) $\mathscr{M} \models \varphi$ *iff* $\mathscr{N} \models \varphi$ *for each sentence* φ *of* $L_{\mathbb{A}P}$ *in the normal form of Theorem* 3.1.4. $\square$

The following consequence characterizes $L_{\mathbb{A}P}$ equivalence in terms of truth values in finite samples. It has no analog in first-order logic.

3.1.6 Theorem. *Let $\mathscr{M}$ and $\mathscr{N}$ be graded probability structures for L. The following are equivalent.*

(a) $\mathscr{M}$ *and* $\mathscr{N}$ *are* $L_{\mathbb{A}P}$*-equivalent.*
(b) *For every sentence* φ *of* $L_{\mathbb{A}P}$ *and* $k \in \mathbb{N}$, *we have*

$$\mu_k\{\bar{a}_k \mid \mathscr{M}(\bar{a}_k) \models \varphi\} = \nu_k\{\bar{b}_k \mid \mathscr{N}(\bar{b}_k) \models \varphi\}.$$

That is, φ *has the same probability in the set of k-element samples of* $\mathscr{M}$ *as in the set of k-element samples of* $\mathscr{N}$.
(c) *For each sentence* φ *of* $L_{\mathbb{A}P}$,

$$\lim_{k\to\infty} \mu_k\{\bar{a}_k \mid \mathscr{M}(\bar{a}_k) \models \varphi\} = 1$$

if and only if

$$\lim_{k\to\infty} \nu_k\{\bar{b}_k \mid \mathscr{N}(\bar{b}_k) \models \varphi\} = 1.$$

That is, φ *has large probability in large finite samples in* $\mathscr{M}$ *iff it does in* $\mathscr{N}$.

Proof. We give a proof using Hoover's normal form theorem. The result can also be proved directly from the Weak Law of Large Numbers for $L_{\mathbb{A}P}$. Now, (a) implies (b), because for each k and ψ there is a formula $\psi(v_1, \ldots, v_k)$ of $L_{\mathbb{A}P}$ which says that a k-element sample satisfies φ. It is trivial that (b) implies (c). Assume then that (c) holds, and let $\varphi(\bar{x})$ be a finite quantifier-free formula. Suppose that $\mathscr{M} \models (P\bar{x} \geq r)\varphi(\bar{x})$, and let $s < r$. By the Weak Law of Large Numbers, we have

$$\lim_{k\to\infty} \mu_k\{\bar{a}_k \mid \mathscr{M}(\bar{a}_k) \models (P\bar{x} > s)\varphi\} = 1.$$

By (c), the same holds in $\mathscr{N}$. Applying the Weak Law again, we thus have $\mathscr{N} \models (P\bar{x} \geq s)\varphi$. Since this holds for all $s < r$, $\mathscr{N} \models (P\bar{x} \geq r)\varphi$. It follows from the Normal Form Theorem that $\mathscr{M} \equiv_{\omega_1 P} \mathscr{N}$. $\square$

3.2. Hyperfinite Models

We will assume throughout this section that L has only finitely many constant symbols. We have seen in Example 1.3.2 that the sentence

$$(Px \geq 1)(Py \geq 1)x \neq y,$$

stating that $\mathscr{M}$ is atomless, has no countable models. In this section, we prove an analogue of the Löwenheim–Skolem theorem for atomless probability structures, but with infinite $*$-finite numbers in place of infinite cardinals. We will show that, for each atomless structure $\mathscr{M}$ and infinite $*$-finite number H, there is an essentially unique hyperfinite probability structure $\mathscr{N} \equiv_{\omega_1 P} \mathscr{M}$ of $*$-cardinal H. We will use a fixed ω_1-saturated nonstandard universe.

3.2.1 Definition. A (uniform) *finite probability structure* is a probability structure $\mathscr{M}$ whose universe M is finite and whose measure is the counting measure $\mu(Y) = |Y|/|M|$. A *$*$-finite probability structure* is a finite probability structure in the sense of the nonstandard universe. A *hyperfinite probability structure* is a probability structure $\mathscr{M}$ such that the universe M is an infinite $*$-finite set and μ is the Loeb measure determined by the $*$-counting measure on M. A *hyperfinite graded structure* is a graded probability structure whose universe M is an infinite $*$-finite set and each μ_n is the Loeb measure determined by the $*$-counting measure on M^n.

3.2.2 Proposition. *Every hyperfinite probability structure or graded structure is atomless.* $\square$

Here is a reformulation of Proposition 2.2.4.

3.2.3 Proposition. *Let $\mathscr{M}_0$ be a first-order structure such that the universe M is an infinite $*$-finite set and each relation of $\mathscr{M}_0$ is Loeb measurable with respect to the $*$-counting measure on M^n. Then there is a unique hyperfinite graded structure with first-order part $\mathscr{M}_0$.* $\square$

We will now introduce an important relation between hyperfinite and $*$-finite structures, called a *lifting*.

3.2.4 Definition. Let $\mathscr{M}$ be a hyperfinite graded structure. A *lifting* of $\mathscr{M}$ is a $*$-finite probability structure $\mathscr{N}$ such that $\mathscr{N}$ has the same universe and constants as $\mathscr{M}$, and for each atomic formula $\varphi(\bar{x})$, the set

$$\{\bar{a} \mid \mathscr{M} \models \varphi[\bar{a}] \quad \text{iff} \quad \mathscr{N} \models \varphi[\bar{a}]\}$$

has μ_n-measure one. By a *lifting* of a hyperfinite probability structure $\mathscr{M}$ we mean a lifting of the unique hyperfinite graded structure $\mathscr{M}'$ which has the same first-order part as $\mathscr{M}$.

3.2.5 Lemma. (i) *Every infinite $*$-finite probability structure is a lifting of some hyperfinite graded structure.*
(ii) *Every hyperfinite graded structure has a lifting.*
(iii) *If $\mathscr{M}$ and $\mathscr{N}$ are hyperfinite graded structures with a common lifting, then $\mathscr{M} = \mathscr{N}$ a.s. and $\mathscr{M} \equiv_{\omega_1 P} \mathscr{N}$.*

Proof. The argument for part (i) follows by Proposition 3.2.3. The argument for part (ii) follows by Theorem 2.1.3. And the argument for part (iii) follows by Lemma 2.3.3. □

3.2.6 Theorem (Existence Theorem for Hyperfinite Models (Keisler [1977b])). *Let $\mathscr{N}$ be an atomless probability structure for L, and let M be an infinite $*$-finite set. Then there exists a hyperfinite probability structure $\mathscr{M}$ with universe M which is $L_{\mathbb{A}P}$-equivalent to $\mathscr{N}$.*

Proof. Assume first that L has no constant symbols. Let S be the set of all infinite sequences $\bar{a}$ of elements of $\mathscr{N}$ such that for every finite existential sentence φ of $L_{\mathbb{A}P}$, if $\mathscr{N} \models \varphi$ then $\mathscr{N}(\bar{a}_k) \models \varphi$, for all but finitely many k. By the Strong Law of Large Numbers, $\nu^{\mathbb{N}}(S) = 1$. Since $\mathscr{N}$ is atomless, $\nu^{\mathbb{N}}$ almost every sequence $\bar{a}$ is one-to-one; and, hence, each $\mathscr{N}(\bar{a}_k)$ is a uniform finite probability structure. Thus, there exists $\bar{a} \in S$ such that $\bar{a}$ is one-to-one. Let K be an infinite hyperinteger. Then $\mathscr{N}(\bar{a}_K)$ is a $*$-finite probability structure of $*$-cardinal K and is a lifting of a hyperfinite graded structure $\mathscr{M}'$. Since $\bar{a} \in S$, for each finite quantifier-free sentence $\psi(\bar{x})$ and each r, $\mathscr{N} \models (P\bar{x} > r)\psi$ implies $\mathscr{M}' \models (P\bar{x} \geq r)\psi$. It follows then that, for each ψ and r, $\mathscr{N} \models (P\bar{x} \geq r)\psi$ iff $\mathscr{M}' \models (P\bar{x} \geq r)\psi$. By the Normal Form Theorem, $\mathscr{M}'$ is $L_{\mathbb{A}P}$-equivalent to $\mathscr{N}$. By Theorem 2.3.4, there is a hyperfinite probability structure $\mathscr{M}$ with $\mathscr{M} = \mathscr{M}'$ a.s. Then $\mathscr{M}$ is $L_{\mathbb{A}P}$-equivalent to $\mathscr{N}$.

The case in which L has finitely many constants is the same except that the measure on $*\mathscr{N}(\bar{a}_K)$ is slightly different from the counting measure since constants have measure zero. □

The Existence Theorem also holds for graded probability structures, with the same proof. For $L_{\mathbb{A}P}$ without equality, the existence theorem holds even without the hypothesis that $\mathscr{N}$ is atomless.

3.2.7 Definition. Let $\mathscr{M}$ and $\mathscr{N}$ be probability structures. An *almost sure isomorphism from $\mathscr{M}$ to $\mathscr{N}$* (in symbols, $h: \mathscr{M} \cong \mathscr{N}$ a.s.) is a bijection $h: M \to N$ such that

(a) h is an isomorphism on the probability spaces, $h: \langle M, \mu\rangle \cong \langle N, \nu\rangle$;
(b) for each atomic formula $\varphi(\bar{x})$,

$$\mathscr{M} \models \varphi[\bar{a}] \quad \text{iff} \quad \mathscr{N} \models \varphi[h\bar{a}]$$

almost surely in $\mu^{(n)}$.

3.2.8 Lemma. *Suppose $h: \mathcal{M} \cong \mathcal{N}$ a.s., then*

(i) *for each formula $\varphi(\bar{x})$* of $L_{\mathbb{A}P}$,

$$\mathcal{M} \models \varphi[\bar{a}] \quad \text{iff} \quad \mathcal{N} \models \varphi[h\bar{a}]$$

almost surely in $\mu^{(n)}$;

(ii) *$\mathcal{M}$ and $\mathcal{N}$ are $L_{\mathbb{A}P}$-equivalent.*

Proof. The proof follows by induction on φ. ☐

The following result is new.

3.2.9 Theorem (Uniqueness Theorem for Hyperfinite Models). *Let $\mathcal{M}$ and $\mathcal{N}$ be hyperfinite probability structures with the same universe M. The following are equivalent:*

(a) *$\mathcal{M}$ and $\mathcal{N}$ are $L_{\mathbb{A}P}$-equivalent.*
(b) *There is an $h: \mathcal{M} \cong \mathcal{N}$ a.s.*
(c) *There is an internal h such that $h: \mathcal{M} \cong \mathcal{N}$ a.s.*

Idea of Proof. We assume that (a) holds and prove that (c) must hold also. Note that any internal bijection preserves measure. Consider an n-tuple of atomic formulas $\varphi_1(\bar{y}), \dots, \varphi_n(\bar{y})$ of L and let $\varepsilon > 0$. Using the Rectangle Approximation Lemma (Lemma 2.3.1), one can find a bijection $h_0: M \to M$ such that

$$\mu^{(m)}\left(\bigcap_{k=1}^{n} \{\bar{a} \mid \mathcal{M} \models \varphi_k[\bar{a}] \quad \text{iff} \quad \mathcal{N} \models \varphi_k[h_0 \bar{a}]\}\right) \geq 1 - \varepsilon.$$

The idea is to use Theorem 2.1.3 in choosing an h_0 which approximately preserves each coordinate of the rectangles which approximate φ_k. Now let $\hat{\mathcal{M}}, \hat{\mathcal{N}}$ be liftings of $\mathcal{M}, \mathcal{N}$. By ω_1-saturation, we can find an internal bijection h so that for all atomic $\varphi(\bar{y})$ and all real $\varepsilon > 0$,

$$\mu^{(m)}(\{\bar{a} \mid \hat{\mathcal{M}} \models \varphi[\bar{a}] \quad \text{iff} \quad \hat{\mathcal{N}} \models \varphi[h\bar{a}]\}) \geq 1 - \varepsilon.$$

It follows then that $h: \mathcal{M} \cong \mathcal{N}$ a.s., and thus (c) holds. ☐

As a consequence of the preceding, we obtain a "soft" characterization of the $L_{\mathbb{A}P}$-equivalence relation, namely

3.2.10 Corollary. *Let $\approx$ be an equivalence relation on the atomless probability structures for L such that:*

(a) *If $\mathcal{M} \cong \mathcal{N}$ a.s., then $\mathcal{M} \approx \mathcal{N}$.*
(b) *For each $\mathcal{M}$ and each infinite $*$-finite set H, there is a hyperfinite probability structure $\mathcal{N}$ with universe H such that $\mathcal{M} \approx \mathcal{N}$.*
(c) *If $\mathcal{M} \approx \mathcal{N}$, then $\mathcal{M} \equiv_{\mathbb{A}P} \mathcal{N}$.*

Then $\approx$ is the relation $\equiv_{\mathbb{A}P}$.

3.2.11 Example (D. Hoover, unpublished). This example shows that the uniqueness theorem (Theorem 3.2.9) is false for hyperfinite graded models. Let M be a hyperfinite set of the form $M = A \cup B \cup C \cup D$ where A, B, C, D are disjoint sets with $*$-cardinalities

$$|A| = \tfrac{1}{2}|M|, \qquad |B| = \tfrac{1}{4}|M|, \qquad |C| = |D| = \tfrac{1}{8}|M|.$$

Let f be an internal bijection from C to D. By using an exponential form of Chebyshev's inequality, it can be shown that there is an internal binary relation $R \subseteq A \times (B \cup C \cup D)$ such that:

(1) For all $y \in B \cup C \cup D$,

$$\mu\{x \mid R(x, y)\} = \tfrac{1}{4}.$$

(2) For all $y \in C$,

$$\{x \mid R(x, y)\} = \{x \mid R(x, fy)\}.$$

(3) For all $y, z \in B \cup C \cup D$ with $z \neq y, z \neq fy$,

$$\mu\{x \mid R(x, y) \wedge R(x, z)\} = \tfrac{1}{8}.$$

Let $\mathscr{M}$ and $\mathscr{N}$ be the graded hyperfinite structures with first-order parts $\mathscr{M}_0 = \langle M, B, R\rangle$, and $\mathscr{N}_0 = \langle M, C \cup D, R\rangle$. The reader can check that $\mathscr{M}$ and $\mathscr{N}$ are $L_{\omega_1 P}$-equivalent but for any internal bijection h on M which maps B onto $C \cup D$, the set

$$\{(x, y) \mid R(x, y) \quad \text{iff} \quad R(hx, hy)\}$$

has measure at most $\frac{31}{32}$.

A weak uniqueness theorem for hyperfinite graded models is given in Keisler [1977, p. 34].

3.3. Robinson Consistency and Craig Interpolation

The results of this section are due to Hoover [1978b]. The hyperfinite structures play the same role that saturated structures play in first-order model theory.

3.3.1 Theorem (Robinson Consistency Theorem for $L_{\mathbb{A}P}$). *Let L^1 and L^2 be two languages with $L^0 = L^1 \cap L^2$. Let $\mathscr{M}^1$, $\mathscr{M}^2$ be probability structures for L^1, L^2 respectively whose reducts $\mathscr{M}^1 \upharpoonright L^0$, $\mathscr{M}^2 \upharpoonright L^0$ are $L^0_{\mathbb{A}P}$-equivalent. Then there exists a probability structure $\mathscr{N}$ for $L^1 \cup L^2$ such that*

$$\mathscr{N} \upharpoonright L^1 \equiv_{L^1_{\mathbb{A}P}} \mathscr{M}^1 \quad \text{and} \quad \mathscr{N} \upharpoonright L^2 \equiv_{L^2_{\mathbb{A}P}} \mathscr{M}^2.$$

Proof. We give the proof when L^1 and L^2 have no constants and $\mathscr{M}^1$, $\mathscr{M}^2$ are atomless. The general case will follow by adding a new relation symbol for each atomic formula, and working with the atomless parts. By Theorem 3.2.6, we may take $\mathscr{M}^1$ and $\mathscr{M}^2$ to be hyperfinite probability structures with the same universe M. By Theorem 3.2.9 there is an internal bijection $h: \mathscr{M}^1 \restriction L^0 \cong \mathscr{M}^2 \restriction L^0$ a.s. Renaming elements, we can take h to be the identity map. By changing the L^0 relations of $\mathscr{M}^2$ on a set of measure zero, we get $\mathscr{M}^1 \restriction L^0 = \mathscr{M}^2 \restriction L^0$. Let $\mathscr{N}$ be the common expansion of $\mathscr{M}^1$ and $\mathscr{M}^2$. $\square$

3.3.2 Theorem (Craig Interpolation Theorem for $L_{\mathbb{A}P}$). *Let $L^0 = L^1 \cap L^2$ and let $\varphi^1 \in L^1_{\mathbb{A}P}$, and $\varphi^2 \in L^2_{\mathbb{A}P}$ be sentences such that $\models \varphi^1 \to \varphi^2$. Then there is a sentence $\varphi^0 \in L^0_{\mathbb{A}P}$ such that $\models \varphi^1 \to \varphi^0$, and $\models \varphi^0 \to \varphi^2$.*

Proof. Suppose there is no such φ^0. By a Henkin construction, there then are weak models $\mathscr{M}^1$ of φ^1 and $\mathscr{M}^2$ of $\neg\varphi^2$ for $L^1_{\mathbb{A}P}$ and $L^2_{\mathbb{A}P}$ such that $\mathscr{M}^1 \restriction L^0$ and $\mathscr{M}^2 \restriction L^0$ are $L^0_{\mathbb{A}P}$-equivalent, and all the axioms of $L^1_{\mathbb{A}P}$, $L^2_{\mathbb{A}P}$ are valid. By the completeness proof, we then obtain probability models $\mathscr{N}^1$ of φ^1, $\mathscr{N}$ of $\neg\varphi^2$, where $\mathscr{N}^1 \restriction L^0$ and $\mathscr{N}^2 \restriction L^0$ are $L^0_{\mathbb{A}P}$-equivalent. By Robinson consistency, $\varphi^1 \wedge \neg\varphi^2$ has a model—contradicting $\models \varphi^1 \to \varphi^2$. $\square$

Since the compactness theorem fails for $L_{\mathbb{A}P}$, we cannot apply the general fact that Robinson consistency and compactness implies Craig interpolation. A separate Henkin construction is thus needed. Mundici, in Chapter VIII, showed that for many logics, Robinson consistency implies compactness. The logic $L_{\mathbb{A}P}$ fails to satisfy several of his hypotheses, including closure under universal quantification and under disjoint unions.

Hoover [198?] has recently proved the following stronger interpolation theorem, thus improving an earlier result which appeared in Hoover [1982].

3.3.3 Theorem (Almost Sure Interpolation Theorem). *Let $L^0 = L^1 \cap L^2$ and suppose the symbols of L^0 have a well-ordering in $\mathbb{A}$. Let $\varepsilon > 0$ and let $\varphi^1(\bar{v}) \in L^1_{\mathbb{A}P}$, and $\varphi^2(\bar{v}) \in L^2_{\mathbb{A}P}$ be formulas such that*

$$\models (P\bar{v} \geq 1 - \varepsilon)(\varphi(\bar{v}) \to \psi(\bar{v})).$$

Then, for every $\delta > \varepsilon^{1/4} + \varepsilon^{1/2}$, there is a formula $\theta(\bar{v}) \in L^0_{\mathbb{A}P}$ such that

$$\models (P\bar{v} \geq 1 - \delta)(\varphi(\bar{v}) \to \theta(\bar{v})) \quad \textit{and} \quad \models (P\bar{v} \geq 1 - \delta)(\theta(\bar{v}) \to \psi(\bar{v})).$$

Moreover, if

$$\models (P\bar{v} \geq 1)(\varphi(\bar{v}) \to \psi(\bar{v})),$$

then there is a formula $\theta(\bar{v}) \in L^0_{\mathbb{A}P}$ such that

$$\models (P\bar{v} \geq 1)(\varphi(\bar{v}) \to \theta(\bar{v})) \quad \textit{and} \quad \models (P\bar{v} \geq 1)(\theta(\bar{v}) \to \psi(\bar{v})). \ \square$$

Hoover proved each of the results Sections 3.2.1–3.2.3 for graded probability structures, and the results for probability structures follow. His proof of the Robinson consistency theorem was somewhat more difficult, because the uniqueness theorem for graded hyperfinite structures is false.

Additional model-theoretic results for $L_{\mathbb{A}P}$ are in Keisler [1977b] and Hoover [1982]. Hoover [1981] gives some applications to probability theory. Kaufmann [1978a] in his thesis gave a back-and-forth criterion which is *sufficient* for two graded structures to be $L_{\mathbb{A}P}$-equivalent, and *necessary* for two hyperfinite graded structures to be $L_{\mathbb{A}P}$-equivalent.

3.4. Logic with Integrals

Properties of random variables are usually easier to express using integrals rather than probability quantifiers. We will now introduce a logic $L_{\mathbb{A}\int}$ (from Keisler [1977]), which is equivalent to $L_{\mathbb{A}P}$. It has no quantifiers, although it does have an integral operator which builds terms with bound variables. Indeed, the logics $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$ correspond to two alternative approaches to integration theory: Lebesgue measure theory and the Daniell integral. The completeness proof for $L_{\mathbb{A}P}$ used Loeb's construction of a measure by non-standard analysis, while the completeness proof for $L_{\mathbb{A}\int}$ will use the construction of the Daniell integral as given in Loeb [1982].

Given a relation $R(\bar{x})$, the *indicator function* $\mathbf{1}(R(\bar{x}))$ is defined by

$$\mathbf{1}(R(\bar{x})) = \begin{cases} 1 & \text{if } R(\bar{x}) \text{ is true,} \\ 0 & \text{if } R(\bar{x}) \text{ is false.} \end{cases}$$

The language $L_{\mathbb{A}\int}$ will have atomic terms interpreted as the indicator functions of the atomic formulas of L, and more complex terms will be built from these by applying continuous real functions and integration. The atomic formulas of $L_{\mathbb{A}\int}$ will be inequalities between terms.

3.4.1 Definition. Let L be an $\mathbb{A}$-recursive set of finitary relation and constant symbols. For each atomic formula

$$R(\bar{x}) \quad \text{or} \quad x = y$$

of the first-order logic L, $L_{\mathbb{A}\int}$ has an *atomic term*

$$\mathbf{1}(R(\bar{x})) \quad \text{or} \quad \mathbf{1}(x = y).$$

The set of *terms* of $L_{\mathbb{A}\int}$ is the least set such that:

(a) Every atomic term is a term.
(b) If τ is a term and x is an individual variable, then $\int \tau \, dx$ is a term.
(c) Each real number $r \in \mathbb{A} \cap \mathbb{R}$ is a term.
(d) If $\tau_1, \ldots, \tau_n$ are terms and F belongs to the set $C_{\mathbb{A}}(\mathbb{R}^n)$ of continuous functions $F: \mathbb{R}^n \to \mathbb{R}$ such that $F \restriction \mathbb{Q}^n \in \mathbb{A}$, then $\mathbf{F}(\tau_1, \ldots, \tau_n)$ is a term.

Clause (c) is just the special case of Clause (d) when $n = 0$. We will usually identify the function F and the corresponding logical symbol $\mathbf{F}$. Note that individual variables and constants are not terms of $L_{\mathbb{A}\int}$. The notion of *free and bound* variables in a term are defined as usual, with the integral $\int \tau \, dx$ binding x. A *closed term* is a term with no free variable.

3.4.2 Definition. The set of *formulas of* $L_{\mathbb{A}\int}$ is the least set such that:

(a) For each term τ of $L_{\mathbb{A}\int}$, $\tau \geq 0$ is an atomic formula.
(b) If φ is a formula, so is $\neg\varphi$.
(c) If Φ is a set of formulas with finitely many free variables and $\Phi \in \mathbb{A}$, then $\bigwedge\Phi$ is a formula.

A *sentence* is a formula with no free variables. The structures for $L_{\mathbb{A}\int}$ are the same as the structures for $L_{\mathbb{A}P}$, namely, the probability structures for L.

3.4.3 Definition. Let $\mathcal{M}$ be a probability structure for L. The *value* $\tau(\bar{a})^{\mathcal{M}}$ of a term $\tau(\bar{v})$ of $L_{\mathbb{A}\int}$ in $\mathcal{M}$ at a tuple $\bar{a}$ in M is defined by:

(a) $\mathbf{1}(\varphi(\bar{a}))^{\mathcal{M}} = \begin{cases} 1 & \text{if } \mathcal{M} \models \varphi[\bar{a}], \\ 0 & \text{if } \mathcal{M} \models \neg\varphi[\bar{a}] \end{cases}$

for each atomic formula $\varphi(\bar{v})$ of $L_{\omega\omega}$.
(b) $(\int \tau(x, \bar{a}) \, dx)^{\mathcal{M}} = \int \tau(b, \bar{a})^{\mathcal{M}} \, d\mu(b)$.
(c) $r^{\mathcal{M}} = r$.
(d) $\mathbf{F}(\tau_1, \ldots, \tau_n)(\bar{a})^{\mathcal{M}} = F(\tau_1(\bar{a})^{\mathcal{M}}, \ldots, \tau_n(\bar{a})^{\mathcal{M}})$.

Since each atomic term has values in $\{0, 1\}$, by induction we see that each function $\tau(\bar{a})^{\mathcal{M}}$ is bounded and $\mu^{(k)}$-measurable. In particular, the integral in Part (b) exists and is finite. The *satisfaction relation* $\mathcal{M} \models \varphi[\bar{a}]$ for $L_{\mathbb{A}\int}$ is defined in the natural way, with the atomic formula rule

$$\mathcal{M} \models (\tau(\bar{u}) \geq 0)[\bar{a}] \quad \text{iff} \quad \tau(\bar{a})^{\mathcal{M}} \geq 0.$$

3.5. Completeness Theorem with Integrals

3.5.1 Definition. The axioms for $L_{\mathbb{A}\int}$, where σ, τ are terms, r, s are elements of $\mathbb{A} \cap \mathbb{R}$, and x, y are individual variables are:

D1. All axiom schemes for $L_{\mathbb{A}}$ without quantifiers, with $\mathbf{1}(x = y) = 1$ in place of $x = y$.
D2. For each atomic term τ, we have $\tau = 0 \vee \tau = 1$.
D3. *Order axioms.* Using abbreviations, we have
 (i) $r \geq r$.
 (ii) $\tau \geq r \rightarrow \tau \geq s$, when $r \geq s$.
D4. For each rational closed rectangle $S \subseteq \mathbb{R}^m$, and each $F \in C_{\mathbb{A}}(\mathbb{R}^m)$ with image $F(S) = [a, b]$, we have

$$\langle \tau_1, \ldots, \tau_m \rangle \in S \rightarrow \mathbf{F}(\tau_1, \ldots, \tau_M) \in [a, b].$$

D5. *Integral axioms.*

(i) $\int r\,dx = r$.

(ii) $\int \tau(x)\,dx = \int \tau(y)\,dy$.

(iii) $\iint \tau(x, y)\,dx\,dy = \iint \tau(x, y)\,dy\,dx$.

(iv) $\int (r \cdot \sigma + s \cdot \tau)\,dx = r \cdot \int \sigma\,dx + s \cdot \int \tau\,dx$.

D6. *Archimedean axiom.*

$$\tau > r \leftrightarrow \bigvee_n \tau \geq r + \frac{1}{n}.$$

D7. *Product measurability.* For each $m \in \mathbb{N}$, we have

$$\bigvee_k \int F_k\left(1 - \int F_m\left(\int |\tau(\bar{x}, \bar{z}) - \tau(\bar{y}, \bar{z})|\,d\bar{z}\right) d\bar{y}\right) d\bar{x} \geq 1 - \frac{1}{m},$$

where $d\bar{x}$ is $dx_1 \ldots dx_n$, and

$$F_k(u) = \begin{cases} 0 & \text{if } u \leq 1/k, \\ \text{linear for} & 1/k \leq u \leq 2/k, \\ 1 & \text{if } u \geq 2/k. \end{cases}$$

This is essentially a translation of the axiom B4 for $L_{\mathbb{A}P}$.

3.5.2 Definition. The rules of inference for $L_{\mathbb{A}\int}$ are:

S1. *Modus ponens*: $\varphi, \varphi \to \psi \vdash \psi$.

S2. *Conjunction*: $\{\varphi \to \psi \mid \psi \in \Psi\} \vdash \varphi \to \bigwedge\Psi$.

S3. *Generalization*: $\varphi \to (\tau(x) \geq 0) \vdash \varphi \to (\int \tau(x)\,dx \geq 0)$, where x is not free in φ.

This set of axioms was motivated by the thesis of Rodenhausen [1982].

3.5.3 Theorem (Soundness Theorem for $L_{\mathbb{A}\int}$). *Any set Φ of sentences for $L_{\mathbb{A}\int}$ which has a model is consistent.* ☐

3.5.4 Theorem (Completeness Theorem for $L_{\mathbb{A}\int}$). *A countable consistent set Φ of sentences of $L_{\mathbb{A}\int}$ has a model.* ☐

Idea of Proof. As in the case of $L_{\mathbb{A}P}$, the main steps are to prove a weak completeness theorem, and then use a construction from non-standard analysis to obtain a graded model of Φ. This done, the product measurability axiom is then used to get a model of Φ.

A *weak structure for* $L_{\mathbb{A}\int}$ is a structure

$$\mathscr{M} = \langle M, R_i, c_j, I \rangle,$$

where $\langle M, R_i, c_j \rangle$ is a first-order structure and I is a positive linear real function on the set of terms of $L_{\mathbb{A}\int}$ with at most one free variable x and parameters from M. That is,

(1) $I(r) = r$.
(2) $I(r \cdot \sigma + s \cdot \tau) = r \cdot I(\sigma) + s \cdot I(\tau)$.
(3) If $\tau(b, \bar{a})^{\mathscr{M}} \geq 0$, for all $b \in M$, then $I(\tau(x, \bar{a})) \geq 0$.

The recursive definition of $\tau(\bar{a})^{\mathscr{M}}$ is the same as for ordinary probability structures but with the integral clause

$$\left(\int \tau(x, \bar{a})\, dx\right)^{\mathscr{M}} = I(\tau(x, \bar{a})).$$

A Henkin argument is used to construct a weak model of Φ in which each axiom of $L_{\mathbb{A}\int}$ is valid. Then the internal structure $\mathscr{M}$ is formed in the non-standard universe. The Daniell integral construction of Loeb [1984] produces a probability measure μ on $^*M$ such that for each $*$-term $\tau(x)$, the standard part of $^*I(\tau)$ is the integral $\int {}^{\circ}\tau(b)^{\mathscr{M}}\, d\mu(b)$. Define measures μ_n on $^*M^n$ using iterated integrals. This yields a graded model of Φ,

$$\hat{\mathscr{M}} = \langle {}^*M, {}^*R_i, {}^*c_j, \mu_n \rangle,$$

which satisfies the produce measurability axiom almost everywhere. Finally, Theorem 2.3.4 is used, as before, to obtain a probability model $\mathscr{N}$ of Φ. $\square$

3.6. Conservative Extension Theorem

We will now show that the logics $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$ are equivalent in a strong sense. This is done by considering their common extension $L_{\mathbb{A}P\int}$.

3.6.1 Definition. $L_{\mathbb{A}P\int}$ is the language which has all the symbols and formation rules of $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$. The satisfaction relation in probability structures is defined as before.

3.6.2 Theorem. *$L_{\mathbb{A}P\int}$ is a conservative definitional extension of both $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$. That is*:

(i) (Conservative): *For any sentence φ in $L_{\mathbb{A}P}$, we have $L_{\mathbb{A}P\int} \models \varphi$ iff $L_{\mathbb{A}P} \models \varphi$. And, for any φ in $L_{\mathbb{A}\int}$, we have $L_{\mathbb{A}P\int} \models \varphi$ iff $L_{\mathbb{A}\int} \models \varphi$.*
(ii) (Definitional): *For each $\varphi(\bar{v})$ in $L_{\mathbb{A}P\int}$, there are $\psi(\bar{v})$ in $L_{\mathbb{A}P}$ and $\theta(\bar{v})$ in $L_{\mathbb{A}\int}$ such that $L_{\mathbb{A}P\int} \models \varphi(\bar{v}) \leftrightarrow \psi(\bar{v})$, and $L_{\mathbb{A}P\int} \models \varphi(\bar{v}) \leftrightarrow \theta(\bar{v})$.*

Remark. The $L_{\mathbb{A}\int}$ case is given in Keisler [1977b] and the $L_{\mathbb{A}P}$ case in Hoover [1978b]. In his work Hoover also gave an axiom set and completeness theorem for $L_{\mathbb{A}P\int}$.

Idea of Proof of Theorem 3.6.2. The proof of part (i) is trivial. Concerning the proof of part (ii), interpretations

$$f: L_{\mathbb{A}P\int} \to L_{\mathbb{A}P} \quad \text{and} \quad g: L_{\mathbb{A}P\int} \to L_{\mathbb{A}\int}$$

can be defined first for atomic formulas $\tau \geq r$ by induction on terms τ, and then by induction on formulas. The idea is to formalize the definitions of integral in terms of measure and vice versa. It is important in this result that $\omega \in \mathbb{A}$, so that the appropriate limits can be expressed in $L_{\mathbb{A}\int}$ and $L_{\mathbb{A}P}$. The finitary analogs of $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$ do not seem to be equivalent. ◻

This theorem often allows one to convert a theorem about $L_{\mathbb{A}P}$ to a similar theorem about $L_{\mathbb{A}\int}$, and vice versa.

3.6.3 Corollary (Keisler [1977b] and Hoover [1978b]). *Let $\mathcal{M}$ and $\mathcal{N}$ be probability structures for L. The following are equivalent:*

(a) $\mathcal{M} \equiv_{\mathbb{A}P} \mathcal{N}$.
(b) $\mathcal{M} \equiv_{\mathbb{A}\int} \mathcal{N}$.
(c) *For each closed term τ of $L_{\mathbb{A}\int}$, $\tau^{\mathcal{M}} = \tau^{\mathcal{N}}$.*

Proof. The proof of this result follows from the conservative extension and normal form theorems. ◻

The Barwise completeness and compactness theorems also hold for $L_{\mathbb{A}\int}$. For these one must check that the interpretation functions f and g in the proof of Theorem 3.6.2 are $\mathbb{A}$-recursive.

3.6.4 Theorem (Finite Compactness Theorem (Keisler [1977b])). *Let Φ be a set of sentences of $L_{\mathbb{A}\int}$ of the form $\tau \in [r, s]$. If every finite subset of Φ has a graded model, then Φ has a graded model.*

Proof. The proof follows by an ultraproduct construction. ◻

The Strong Law of Large Numbers takes a particularly nice form for $L_{\mathbb{A}\int}$.

3.6.5 Theorem (Strong Law of Large Numbers for $L_{\mathbb{A}\int}$). *For any (graded) probability structure $\mathcal{M}$ and term τ with no variables in $L_{\mathbb{A}\int}$,*

$$\lim_{k \to \infty} \tau^{\mathcal{M}(\bar{a}_k)} = \tau^{\mathcal{M}}$$

for $\mu_{\mathbb{N}}$-almost all sequences $\bar{a} \in M^{\mathbb{N}}$.

3.6.6 Definition. When the product measurability axiom, (Axiom D7), is omitted, we obtain the logic *graded* $L_{\mathbb{A}\int}$. Satisfaction in graded probability structures is defined in the natural way.

All the results in Sections 3.5 and 3.6 hold for graded $L_{\mathbb{A}\int}$ and graded $L_{\mathbb{A}P}$.

4. Logic with Conditional Expectation Operators

The logic $L_{\mathbb{A}P}$ is not rich enough to express many basic notions from probability theory, notions such as martingale, Markov process, and stopping time. The missing ingredient here is the concept of conditional expectation. In this section, we will develop an enriched language in which these notions can be expressed. It is easier to work with logic having integral operators rather than with probability quantifiers when we add conditional expectations.

4.1. Random Variables and Integrals

We first prepare to extend our logic by introducing a random variable form of $L_{\mathbb{A}\int}$ which is equivalent to the random variable logic $L_{\mathbb{A}P}(\mathbb{R})$ of Section 2.5. In $L_{\mathbb{A}\int}$, each term $\tau(\bar{v})$ is interpreted by an n-fold random variable $\tau^{\mathcal{M}}[\bar{a}]$, and the atomic terms have values in $\{0, 1\}$. In the new logic $L_{\mathbb{A}\int}(\mathbb{R})$ the atomic terms are allowed to have values in $\mathbb{R}$. Let L be the language $L = \{X_i, c_j \mid i \in I, j \in J\}$.

4.1.1 Definition. The logic $L_{\mathbb{A}\int}(\mathbb{R})$ has atomic terms

$$[X_i(\bar{u}) \upharpoonright r], \qquad \mathbf{1}(u_1 = u_2),$$

where $\bar{u}$ is a tuple of constants or variables and $r \in \mathbb{Q}^+$. The set of *terms* and *formulas* of $L_{\mathbb{A}\int}(\mathbb{R})$ is defined exactly as for $L_{\mathbb{A}\int}$.

The structures for $L_{\mathbb{A}\int}(\mathbb{R})$ are the random variable structures

$$\mathcal{M} = \langle M, X_i^{\mathcal{M}}, c_j^{\mathcal{M}}, \mu\rangle$$

as defined in Section 2.5.

4.1.2 Definition. The *value* $\tau(\bar{a})^{\mathcal{M}}$ of a term $\tau(\bar{v})$ of $L_{\mathbb{A}\int}(\mathbb{R})$ in a random variable structure $\mathcal{M}$ is defined as for $L_{\mathbb{A}\int}$ except for the following new rule for atomic terms:

$$[X_i(\bar{a}) \upharpoonright r]^{\mathcal{M}} = \begin{cases} r & \text{if } X_i^{\mathcal{M}}(\bar{a}) \geq r, \\ -r & \text{if } X_i^{\mathcal{M}}(\bar{a}) \leq -r, \\ X_i^{\mathcal{M}}(\bar{a}) & \text{otherwise.} \end{cases}$$

Thus, $[X_i(\bar{a}) \upharpoonright r]^{\mathcal{M}}$ is equal to $X_i^{\mathcal{M}}(\bar{a})$ truncated at r. The reason the atomic terms are truncated is so that each term will be interpreted by a bounded, and hence integrable, random variable.

4.1.3 Definition. *The axioms and rules of inference of* $L_{\mathbb{A}\int}(\mathbb{R})$ are exactly the same as for $L_{\mathbb{A}\int}$ except that the atomic term axiom, (Axiom D2), is replaced by the following list of axioms, where $\bar{u}$ is an n-tuple of constants or variables.

E1. $\mathbf{1}(u = v) = 0 \vee \mathbf{1}(u = v) = 1$.
E2. $[X_i(\bar{u}) \upharpoonright s] = \min(s, \max(-s, [X(\bar{u}) \upharpoonright r]))$ when $0 \leq s \leq r$.
E3. $\bigvee_k (|[X_i(\bar{u}) \upharpoonright k + 1]| \leq k)$.

This says that $X_i(\bar{u})$ is finite.

E4. For each $m \in \mathbb{N}$,

$$\bigvee_k \int |[X_i(\bar{u}) \upharpoonright k + 1] - [X_i(\bar{u}) \upharpoonright k]| \, d\bar{u} \leq \frac{1}{m}.$$

(The probability that $|X_i(\bar{u})| \geq k$ approaches zero as $k \to \infty$.)

We state the main facts without proof.

4.1.4. Theorem (Soundness and Completeness Theorem). *A countable set* Φ *of sentences of* $L_{\mathbb{A}\int}(\mathbb{R})$ *has a model if and only if it is consistent.* ∎

4.1.5 Theorem. *The logics* $L_{\mathbb{A}\int}(\mathbb{R})$ *and* $L_{\mathbb{A}P}(\mathbb{R})$ *for random variable structures have a common conservative definitional extension.* ∎

In other words, $L_{\mathbb{A}\int}(\mathbb{R})$ and $L_{\mathbb{A}P}(\mathbb{R})$ are equivalent logics. Those logics may be generalized to study random variables with values in a Polish space $\mathbb{S}$ instead of in $\mathbb{R}$. The only changes needed are in the definition and axioms for atomic formulas of $L_{\mathbb{A}P}(\mathbb{S})$ and atomic terms of $L_{\mathbb{A}\int}(\mathbb{S})$.

4.2. Conditional Expectations

We will introduce the logic $L_{\mathbb{A}E}(\mathbb{R})$ by adding a conditional expectation operator E to the logic $L_{\mathbb{A}\int}(\mathbb{R})$. A structure for $L_{\mathbb{A}E}(\mathbb{R})$ has the form $(\mathscr{M}, \mathscr{F})$ where $\mathscr{M}$ is a random variable structure and $\mathscr{F}$ is a σ-algebra of measurable subsets of M. We first review the notion of conditional expectation.

4.2.1 Definition. Let $\langle M, S, \mu \rangle$ be a probability space, let $\mathscr{F}$ be a σ-subalgebra of S, and let $g: M \to \mathbb{R}$ be bounded and measurable. A *conditional expectation of* g *with respect to* $\mathscr{F}$ is an $\mathscr{F}$-measurable function $h: M \to \mathbb{R}$ such that for all $B \in \mathscr{F}$, $\int_B g \, d\mu = \int_B h \, d\mu$. It is denoted by $h = E[g | \mathscr{F}]$, or $h(x) = E[g(\cdot) | \mathscr{F}](x)$.

4.2.2 Proposition. (i) *The conditional expectation* $h(x) = E[g(\cdot)|\mathscr{F}](x)$ *exists and is almost surely unique in the sense that any two such functions are equal except on a null set.* ∎

This is a standard consequence of the Radon–Nikodym theorem. Here, h is the Radon–Nikodym derivative of the measure $\nu(B) = \int_B g\, d\mu$, for $B \in \mathscr{F}$, with respect to $\mu \restriction \mathscr{F}$.

We now introduce the logic $L_{\mathbb{A}E}(\mathbb{R})$.

4.2.3 Definition. The logic $L_{\mathbb{A}E}(\mathbb{R})$ has all the formation rules of $L_{\mathbb{A}\int}(\mathbb{R})$ plus the *a* term-builder, the *conditional expectation operator*: If $\tau(u, \bar{v})$ is a term and u, w are individual variables (with u not in $\bar{v}$), then

$$E[\tau(u, \bar{v})\,|\,u](w)$$

is a term in which the occurrences of u are bound and w is free.

This logic has not been considered before in the literature.

4.2.4 Definition. We will use the abbreviation:

$$E[\tau(u, \bar{v})\,|\,u] \quad \text{for} \quad E[\tau(u, \bar{v})\,|\,u](u).$$

Thus, u is free in $E[\tau(u, \bar{v})\,|\,u]$.

The values of a term of $L_{\mathbb{A}E}(\mathbb{R})$ are only almost surely unique in a structure. Here are the details.

4.2.5 Definition. A *conditional expectation structure* for L is a pair $\mathscr{M} = (\mathscr{M}_0, \mathscr{F})$, where $\mathscr{M}_0$ is a random variable structure and $\mathscr{F}$ is a σ-field of μ-measurable sets.

4.2.6 Definition. An *interpretation* of $L_{\mathbb{A}E}(\mathbb{R})$ in a conditional expectation structure $\mathscr{M}$ assigns to each term $\tau(u_1, \ldots, u_n)$ a $\mu^{(n)}$-measurable function $\tau^{\mathscr{M}}: M^n \to \mathbb{R}$ such that

(a) The clauses of the definition of $\tau(\bar{a})^{\mathscr{M}}$ for $L_{\mathbb{A}\int}(\mathbb{R})$ hold.
(b) $(E[\tau(\bar{a}, v)\,|\,v](b))^{\mathscr{M}}$ is $\mu^{(n)} \times \mathscr{F}$-measurable and, for each $\bar{a} \in M^n$,

$$(E[\tau(\bar{a}, v)\,|\,v](b))^{\mathscr{M}} = E[\tau(\bar{a}, \cdot)^{\mathscr{M}}\,|\,\mathscr{F}](b)$$

for μ-almost all b.

4.2.7 Lemma. *For every conditional expectation structure $\mathscr{M}$ for L, there exists an interpretation of $L_{\mathbb{A}E}(\mathbb{R})$ in $\mathscr{M}$, and two interpretations agree almost surely on each term. The values of closed terms and sentences in $\mathscr{M}$ are the same for all interpretations.* ☐

4.2.8 Definition. The logic $L_{\mathbb{A}E}(\mathbb{R})$ has all the axiom schemes and rules of inference of $L_{\mathbb{A}\int}(\mathbb{R})$ as well as:

F1. $E[\tau(x, \bar{v})\,|\,x](w) = E[\tau(y, \bar{v})\,|\,y](w)$ where x and y do not occur in $\bar{v}$.
F2. $\int E[\sigma(u)\,|\,u] \cdot \tau(u)\, du = \int E[\sigma(u)\,|\,u] \cdot E[\tau(u)\,|\,u]\, du$. This formalizes the definition of conditional expectation.

4.2.9 Definition. The *bound* $\|\tau\|$ of a term τ of $L_{\mathbb{A}E}$ is defined by:

(i) $\|[X_i(\bar{u}) \restriction r]\| = r$.
(ii) $\|\mathbf{1}(x = y)\| = 1$.
(iii) $\|\int \tau \, dx\| = \|\tau\|$.
(iv) $\|E[\tau | u](w)\| = \|\tau\|$.
(v) $\|F(\tau_1, \ldots, \tau_n)\| = \sup\{|F(s_1, \ldots, s_n)| \mid |s_i| \le \|\tau_i\|\}$.
(vi) $\|r\| = r$.

4.2.10 Lemma. $|\tau(\bar{a})^{\mathscr{M}}| \le \|\tau\|$. □

4.2.11 Theorem (Soundness and Completeness Theorem for $L_{\mathbb{A}E}(\mathbb{R})$). *A countable set of sentences of $L_{\mathbb{A}E}(\mathbb{R})$ has a model if and only if it is consistent.*

Proof. The proof of soundness is easy. Let Φ be consistent. Form a new language $K \supseteq L$ by adding a new random variable symbol $X_\tau(\bar{v})$ for each term $\tau(\bar{v})$ of $L_{\mathbb{A}E}(\mathbb{R})$ of the form $E[\sigma | u](w)$. Each such term $\tau(\bar{v})$ translates to the atomic term $[X_\tau(\bar{v}) \restriction n]$ where $n \ge \|\tau\|$; and, hence, each term and sentence of $L_{\mathbb{A}E}(\mathbb{R})$ has a translation in $K_{\mathbb{A}\int}(\mathbb{R})$. Let Ψ be the theory in $K_{\mathbb{A}\int}(\mathbb{R})$ consisting of: all translations of sentences of Φ, all translations of theorems of $L_{\mathbb{A}E}(\mathbb{R})$, and

$$(P\bar{v} \ge 1)[X_\tau(\bar{v}) \restriction r] = [X_\tau(\bar{v}) \restriction s], \qquad r, s \ge \|\tau\|.$$

Ψ is consistent in $K_{\mathbb{A}\int}(\mathbb{R})$ and has a random variable model $\mathscr{M}$. Let $\mathscr{F}$ be the σ-algebra on M generated by the sets

$$\{d \in M \mid X_\tau(\bar{c}, d)^{\mathscr{M}} \ge r\},$$

where $\tau(\bar{u}, w)$ has the form $E[\sigma(\bar{u}, v) | v](w)$ and $\bar{c}$ is in M. Let $\mathscr{M}_0$ be the reduct of $\mathscr{M}$ to L and $\mathscr{N} = (\mathscr{M}_0, \mathscr{F})$. Using the axioms, it can be shown by induction on τ that if τ has translation σ, then $\tau(\bar{a})^{\mathscr{N}} = \sigma^{\mathscr{M}}(\bar{a})$ is an interpretation of $L_{\mathbb{A}E}(\mathbb{R})$ in $\mathscr{N}$, and thus $\mathscr{N}$ is a model of Φ. □

Our treatment of $L_{\mathbb{A}E}(\mathbb{R})$ can be readily extended to logics with two or more conditional expectation operators and to logics with conditional expectation operators on n variables. A case of particular interest is two operators E_1 and E_2 where one σ-algebra is to be contained in another. The author's student, S. Fajardo, has proved the following.

4.2.12 Theorem. *Let Φ be a countable set of sentences in $L_{\mathbb{A}E}(\mathbb{R})$ with two conditional expectation operators. Φ has a model $\mathscr{M} = (\mathscr{M}_0, \mathscr{F}_1, \mathscr{F}_2)$, with $\mathscr{F}_1 \subseteq \mathscr{F}_2$ if and only if Φ is consistent in $L_{\mathbb{A}E}$ with the additional axiom scheme*

$$E_1[\tau(u, \bar{v}) | u] = E_2[E_1[\tau(u, \bar{v}) | u] | u]. \quad \square$$

4.3. Adapted Probability Logic

We now consider a special case of the logic $L_{\mathbb{A}E}(\mathbb{R})$, a case that is appropriate for the study of stochastic processes. Throughout this section we will assume that μ is a probability measure on M and β is the Borel measure on $[0, 1]$. By a (continuous time) *stochastic process* we mean a $(\mu \otimes \beta)$-measurable function

$$X: M \times [0, 1] \to \mathbb{R}.$$

In probability theory, the evolution of a stochastic process over time is studied by means of an adapted probability space as defined below.

4.3.1 Definition. If $B \subseteq M \times [0, 1]$ and $t \in [0, 1]$, the *section* B_t is the set $B_t = \{w \in M \mid \langle w, t\rangle \in B\}$.

4.3.2 Definition. An *adapted* (*probability*) *space* (or stochastic base) is a structure

$$\mathscr{S} = \langle M, \mu, \mathscr{F}_t\rangle_{t \in [0,1]},$$

where:

(a) μ is a probability measure on M.
(b) Each $\mathscr{F}_t$ is a σ-algebra of μ-measurable subsets of M.
(c) For each $t \in [0, 1]$, $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$, that is, $\mathscr{F}_t$ is increasing and right continuous.

The family of σ-algebras $\langle \mathscr{F}_t \mid t \in [0, 1]\rangle$ is called the *filtration* of $\mathscr{S}$. Intuitively, M is the set of possible states of the world, and a set $B \subseteq M$ belongs to $\mathscr{F}_t$ if B is an event whose outcome is determined at or before time t.

Adapted spaces have been extensively studied in the literature (see, for example, Dellacherie–Meyer [1981], or Metivier–Pellaumail [1980]).

4.3.3 Definition. Let L be a set of "stochastic process" symbols X_i, $i \in I$. An *adapted* (*probability*) *structure* for L is a structure

$$\mathscr{M} = \langle M, X_i, \mu, \mathscr{F}_t\rangle_{i \in I, t \in [0,1]},$$

such that $\langle M, \mu, \mathscr{F}_t\rangle$ is an adapted space and each $X_i^{\mathscr{M}}: M \times [0, 1] \to \mathbb{R}$ is a stochastic process on $\langle M, \mu\rangle$.

4.3.4 Definition. The *adapted probability logic* $L_{\mathbb{A}\text{ad}}(\mathbb{R})$, or more briefly L_{ad}, is a two-sorted form of $L_{\mathbb{A}E}(\mathbb{R})$ with just one variable w of the first sort, and countably many variables $t_1, t_2, \ldots$ of the second sort, called *time variables*. The non-logical symbols of L are stochastic process symbols X_i with just one argument place of each sort. L_{ad} has no equality symbol. This logic was introduced in Keisler [1979] and has been studied further in Rodenhausen [1982].

4.3.5 Definition. The *terms* of L_{ad} are as follows, where s, t are time variables.

(a) For each $r \in \mathbb{Q}^+$, $[X_i(w, t){\restriction}r]$ is an atomic term.
(b) Each time variable t is a term.
(c) Each real $r \in \mathbb{A} \cap \mathbb{R}$ is a term.
(d) If τ is a term, so are

$$\int \tau \, dw, \quad \int \tau \, ds, \quad E[\tau \mid s](w, t).$$

(e) If $\tau_1, \ldots, \tau_n$ are terms and $F \in C_{\mathbb{A}}(\mathbb{R}^n)$, then $F(\tau_1, \ldots, \tau_n)$ is a term.

For each term τ, $\tau \geq 0$ is an atomic formula, and formulas are closed under $\neg$ and $\wedge$.

4.3.6 Definition. The adapted structure

$$\mathscr{M} = \langle M, X_i, \mu, \mathscr{F}_t \rangle$$

for L is identified with the two-sorted conditional expectation structure

$$\mathscr{M} = \langle M, [0, 1], X_i^{\mathscr{M}}, \mu, \beta, \mathscr{F} \rangle.$$

Here, β is Borel measure on $[0, 1]$, and $\mathscr{F}$ is the σ-algebra on $M \times [0, 1]$ generated by the set of $(\mu \otimes \beta)$-measurable sets B such that for each t, $B_t \in \mathscr{F}_t$ and $B_t = \bigcap_{s>t} B_s$. $\mathscr{F}$ is called the *optional σ-algebra*.

4.3.7 Definition. The notion of an interpretation $\tau^{\mathscr{M}}$ of a term $\tau(w, \bar{t})$ in an adapted structure $\mathscr{M}$ is defined as in Definition 4.2.6 for $L_{\mathbb{A}E}(\mathbb{R})$, but with the following stronger clause for the conditional expectation operator.

For each term $\tau(w, s, \bar{b})$ with n parameters $\bar{b}$ from $[0, 1]$, (b1) through (b3) hold:

(b1) $(E[\tau(w, s, \bar{b}) \mid s](w, a))^{\mathscr{M}}$ *is* $\mathscr{F} \otimes \beta^n$-measurable.
(b2) For each $\bar{b}$, $(E[\tau(w, s, \bar{b}) \mid s](w, a))^{\mathscr{M}} = E[\tau(\cdot, \cdot, \bar{b})^{\mathscr{M}} \mid \mathscr{F}](w, a)$ $(\mu \otimes \beta)$-almost surely.
(b3) For each $\bar{b}$ and $a \in [0, 1]$, $(E[\tau(w, s, \bar{b}) \mid s](w, a))^{\mathscr{M}} = E[\tau(\cdot, a, \bar{b})^{\mathscr{M}} \mid \mathscr{F}_a](w)$ μ-almost surely.

4.3.8 Lemma. *Every adapted structure $\mathscr{M}$ has an interpretation $\tau \mapsto \tau^{\mathscr{M}}$. For each term $\tau(w, \bar{s})$ and all tuples $\bar{a}$ in $[0, 1]$, any two interpretations agree at $\tau(w, \bar{a})$ for μ-almost all w. In particular, if w is not free in $\tau(\bar{a})$, then any two interpretations in $\mathscr{M}$ agree at $\tau(\bar{a})$ for all $\bar{a}$ in $[0, 1]$.*

Idea of Proof. The main difficulty here lies in proving the existence of an interpretation by induction on τ, at the conditional expectation step. We use the fact that for any random variable $f(w)$, $E[f(\cdot) \mid \mathscr{F}_t](w)$ has a right continuous version, and any right continuous process is measurable in the optional σ-algebra $\mathscr{F}$ (see Dellacherie–Meyer [1981]). This done, we then show that $E[g(\cdot, t) \mid \mathscr{F}_t](w)$ is $\mathscr{F}$-measurable by applying the monotone class theorem. $\square$

Remark. In case L is the empty language, the adapted structures for L are just the adapted spaces. In this case, the value of each term $\tau^{\mathcal{M}}(w, \bar{a})$ depends only on $\bar{a}$ and not on w or $\mathcal{M}$.

4.4. Examples

As an indication of the expressive power of the logic L_{ad}, we formalize some central notions from the theory of stochastic processes. In each example, the process $\tau^{\mathcal{M}}$ has the stated property if and only if the formula holds for all s, t in $[0, 1]$. We use the abbreviations

$$E[\tau|s] \quad \text{for} \quad E[\tau|s](w, s),$$

and

$$\sigma(s) = \tau(s) \text{ a.s.} \quad \text{for} \quad \int |\sigma(s) - \tau(s)|\, dw \leq 0.$$

(1) $\sigma(s)$ is a version of $\tau(s)$: $\sigma(s) = \tau(s)$ a.s.
(2) $\tau(s)$ is adapted: $\tau(s) = E[\tau(s)|s]$ a.s.
(3) $\tau(s)$ is a martingale: $s \leq t \rightarrow \tau(s) = E[\tau(t)|s]$ a.s.; recall that s and t are terms of L_{ad}.
(4) $\tau(s)$ is a submartingale: $\tau(s) \leq E[\tau(t)|s]$ a.s.
(5) $\tau(s)$ is Markov process with continuous transition function $F \mapsto T_F$ (a Feller process): For each $F \in C_{\mathbb{A}}(\mathbb{R})$,

$$s \leq t \rightarrow E[F(\tau(t))|s] = T_F(s, t, \tau(s)) \text{ a.s.}$$

(6) $\tau(w)$ is a stopping time: $\min(\tau, s) = E[\min(\tau, s)|s]$ a.s.
(7) X is a Brownian motion (X is not bounded, so this would have to be modified to fit within the language L_{ad}):
 (a) X is a martingale
 (b) $s = 0 \rightarrow X(s) = 0$ a.s.
 (c) $s \leq t \rightarrow E[(X(t) - X(s))^2|s] = t - s$ a.s.
 (d) $s \leq t \rightarrow E[F(X(t) - X(s))|s] = \int F(X(t) - X(s))\, dw$ a.s.; that is, $X(t) - X(s)$ is independent of $\mathcal{F}_s$.

4.5. Axioms and Completeness

4.5.1 Definition. The logic L_{ad} has all the axiom schemes and rules of inference for two-sorted $L_{\mathbb{A}E}(\mathbb{R})$ (with only one variable of the first sort, and E applied to one variable of each sort) as well as:

G1. For any $F \in C_{\mathbb{A}}([0, 1])$ with $\int_0^1 F(x)\, dx = r$,

$$\int F(t)\, dt = r.$$

G2. $s \le t \to E[\tau|s] = E[E[\tau|s]|t]$; that is, $s \le t$ implies $\mathscr{F}_s \subseteq \mathscr{F}_t$.
G3. $\bigwedge_m \bigvee_n \iiint |\tau(w, s) - \tau(w, t)| \cdot \max(0, 1 - |s - t| \cdot n)\, ds\, dt\, dw \le 1/m \cdot n$. That is, τ is $(\mu \otimes \beta)$-measurable. Intuitively, on a small diagonal strip $\{\langle w, s, t\rangle : |s - t| \le 1/n\}$, $\tau(w, s)$ is usually close to $\tau(w, t)$.

This set of axioms is essentially due to Rodenhausen [1982].

4.5.2 Theorem (Soundness and Completeness Theorem for L_{ad} (Rodenhausen [1982])). *A countable set Φ of sentences of L_{ad} has a model if and only if it is consistent.* ☐

The proof of Rodenhausen is direct and quite long. A fairly short alternative proof can be given using the completeness theorem for the two-sorted logic $L_{\mathbb{A}E}(\mathbb{R})$. The idea is to add an extra stochastic process symbol $I(t)$ to L to represent the term t. A two-sorted model $\mathscr{M}$ for $L_{\mathbb{A}E}$ is made into an adapted model by using $I^{\mathscr{M}}$ to replace the second universe of $\mathscr{M}$ by $[0, 1]$. The extra axioms G1 through G3 are needed at that point.

The Barwise completeness and compactness theorems, and the finite compactness theorem, carry over to L_{ad}.

4.6. *Elementary Equivalence in Adapted Probability Logic*

There are two natural notions of elementary equivalence in L_{ad}.

4.6.1 Definition. Let $\mathscr{M}$ and $\mathscr{N}$ be adapted probability structures for L.

(i) $\mathscr{M}$ and $\mathscr{N}$ are *weakly L_{ad}-equivalent*, in symbols,

$$\mathscr{M} \equiv^{w} \mathscr{N},$$

if $\mathscr{M}$ and $\mathscr{N}$ satisfy the same sentences of L_{ad}.

(ii) $\mathscr{M}$ and $\mathscr{N}$ are *strongly L_{ad}-equivalent*,

$$\mathscr{M} \equiv^{s} \mathscr{N},$$

if for each tuple $\bar{a}$ in $[0, 1]$ and formula $\varphi(\bar{t})$ of L_{ad} in which w is not free,

$$\mathscr{M} \models \varphi[\bar{a}] \quad \text{iff} \quad \mathscr{N} \models \varphi[\bar{a}].$$

4.6.2 Proposition. *Any two adapted spaces (adapted structures for $L = \varnothing$) are strongly L_{ad}-equivalent.* ☐

The strong L_{ad}-equivalence relation is more important than weak L_{ad}-equivalence, because each adapted structure has the same second universe $[0, 1]$. Each notion in Example 4.4 is preserved under strong L_{ad}-equivalence but not under weak L_{ad}-equivalence. Following are some useful characterizations of these relations.

4.6.3 Proposition (Hoover and Keisler [1984]). *The following ar equivalent:*

(a) $\mathcal{M} \equiv^{w} \mathcal{N}$.
(b) *There is a set* $T \subseteq [0, 1]$ *of measure one such that for each* $\bar{a}$ *in* T *and formula* $\varphi(s)$ *of* L_{ad}, $\mathcal{M} \models \varphi[\bar{a}]$ *iff* $\mathcal{N} \models \varphi[\bar{a}]$.
(c) *For each term* $\tau(\bar{s})$ *of* L_{ad} *with no integrals over time variables,* $\tau(\bar{a})^{\mathcal{M}} = \tau(\bar{a})^{\mathcal{N}}$, *for almost all* $\bar{a}$ *in* $[0, 1]$. ☐

4.6.4 Proposition. *The following are equivalent:*

(a) $\mathcal{M} \equiv^{s} \mathcal{N}$.
(b) *For each term* $\tau(\bar{s})$ *with no integrals over time variables, and all* $\bar{a}$ *in* $[0, 1]$, $\tau(\bar{a})^{\mathcal{M}} = \tau(\bar{a})^{\mathcal{N}}$. ☐

The function $\tau(\bar{a}) \mapsto \tau(\bar{a})^{\mathcal{M}}$ is called the *adapted distribution* of $\mathcal{M}$ and it is analogous to the distribution of a random variable. Most stochastic processes which arise naturally are right continuous (in t for almost all w). For right continuous processes the two notions of L_{ad}-equivalence coincide.

4.6.5 Theorem (Hoover and Keisler [1984]). *If* $\mathcal{M} \equiv^{w} \mathcal{N}$ *and each stochastic process* $X_i^{\mathcal{M}}$ *and* $X_i^{\mathcal{N}}$ *is right continuous, then* $\mathcal{M} \equiv^{s} \mathcal{N}$. ☐

Brownian motion plays a central role in the study of stochastic processes. The following result shows that the L_{ad}-theory of independent Brownian motions is complete.

4.6.6 Theorem (Keisler [1984]). *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be adapted structures for L whose stochastic processes are mutually independent Brownian notions. Then* $\mathcal{M} \equiv^{s} \mathcal{N}$. ☐

4.7. Robinson Consistency and Craig Interpolation

The results of this section are all from the paper Hoover and Keisler [1982], a paper which studies L_{ad}-equivalence and which gives its applications to the theory of stochastic processes. The following notion corresponds to saturated structures in first-order model theory, except that stochastic processes take the place of both relations and constants.

4.7.1 Definition. An adapted space

$$\mathcal{S} = \langle M, \mu, \mathcal{F}_t \rangle$$

is *saturated* if whenever $L^1 \subseteq L^2$, $\mathcal{M}^1$ is an expansion of $\mathcal{S}$ to L^1, $\mathcal{N}^1 \equiv^{s} \mathcal{M}^1$, and $\mathcal{N}^2$ is an expansion of $\mathcal{N}^1$ to L^2, there exists an expansion $\mathcal{M}^2$ of $\mathcal{M}^1$ to L^2, such that $\mathcal{N}^2 \equiv^s \mathcal{M}^2$. The space $\mathcal{S}$ is *weakly saturated* if the above condition holds with weak L_{ad}-equivalence instead of strong L_{ad}-equivalence.

4.7.2 Proposition. *Every saturated adapted space $\mathcal{S}$ is universal; that is, for every adapted structure $\mathcal{N}$, there is an expansion $\mathcal{M}$ of $\mathcal{S}$ with $\mathcal{M} \equiv^s \mathcal{N}$. Furthermore, every weakly saturated adapted space is weakly universal.*

Proof. Take $L^1 = \varnothing$. □

4.7.3 Proposition (Hoover and Keisler [1984]). *Every saturated adapted space is weakly saturated.* □

4.7.4 Definition. An *adapted Loeb space* is an adapted space

$$\langle M, \mu, \mathcal{F}_t \rangle_{t \in [0,1]}$$

such that for some internal $*$-adapted space

$$\langle M, \nu, \mathcal{G}_s \rangle_{s \in {}^*[0,1]},$$

with universe M, μ is the completion of the Loeb measure of ν and $\mathcal{F}_t$ is the σ-algebra generated by

$$\bigcup_{{}^\circ s = t} \mathcal{G}_s \cup (\text{null sets of } \mu).$$

The following theorem is the main result in Hoover and Keisler [1984].

4.7.5 Theorem. *Every adapted Loeb space which admits a Brownian motion is saturated.* □

Remark. Anderson [1976] constructed an adapted Loeb space which admits a Brownian motion. Hence, saturated adapted probability spaces exist.

4.7.6 Theorem (Robinson Consistency Theorem for L_{ad}). *Let $L^0 = L^1 \cap L^2$, and let $\mathcal{M}^1$, $\mathcal{M}^2$ be adapted structures for L^1 and L^2 such that $\mathcal{M}^1 \restriction L^0 \equiv^s \mathcal{M}^2 \restriction L^0$. Then there is an adapted structure $\mathcal{N}$ for $L^1 \cup L^2$ such that $\mathcal{N} \restriction L^1 \equiv^s \mathcal{M}^1$, and $\mathcal{N} \restriction L^2 \equiv^s \mathcal{M}^2$. A similar result holds for weak L_{ad}-equivalence.*

Proof. There is an adapted structure $\mathcal{N}^1 \equiv^s \mathcal{M}^1$ on any saturated space. Then $\mathcal{N}^1 \restriction L^0 \equiv^s \mathcal{M}^2 \restriction L^0$; so, by saturation, there is an expansion $\mathcal{N}^2$ of $\mathcal{N}^1 \restriction L^0$ with $\mathcal{N}^2 \equiv^s \mathcal{M}^2$. Let $\mathcal{N}$ be the common expansion of $\mathcal{N}^1$ and $\mathcal{N}^2$. □

4.7.7 Theorem. *The Craig interpolation theorem holds for L_{ad}, with or without time constants from* [0, 1].

Proof. As in Theorem 3.6.2 for $L_{\mathbb{A}P}$, we first use a Henkin construction and then apply Robinson consistency. □

Following is a characterization of $\mathcal{M} \equiv^s \mathcal{N}$ as a coarsest equivalence relation in the style of soft model theory.

4.7.8 Theorem (Hoover). *Let $\approx$ be an equivalence relation on adapted structures for L with the following properties for all adapted structures $\mathcal{M}$, $\mathcal{N}$ for L:*

(a) *If $L^0 \subseteq L$ and $\mathcal{M} \approx \mathcal{N}$, then $\mathcal{M} \upharpoonright L^0 \approx \mathcal{N} \upharpoonright L^0$.*
(b) *If $\mathcal{M} \approx \mathcal{N}$, then for each term $\tau(w, \bar{t})$ with no integral or conditional expectation operators and all $\bar{a}$ in $[0, 1]$, $(\int \tau(w, \bar{a})\, dw)^{\mathcal{M}} = (\int \tau(w, \bar{a})\, dw)^{\mathcal{N}}$.*
(c) *If $X_t^{\mathcal{M}}$ is a martingale and $\mathcal{M} \approx \mathcal{N}$, then $X_t^{\mathcal{N}}$ is a martingale.*
(d) *The relation $\approx$ has the Robinson consistency property.*

Then $\mathcal{M} \approx \mathcal{N}$ implies $\mathcal{M} \equiv^s \mathcal{N}$. □

Aldous [198?] introduced the notion of *synonymous* adapted structures. $\mathcal{M}$ and $\mathcal{N}$ are *synonymous* if $\tau(\bar{a})^{\mathcal{M}} = \tau(\bar{a})^{\mathcal{N}}$, for each term $\tau(\bar{v})$ with at most one conditional expectation operator and each $\bar{a}$ in $[0, 1]$. He showed that each property in Section 4.4 is preserved under synonymity. In Hoover–Keisler [1982] there is an example of two synonymous adapted structures which are not weakly L_{ad}-equivalent. It follows from Theorem 4.7.8 that the Robinson consistency property fails for synonymity.

A theory of hyperfinite adapted structures has been developed in Keisler [1979] and Rodenhausen [1982] with results that parallel those on hyperfinite probability structures in Section 3.5, for both $\equiv^w$ and $\equiv^s$.

The adapted Loeb structures have a number of applications to standard probability theory, this is particularly true of existence theorems for stochastic differential equations where the richness of the space is necessary. See Cutland [1982], Hoover–Perkins [1983a, b], Keisler [1984], Kosciuck [1982], T. Lindstrom [1980a–d], and Perkins [1982].

Our treatment of adapted probability logic can be extended in several ways such as the following:

(a) The optional σ-algebra $\mathcal{F}$ may be replaced by any σ-algebra $\mathcal{G} \supseteq \mathcal{F}$ of $(\mu \otimes \beta)$-measurable sets such that for each $U \in \mathcal{G}$ and $t \in [0, 1]$, $U_t \in \mathcal{F}_t$. Each interpretation in $(\mathcal{M}, \mathcal{F})$ is then an interpretation in $(\mathcal{M}, \mathcal{G})$, and hence $(\mathcal{M}, \mathcal{F}) \equiv (\mathcal{M}, \mathcal{G})$.
(b) The language L has constant time symbols c_r, $r \in \mathbb{A} \cap [0, 1]$, which occur in place of time variables (only finitely many in a single formula). The additional axiom scheme is

$$\text{G4. } c_r = r.$$

(c) The time variables range over $[0, \infty)$ instead of $[0, 1]$. Changes must be made since β is no longer a probability measure.

5. *Open Questions and Research Problems*

Following is a list of questions and problems which suggest some fruitful areas of research with respect to some of the notions and relationships that were examined in this chapter.

Problem 1. Develop a form of $L_{\mathbb{A}P}$ which has the universal quantifier $(\forall x)$.

Three ways to add $(\forall x)$ so that the satisfaction relation behaves properly are:

(a) Restrict to absolutely Borel structures as indicated at the end of Section 2.
(b) Add $(\forall x)$ to $L_{\mathbb{A}P}$ with the restriction that no universal quantifier may occur within the scope of a probability quantifier.
(c) Add $(\forall x)$ to $L_{\mathbb{A}\int}$ with no restrictions.

None of our major proofs carry over to these logics, because the Loeb measure construction does not preserve truth values involving $(\forall x)$.

Problem 2. Develop a logic with $(\forall x)$ and quantifiers for inner measure at least r and outer measure at least r.

Since inner and outer measure are defined for all subsets of M, there is no difficulty in defining the satisfaction relation.

Problem 3. Study a logic such as $L_{\mathbb{A}P}$ for structures with infinite measures instead of probability measures.

Problem 4. Study a logic such as $L_{\mathbb{A}P}$ for structures with two measures (and corresponding quantifiers). Obtain completeness theorems for structures with two measures μ, ν such that:

(a) μ is orthogonal to ν.
(b) μ is absolutely continuous with respect to ν.

Problem 5. Define hyperfine conditional expectation structures appropriately and prove an existence and uniqueness theorem for $L_{\mathbb{A}E}$.

Problem 6. Does $L_{\mathbb{A}E}$ have the Robinson consistency and/or the Craig interpolation property?

Problem 7. Extend the results for adapted probability logic to allow universal quantifiers $(\forall t)$ for the second sort $[0, 1]$.

Problem 8. Study various operations on probability structures from the viewpoint of the logics examined in this chapter.

A small beginning for $L_{\mathbb{A}\int}$ is in Keisler [1977b].

Problem 9. The results on graded $L_{\mathbb{A}P}$ carry over without difficulty when L has function symbols (Hoover [1978b]). Do the results on $L_{\mathbb{A}P}$ carry over when L has function symbols?

The difficulty lies in the proof of Theorem 2.3.4.

Problem 10. Reexamine abstract model theory in the light of logics such as $L_{\mathbb{A}P}$.

The hypotheses for a logic in the enriched abstract model theory of Mundici, Chapter VIII, fail badly for $L_{\mathbb{A}P}$ and the other logics of this chapter. Mundici proved (under the set-theoretic assumption ♮) that every logic with relativization which has the Robinson consistency property is compact. Since $L_{\mathbb{A}P}$ is not compact, it follows that no extension of $L_{\mathbb{A}P}$ which is a logic with relativization in the sense of Mundici has the Robinson consistency property.

The logic $L_{\mathbb{A}P}$ does not have universal quantifiers and does not allow function symbols. The relativization property holds only for relativizing to a set of positive measure. Moreover, there does not seem to be a way to make the class of probability structures into a semantic domain in the sense of Mundici. Closure under strict expansion fails. The natural notions of isomorphic embedding which come to mind fail to satisfy either factorization, or existence and closure under disjoint union.

An essential characteristic of $L_{\mathbb{A}P}$ is that sets of measure zero are unimportant. It appears that to prove that Robinson consistency implies compactness, constructions are needed which make sets of measure zero important.

Problem 11. Is there any equivalence relation $\approx$ on adapted structures which satisfies conditions (a)–(d) of Theorem 4.7.8, is strictly finer than $\equiv^s$, and is strictly coarser than $\cong$? Here $h: \mathscr{M} \cong \mathscr{N}$ means that h sends μ to ν modulo null sets, and for all t, $h(\mathscr{F}_t) = \mathscr{G}_t$ modulo null sets, and $X_i(w, t) = X_i(hw, t)$ for μ-almost all w.

Added in proof: Problems 3, 4, 5, and 6 were solved while this article was in press. M. Rašković solved Problem 3 in the forthcoming paper "Model Theory for $L_{\mathbb{A}M}$ Logic". M. Rašković and R. Zivaljevič solved Problem 4 in "Barwise Completeness for Biprobability Logics". S. Fajardo will publish affirmative solutions to Problems 5 and 6 in "Probability Logic with Conditional Expectation".

Chapter XV
Topological Model Theory

by M. ZIEGLER

1. Topological Structures

A (one-sorted) topological structure $\tilde{\mathfrak{A}} = (\mathfrak{A}, \alpha)$ with vocabulary τ consists of a τ-structure $\mathfrak{A}$ and a topology α on A. Familiar examples are topological spaces ($\tau = \emptyset$), and topological groups and fields. Note that in general we do not assume that the relations and operations of $\mathfrak{A}$ are compatible with α. This in contrast to Robinson [1974].

A *logic* for topological structures is a pair $(\mathscr{L}, \models)$, where $\mathscr{L}[\tau]$ is a class (of "$\mathscr{L}$-sentences") for each vocabulary τ and $\models$ is a relation between topological structures and $\mathscr{L}$-sentences. We will now assume that the axioms of a regular logic hold for topological structures (see Examples 1.1.1 and Discussion 1.2). The relativization axiom is, of course, an exception to this general assumption. The reader should consult Section 2 for a description of the many-sorted case.

1.1. Three Logics for Topological Structures

We first consider *quantification over α and the logic $\mathscr{L}^t_{\text{mon}}$*. We say that an $\mathscr{L}^t_{\text{mon}}[\tau]$-formula is built up from atomic $\mathscr{L}_{\omega\omega}[\tau]$-formulas and atomic formulas

$$t \in X,$$

where t is a τ-term and X a "set variable" (which ranges over α), using $\neg$, $\wedge$, $\vee$, $\forall x$, $\exists x$, $\forall X$, $\exists X$. The semantics are self-explanatory. A logic (for $\tau = \emptyset$) equivalent to $\mathscr{L}^t_{\text{mon}}$ was introduced in Grzegorczyk [1951] and Henson *et al.* [1977].

1.1.1 Examples. (i) $(A, \alpha) \models \forall X\ \forall Y(\exists x\ \exists y(x \in X\ \wedge\ y \in Y) \to \exists x((x \in X\ \wedge\ x \in Y)\ \vee\ (\neg x \in X\ \wedge\ \neg x \in Y)))$ or, more briefly, $(A, \alpha) \models \forall X,\ Y(X \neq \emptyset\ \wedge\ Y \neq \emptyset \to (X \cap Y \neq \emptyset\ \vee\ X \cup Y \neq \text{universe}))$ which holds iff (A, α) is connected.

(ii) $(A, F, \alpha) \models \forall X\ \exists Y\ Y = f^{-1}(X)$ iff $F\colon A \to A$ is continuous with respect to α.

(iii) $(A, B, \alpha) \models \exists X\ \forall x(P(x) \leftrightarrow x \in X)$ iff B is open, i.e., $B \in \alpha$.

The next idea is that of restricted quantification over α and the logic $\mathscr{L}^t_{\omega\omega}$. We say that the formulas of $\mathscr{L}^t_{\omega\omega}$ are those $\mathscr{L}^t_{\text{mon}}$-formulas in which quantification over set variables is allowed only in the form

$$\exists X(t \in X \wedge \varphi) \qquad \text{(more briefly, } \exists X \ni t\varphi\text{)},$$

where X (that is, any atomic formula $s \in X$) occurs only negatively in φ, and dually in the form

$$\forall X(t \in X \rightarrow \varphi) \qquad \text{(more briefly, } \forall X \ni t\varphi\text{)},$$

where X occurs only positively in φ. $\mathscr{L}^t_{\omega\omega}$ was introduced by McKee [1975], [1976] and developed in Garavaglia [1978a] and Flum–Ziegler [1980]. Indeed, most of the material in the present chapter is explored in greater detail in Flum–Ziegler [1980a].

1.1.2 Examples. (i) $(A, F, \alpha) \models \forall x\, \forall Y \ni f(x)\, \exists X \ni x\, \forall z(z \in X \rightarrow f(z) \in Y)$ iff F is continuous.

(ii) $(A, B, \alpha) \models \forall x(P(x) \rightarrow \exists X \ni x\, \forall y(y \in X \rightarrow P(y)))$ iff B is open.

(iii) $(A, \alpha) \models \forall x\, \forall X \ni x\, \exists Y \ni x\, \forall y(y \in X \vee \exists Z \ni y \wedge Z \cap Y = \varnothing)$ or, more shortly, iff (A, α) is regular—regular meaning simply that every point has a base of closed neighborhoods.

Finally, we consider the interior operator and the logic $\mathscr{L}_{\omega\omega}(I^n)$ for $n \geq 1$. We pass from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}_{\omega\omega}(I^n)$, adding the formation rule that if φ is a formula and $x_1 \cdots x_n$ are distinct variables, then

$$I^n x_1 \ldots x_n \varphi$$

is a formula the free variables of which are $x_1 \ldots x_n$ and the free variables of φ. The semantics is given by

$$\begin{aligned}&\mathfrak{A} \models I^n x_1 \ldots x_n\, \varphi(\vec{x}, \vec{y})[a_1 \ldots a_n, \vec{b}] \quad \text{iff}\\ &\vec{a} \text{ is in the interior of } \{\vec{c} \in A^n \mid \mathfrak{A} \models \varphi(\vec{c}, \vec{b})\}.\end{aligned}$$

$\mathscr{L}_{\omega\omega}(I^n)$ was investigated in Sgro [1980a] and Makowsky–Ziegler [1981].

1.1.3 Examples. (i) $(A, B, \alpha) \models \forall x(P(x) \rightarrow I^1 x P(x))$ iff B is open.

(ii) $(A, \alpha) \models \forall x, y(x = y \vee I^2 xy \neg x = y)$ iff (A, α) is a Hausdorff space.

1.2. Discussion

From the preceding developments, we clearly have that $\mathscr{L}^t_{\omega\omega} \leq \mathscr{L}^t_{\text{mon}}$. Also, $\mathscr{L}_{\omega\omega}(I^n) \leq \mathscr{L}^t_{\omega\omega}$, since $I^n x_1 \ldots x_n \varphi$ can be expressed by

$$\exists X_1 \ni x_1, \ldots, \exists X_n \ni x_n\, \forall x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} x_i \in X_i \rightarrow \varphi \right).$$

We will now prove that

$$\mathscr{L}_{\omega\omega}(I^n)_{n<\omega} < \mathscr{L}^t_{\omega\omega} < \mathscr{L}^t_{\text{mon}},$$

the first inequality following from

1.2.1 Lemma. *Regularity is not expressible in $\mathscr{L}_{\omega\omega}(I^n)$.*

Proof. By an easy induction on φ, we show that for every $\mathscr{L}_{\omega\omega}(I^n)[\varnothing]$-formula φ there is an quantifier-free $\mathscr{L}_{\omega\omega}[\varnothing]$-formula which is equivalent to φ in any Hausdorff space having no isolated points. Whence, all such spaces are $\mathscr{L}_{\omega\omega}(I^n)$-equivalent. But there are regular and non-regular examples of such spaces. ▯

Remarks. (a) $\mathscr{L}_{\omega\omega}(I^n) < \mathscr{L}_{\omega\omega}(I^{n+1})$.

(b) Continuity is not expressible in $\mathscr{L}_{\omega\omega}(I^n)$.

(c) Sgro [1977a] initiated the study of topological model theory by proving a completeness theorem for $\mathscr{L}_{\omega\omega}(Q)$, where $\mathscr{L}_{\omega\omega}(Q)$ is obtained from $\mathscr{L}_{\omega\omega}$ by adding the quantifier $Qx\ \varphi$ whose meaning is "$\{x \mid \varphi(x)\}$ is open." $\mathscr{L}_{\omega\omega}(Q)$ is weaker than $\mathscr{L}_{\omega\omega}(I^1)$, and does not have the interpolation property even though $\mathscr{L}_{\omega\omega}(I^n)$ does.

To see that $\mathscr{L}^t_{\text{mon}}$ is strictly stronger that $\mathscr{L}^t_{\omega\omega}$, we first observe that $\mathscr{L}^t_{\text{mon}}$ is not $\aleph_0$-compact and does not have the Löwenheim–Skolem property down to $\aleph_0$. (We say that $(\mathfrak{A}, \alpha)$ is *countable* if $\mathfrak{A}$ is countable and α has a countable base.) Moreover, $\mathscr{L}^t_{\text{mon}}$ is not recursively axiomatizable. To see these facts, we will let α be the natural topology on $\mathbb{R}$. Then $(\mathbb{R}, 0, 1, +, -, \cdot, <, \alpha)$ is characterized by the $\mathscr{L}^t_{\text{mon}}$-sentence

$$\theta \equiv \text{"ordered field with connected order topology."}$$

This proves the first two assertions. For the third, we observe that for discrete α $\mathscr{L}^t_{\text{mon}}$ reduces to monadic second-order language, which we can use to characterize $(\mathbb{N}, +, \cdot)$. On the other hand we have:

1.2.2 Theorem. *The logic $\mathscr{L}^t_{\omega\omega}$*

(i) *is compact;*
(ii) *has the Löwenheim–Skolem property down to $\aleph_0$, and*
(iii) *is recursively axiomatizable.*

We use the notion of a *weak structure* to prove this result, such a structure being a pair $(\mathfrak{A}, \beta)$, where β is a set of subsets of A. If we consider $\mathscr{L}^t_{\text{mon}}$ as a logic for weak structures, we have—by first-order model theory—compactness, the Löwenheim–Skolem property, and recursive axiomatizability. But the sentences of $\mathscr{L}^t_{\omega\omega}$ are just designed to be *basis-invariant*:

1.2.3 Lemma. *If $\varphi \in \mathscr{L}^t_{\omega\omega}$, $(\mathfrak{A}, \alpha)$ is a topological structure and β is a base of α, then*

$$(\mathfrak{A}, \alpha) \models \varphi \quad \textit{iff} \quad (\mathfrak{A}, \beta) \models \varphi.$$

This is a fact familiar from $\varepsilon - \delta$-calculus. The proof follows immediately from an easy induction on φ.

Finally, consider the $\mathscr{L}^t_{\omega\omega}$-sentence

$$\varphi_{\text{bas}} = \forall x\, \exists X \ni x \wedge \forall x\, \forall X \ni x\, \forall Y \ni x\, \exists Z \ni x \quad Z \subset X \cap Y.$$

Clearly, we have that $(A, \beta) \models \varphi_{\text{bas}}$ iff β is a base of a topology.

1.2.4 Corollary. *$T \subset \mathscr{L}^t_{\omega\omega}$ has a topological model iff $T \cup \{\varphi_{\text{bas}}\}$ has a weak model.*

Remark. This can be rephrased as $T \models_t \varphi$ iff $T \cup \{\varphi_{\text{bas}}\} \models \varphi$. ("$\models$" for weak models).

Theorem 1.2.2 thus follows immediately from Corollary 1.2.4.

In the next section we will prove that for topological structures $\mathscr{L}^t_{\omega\omega}$ is a maximal logic for which is compact and has the Löwenheim–Skolem property. We thus can regard $\mathscr{L}^t_{\omega\omega}$ as the logic which is to topological structure as $\mathscr{L}_{\omega\omega}$ is to ordinary structures. Interestingly enough, Robinson [1973] asked for just such a logic.

The weaker logic $\mathscr{L}_{\omega\omega}(I^n)$ is important because, in some respects at least, it is better behaved than $\mathscr{L}^t_{\omega\omega}$: There is an omitting types theorem—a theorem which is false for $\mathscr{L}^t_{\omega\omega}$, as was shown by Flum–Ziegler [1980, Chapter I, Section 9]—and there is a useful notion of elementary extension.

In subsequent sections we will present results on interpolation, preservation, and definability. That done, we will treat $\mathscr{L}^t_{\omega\omega}$, and, in Section 5, examine the model theory of some special $\mathscr{L}^t_{\omega\omega}$-theories. A series of examples will be given at the end of the chapter, a series that will illustrate how to obtain logics for structures that are similar to topological structures—for example, for uniform structures or for proximity structures. We refer the reader to Flum–Ziegler for more detailed information on these notions.

2. *The Interpolation Theorem*

We discuss the notion of partially isomorphic topological structures and its finite approximations. The methods of Chapter II yield the interpolation theorem and a Lindström theorem for $\mathscr{L}^t_{\omega\omega}$. We will use the interpolation theorem to show that basis-invariant $\mathscr{L}^t_{\text{mon}}$-sentences are equivalent to $\mathscr{L}^t_{\omega\omega}$-sentences. Finally, we will prove that two topological structures are $\mathscr{L}^t_{\omega\omega}$-equivalent iff they have isomorphic ultrapowers. The results stem from Garavaglia [1978a] and Flum–Ziegler [1980].

A many-sorted topological structure is a many-sorted structure with a family of topologies on every sort. Thus, a many-sorted vocabulary for topological structures consists of sort symbols, relation symbols, function symbols, constants, and topology-sort symbols, which are equipped with sort symbols for the universe on which the topology is defined. Thus, we see that the set variables are themselves sorted.

We will often give definitions, theorems, or proofs for only the one-sorted case. However, this is only for the sake of notational simplicity.

2.1. Partial Isomorphisms

We begin our discussion with the notions contained in

2.1.1 Definition. Let $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ be topological structures.

(i) A *partial isomorphism* between $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ is a triple $\bar{\pi} = (\pi_0, \pi_1, \pi_2)$, where
  (a) $\pi_0 \subset A \times B$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$;
  (b) $\pi_1 \subset \alpha \times \beta$ satisfies $a\, \pi_0\, b$, $U\, \pi_1\, V$ and $a \in U$ imply $b \in V$;
  (c) $\pi_2 \subset \alpha \times \beta$ satisfies $a\, \pi_0\, b$, $U\, \pi_2\, V$ and $b \in V$ imply $a \in U$;

(ii) $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are n-isomorphic ($\simeq_p^n$), if there is a sequence $I_0 \ldots I_n$ of non-empty sets of partial isomorphisms such that for all $\bar{\rho} \in I_{i+1}$ $(i < n)$ the following holds
  (a) For all $b \in B$ there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ that is, $\pi_i \supseteq \rho_i$, for $i = 0, 1, 2$ such that $b \in \operatorname{Rng} \pi_0$;
  (b) For all $a \in A$ there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ such that $a \in \operatorname{Dom} \pi_0$.
  Furthermore, for all $(a, b) \in \rho_0$, we have
  (c) For all neighborhoods V' of b, there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ and a pair $(U, V) \in \pi_1$ such that $a \in U$ and $b \in V \subset V'$.
  (d) For all neighborhoods U' of a, there is an extension $\bar{\pi} \in I_i$ of $\bar{\rho}$ and a pair $(U, V) \in \pi_2$ such that $b \in V$ and $a \in U \subset U'$.

(iii) $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are partially isomorphic ($\simeq_p$), if they are 1-isomorphic with $I_0 = I_1$.

2.1.2 Proposition. *Isomorphic topological structures are partially isomorphic. The converse is true for countable topological structures.*

Proof. If $f: \bar{\mathfrak{A}} \to \bar{\mathfrak{B}}$ is an isomorphism, set $I = \{(f, \pi, \pi)\}$, where

$$\pi = \{(U, f(U)) \mid U \in \alpha\}.$$

Then $\bar{\mathfrak{A}} \simeq_p \bar{\mathfrak{B}}$ via I.

If, conversely, $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are countable and partially isomorphic via I, we construct an ascending sequence $\bar{\pi}^i$ $(i \in \omega)$ of elements of I such that $\bar{\mathfrak{A}} \simeq_p \bar{\mathfrak{B}}$ via $\{\bar{\pi}^i \mid i \in \omega\}$. (Note that in Definition 2.1.1((ii)(c), (d)) it is enough to let U' and V' range over a countable base of α and β.) But now $\bigcup \{\pi_0^i \mid i \in \omega\}$ is an isomorphism of $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$. ◻

2.1.3 Proposition. *Suppose τ is finite. Then for every n and every topological τ-structure $\bar{\mathfrak{B}}$ there is an $\mathscr{L}^t_{\omega\omega}[\tau]$-sentence $\psi^n_{\bar{\mathfrak{B}}}$ such that*

$$\bar{\mathfrak{A}} \simeq^n_p \bar{\mathfrak{B}} \quad \textit{iff} \quad \bar{\mathfrak{A}} \models \psi^n_{\bar{\mathfrak{B}}}.$$

Proof. Let $\bar{\mathfrak{B}} = (\mathfrak{B}, \beta)$. We define for $b_0 \ldots b_{k-1} \in B$ and $V_0 \ldots V_{l-1}, V'_0 \ldots V'_{m-1} \in \beta$ the formula

$$\psi^0_{\vec{b};\vec{V};\vec{V}'}(x_0 \ldots x_{k-1}; X_0 \ldots X_{l-1}; Y_0 \ldots Y_{m-1})$$

as the conjunction of all reduced basic $\vartheta(\vec{x})$, where $\bar{\mathfrak{B}} \models \vartheta(\vec{b})$, $\neg x_i \in X_j$, where $b_i \notin V_j$, and $x_i \in Y_j$, where $b_i \in V'_j$.

Using induction, we define

$$\psi^{i+1}_{\vec{b};\vec{V};\vec{V}'}(\vec{x}; \vec{X}; \vec{Y})$$

to be the conjunction of the following four formulas which correspond to Definition 2.1.1(ii)(a), (b), (c), (d):

$$\bigwedge_{b \in B} \exists x\, \psi^i_{\vec{b},b;\vec{V};\vec{V}'}(\vec{x}, x; \vec{X}; \vec{Y}),$$

$$\forall x \bigvee_{b \in B} \psi^i_{\vec{b},b;\vec{V};\vec{V}'}(\vec{x}, x; \vec{X}; \vec{Y}),$$

$$\bigwedge_{j<m} \bigwedge_{b_j \in V \in \beta} \exists X \ni x_j \psi^i_{\vec{b};\vec{V};V;\vec{V}'}(\vec{x}; \vec{X}, X; \vec{Y}),$$

and

$$\bigwedge_{j<m} \forall Y \ni x_j \bigvee_{b_j \in V' \in \beta} \psi^i_{\vec{b};\vec{V};\vec{V}',V'}(\vec{x}; \vec{X}; \vec{Y}, Y).$$

Note that we can prove by induction that all conjunctions and disjunctions are in fact finite and that the X_j (Y_j) occur only negatively (positively) in $\psi^i \ldots$. We set

$$\psi^n_{\bar{\mathfrak{B}}} = \psi^n_{\varnothing;\varnothing;\varnothing}.$$

If $\bar{\mathfrak{A}} \simeq^n_p \bar{\mathfrak{B}}$ via $I_0 \ldots I_n$, then we show by induction on i that

$$\bar{\mathfrak{A}} \models \psi^i_{\vec{b};\vec{V};\vec{V}'}(\vec{a}; \vec{U}; \vec{U}'),$$

whenever $a_j\, \pi_0\, b_j$, $U_j\, \pi_1\, V_j$ and $U'_j\, \pi_2\, V'_j$, for some $\bar{\pi} \in I_i$.

For the converse, for $\bar{\mathfrak{A}} = (\mathfrak{A}, \alpha)$ define

$$I_i = \{(\{(a_0, b_0) \cdots (a_{k-1}, b_{k-1})\}, \{(U_0, V_0), \ldots\}, \{\ldots (U'_{m-1}, V'_{m-1})\}) \mid \bar{\mathfrak{A}} \models \psi^i_{\vec{b};\vec{V};\vec{V}'}(\vec{a}, \vec{U}, \vec{U}'), U_j \in \alpha, U'_j \in \alpha\}.$$

Then, $\bar{\mathfrak{A}} \simeq_p^n \bar{\mathfrak{B}}$ via $I_0 \ldots I_n$, to see that I_n is not empty we notice that $\bar{\mathfrak{A}} \models \psi_{\mathfrak{B}}^n$ implies $(\emptyset, \emptyset, \emptyset) \in I_n$. $\square$

Remark. In fact, $\bar{\mathfrak{A}} \equiv_{\mathscr{L}^t_{\omega\omega}} \bar{\mathfrak{B}}$ iff $\bar{\mathfrak{A}} \simeq_p^n \bar{\mathfrak{B}}$ for all n.

2.2. The Interpolation Theorem

2.2.1 Theorem. *$\mathscr{L}^t_{\omega\omega}$ has the interpolation property.*

To prove this result we need the following

2.2.2 Lemma (See Chapter II, Section 5.5). *For finite τ $\simeq_p$ is an* RPC-*relation with definable approximations $\simeq_p^n$. This, in effect, means that there is an extension τ^* of τ containing a new copy of τ and a new relation symbol $<$ and there is $\Sigma \in \mathscr{L}^t_{\omega\omega}[\tau^*]$ such that for all topological τ-structures $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}$:*

(i) $\bar{\mathfrak{A}} \simeq_p \bar{\mathfrak{B}}$ *iff the pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ can be expanded to a model of Σ, where $<$ defines a non-well-ordering.*
(ii) $\bar{\mathfrak{A}} \simeq_p^n \bar{\mathfrak{B}}$ *iff the pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ can be expanded to a model of Σ, where $<$ defines a linear ordering of its field with more than n elements.*

We leave the proof to the reader.

To prove Theorem 2.2.1 we let κ_1 and κ_2 be two disjoint RPC-classes in $\mathscr{L}^t_{\omega\omega}[\tau]$. Let the $\psi_{\mathfrak{B}}^n$ be as in Proposition 2.1.3. For every n, we have

$$\kappa_1 \models \bigvee \{\psi_{\mathfrak{B}}^n \mid \bar{\mathfrak{B}} \in \kappa_1\}.$$

Thus, by compactness $\kappa_1 \models \chi^n$ for a finite disjunction χ^n of the $\psi_{\mathfrak{B}}^n$ $(\bar{\mathfrak{B}} \in \kappa_1)$.

We want to show that $\kappa_2 \models \neg\chi^n$, for some n. If not, then there is $\bar{\mathfrak{A}}_n \in \kappa_2$, $\bar{\mathfrak{B}}_n \in \kappa_1$ such that $\bar{\mathfrak{A}}_n \models \psi_{\mathfrak{B}_n}^n$. Whence, $\bar{\mathfrak{A}}_n \simeq_p^n \bar{\mathfrak{B}}_n$, for every n. By Lemma 2.2.2, compactness and the Löwenheim–Skolem property, there are countable $\bar{\mathfrak{A}} \in \kappa_2$, $\bar{\mathfrak{B}} \in \kappa_1$ such that $\bar{\mathfrak{A}} \simeq_p \bar{\mathfrak{B}}$. But then $\bar{\mathfrak{A}} \cong \bar{\mathfrak{B}}$ and κ_1 and κ_2 are not disjoint—a contradiction. $\square$

2.2.3 Corollary (Flum–Ziegler [1980]). *$\mathscr{L}^t_{\omega\omega}$ is a maximal logic for (many-sorted) topological structures which is compact and has the Löwenheim–Skolem property down to $\aleph_0$.*

(See Chapter II) *Proof.* Let $\mathscr{L}$ be a compact extension of $\mathscr{L}^t_{\omega\omega}$, with the Löwenheim–Skolem property. The above proof shows how to separate disjoint $\mathrm{EC}_{\mathscr{L}}$-classes by an $\mathrm{EC}_{\mathscr{L}^t_{\omega\omega}}$-class. $\square$

2.2.4 Corollary. *The basis-invariant sentences of $\mathscr{L}^t_{\mathrm{mon}}$ are equivalent to $\mathscr{L}^t_{\omega\omega}$-sentences.*

Proof. This follows directly from Corollary 2.2.3, since invariant sentences form a compact logic with the Löwenheim–Skolem property. Instead of proceeding on the basis of Corollary 2.2.3 we give a derivation which stems from Theorem 2.2.1.

Let $(\mathfrak{A}, \alpha)$ be a topological structure and let β_1, β_2 be two bases of α. We code β_1 and β_2 in the structure

$$(\mathfrak{A}, \alpha, B_1, B_2, E_1, E_2)$$

using two new sorts B_1, B_2 and two relations $E_i \subset A \times B_i$ such that

$$\beta_i = \{E_i b \mid b \in B_i\},$$

where $E_i b = \{a \mid aE_i b\}$. If φ is an $\mathscr{L}^t_{\mathrm{mon}}$-sentence, let φ_i denote the $\mathscr{L}_{\omega\omega}$-sentence obtained by replacing the set variables $X, Y, \ldots$ in φ by $x^i, y^i, \ldots$ of sort i and the atomic sentences $t \in X$ by $t\mathbf{E}_i x^i$, where $\mathbf{E}_i$ is the symbol for E_i.

If φ is basis-invariant (in the vocabulary of $\mathfrak{A}$), then we have

$$\models_t (\mathbf{E}_1 \text{ codes a base} \wedge \varphi_1) \to (\mathbf{E}_2 \text{ codes a base} \to \varphi_2)$$

By Theorem 2.2.1, we find an interpolant ψ in $\mathscr{L}^t_{\omega\omega}$. But then $\models_t \varphi \leftrightarrow \psi$. $\square$

The final result in this section makes use of the notion of the *ultrapowers*, in particular the ultrapower $(\mathfrak{A}, \alpha)^I/_U$ of $(\mathfrak{A}, \alpha)$ is $(\mathfrak{A}^I/_U, \gamma)$, where γ is the topology with base $\alpha^I/_U$.

2.2.5 Corollary. *Two topological structures are $\mathscr{L}^t_{\omega\omega}$-equivalent iff they have isomorphic ultrapowers.*

Proof. Since $\mathscr{L}^t_{\omega\omega}$-sentences are basis-invariant, a topological structure is $\mathscr{L}^t_{\omega\omega}$-equivalent to its ultrapower. This proves one direction.

Suppose $(\mathfrak{A}_1, \alpha_1) \equiv_{\mathscr{L}^t_{\omega\omega}} (\mathfrak{A}_2, \alpha_2)$. Expand the vocabulary τ by two new sorts and two new relation symbols as in the proof of Corollary 2.2.4. Code a base of α_i in $\mathfrak{C}_i = (\mathfrak{A}_i, B_i, E_i)$. By assumption and Theorem 2.1.1 the $\mathscr{L}^t_{\omega\omega}$-theory

$$T = \mathrm{Th}_{\mathscr{L}_{\omega\omega}}(\mathfrak{C}_1) \cup \mathrm{Th}_{\mathscr{L}_{\omega\omega}}(\mathfrak{C}_2) \cup \{\mathbf{E}_1 \text{ codes a base}\} \cup \{\mathbf{E}_2 \text{ codes a base}\}$$

is consistent. Whence there is a model $(\mathfrak{A}, \alpha, B'_1, B'_2, E'_1, E'_2)$ of T. Moreover, by the Keisler–Shelah theorem (see Chang–Keisler [1977]) there is an ultrafilter U such that

$$(\mathfrak{A}_i, B_i, E_i)^I/_U \cong (\mathfrak{A}, B'_i, E'_i)^I/_U.$$

But this implies that

$$(\mathfrak{A}_i, \alpha_i)^I/_U \cong (\mathfrak{A}, \alpha)^I/_U. \quad \square$$

Remark. It is easy to construct compact logics for topological models having the Löwenheim–Skolem property and which extend $\mathscr{L}_{\omega\omega}$ but are not contained in $\mathscr{L}^t_{\omega\omega}$. However, these examples are not natural.

3. Preservation and Definability

In Section 3.1 we give some examples which will show how to extend preservation theorems from $\mathscr{L}_{\omega\omega}$ to $\mathscr{L}^t_{\omega\omega}$. Here the classical theorem characterizing the $\mathscr{L}_{\omega\omega}$-sentences which are preserved under substructures as the sentences equivalent to universal formulas splits into two.Thus, in this discussion we will use two notions of topological substructure: the just "substructure" (with the subspace topology) appearing in Theorem 3.1.1 and the "open substructure" in Theorem 3.1.2.

In Section 3.2 we prove the topological Feferman–Vaught theorem by an adaptation of the classical proof. This result asserts, in effect, that $\prod_{i\in I}\bar{\mathfrak{A}}_i$ and $\prod_{i\in I}\bar{\mathfrak{B}}_i$ are $\mathscr{L}^t_{\omega\omega}$-equivalent if, for all $i\in I$, $\bar{\mathfrak{A}}_i$ and $\bar{\mathfrak{B}}_i$ are $\mathscr{L}^t_{\omega\omega}$-equivalent. Interestingly enough, a new feature comes into the picture in the case of Beth's theorem. For, according to Definition 2.1.1 an $\mathscr{L}^t_{\omega\omega}$-theory defines a new relation symbol explicitly (by an $\mathscr{L}^t_{\omega\omega}$-formula), if it defines the relation implicitly. But we can now ask what happens if T defines a topology implicitly. If there is no other topology in the vocabulary, then T defines the topology by an $\mathscr{L}_{\omega\omega}$-formula (see Theorem 3.3.2). If not, then no such theorem exists (see Remark 3.3.4)

3.1. Substructures

$(\mathfrak{A}, \alpha)$ is a *substructure* of $(\mathfrak{B}, \beta)$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and α is the restriction of β to A. If $A \in \beta$, then $\bar{\mathfrak{A}}$ is called an open substructure of $\bar{\mathfrak{B}}$.

An $\mathscr{L}^t_{\omega\omega}$-formula in negational normal form (that is, built up from atomic and negated atomic formulas using $\wedge$, $\vee$, $\forall$, $\exists$) is *universal* if it contains no existential individual quantifier. An example of this is the sentence "regular" in Section 1.1.)

3.1.1 Theorem (Flum–Ziegler [1980], Garavaglia [1978a]). *An $\mathscr{L}^t_{\omega\omega}$-sentence is preserved under substructures iff it is equivalent to an universal sentence.*

Proof. Let $\bar{\mathfrak{A}} \subset^n_p \bar{\mathfrak{B}}$ mean that there is a family $I_0 \cdots I_n$ of non-empty sets of partial isomorphisms between $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ such that for all $\bar{p} \in I_{i+1}$ $(i < n)$ assertions (b), (c), (d) of Definition 2.1.1(ii) hold. If the above holds for $I_0 = I_1$, we write $\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$.

The following facts can be shown as Propositions 2.1.2 and 2.1.3 and Lemma 2.2.2:

(a) If $\bar{\mathfrak{A}}$ is a substructure of $\bar{\mathfrak{B}}$, then $\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$.
(b) If $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are countable and if $\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$, then $\bar{\mathfrak{A}}$ is isomorphic to a substructure of $\bar{\mathfrak{B}}$.
(c) For every n and every $\bar{\mathfrak{B}}$, there is an universal $\mathscr{L}^t_{\omega\omega}$-sentence $\varphi^n_{\bar{\mathfrak{B}}}$ such that $\bar{\mathfrak{A}} \models \varphi^n_{\bar{\mathfrak{B}}}$ iff $\bar{\mathfrak{A}} \subset^n_p \bar{\mathfrak{B}}$, where τ is finite.
(d) "$\bar{\mathfrak{A}} \subset_p \bar{\mathfrak{B}}$" is an RPC-relation with definable approximations $\subset^n_p$, where τ is finite.

Suppose now that φ is preserved under substructures, or—even more generally—that φ holds in all substructures of models of ψ. Set $\kappa_1 = \operatorname{Mod}\psi$ and $\kappa_2 = \operatorname{Mod}\neg\varphi$. Now the proof of Theorem 2.2.1 shows that there is a universal χ (equal to a finite disjunction of some $\varphi^n_{\bar{\mathfrak{B}}}$, $\bar{\mathfrak{B}} \in \kappa_1$) such that $\kappa_1 \models_t \chi$, $\kappa_2 \models_t \neg\chi$. $\square$

We next treat open substructures. The $\mathscr{L}^t_{\omega\omega}$-sentences that are preserved here are the Π-sentences: In negation normal form the existential individual quantifier can only occur in *bounded* form: $\exists x(x \in X \wedge \varphi)$. The next theorem is related to the Feferman–Kreisel theorem on end extensions (See Section 6) and clarifies the idea of a "local" property.

3.1.2 Theorem (Flum–Ziegler [1980]). *An $\mathscr{L}^t_{\omega\omega}$-sentence is preserved under open substructures iff it is equivalent to a Π-sentence.*

Proof. As the proof of Theorem 3.1.1, we need the proper definition of "$\subset^p_n$ (open)". Thus, we use in conditions (b), (c), and (d) of part (ii) of Definition along with

(a′) For all $(U, V) \in \rho_2$ and all $b \in V$, there is an extension $\bar{\pi} \in I_i$ of $\bar{p}$ such that $b \in \operatorname{Rng} \pi_0$. ☐

Remark. The $\mathscr{L}^t_{\omega\omega}$-sentences preserved under continuous images are the positive sentences without existential set quantification.

3.2. The Feferman–Vaught Theorem

Let $\bar{\mathfrak{A}}_i = (\mathfrak{A}_i, \alpha_i)$, for $i \in I$ be a family of topological structures. The product

$$\prod_{i \in I} \bar{\mathfrak{A}}_i$$

is $(\prod_{i \in I} \mathfrak{A}_i, \alpha)$, where α is the product topology. Furthermore, let σ be the vocabulary of the structure $(\mathscr{P}(I), \cap, \cup, \mathrm{Fin})$, where Fin is the set of finite subsets of I, we can now consider

Theorem (Flum–Ziegler [1980]). *For every $\mathscr{L}^t_{\omega\omega}$-sentence φ there are $\mathscr{L}^t_{\omega\omega}$-sentences $\vartheta_1 \ldots \vartheta_n$ and an $\mathscr{L}_{\omega\omega}[\sigma]$-formula χ such that for all families $(\bar{\mathfrak{A}}_i)_{i \in I}$*

$$\prod_{i \in I} \bar{\mathfrak{A}}_i \models \varphi \quad \textit{iff} \quad (\mathscr{P}(I), \cap, \cup, \mathrm{Fin})$$
$$\models \chi(\{i \mid \bar{\mathfrak{A}}_i \models \vartheta_1\}, \ldots, \{i \mid \bar{\mathfrak{A}}_i \models \vartheta_n\}).$$

Proof. Suppose that the X_i only occur negatively in $\varphi(x, \vec{X}^-, \vec{Y}^+)$ and the Y_i only positively. Then we can show by induction on φ that there are $\vartheta_1(\vec{x}, \vec{X}^-, \vec{Y}^+), \ldots, \vartheta_n(\vec{x}, \vec{X}^-, \vec{Y}^+)$ and $\chi(y_1, \ldots, y_n)$ such that χ is monotone in all variables and

$$\prod_{i \in I} \bar{\mathfrak{A}}_i \models \varphi(\vec{a}, \vec{U}, \vec{V}) \quad \text{iff} \quad (\mathscr{P}(I), \ldots)$$
$$\models \chi(\{i \mid \bar{\mathfrak{A}}_i \models \vartheta_1(\vec{a}_i, \vec{U}_i, \vec{V}_i)\}, \ldots,),$$

for all $\vec{a} \in (\prod_{i \in I} A_i)^k$ and for all $\vec{U}, \vec{V} \in \alpha^k$. ☐

3.3. Definability

First of all, we note that interpolation implies the Beth definability theorem:

3.3.1 Theorem. *Let $\tau \subset \tau^*$ be vocabularies, $T \subset \mathscr{L}^t_{\omega\omega}[\tau^*]$, and $R \in \tau^*$. If in all models $\mathfrak{A}$ of T the interpretation of R is determined by $\mathfrak{A} \upharpoonright \tau$, then there is an $\mathscr{L}^t_{\omega\omega}[\tau]$-formula $\varphi(\vec{x})$ such that $T \models_t \forall\vec{x}(\varphi(\vec{x}) \leftrightarrow R(\vec{x}))$.*

We will now try to define the topology explicitly. Let $(\mathfrak{A}, \alpha)$ be a topological structure. A formula $\varphi(x, \vec{y})$ *defines* α, if

$$\{\{a \in A \mid \mathfrak{A} \models \varphi(a, \vec{b})\} \mid \vec{b} \in A\}$$

is a base of α. If, for example, α is the order topology of $(A, <^{\mathfrak{A}})$, then α is defined by $y_1 < x \wedge x < y_2$. In general, however, a topology is not definable. But we have:

3.3.2 Theorem (Flum–Ziegler [1980]). *Let T be an $\mathscr{L}^t_{\omega\omega}$-theory, then the following are equivalent:*

(a) *T defines the topology implicitly; that is, $(\mathfrak{A}, \alpha_i) \models T$ implies $\alpha_1 = \alpha_2$.*
(b) *There is an $\mathscr{L}_{\omega\omega}$-formula which defines the topology in all models of T.*

Proof. The reader should consult Flum–Ziegler [1980] for a more detailed proof of this result. The assertion that (b) implies (a) is clear. To prove the other implication, we assume that (a) is true. The interpolation theorem implies:

Claim 1. Every $\mathscr{L}^t_{\omega\omega}$-formula is equivalent (modulo T) to an $\mathscr{L}_{\omega\omega}$-formula.

Now we will further suppose that (b) does not hold and thus derive a contradiction. To this end, we assert

Claim 2. There is a countable model $(\mathfrak{A}, \alpha)$ of T, an element a_0 of A and an open neighborhood P of a_0 which contains no $\mathscr{L}_{\omega\omega}$-definable neighborhood of a_0.

Otherwise, there are $\mathscr{L}_{\omega\omega}$-formulas $\vartheta_1(x, y), \ldots, \vartheta_n(x, y)$ such that in every model $(\mathfrak{A}, \alpha)$ of T every $a_0 \in A$ has a base of neighborhoods of the form

$$\{a \mid \mathfrak{A} \models \vartheta_i(a, \vec{b})\}.$$

We can thus code the ϑ_i in one formula and so assume that $n = 1$. But then

$$\varphi'(x, \vec{y}) = I^1 x \vartheta_1(x, \vec{y})$$

defines the topology in all models of T. By Claim 1 φ' is equivalent to an $\mathscr{L}_{\omega\omega}$-formula $\varphi(x, \vec{y})$. Whence (b) must hold. Contradiction. We now make

Claim 3. There is a topological structure $(\mathfrak{A}^*, P^*, \alpha^*)$ such that $(\mathfrak{A}^*, \alpha^*) \models T$, $(\mathfrak{A}, P) \prec (\mathfrak{A}^*, P^*)$ and P^* is not a neighborhood of a_0.

This is a contradiction of Claim 1, because "S is a neighborhood of a" is an $\mathscr{L}^t_{\omega\omega}$-expression, and so the theorem will follow.

Proceeding with the argument we add a new sort C and a relation $E \subset A \times C$ such that (A, C, E) codes a countable base of α. We need, however,

Claim 4. Let $E_c = \{a: \mathfrak{A} \models a\mathbf{E}c\}$, for $c \in C$, be a neighborhood of a_0. Then there is an extension $(\mathfrak{A}', P', C', E')$ of $(\mathfrak{A}, P, C, E)$ such that $(\mathfrak{A}, P) \prec (\mathfrak{A}', P')$, $(\mathfrak{A}, C, E) \prec (\mathfrak{A}', C', E')$ and $E'c \not\subset P'$.

Otherwise,

$$\mathrm{Th}(\mathfrak{A}, P, a)_{a \in A} \cup \mathrm{Th}(\mathfrak{A}, C, E, a, d)_{a \in A, d \in C} \vdash \forall x(x\mathbf{E}c \to \mathbf{P}(x)).$$

By interpolation, there is an $\mathscr{L}_{\omega\omega}$-formula $\vartheta(x, \vec{a})$ ($\vec{a} \in A$) such that $(\mathfrak{A}, C, E) \models \forall x(x\mathbf{E}c \to \vartheta(x, \vec{a}))$ and $(\mathfrak{A}, P) \models \forall x(\vartheta(x, \vec{a}) \to P(x))$. But then $\vartheta(x, \vec{a})$ defines a neighborhood of a_0, which is contained in P. This contradicts Claim 2.

We can now continue the proof of Claim 3. Starting with $(\mathfrak{A}, P, C, E)$, we can iterate the construction of Claim 4 so as to construct an ascending sequence of countable structures with union $(\mathfrak{A}^*, P^*, C^*, E^*)$ such that $(\mathfrak{A}, P) \prec (\mathfrak{A}^*, P^*)$, $(\mathfrak{A}, C, E) \prec (\mathfrak{A}^*, C^*, E^*)$ and $E^*c \not\subset P^*$, whenever $c \in C^*$, $a_0 E^*c$. Let α^* be the topology generated by $\{E^*c \mid c \in C^*\}$.

3.3.3 Remark. Theorem 3.3.2 can be generalized to a Chang–Makkai type theorem: that is, for an $\mathscr{L}^t_{\omega\omega}$-theory T the following are equivalent:

(a) For all countable $\mathfrak{A}$, $\{\alpha \mid (\mathfrak{A}, \alpha) \models T\}$ is countable.
(b) For all countable models $(\mathfrak{A}, \alpha)$ of T,

$$|\{\beta \mid (\mathfrak{A}, \beta) \cong (\mathfrak{A}, \alpha)\}| < 2^{\aleph_0}.$$

(c) There is an $\mathscr{L}_{\omega\omega}$-formula $\vartheta(x, \vec{y}, \vec{z})$ such that in every model $(\mathfrak{A}, \alpha)$ of T there are $\vec{a} \in A$ for which $\vartheta(x, \vec{y}, \vec{a})$ defines a base of α.

3.3.4 Remark. In concluding this section, we point out two interesting facts about the notions we have discussed. First, we note that there is no Chang–Makkai version of Theorem 3.3.1; and, second, if T is an $\mathscr{L}^t_{\omega\omega}$-theory of structures with two topologies on it, and if we know that $(\mathfrak{A}, \alpha, \beta_i) \models T$ implies that $\beta_1 = \beta_2$, then in general we cannot conclude that B is definable in $(\mathfrak{A}, \alpha)$. The reader should consult Flum–Ziegler [1980] for a more detailed examination of this material.

4. *The Logic* $\mathscr{L}^t_{\omega_1\omega}$

Much of the theory of $\mathscr{L}_{\omega_1\omega}$ and $\mathscr{L}^t_{\omega\omega}$ can be transferred to $\mathscr{L}^t_{\omega_1\omega}$, the latter being equal to $\mathscr{L}^t_{\omega\omega}$ with countable conjunctions and disjunction. For example, the $\mathscr{L}^t_{\omega\omega}$-sentences are (up to equivalence) the basis-invariant $\mathscr{L}^t_{\mathrm{mon}\,\omega_1\omega}$-sentences, where $\mathscr{L}^t_{\mathrm{mon}\,\omega_1\omega}$ is $\mathscr{L}^t_{\mathrm{mon}}$ with countable disjunctions and conjunctions. Moreover, the interpolation theorem, the preservation theorems, and the definability theorem of Section 3, where α is defined by a sequence of formulas, are all true for $\mathscr{L}^t_{\omega_1\omega}$. In the present discussion, we will present the covering theorem (see Chapter X), a theorem which immediately implies the interpolation theorem.

4.1 Theorem. *Let $\tau \subset \tau^*$ be countable vocabularies, and let ψ be a sentence of $\mathscr{L}^t_{\omega_1\omega}[\tau^*]$. Then there is a sequence ϑ_α ($\alpha < \omega_1$) of $\mathscr{L}^t_{\omega_1\omega}[\tau]$-sentences such that*

(i) $\psi \models_t \vartheta_\alpha$;

(ii) *for all countable τ-structures $\mathfrak{A}$: if $\mathfrak{A} \models \bigwedge_{\alpha<\omega_1} \vartheta_\alpha$, then $\mathfrak{A}$ is the reduct of a model of ψ;*

(iii) *if $\tau^+ \cap \tau^* = \tau$, $\varphi \in \mathscr{L}^t_{\omega_1\omega}[\tau^+]$ and $\psi \models_t \varphi$, then $\vartheta_\alpha \models_t \varphi$, for some $\alpha < \omega_1$.*

Before undertaking the proof of the theorem, we will consider

4.2 Example. Let τ be empty, $\tau^* = \{P\}$, P a unary predicate, and ψ = "P is perfect." Then, for ϑ_α we can take the sentence which says that the α-th Cantor–Bendixson derivative is non-empty.

Proof. We will indicate the proof of the special case in which τ is one-sorted, $\tau^* = \tau \cup \{P\}$, and $\psi \in \mathscr{L}^t_{\omega\omega}$. It is easy to supply the details a proof of the general result (see Chapter VIII).

First, we observe that ψ can be put in the form

$$\begin{aligned}&\forall x_1\,\forall X_1 \ni x_1\,\exists y_1\,\exists Y_1 \ni y_1\,\forall x_2, \ldots, \exists Y_n \ni y_n\\ &\bigvee_{k<m}\left(\pi_k(x_1, X_1^+, \ldots, y_n, Y_n^-) \wedge \bigwedge_{j<r_k} P(t_j(\vec{x}, \vec{y})) \wedge \bigwedge_{j'<r'_k} \neg P(t'_{j'}(\vec{x}, \vec{y}))\right)\end{aligned}$$

We now associate to ψ a game sentence. First, we choose a 1–1 enumeration $(s_i)_{i<\omega}$ of $\bigcup_{0<i\le n} {}^i\omega$ such that $s_i \subset s_j$ implies $i \le j$. Set

$$\Gamma = \forall u_0\,\forall U_0 \ni u_0\,\exists v_0\,\exists V_0 \ni v_0 \bigvee_{k_0<m} \forall u_1, \ldots, \bigwedge \Phi,$$

where Φ is the union of

$$\{\pi_{k_{i_n}}(u_{i_1}, U_{i_1}, \ldots, V_{i_n}) \mid s_{i_1} \subsetneqq s_{i_2} \subsetneqq \cdots \subsetneqq s_{i_n}\}$$

and of

$$\begin{aligned}\{t_j(x_{i_1}, \ldots, y_{i_n}) \neq t'_{j'}(x_{i'_1}, \ldots, y_{i'_n}) \mid s_{i_1} \subsetneqq \cdots \subsetneqq s_{i_n},\\ s_{i'_1} \subsetneqq \cdots \subsetneqq s_{i'_n}, j < r_{k_{i_n}}, j' < r'_{k_{i'_n}}\}.\end{aligned}$$

Now it is easy to see that $\psi \models_t \Gamma$ and that every countable model $(\mathfrak{A}, \alpha)$ of Γ can be expanded to a model of ψ. For the ϑ_α, we take the approximations of Γ:

$$\begin{aligned} &\vartheta_0^{k_0,\ldots,k_{i-1}}(u_0, U_0, v_0, V_0, \ldots, V_{i-1}) \\ &\quad = \bigwedge \{\varphi \in \Phi \mid \varphi \text{ contains only } k_0, \ldots, k_{i-1}, u_0, \ldots, V_{i-1}\}, \end{aligned}$$

$$\begin{aligned} &\vartheta_\alpha^{k_0,\ldots,k_{i-1}}(u_0, \ldots,) \\ &\quad = \forall u_i\, \forall U_i \ni u_i\, \exists v_i\, \exists V_i \ni v_i \bigvee_{k_i < m} \bigwedge_{\beta < \alpha} \vartheta_\beta^{k_0,\ldots,k_i}(u_0, \ldots, V_i). \end{aligned}$$

Finally, one shows that $\Gamma \models_t \bigwedge_{\alpha<\omega_1} \vartheta_\alpha \models_t' \Gamma$ and that $\Gamma \models_t \varphi$ implies $\vartheta_\alpha \models_t \varphi$, for some $\alpha < \omega_1$, where $\models_t'$ means $\models_t$ for countable models. $\square$

5. *Some Applications*

In the following discussions we will give four examples of the expressive power of $\mathscr{L}^t_{\omega\omega}$. In Section 5.1 we show that the theory of T_2-spaces is undecidable while the theory of T_3-spaces is decidable. We will also give invariants that determine the elementary type of T_3-spaces. In Section 5.2 we will show that the theory of torsion free locally pure abelian groups is decidable, although the theory of all topological groups is not. In Section 5.3 we present a complete axiomatization of the theory of the topological field of complex numbers. And finally, in Section 5.4, we show that all infinite dimensional, locally bounded real topological vector spaces are $\mathscr{L}^t$-equivalent: They are, in fact, models of an explicitly given complete theory. The results given in Section 5.1 are explored in Flum–Ziegler [1980].

5.1. Topological Spaces

Let T_2 be the theory of Hausdorff spaces; that is, the set

$$\forall x\, \forall y(x \neq y \rightarrow \exists X \ni x\, \exists Y \ni y \quad X \cap Y = \varnothing),$$

then we can consider

5.1.1 Theorem. *T_2 is hereditarily undecidable.*

Proof. Let $\varphi(x, y)$ be the formula $\neg(\exists X \ni x\, \exists Y \ni y \quad \bar{X} \cap \bar{Y} = \varnothing)$, then, for Hausdorff spaces $\mathfrak{A}$, we can make

$$(U, \{(a, b) \in U^2 \mid \mathfrak{A} \models \varphi(a, b)\}),$$

where $U = \{a \in A \mid \mathfrak{A} \models \exists y \neq a\varphi(a, y)\}$ is isomorphic to any graph without isolated points. But the theory of these graphs is known to be hereditarily undecidable. Thus, the assertion in the theorem is established. $\square$

Remark. We recall that totally disconnected spaces are spaces in which any two points can be separated by a clopen set. Let T_ω be the theory of all totally disconnected spaces (that is, T_ω is the set of all $\mathscr{L}^t_{\omega\omega}$ sentences true in all these spaces), then every finite subtheory of T_ω is hereditarily undecidable (for example, the $T_{2,5}$ separation axiom). However, relative to T_ω, every formula is equivalent to a boolean combination of formulas $x = y$ and of formulas having only one free variable. Whether T_ω is decidable, remains an open question.

Let T_3 be the theory of regular Hausdorff spaces, then we have

5.1.2 Theorem. *T_3 is decidable.*

Proof. Every T_3-space is $\mathscr{L}^t_{\omega\omega}$-equivalent to a countable T_3-space. But the countable T_3-spaces are just the topological spaces which come from a countable linear order. Therefore, our result follows from the decidability of the elementary theory of linear orders.

Remark. T_ω is a subtheory of T_3, since in countable regular spaces disjoint closed sets can be separated by clopen sets.

In order to define elementary invariants of a T_3-space $\mathfrak{A}$ we divide A into sets A^s of all points of "type s", where s is an element of

$$S = \bigcup \{S^n \mid n \in \mathbb{N}\}, \quad \text{where} \quad S^0 = \{*\} \text{ and } S^{n+1} = \mathscr{P}(S^n).$$

We set $A^* = A$ and, for $s \in S^{n+1}$, we set

$$A^s = \{a \in A \mid a \text{ is an accumulation point of } A^r \text{ iff } r \in s, \text{ for all } r \in S^n\}.$$

5.1.3 Theorem. *Two T_3-spaces $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathscr{L}^t_{\omega\omega}$-equivalent iff $|A^s| = |B^s|$ (mod $\aleph_0$) for all $s \in S$.*

Example. All T_3-spaces without isolated points are $\mathscr{L}^t_{\omega\omega}$-equivalent. For then $A^s = A$, if s is of the form $*$, $\{*\}$, $\{\{*\}\}, \ldots$, and $A^s = \varnothing$ otherwise.

Proof. One direction follows from the observation that the A^s are $\mathscr{L}^t_{\omega\omega}$- definable in $\mathfrak{A}$. For the converse, we can assume that $\mathfrak{A}$ and $\mathfrak{B}$ have bases α and β of clopen sets such that (A, α) and (B, β) are $\aleph_0$-saturated. It is then easily proved that $(\mathfrak{A}, A^s)_{s\in S}$ and $(\mathfrak{B}, B^s)_{s\in S}$ are partially isomorphic via the system I which consists of all finite partial isomorphisms (π_0, π_1, π_2), where $\pi_1 = \pi_2 = \{(U_i, V_i) \mid i < n\}$, the $U_i \in \alpha$ (respectively the $V_i \in \beta$) form a clopen partition of A (respectively B), and $|U_i^s| = |V_i^s|$ (mod $\aleph_0$) for all $i < n$. $\square$

5.2. Topological Abelian Groups

We will now consider Hausdorff topological abelian groups. Noting first that this is an $\mathscr{L}^t_{\omega\omega}$-elementary class, we proceed to examine several interesting results, the first of which is

5.2.1 Theorem (Cherlin–Schmitt [1981]). *The theory of Hausdorff abelian groups is undecidable.*

Proof. Let p be a prime and $q = p^9$. Baur [1976] has proven that the theory of all abelian groups (no topology) A of exponent q with a distinguished subgroup B is undecidable. But such a pair can be interpreted in a suitable topological group $\mathfrak{C}$ by letting

$$A = C/qC \quad \text{and} \quad B = \overline{qC}/qC. \quad \square$$

Call a group *locally pure*, if (partial) division by n is continous at 0. That is, a group is locally pure if the following $\mathscr{L}^t_{\omega\omega}$-sentence holds for every n

$$\forall X \ni 0\, \exists Y \ni 0\, \forall x(nx \in Y \to \exists y \in X \quad ny = nx).$$

Cherlin–Schmitt [1981] also proved that the theory of all locally pure groups is hereditarily undecidable. Furthermore, we have

5.2.2 Theorem (Cherlin–Schmitt [1980]). *The theory of all torsion free, locally pure groups is decidable.*

Proof. Since the theory of all (discrete) torsion free groups is decidable and the theory of all non-trivial ordered abelian groups is decidable (see Gurevic [1964]), the theorem follows from

5.2.3 Lemma. *A topological abelian group is torsion free, locally pure, and non-discrete iff it is $\mathscr{L}^t_{\omega\omega}$-equivalent to a non-trivial group with the order topology.*

Proof. One direction is easy to establish. For the converse suppose that $\tilde{\mathfrak{A}}$ is torsion free, locally pure and non-discrete. We choose an $\mathscr{L}^t_{\omega\omega}$-equivalent group $(\mathfrak{A}_1, \alpha_1)$ where α_1 has a basis β_1 such that $(\mathfrak{A}_1, \beta_1)$ is $\aleph_1$-saturated. Then, as can be easily shown, α_1 is closed under countable intersections. Starting with an arbitrary U_0, we construct a sequence $(U_i)_{i<\omega}$ of open neighbourhoods of 0 such that for all $i = 0, 1, 2, \ldots$

$$U_{i+1} - U_{i+1} \subset U_i \quad \text{and} \quad nx \in U_{i+1} \to \exists y \in U_i \quad ny = nx.$$

Then the intersection of the U_i is an open pure subgroup of $\mathfrak{A}_1$. Thus, $\tilde{\mathfrak{A}}_1$ has a base γ_1 of neighborhoods of 0 consisting of pure subgroups. Choose a countable $(\mathfrak{A}_2, \gamma_2)$ that is elementarily equivalent to $(\mathfrak{A}_1, \gamma_1)$. Then $\{a + U \mid a \in A_2, U \in \gamma_2\}$ is a base of a topology α_2 on A_2 such that $\tilde{\mathfrak{A}}$ and $(\mathfrak{A}_2, \alpha_2)$ are $\mathscr{L}^t_{\omega\omega}$-equivalent.

From γ_2 we now choose a descending base for the neighborhood filter of 0, writing $U = U_0 \supset U_1 \supset \cdots$. We then fix an ordering $<_i$ of the torsion free group U_i/U_{i+1} ($i \in N$). If we define $x < y$ iff $x, y \in U_i$, $x + U_{i+1} <_i y + U_{i+1}$, for some i, we then obtain an ordering of $\mathfrak{A}_2$ which generates α_2.

5.3. Topological Fields

Theorem (Prestel–Ziegler [1978]). *The $\mathscr{L}^t_{\omega\omega}$-theory of the topological field of complex numbers is axiomatized by the sentences asserting*

(a) "*algebraically closed field of characteristic* 0";
(b) "*non-discrete Hausdorff topological ring*";
(c) "*V-topology*"; *that is, in symbols, we have*

$$\forall X \ni 0\, \exists Y \ni 0\, \forall x, y(xy \in Y \rightarrow x \in X \vee y \in X)$$

Proof. Let $\mathfrak{A}$ be a model of the axioms. Choose $(\mathfrak{B}, \beta)$ $\mathscr{L}^t_{\omega\omega}$-equivalent of $\mathfrak{A}$, where β is closed under countable intersections. Choose a sequence $(U_i)_{i<\omega}$ of neighborhoods of 0 such that $(i + 1) \notin U_i$, $U_{i+1}U_{i+1} \subset U_i$, $U_{i+1} - U_{i+1} \subset U_i$ and $x, y \in U_{i+1} \rightarrow x \in U_i$ or $y \in U_i$. Then the intersection U of the U_i is a neighborhood of 0 and has the following properties:

(1) $\mathbb{N} \cap U = \{0\}$.
(2) $UU \subset U$.
(3) $U - U \subset U$.
(4) $x, y \in U \Rightarrow x \in U$ or $y \in U$.

Set

(5) $R = \{b \in B \mid bU \subset U\}$.

Because of Property (3), R is a subring of B. In fact, we prove that R is a valuation ring of B. That is, we can prove that for all $b \in B$, either $b \in R$ or $b^{-1} \in R$. For, otherwise there are $u_i \in U$ such that $bu_1 \notin U$ and $b^{-1}u_2 \notin U$. But by (4) this implies that $u_1u_2 = bu_1b^{-1}u_2 \notin U$—a contradiction to (2).

By (3) U is an ideal of R and is is proper by (1) and prime by (4). But then (5) can hold only if U is the maximal ideal of the valuation ring R. Since $U \neq 0$, we must have that $R \neq B$. Furthermore, (1) implies that R/U has characteristic zero.

By Robinson [1956b], all $(\mathfrak{B}, R)$ are elementarily equivalent, where $\mathfrak{B}$ is algebraically closed and R is a proper valuation ring of $\mathfrak{B}$ with residue class of characteristic 0. Therefore, in order to show the completeness of our axioms, it remains to show that β is the valuation topology of $(\mathfrak{B}, R)$; that is, that $\{rU \mid r \in R\backslash\{0\}\}$ is a base for the neighborhoods of 0.

To that end, we now assume that V is a neighborhood of 0 and choose another neighborhood-W of 0 such that $x, y \notin V \cap U \Rightarrow xy \notin W$. Then $rU \subset V$, for any $r \in W$. For $u \in U$ implies $u^{-1} \notin U$ by (1) and (2). Therefore, $ru \notin V$ would imply that $r = ruu^{-1} \notin W$. ☐

The methods used in the above proof can be used to prove the following result, a theorem due to Stone [1969].

Approximation Theorem. *Let $\alpha_1, \ldots, \alpha_n$ be different V-topologies of the field K. Then the intersection of any sequence of non-empty open sets $U_i \in \alpha_i$ is non-empty.*

Proof. The theorem claims that $(K, \alpha_1, \ldots, \alpha_n)$ has a certain $\mathscr{L}^t_{\omega\omega}$-property. But we have seen that $(K, \alpha_1, \ldots)$ is $\mathscr{L}^t$-equivalent to a structure $(L, \beta_1, \ldots)$, where the β_i are defined by valuations. In this case, the theorem is well known from valuation theory. ☐

5.4. Topological Vector Spaces

We look at topological vector spaces as two sorted topological structures (R, V, α), where R is an ordered field, V is an R-vector space with a compatible non-discrete Hausdorff topology α. We let x, y range over V and ξ range over R.

Theorem (Sperschneider [1979]). *The $\mathscr{L}^t_{\omega\omega}$-theory of locally bounded real vector spaces of infinite dimension is complete and can be axiomatized by sentences asserting:*

"*infinite dimensional topological vector space over an ordered real closed field*";

"*locally bounded*": $\exists X \ni 0\ \forall Y \ni 0\ \exists \xi \quad X \subset \xi Y$;

"*the Riesz Lemma*": *For all n*, $\forall X \ni 0\ \exists Y \ni 0$ *such that for all subspaces F of dimension* $\leq n$ *and all* $x \notin F\ \exists y \quad y \in F + \langle x \rangle \wedge y \in X \wedge y \notin (F + Y)$.

Proof. It is easy to see that locally bounded real vector spaces satisfy our axioms. (If V is normed, the last axioms follow directly from the Riesz lemma.) Since all infinite dimensional vector spaces over a real closed field with a distinguished Euclidean bilinear form are elementarily equivalent, it is enough to show that every model (R, V, α) of our axioms is $\mathscr{L}^t_{\omega\omega}$-equivalent to a topological vector space whose topology is defined by an Euclidean norm.

We can suppose that α is closed under countable intersections. Then, taking the intersection of a suitable descending chain, we find a bounded neighborhood U of 0; (that is, $\{rU \mid r \in R \setminus \{0\}\}$ is a basis for the neighborhoods of 0) and an infinitesimal $r > 0$ such that $U - U \subset U$, $[-1, 1]U \subset U$ and for all finite dimensional F and $x \notin F$, there is $y \in F + \langle x \rangle$ such that $y \in U$ and $y \notin (F + rU)$. Finally, we choose a neighborhood V of 0 that is contained in all $r^n U$ $(n \leq N)$.

Now (proceed to an elementarily equivalent situation) we drop the assumption that α is closed under countable intersections, and instead assume that V is countable. We can then construct a basis $(x_i)_{i<\omega}$ of V such that $x_i \in U$ and $x_i \notin (\langle x_0, x_1, \ldots, x_{i-1} \rangle + rU)$. Define an Euclidean bilinear form on V such that $(x_i)_{i<\omega}$ becomes an orthonormal basis. Now set $B = \{x \in V \mid (x, x) \leq 1\}$. We will complete the proof by showing that $V \subset B \subset U$.

If $r_0 x_0 + r_1 x_1 + \cdots + r_n x_n \in V \subset r^{n+2} U$, we can conclude that $|r_n| < r^{n+1}$ and $r_0 x_0 + \cdots + r_{n-1} x_{n-1} \in r^{n+1} U$, etc. Whence, we have that $|r_i| < r^{i+1} \leq r$, for all $i = 0, 1, \ldots$. It now follows that $r_0 x_0 + \cdots + r_n x_n \in B$. This again implies that $|r_i| \leq 1$, for all i. Whence, $r_i x_i \in U$ and $r_0 x_0 + \cdots + r_n x_n \in U$. ☐

6. Other Structures

As a logic for topological structures, $\mathscr{L}^t_{\omega\omega}$ was constructed in the following three steps

(1) The second-order notion of a topology was replaced by the first-order notion of a base of topology.
(2) An appropriate logic ($\mathscr{L}^t_{\text{mon}}$) for the "weak structures" $(\mathfrak{A}, \beta)$ was chosen, where β is a base of a topology.
(3) That the $\mathscr{L}^t_{\omega\omega}$-sentences are (up to equivalence) just the base-invariant sentences of $\mathscr{L}^t_{\text{mon}}$ was shown.

There are many other cases in which this philosophy is successful. In the following examples, all of the general theorems given in Sections 1, 2, 3.3, and 4 hold true.

6.1. Quasitopologies

A set of subsets of A is a quasitopology on A, if it is closed under arbitrary unions. Every set β of subsets of A is the base of a quasi-topology α on A since it is possible to set $\alpha = \{\bigcup s \mid s \subset \beta\}$. Thus, a weak structure $(\mathfrak{A}, \beta)$ consists of a structure $\mathfrak{A}$ and a set of subsets of A. The appropriate logic for weak structures is $\mathscr{L}^t_{\text{mon}}$. The sentences of $\mathscr{L}^t_{\text{mon}}$ are basis-invariant are also, up to equivalence, the sentences of $\mathscr{L}^t_{\omega\omega}$. Thus, $\mathscr{L}^t_{\omega\omega}$ can also serve as a natural logic for quasi-topological structures. Topological structures form an elementary class of quasi-topological structures. It is now clear why φ_{bas} (see Corollary 1.2.4) was taken as an $\mathscr{L}^t_{\omega\omega}$-sentence.

6.2. Monotone Systems

Let n be a non-zero natural number. An n-monotone system on A is a system of subsets of A^n which is closed under supersets. A set β of subsets of A^n is the base of the n-monotone system

$$\{C \subset A^n \mid B \subset C \text{ for some } B \in \beta\}.$$

Thus, a weak structure $(\mathfrak{A}, \beta)$ is a structure $\mathfrak{A}$ with a set β of subset of A^n. The logic $\mathscr{L}$ for these weak structures adds set variables $X, Y, \ldots$ and atomic formulas $(t_1 \ldots t_n) \in X$ to $\mathscr{L}_{\omega\omega}$.

Now, up to equivalence, the base invariant $\mathscr{L}$-sentences are the sentences in which set quantification $\exists X\ \varphi$ (respectively $\forall X\ \varphi$) is allowed only if X occurs only negatively (respectively positively in φ).

We use these sentences as a logic $\mathscr{L}^*$ for n-monotone structures. We observe in passing that the same can be done for antitone systems.

Example. ν is a uniformity on A iff (A, ν) is a 2-monotone structure which satisfies the following $\mathscr{L}^*$-axioms:

$$\exists X(\text{true}), \qquad (\text{that is, } \nu \text{ is non-empty};$$

$$\forall X\, \forall x(x, x) \in X;$$

$$\forall X\, \forall Y\, \exists Z\, \forall x\, \forall y(x, y) \in Z \to ((x, y) \in X \wedge (x, y) \in Y);$$

$$\forall X\, \exists Y\, \forall x\, \forall y\, \forall z((x, y) \in Y \wedge (x, z) \in Y) \to (y, z) \in X).$$

It is easy to prove that ν is an uniformity on A iff (A, ν) is $\mathscr{L}^*$-equivalent to a 2-monotone structure (B, μ) where μ is closed under finite intersections and has a base of equivalence relations.

6.3. *Point Monotone Systems*

A point monotone system μ on A assigns to every $a \in A$ an 1-monotone system $\mu(a)$ on A. The function $\beta: A \to \mathscr{P}(A)$ is a base of the point monotone system (monotone system with base $\beta(a) \mid a \in A$).

Precisely what constitutes a logic for these structures? Letting $\mathscr{L}$ denote the logic for such structures, we use sentences that are built-up like $\mathscr{L}_{\omega\omega}$-sentences along with set variables $X, Y, \ldots,$ atomic formulas $t \in X$, and quantification $\exists X(t)\varphi$ and $\forall X(t)\varphi$ as the constituents of $\mathscr{L}$. The interpretation of these last two formulas is $X \in \beta(t)$ such that φ and for all $X \in \beta(t)$, φ. Now, the quantification $\exists X(t)\varphi$ (respectively, $\forall X(t)\varphi$) is only allowed in $\mathscr{L}^*$-sentences if X occurs only negatively (respectively, positively) in φ. These are, up to equivalence, the base invariant $\mathscr{L}$-sentences. Thus, we can use $\mathscr{L}^*$ as a logic for point monotone structures.

Example. We can interpret a topology on A as a point monotone structure (A, μ), where $\mu(a)$ is the neighborhood filter of a. Moreover, we can formulate Hausdorff's axioms in $\mathscr{L}^*$ as follows: A point monotone structure (A, μ) is a topological space iff the following $\mathscr{L}^*$-axioms are satisfied:

$$\forall x\, \exists X(x)\, (\text{true});$$

$$\forall x\, \forall X(x)\; x \in X;$$

$$\forall x\, \forall X(x)\, \forall Y(x)\, \exists Z(x)\, \forall y \quad y \in Z \to (y \in X \wedge y \in Y);$$

$$\forall x\, \forall X(x)\, \exists Y(x)\, \forall y(y \in Y \to \exists Z(y)\, \forall z \quad z \in Z \to z \in X).$$

The resulting logic for topological structures is, of course, equivalent to $\mathscr{L}^t_{\omega\omega}$.

Remark. Call the point monotone structure $(\mathfrak{A}, \mu)$ an open substructure of the point monotone structure $(\mathfrak{B}, \nu)$, if $\mathfrak{A}$ is an substructure of $\mathfrak{B}$ and every $\mu(a)$ is a base of $\nu(a)$. Then, up to equivalence, the $\mathscr{L}^*$-sentences preserved under open substructures are the Π-sentences (which are similarly defined as in Theorem 3.1.2). This result generalizes both Theorem 3.1.2 and the Feferman–Kreisel theorem on end extensions.

6.4. *Antitone Systems of Pairs of Sets*

A set δ of pairs of subsets of A is *antitone*—and, for the sake of brevity, we write ASPS on A—if $(B_1, B_2) \in \delta$, $C_1 \subset B_1$, $C_2 \subset B_2$ implies $(C_1, C_2) \in \delta$. Every set of pairs of subsets of A is a base of an ASPS in the obvious way. This notion clear, we can arrive at the logic $\mathscr{L}^*$ for ASPS-structures $(\mathfrak{A}, \delta)$ as follows: We extend $\mathscr{L}_{\omega\omega}$ by set variables $X, Y, \ldots$ (for pairs of sets) and new atomic sentences $t \in_1 X$, $t \in_2 X$ whose meaning is that t is in the first (respectively, the second) component of X, and we allow quantification $\exists X\varphi$ (respectively, $\forall X\varphi$) only if X occurs only positively (respectively, negatively) in φ.

Example. A *proximity space* is an ASPS-structure (A, δ) with the following properties:

(a) if $B \,\delta\, C$, then $C \,\delta\, B$;
(b) if $B_1 \,\delta\, C$ and $B_2 \,\delta\, C$, then $B_2 \cup B_1 \,\delta\, C$;
(c) for no $a \in A$ $\{a\} \,\delta\, \{a\}$;
(d) $\emptyset \,\delta\, A$;
(e) if $B \,\delta\, C$, then there are B', C' such that $B \subset B'$, $C \subset C'$, $B' \cap C' = \emptyset$, $B \,\delta\, (A \backslash B')$, and $(A \backslash C') \,\delta\, C$.

Each of the properties can be formulated in $\mathscr{L}^*$. Thus, for example, property (e) reads

$$\forall X \, \exists Y \, \exists Z(\forall x(x \in_2 Y \vee x \in_2 Z) \wedge \forall x(x \in_1 X \rightarrow x \in_1 Y) \\ \wedge \forall x(x \in_2 X \rightarrow x \in_1 Z))$$

Finally, in concluding this discussion, we briefly note that we write $B \,\delta\, C$, for $(B, C) \in \delta$ to mean that Band C are *not proximate*.

Chapter XVI

Borel Structures and Measure and Category Logics

by C. I. STEINHORN

Two very significant ways in which the theory of models has been extended beyond first-order logic are the enrichment of the syntax to include additional quantifiers and the restriction of the class of structures to be considered. These two means will be brought together in this chapter. The focus here will be on the model theory of structures whose domain and some subset of whose definable relations and functions can be built from the subsets of $\mathbb{R}^n$ that are most frequently encountered in analysis and topology: the Borel sets (see Section 1.1 for precise definitions). Such first-order structures are studied in Sections 1.2 and 1.3. The most widely-used notions of size for Borel sets are category, measure, and uncountability. The model theory of "Borel structures," when the syntax is expanded to allow quantifiers capable of expressing one or more of these concepts will be explored in the final two sections of the chapter.

Friedman initiated the study of the structures and logics which are the subject of this chapter in the series of abstracts (Friedman [1978], [1979a] and [1979b]). Most of the major results presented here are due to him. Friedman has expressed the hope that by restricting the available class of structures for a theory to those which are in some sense Borel, the negative results obtained by using arbitrary uncountable or non-separable structures can be largely eliminated. That is to say, the abundance of positive results found in many areas of mathematical practice for countable, separable, or even well-behaved uncountable and non-separable structures may also be discovered for the classes of structures to be discussed here.

At present it is not at all clear that the study of these structures and logics can quite realize the aims sketched above. Nevertheless, the techniques and notions that have already been developed seem powerful, and the wealth of interesting problems that arise in this area surely warrants our further attention.

1. Borel Model Theory

1.1. Measure and Category Logic, and Borel Structures

In this chapter, all theories will be built from a countable vocabulary τ, even though we will usually suppress explicit reference to τ. First-order logic and several finitary extensions of it obtained by adjoining various combinations of the new quantifiers Q, Q_c, and Q_m will be considered in this chapter. The intended interpretation of Q is "there exist uncountably many," and that of Q_c, "there exist non-meager (= not first-category) many," while that of Q_m is "there exist non-measure 0 many." Thus, for example, the logic $\mathscr{L}(Q, Q_c)$ whose formulas consist of those finitary formulas that are constructed from the symbols of first-order logic and the additional first-order quantifiers Q and Q_c will be studied.

The domain of any structure mentioned in this chapter will be a subset of $\mathbb{R}$, the set of real numbers. The extra clauses in the definition of satisfaction dealing with the new quantifiers are then given naturally. The definition is as usual for Q (see Chapter II). For Q_c and Q_m, if $\mathscr{M}$ is a τ-structure, $\mathrm{dom}(\mathscr{M}) = M \subseteq \mathbb{R}$, and $\bar{a} \in {}^nM$, then

$$\mathscr{M} \models Q_c x\, \psi(x, \bar{a}) \quad \text{iff} \quad \{x \in M : \mathscr{M} \models \psi(x, \bar{a})\} \text{ is non-meager,}$$

and

$$\mathscr{M} \models Q_m x\, \psi(x, \bar{a}) \quad \text{iff} \quad \{x \in M : \mathscr{M} \models \psi(x, \bar{a})\} \text{ is not of measure 0.}$$

Structures whose domain and/or relations and functions are arbitrary subsets of $\mathbb{R}$ or $\mathbb{R}^m$ will not be considered here. Rather, this chapter will focus attention on various specializations that are obtained in different ways when the subsets to be considered are required to be Borel.

1.1.1 Definition. A *Borel structure* will be a structure whose domain is a non-empty Borel subset of $\mathbb{R}$ and whose relations and functions are all Borel. A structure for one of the logics $\mathscr{L}$ just described is said to be *totally Borel* if all relations definable by $\mathscr{L}$-formulas with parameters are Borel. Moreover, if φ is a formula in one of these logics then an $\mathscr{L}$-structure $\mathscr{M}$ is *Borel for* φ if it is Borel and every relation definable over $\mathscr{M}$ from a subformula of φ is also Borel. If T is an $\mathscr{L}$-theory, then the $\mathscr{L}$-structure $\mathscr{M}$ is *Borel for* T if it is Borel for every $\varphi \in T$.

We now illustrate the expressive capabilities of some of the logics to be considered. The reader is encouraged to produce further examples.

Example 1. A definable instance of the property that a countable union of meager sets is meager can be expressed by the $\mathscr{L}(Q, Q_c)$ formula

$$\forall y \neg Q_c x\, \varphi(x, y) \wedge \neg Q y\, \exists x\, \varphi(x, y) \rightarrow \neg Q_c x\, \exists y\, \varphi(x, y).$$

Obviously, this formula is true in any $\mathscr{L}(Q, Q_c)$-structure.

Example 2. The following $\mathscr{L}(Q_m)$-formula expresses a definable form of the Fubini theorem:

$$Q_m x\, Q_m y\, \varphi(x, y) \leftrightarrow Q_m y\, Q_m x\, \varphi(x, y).$$

This formula certainly is valid for all totally Borel $\mathscr{L}(Q_m)$-*structures.*

Example 3. If "Q_m" is replaced everywhere in the above formula by "Q_c," then the resulting formula asserts a definable form of the Kuratowski–Ulam theorem (see Oxtoby [1971, Theorem 15.1]), which is true in all totally Borel $\mathscr{L}(Q_c)$-structures.

1.2. The Borel Completeness Theorem and Some Classical Applications

The primary result to be examined in this section is

1.2.1 Borel Completeness Theorem (Friedman [1978]). *A first-order theory T has an uncountable totally Borel model iff it has an infinite model.* ☐

Remarks. Before we prove this theorem, some comments are in order. First, since every countable subset of $\mathbb{N}$ is Borel, the ordinary completeness theorem implies that every consistent first-order theory has a totally Borel model. Thus, the real content of Theorem 1.2.1 lies in the construction of an *uncountable* totally Borel structure for T. Second, this result can be considerably sharpened. It is well known that every uncountable Borel subset of $\mathbb{R}$ is Borel isomorphic to $\mathbb{R}$ itself. Consequently, if T has an infinite model, then T has a totally Borel model whose domain is $\mathbb{R}$.

Proof of Theorem 1.2.1. All that requires proof is the assertion that if T has an infinite model, then T has an uncountable totally Borel model. So we assume that T has an infinite model. The uncountable totally Borel model that we will construct will be the Skolem hull of a sequence of indiscernibles.

First, extend the theory T to a theory T^* in an expanded language so that T^* has built-in Skolem functions. As usual, T^* has a model with an infinite sequence of indiscernibles $(I, <)$ having the same order type as the rational numbers. This sequence of indiscernibles may then be stretched to obtain a sequence of indiscernibles $(J, <)$ which also has the same order type as the irrational numbers. We will construct the totally Borel model of T from $\mathscr{H}\langle J\rangle$, the Skolem hull of J.

Any element of $H\langle J\rangle$ can be generated as $t(i_1, \ldots, i_n)$, where $t(v_1, \ldots, v_n)$ is a term having exactly the free variables $v_1, \ldots, v_n$ and $i_1 < i_2 < \cdots < i_n$ are distinct elements of J. We will restrict our attention to such representations for the remainder of the proof. The following two statements may be verified with only a little effort:

(1) For any term $t(v_1, \ldots, v_n)$, there exists a smallest $S \subseteq \{1, \ldots, n\}$, so that for any $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$ from J, $t(i_1, \ldots, i_n) = t(j_1, \ldots, j_n)$ iff $i_k = j_k$, for every $k \in S$.
(2) Suppose $t(v_1, \ldots, v_n)$ and $t'(u_1, \ldots, u_m)$ are two terms with associated $S_t \subseteq \{1, \ldots, n\}$ and $S_{t'} \subseteq \{1, \ldots, m\}$. If for some $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_m$ from J, it is true that $t(i_1, \ldots, i_n) = t'(j_1, \ldots, j_m)$, then $\{k: i_k = j_l$ for some $l\} = S_t$ and $\{l: j_l = i_k$, for some $k\} = S_{t'}$.

Also, we fix an enumeration $\langle t_k : k < \omega \rangle$ of terms, each with its corresponding S_k given by (1) above.

As in descriptive set theory, we will identify the irrational numbers with ${}^{\omega}\omega$. Furthermore, for any k, $({}^{\omega}\omega)^k$ is homeomorphic to ${}^{\omega}\omega$. Since the irrational numbers are homeomorphic to their intersection with any open interval of $\mathbb{R}$, it follows that for any k, $({}^{\omega}\omega)^k$ is homeomorphic to the intersection of the irrationals and any open interval contained in $\mathbb{R}$. With these facts in hand, we can now undertake the construction of the totally Borel model of T.

Our main task is to map $H\langle J \rangle$ properly to a Borel subset of $\mathbb{R}$. To accomplish this, we first map $(J, <)$ ordermorphically onto ${}^{\omega}\omega \cap (-1, 0)$. Then, by induction on k, this mapping will be extended to include all elements generated by t_k. Thus, assume that we have extended the mapping to include those elements generated by t_l, $l < k$. We show how to extend it to include those elements generated by t_k.

The construction for t_k splits into two cases according to whether or not there is some $l < k$ and sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$ from T so that $t_l(i_1, \ldots, i_m) = t_k(j_1, \ldots, j_n)$. Suppose first that there are such an l and sequences $\vec{i}$ and $\vec{j}$. Let l_0 be the least such l. From (2), $|S_{l_0}| = |S_k| = p$. Then, from (1) it follows that the value of t_k, for any increasing n-tuple from J and that of t_{l_0} for any increasing m-tuple from J, depend only on the p coordinates in S_k and S_{l_0}, respectively. To simplify notation, we assume that both S_k and S_{l_0} are $\{1, \ldots, p\}$. It then follows that

$$t_k(i_1, \ldots, i_p, i_{p+1}, \ldots, i_n) = t_{l_0}(i_1, \ldots, i_p, j_{p+1}, \ldots, j_m),$$

for any $i_1 < \cdots < i_n$ and $i_p < j_{p+1} < \cdots < j_m$ from J. We may then naturally identify the elements generated from t_k with that subset of $\mathbb{R}$ to which the terms generated from t_{l_0} have already been mapped via the correspondence

$$t_k(i_1, \ldots, i_n) \mapsto t_{l_0}(i_1, \ldots, i_p, j_{p+1}, \ldots, j_m)$$

Notice that if $S_k = \varnothing$, then t_k is constant for all increasing n-tuples from J. In this case, t_k (and hence t_{l_0}) generates only one element of $H\langle J \rangle$.

Let us now carry out the construction in the second case. Hence, assume that there is no $l < k$ and sequences $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$ from J such that $t_l(i_1, \ldots, i_m) = t_k(j_1, \ldots, j_n)$. Suppose also that $|S_k| = p$. In this case we will map the elements generated by t_k into the intersection of the irrationals and $(k, k + 1)$. Since the value of t_k depends only upon the p coordinates of S_k, we may map the elements generated by t_k canonically to

$$\begin{aligned} \mathcal{O} = \{(x_1, \ldots, x_p) \colon x_l \in {}^{\omega}\omega \quad &\text{for each} \quad l \leq p, \\ \text{and} \quad x_1 < x_2 < \cdots < x_p\}. \end{aligned}$$

Then, since $({}^{\omega}\omega)^p$ is homeomorphic to $({}^{\omega}\omega) \cap (k, k + 1)$, the open set $\mathcal{O}$ of "elements" can be identified with a relatively open subset of $({}^{\omega}\omega) \cap (k, k + 1)$. Again, if $S_k = \varnothing$, then t_k generates one element of $H\langle J \rangle$, which can be mapped to the real number $(2k + 1)/2$. Regardless, the elements of $H\langle J\rangle$ have been identified with a subset of $\mathbb{R}$ that is at worst π_2^0.

To complete the proof of the theorem, we must show that every definable relation of $\mathscr{H}\langle J\rangle$— the domain of $\mathscr{H}\langle J\rangle$ being identified with the Borel subset of $\mathbb{R}$ constructed above—also is Borel. To this end, observe that in $\mathscr{H}\langle J\rangle$ every formula is equivalent to a quantifier free one. Then it is routine, but tedious, to check that the model just constructed is totally Borel. ☐

Some additional information may be extracted from the proof of the theorem. First, it is apparent that all of the definable Borel relations of the model constructed fall within the first ω levels of the Borel hierarchy. Second, we could begin the construction with a Borel subset of arbitrary complexity, and thereby build a Borel model of complexity as high as we please. We must pay the price, unfortunately, for such simultaneous control and flexibility: it is not always easy to build models with desired properties from a Skolem hull of indiscernibles. This point is emphasized by the proof of the first application of the Completeness theorem given as Theorem 1.2.2. A more versatile technique for building models will be introduced in Section 3.

The first use of the completeness theorem establishes the existence of recursively saturated totally Borel structures. That there are limits to the saturation of Borel structures beyond the *a priori* ones will be shown by Theorem 1.3.3. The proof of the next theorem is essentially an adaptation of that given by Barwise [1975] in proving that any structure has a recursively saturated elementary extension of the same cardinality.

1.2.2 Theorem. *Every countably infinite $\mathscr{L}$-structure $\mathscr{M} = \langle M, R_1, \ldots, R_n\rangle$ has an uncountable totally Borel elementary extension that is recursively saturated.*

Proof. We will use the notation and definitions from Barwise [1975]. Hence, let $\mathscr{M}$ be as in the hypothesis of the theorem. To $\mathbb{H}\mathrm{YP}_{\mathscr{M}}$, considered as the one-sorted structure $\mathfrak{A} = \langle M \cup A, M, A, R_1, \ldots, R_n, \in\rangle$, add a one-to-one function $F: M \cup A \to M$. By the compactness theorem, we obtain a countable $\langle \mathfrak{B}, G\rangle \succ \langle \mathfrak{A}, F\rangle$ having non-standard natural numbers. Applying the Borel completeness theorem to $\langle \mathfrak{B}, G\rangle$, there exists an uncountable totally Borel

$$\langle \mathfrak{C}, H\rangle = \langle N \cup C, N, C, R_1, \ldots, R_n, E, H\rangle$$

The well-founded part of $\mathfrak{C}$, $\mathfrak{C}'$, is admissible and $N \in \mathfrak{C}'$. Consequently, $o(\mathfrak{C}') = \omega$, and $\mathscr{N}$ is recursively saturated. Furthermore, $\mathscr{N}$ clearly is totally Borel, and H insures that $\mathscr{N}$ is uncountable. ☐

The second application of Theorem 1.2.1 that we will present deals with the existence of Borel two-cardinal models.

1.2.3 Theorem. *Let T be a stable first-order theory in a vocabulary have a distinguished unary predicate P. If T has a model $\mathscr{M}$ so that $|M| > |P(\mathscr{M})|$, where $P(\mathscr{M}) = \{a \in M : \mathscr{M} \models P(a)\}$, then T has an uncountable totally Borel model $\mathscr{N}$ with $|P(\mathscr{N})| = \aleph_0$.* ☐

Before proving this result, we observe that it is best possible, as an infinite Borel subset of $\mathbb{R}$ has power $\aleph_0$ or $2^{\aleph_0}$. The proof uses the following proposition.

1.2.4 Lemma. *Any first-order theory T with distinguished unary predicate P that has a model $\mathscr{M}$ of power $\beth_\omega$ in which $|P(\mathscr{M})| = \aleph_0$ has an uncountable totally Borel model $\mathscr{N}$ with $|P(\mathscr{N})| = \aleph_0$.*

Proof. Expand T to a theory T^* with built-in Skolem functions in an expanded vocabulary τ^*. Then make $\mathscr{M}$ into a model of T^*, and let $\{t_n(v_1, \ldots, v_{m_n}): n < \omega\}$ be an enumeration of the terms in τ^*. By the Erdös–Rado theorem, the theory consisting of sentences of the kinds given in (1) through (4) below in the vocabulary $\tau^* \cup \{c_i: i < \omega\}$ can be seen to be consistent:

(1) all sentences of T^*;

(2) $c_i \neq c_j$, for $i \neq j$;

(3) $\varphi(c_{i_1}, \ldots, c_{i_p}) \leftrightarrow \varphi(c_{j_1}, \ldots, c_{j_p})$, where φ is a τ^*-formula and $i_1 < \cdots < i_p$, $j_1 < \cdots < j_p$;

(4) $P(t_n(c_{i_1}, \ldots, c_{i_{m_n}})) \to t_n(c_{i_1}, \ldots, c_{i_{m_n}}) = t_n(c_{j_1}, \ldots, c_{j_{m_n}})$, where $n < \omega$ and $i_1 < \cdots < i_{m_n}, j_1 < \cdots < j_{m_n}$.

It is clear that the interpretation of $\{c_i: i < \omega\}$ will be a sequence of indiscernibles in a model of the set of sentences above. Furthermore, the Skolem hull of a stretched sequence of such indiscernibles having the order type of the irrationals will be a model of T of power $2^{\aleph_0}$ in which the interpretation of P has power $\aleph_0$. This structure can be turned into a totally Borel model of T by repeating the relevant part of the proof of Theorem 1.2.1. □

Proof of Theorem 1.2.3. From Lachlan [1973] the stability of T implies that T has a model $\mathscr{M}$ of power $\beth_\omega$ in which the interpretation of P has power $\aleph_0$. The result then follows from Lemma 1.2.4. □

1.3. Two Theorems on Borel Structures

The results in the preceding sections do not seem to support the reasons given in the introduction for studying Borel structures. The two theorems of this section by contrast reveal in striking ways the effect of restricting one's model theory to Borel structures.

1.3.1 Theorem (Shelah). *A Borel linear order is either separable or has uncountably many pairwise disjoint open intervals.*

A linear order which is not separable and which has no uncountable collection of pairwise disjoint open intervals is known as a *Suslin line* (See Kunen [1980] for a detailed discussion). The statement asserting the non-existence of Suslin lines is known as Suslin's hypothesis, and it is of interest since it is tantamount to asking whether or not $(\mathbb{R}, <)$ is, up to isomorphism, the only unbounded, dense, complete linear order that does not have an uncountable collection of pairwise disjoint open intervals. It is now well known that Suslin's hypothesis is independent of Zermelo–Fraenkel set theory. In contrast, the theorem shows that the Borel version of Suslin's hypothesis is true.

Proof of Theorem 1.3.1. (For unexplained notions from set theory, refer to Jech [1978] or Kunen [1980]). Work in a countable transitive model of set theory. To obtain a contradiction, we will suppose that the conclusion fails. Assume, then, that there is a Borel linear order $\langle B, <\rangle$ which is a Suslin line. In the standard way, a *Suslin tree* (that is, a tree of height ω_1 having neither an uncountable anti-chain nor an uncoutable branch) can be constructed from $(B, <)$. The nodes of the tree can be taken to be open intervals of $(B, <)$ that are ordered by reverse inclusion so that incomparable intervals are disjoint. In the generic extension obtained by forcing with the partial order taken from the tree, the linear order $\langle B^*, <^*\rangle$ given by the Borel code for $\langle B, <\rangle$ has uncountably many pairwise disjoint open intervals. This last assertion may be written as a sentence of $\mathscr{L}_{\omega_1\omega}(Q)$, where Q means "there exist uncountably many." Since the sentence is consistent (in the generic extension), the completeness theorem for $\mathscr{L}_{\omega_1\omega}(Q)$ in Keisler [1970] implies that the sentence is satisfiable in the ground model by a structure $\langle C, <c\rangle$. Then, since $\langle C < c\rangle$ looks sufficiently like $\langle B, <\rangle$ (it will actually be a submodel!), it can be seen that $\langle B, <\rangle$ must have uncountably many pairwise disjoint open intervals contrary to the hypothesis. □

Theorem 1.3.1 has been strengthened in Friedman [1979a]:

1.3.2 Theorem. *A Borel linear order is either separable or contains a perfect totally isolated set (that is, a perfect set A such that, for any $a \in A$, there is an open I containing a with $I \cap A = \varnothing$).* □

The final result of this section demonstrates that the restriction to Borel structures does not permit the full saturation of models that we can obtain in classical model theory. Indeed, if we adopt the view that uncountable chains in a linear order reflect pathology and prefer to work with separable orders instead, then Theorem 1.3.3 points out the naturalness of the class of Borel structures.

1.3.3 Theorem (Harrington and Shelah [1982]). *A Borel linear order cannot have an uncountable increasing or decreasing chain.* □

The proof of Theorem 1.3.3 uses forcing and may be found in Harrington and Shelah [1982]. Since the proof is indirect and, consequently, does not say much about Borel linear orders, we can ask:

1.3.4 Problem. Is there a structure theorem for Borel linear orders that accounts for Theorem 1.3.3?

2. Axiomatizability and Consequences for Category and Measure Logics

2.1. Axiomatizability of $\mathscr{L}(Q, Q_c)$

The first subsection is devoted to the proof of axiomatizability, while the second contains a survey of consequences. We remark that an explicit set of axioms for $\mathscr{L}(Q, Q_c)$ will be given in Section 3. The main result of the present discussion is due to Friedman.

2.1.1 Theorem (Friedman [1978]). *The set of sentences of $\mathscr{L}(Q, Q_c)$ that are valid in all totally Borel $\mathscr{L}(Q, Q_c)$-structures is recursively enumerable.*

Proof. We will work both *with* and *in* second-order arithmetic Z_2 (see Apt–Marek [1974] for the axioms of Z_2 as well as other facts about second-order arithmetic). It is known that the theory of Borel sets and functions and category and measure for the same can be carried out in Z_2. In particular, any model of Z_2 will have its own version of the syntax and semantics for $\mathscr{L}(Q, Q_c)$. Let φ be an $\mathscr{L}(Q, Q_c)$-sentence. The theorem will be proved if we show

(1) There is a totally Borel model of φ iff there is a model $\mathscr{M}$ of Z_2 that also satisfies "there is a totally Borel model of φ."

The left-to-right implication in (1) is trivial, and it remains to prove the other direction.

Consequently, we assume that $\mathscr{M} \models Z_2 +$ that "there exists a totally Borel model of φ." To ease the exposition, we further assume that $\mathscr{M}$ is a countable, ω-standard model. The argument to be given can be modified to deal with the possibility that $\mathscr{M}$ is a non-standard model.

In what follows, we will make liberal use of forcing over $\mathscr{M}$. We will obtain $\mathscr{M}$-generic extensions of $\mathscr{M}$ by adding $\mathscr{M}$-generic subsets of ω, or "reals" to $\mathscr{M}$. The reader can check that this may be done just as for set theory and that all the usual facts about forcing—for example, the Generic Model Theorem—hold in this context.

First, we will construct a certain extension, $\mathscr{M}[T]$, of $\mathscr{M}$. To do this, we observe that by standard techniques, we can build a perfect

$$T \subseteq 2^{<\omega} = \bigcup_{n \in \omega} {}^n 2,$$

considered as a tree ordered by inclusion, such that any path through T is a *Cohen generic real* over $\mathscr{M}$. And, moreover, any finite number of such reals are mutually generic. Let $[T]$ be the set of all infinite paths through T. We then let

$$\mathscr{M}[T] = \bigcup_{\substack{G \subseteq [T] \\ G \text{ finite}}} \mathscr{M}[G],$$

where each $\mathscr{M}[G]$ is just the generic extension obtained by iterating k times the construction used to add a single Cohen real to a model of Z_2. It can be shown that

$\mathscr{M}[T]$ is the model constructed from $\mathscr{M}$ and $[T]$ in the sense appropriate for arithmetic (for a detailed discussion, see Halpern–Levy [1971]). It may be shown (the reader should consult Friedman [1975b]) that $\mathscr{M}[T]$ will be a model of Z_2. For what follows, it is crucial to observe that each element of $\mathscr{M}[T]$ is really constructed from finitely many elements of $[T]$.

For some $a \in M$, we have that $\mathscr{M} \models$ "a is a totally Borel model of φ." We will have to show that this fact is absolute. Without loss of generality, a Borel structure may be thought of as having as its domain a subset of ${}^{\omega}2$. Since there are only countably many formulas in $\mathscr{L}(Q, Q_c)$, a totally Borel $\mathscr{L}(Q, Q_c)$-structure $\mathscr{N}$ can be considered as a Borel subset $B \subseteq \bigcup_{n \in \omega} \omega \times ({}^{\omega}2)^n$, where $\mathscr{N} \models \theta(a_1, \ldots, a_n)$ iff $(\ulcorner \theta(v_1, \ldots, v_n) \urcorner, a_1, \ldots, a_n) \in B(\ulcorner . \urcorner$ represents some fixed coding of the formulas of $\mathscr{L}(Q, Q_c)$ on ω). In other words, that $\mathscr{N}$ is totally Borel implies that its satisfaction relation is totally Borel. Let $b_a \in M$ be the version in $\mathscr{M}$ of the B that corresponds to a. It is well known that Borel subsets admit codings by single reals c so that the property "c codes a Borel set" is a π^1_1-statement (see Jech [1978]). If we let c_a be the code for b_a, then

(2) $\mathscr{M} \models$ "c_a codes a Borel set"
$\wedge$ "the set that c_a codes is the complete diagram of an $\mathscr{L}(Q, Q_c)$-structure"
$\wedge$ "φ is true in this structure."

Hence, we must analyze the statement in (2) to see that it is absolute. The first conjunct in (2), is π^1_1, as we have already observed, and the last is elementary. If we show that the second conjunct is no worse than π^1_2, then by the Schoenfield Absoluteness Lemma (see Jech [1978]) it follows that the statement in (2) will be absolute, as we wish to establish. To prove that the second conjunct is as claimed, we simply write down the conjunction of the clauses in the definition of $\mathscr{L}(Q, Q_c)$-satisfaction in a π^1_2 manner. We only indicate how this is done for Q_c clause, as the other clauses are easy. (For the Q clause, we use the fact that a countable set of reals may be coded by a single real and that any uncountable Borel set contains a perfect subset). We must show that the statement

$$S \equiv \text{"}\{e : (\ulcorner \psi(x, v_1, \ldots, v_n) \urcorner, e, a_1, \ldots, a_n) \in b_a\} \text{ is not meager"}$$

has both a π^1_2 and a Σ^1_2 form so that

$$(\ulcorner Q_c x \psi(x, v_1, \ldots, v_n) \urcorner, a_1, \ldots, a_n) \in b_a \leftrightarrow S$$

will be π^1_2.

For the π^1_2 version, observe that

$S \leftrightarrow \neg \exists c$["$c$ codes a Borel set"
$\wedge$ "the set coded by c" $= \bigcup_{n \in \omega}$ "set coded by c_n"
$\wedge$ $\forall n$("the set coded by c_n is closed"
$\wedge$ "the complement of the set coded by c_n is dense and open")
$\wedge$ $\forall e((\ulcorner \psi(x_1, v_1, \ldots, v_n) \urcorner, e, a_1, \ldots, a_n) \in b_a \rightarrow$ the set coded by c")].

The expression under the scope of the outermost quantification can be seen to be π_1^1 (For instance, the property of being open and dense involves only quantification over ω if one uses codings of intervals with rational endpoints; see Jech [1978] for further details). Hence, the entire expression is π_2^1, as desired. To produce a Σ_2^1 rendition of S, we invoke the fact that Borel subsets have the property of Baire. Now,

$$\begin{aligned} S \leftrightarrow \exists c\, \exists e[&\text{“the set coded by } c \text{ is a non-empty open set”} \\ &\wedge \text{“the set coded by } e \text{ is of first category”} \\ &\wedge \forall h(\text{“} h \in (\text{set coded by } c) - (\text{set coded by } e)\text{”} \\ &\rightarrow (\ulcorner \psi(x, v_1, \ldots, v_n) \urcorner, h, a_1, \ldots, a_n) \in b_a)]. \end{aligned}$$

It is easy to check that the right hand side of the equivalence is Σ_2^1.

Consequently, $\mathscr{M}[T] \models$ “c_a is a Borel code for a totally Borel model of φ” by absoluteness. The remainder of the argument is given to the extraction of a real totally Borel model of φ from that which $\mathscr{M}[T]$ understands to be coded by c_a. To this end, we first will define a mapping $f: {}^{\omega}2 \cap \mathscr{M}[T] \xrightarrow{1-1} \omega_2$ so that the image of any $\mathscr{M}[T]$-Borel set is Borel, and moreover,

(3) an $\mathscr{M}[T]$-Borel subset of $\mathscr{M}[T]$ is $\mathscr{M}[T]$ meager iff its image under f is meager.

Let $S = \{\sigma_n : 2^{<\omega} \rightarrow 2^{<\omega} : n < \omega\}$ be a list, which includes the identity, of order-preserving permutations, σ, of $2^{<\omega}$ so that:

(4) for some k, if $lh(s) = k$ then $\sigma(s^\frown t) = \sigma(s)^\frown t$ (i.e. each such σ is determined by its restriction to some ${}^k2$);
(5) for the least k as in (4), there is no $s \in {}^k2$ and $m < k$ such that $\sigma(s) = \sigma(s \restriction m)^\frown s(m)^\frown \cdots ^\frown s(k-1)$;
(6) for each k, there is exactly one $\sigma \in S$ satisfying (4) and (5); and
(7) For any k and any distinct $s, t \in {}^k2$, there is $\sigma \in S$ with $\sigma(s) = t$.

The permutations all will be in $\mathscr{M}$. Let

$$[T_n] = \{g : \omega \rightarrow 2 : (\exists h \in [T])(\forall m) g \restriction m = \sigma_n(h \restriction m)\}.$$

Clearly, each $[T_n]$ is a perfect subtree of ${}^{\omega}2$. More importantly, if $\sigma_n \neq \sigma_m$, then $[T_n] \cap [T_m] = \varnothing$. Indeed, suppose (with a permissible abuse of notation) that $\sigma_n g = \sigma_m h$. Since all the elements of $[T]$ are mutually generic, we have that $g = h$, and so $\sigma_n g = \sigma_m g$. But then the requirements imposed on S imply that $\sigma_m = \sigma_n$.

The mapping f satisfying (3) will be defined separately for $\bigcup_{n<\omega} [T_n]$ and for

$$R = ({}^{\omega}2 \cap \mathscr{M}[T]) \setminus \bigcup_{n<\omega} [T_n].$$

Before defining f, though, we must make one more observation. Each $r \in \mathscr{M}[T] \cap {}^{\omega}2$, as noted earlier, is generated by some unique smallest finite $G = \{g_1, \ldots, g_n\} \subseteq [T]$, where $g_1 < \cdots < g_n$ under the lexicographic order on ${}^{\omega}2$. That is, r is given by the G-interpretation, $t(g_1, \ldots, g_n)$ of a name t consisting of order pairs of elements of ω and n-tuples of forcing conditions. It can be seen that the mapping

$$f_t(g_1, \ldots, g_n) = t(g_1, \ldots, g_n)$$

defined on the Borel domain

$$\{(g_1, \ldots, g_n): g_1 < \cdots < g_n\} \subseteq ({}^{\omega}2)^n$$

has a Borel image in ${}^{\omega}2$. Then, since for each n, there are only countably many names ($\mathscr{M}$ is countable), we conclude that $\mathscr{M}[T] \cap {}^{\omega}2$ is Borel.

We may now define f. Let $Q \subseteq {}^{\omega}2$ be a nowhere dense perfect set and, for each $n < \omega$ let $P_n = \{g \in {}^{\omega}2: g \restriction (n + 2) = \langle 0 \ldots 01 \rangle\}$. For each n, $P_n \backslash Q$, and with the exception of one point, $[T_n]$, can be written as the disjoint union of basic open sets. Hence, $f \restriction [T_n]: [T_n] \xrightarrow{1-1} P_n \backslash Q$ can be defined so that the image of every relatively open subset of $[T_n]$ is not meager in ${}^{\omega}2$. Finally, since ${}^{\omega}2 \cap \mathscr{M}[T]$ is Borel, we can define $f \restriction R: R \xrightarrow{1-1} Q$ so as to insure that f sends Borel sets to Borel sets, as required.

We still must prove that f satisfies (3). First, suppose that B is an $\mathscr{M}[T]$-meager, $\mathscr{M}[T]$-Borel set. A Cohen generic real is characterized (see Jech [1978, Section 42]) by not belonging to any meager Borel set coded in the ground model over which the real is generic. It follows then that B cannot contain any reals that are generic with respect to the finitely many reals of $[T]$ that generate the Borel codes for B and the countable union of closed, nowhere dense sets that compel B to be meager. Consequently, for every n, $B \cap [T_n]$ is finite, whence $B \cap \bigcup_{n<\omega} [T_n]$ is countable, and so $f(B)$ must be meager. On the other hand, suppose that B is an $\mathscr{M}[T]$-non-meager, $\mathscr{M}[T]$-Borel set. Then, for some $s \in 2^{<\omega}$, relative to

$$[s] = \{f \in {}^{\omega}2: f \restriction n = s, \text{ for } n = lh(s)\},$$

the set $B \cap [s]$ is $\mathscr{M}[T]$-comeager. For some $\sigma_n \in S$, a basic open subset of $[T_n]$ lies in $[s]$. As before, by Cohen genericity, only finitely many elements of $[T_n] \cap [s]$ may lie outside B. Thus a basic open subset of $[T_n]$ must be contained in $B \cap [s]$; and, finally, $f(B)$ itself must be non-meager.

Consequently, we see that the $\mathscr{M}[T]$-totally Borel structure that c_a codes in $\mathscr{M}[T]$ becomes a real totally Borel structure under f. By (3), Q_c is preserved under this transformation. Also, Q is preserved because $\mathscr{M}[T] \cap {}^{\omega}2$ is uncountable, and internally $\mathscr{M}[T]$ will reflect the property that any uncountable Borel set contains a perfect subset which must be isomorphic to its understanding of ${}^{\omega}2$. Therefore, the real totally Borel structure will satisfy φ, completing the proof of the theorem. $\square$

2.2. *Consequences of Theorem 2.1.1 and Its Proof*

We first remark that a random real is characterized by not being an element of any measure 0 Borel set with code in the ground model over which it is generic. Thus, if the proof of Theorem 2.1.1 is carried out using random reals instead of Cohen generic reals, the result is:

2.2.1 Theorem (Friedman [1978]). *The set of sentences of $\mathscr{L}(Q, Q_m)$ that are valid in all totally Borel $\mathscr{L}(Q, Q_m)$-structures is recursively enumerable.* □

The definition of f in the proof of Theorem 2.1.1 could be modified in another way. That is, Q could be chosen to be of measure 0 as well as meager, and each relatively open subset of $[T_n]$ could be mapped to a set of positive measure. With these changes, we are able to prove the left-to-right implication in:

2.2.2 Theorem (Friedman [1978]). *For any sentence φ of $\mathscr{L}(Q, Q_c)$ let φ^* be the $\mathscr{L}(Q, Q_m)$-sentence obtained from φ by replacing each "Q_c" by "Q_m". Then, φ has a totally Borel $\mathscr{L}(Q, Q_c)$-model iff φ^* has a totally Borel $\mathscr{L}(Q, Q_m)$-model.* □

The reverse implication can be shown by appropriately modifying the proof of Theorem 2.2.1. Theorem 2.2.2 might be seen as a transfer theorem between $\mathscr{L}(Q, Q_c)$ and $\mathscr{L}(Q, Q_m)$. Of further interest would be a duality theorem for $\mathscr{L}(Q, Q_c, Q_m)$ that parallels Erdös–Sierpinski duality on the real line (see Oxtoby [1971], Theorem 19.5). More explicitly, for an $\mathscr{L}(Q, Q_c, Q_m)$-sentence φ, let φ^* be the $\mathscr{L}(Q, Q_c, Q_m)$-sentence obtained by interchanging "Q_c" and "Q_m". Such a duality principle would state that φ has a totally Borel model iff φ^* does.

2.2.3 Problem. Is there such a duality principle? Also, are the validities of $\mathscr{L}(Q, Q_c, Q_m)$ recursively enumerable?

The proof of Theorem 2.1.1 can be adapted in yet another way to yield results for sets of sentences $\Phi = \{\varphi_i : i < \omega\}$ of $\mathscr{L}(Q, Q_c)$. It can be shown that Φ has a totally Borel model iff there is $\mathscr{M} \models Z_2$ and an $\mathscr{M}$-non-standard formula φ^* having each $\varphi_i \in \Phi$ as a conjunct so that $\mathscr{M} \models$ "φ^* has a totally Borel model"—the left-to-right implication follows from the ordinary compactness theorem. Therefore, we have:

2.2.4 Theorem (Friedman [1978]). *The logic $\mathscr{L}(Q, Q_c)$ is countably compact.* □

Again, by suitably interchanging the roles of category and measure, the same result follows for $\mathscr{L}(Q, Q_m)$.

2.2.5 Theorem (Friedman [1978]). *The logic $\mathscr{L}(Q, Q_m)$ is countably compact.* □

A central thesis of this chapter is that Borel structures permit a more manageable model theory. We offer some further evidence of this with the observation given below. Let Q^* be interpreted by "there exist $2^{\aleph_0}$ many," then we have

2.2.6 Theorem. *Given an $\mathscr{L}(Q)$ sentence φ, let φ^* be the $\mathscr{L}(Q^*)$ sentence obtained by replacing each "Q" by "Q^*". Then φ is valid on all totally Borel structures iff φ^* is.* □

The proof rests simply on the fact that any Borel set has power either $\aleph_0$ or $2^{\aleph_0}$. Observe in support of our claim, that for arbitrary structures, the analogue of Theorem 2.2.6 depends on set theory: It is true if the continuum hypothesis holds, but false if, for example, $2^{\aleph_0} = \aleph_{\omega_2}$. This raises the following question:

2.2.7 Problem. How badly behaved can the set of valid sentences of $\mathscr{L}(Q^*)$ be?

We remark that further refinements of the results above (they can be proved using similar techniques) are announced in Friedman [1979a]. In particular, all the results above are true, if suitably modified, for countable admissible languages.

3. Completeness Theorems

Although the results of Section 2 suffice to establish that $\mathscr{L}(Q, Q_c)$ and $\mathscr{L}(Q, Q_m)$ have recursively enumerable validities, explicit sets of axioms are not exhibited and the proofs of the theorems do not contribute very much towards building a model theory for these logics. Here we present simple complete sets of axioms for these logics and various sublogics. Furthermore, in proving these theorems a genuine model building tool which we might call a "continuous" Henkin construction is developed.

3.1. The Completeness Theorem for $\mathscr{L}(Q_m)$

The axioms for $\mathscr{L}(Q_m)$ are as follows:

(A) All the usual axiom schemas for first-order logic (as in Chang–Keisler [1973], for example).

(M0) $\neg(Q_m x)(x = y)$.

(M1) $(Q_m x)\psi(x, \ldots) \leftrightarrow (Q_m x)\psi(y, \ldots)$, where $\psi(x, \ldots)$ is an $\mathscr{L}(Q_m)$-formula in which y does not occur and $\psi(y, \ldots)$ is the result of replacing each free occurrence of x by y.

(M2) $(Q_m x)(\varphi \vee \psi) \rightarrow (Q_m x)\varphi \vee (Q_m x)\psi$.

(M3) $[(Q_m x)\varphi \wedge (\forall x)(\varphi \rightarrow \psi)] \rightarrow (Q_m x)\psi$.

(M4) $(Q_m x)(Q_m y)\varphi \rightarrow (Q_m y)(Q_m x)\varphi$.

Notice that axiom (M4) represents a definable form of Fubini's theorem. The rules of inference for $\mathscr{L}(Q_m)$ are the same as for first-order logic: *modus ponens* and generalization. Let the system just described be denoted by K_m.

3.1.1 Theorem (Friedman [1979a]). *A set of sentences T in $\mathscr{L}(Q_m)$ has a totally Borel model iff T is consistent in K_m.* ☐

Before proving the theorem, we require one further notion. An *equivalence Borel structure* is a Borel structure equipped with a Borel equivalence relation defined on its domain so that in addition:

(a) equality is interpreted by E;
(b) each E-equivalence class is both meager and null (that is, it is of measure 0);
(c) the quantifier Q counts the number of E-equivalence classes; and
(d) the relations and functions of the structure are preserved by E.

This last clause, (d), simply means that if $\mathscr{M} \models E(a_i, b_i)$ for $i = 0, \ldots, n-1$, R is an n-place relation symbol and F an n-place function symbol, then

$$\mathscr{M} \models R(a_0, \ldots, a_{n-1}) \leftrightarrow R(b_0, \ldots, b_{n-1})$$

and

$$\mathscr{M} \models E(F(a_0, \ldots, a_{n-1}), F(b_0, \ldots, b_{n-1})).$$

In other words, E is a congruence relation.

Proof of Theorem 3.1.1. To simplify the presentation, we assume that no sentence in T contains a universal or existential quantifier. The argument sketched here can be modified to yield the theorem in full generality; and, along the way, we indicate the changes that must be made. Clearly, only the direction from right-to-left requires proof. Moreover, we assume that "$(Q_m x)(x = x)$" $\in T$. For if $T \cup \{(Q_m x)(x = x)\}$ were not consistent in K_m, then $T \vdash \neg(Q_m x)(x = x)$. In this case, any countable model of the $\mathscr{L}$-theory T' obtained from T by replacing the outermost subformulas of members of T of the form $Q_m x\varphi$ by $(\exists x)(x \neq x)$ will suffice. It can be verified that T' could be derived from T within K_m in this case.

The proof will be carried out in two steps: An equivalence Borel model for T will be constructed first, and from this a totally Borel model for T will be built.

Let us fix an enumeration $\{\varphi_i : i < \omega\}$ of the formulas in $\mathscr{L}(Q_m)$ without first-order quantifiers. We add a new set of variables $V = \{x_s : s \in \omega^{<\omega} \wedge (\forall n > 0)s(n) \in \{0, 1\}\}$ to the vocabulary of $\mathscr{L}(Q_m)$. Next, by induction on n, we define sets $V_n \subseteq V$ and F_n contained in the set of $\mathscr{L}(Q_m)(V)$-formulas in which only variables in V_n are free and only $\mathscr{L}(Q_m)$-variables are bound.

Let $V_0 = \{x_{\langle 0 \rangle}\}$ and $F_0 = \varnothing$. Given V_n and F_n, we define V_{n+1} and F_{n+1} so that the following conditions are met:

(1) $x_s \neq x_t \in F_{n+1}$ for all $x_s, x_t \in V_n$, $s \neq t$;
(2) if $\varphi(x_{s_1}, \ldots, x_{s_k}) \in F_n$, then for all $i_1, \ldots, i_k \in \{0, 1\}$, $\varphi(x_{s_1 \wedge i_1}, \ldots, x_{s_k \wedge i_k}) \in F_{n+1}$, where " $\wedge$ " represents concatenation;
(3) for all $i \leq n$, if the free variables of φ_i are $v_1, \ldots, v_k$, $k \leq n$, then for all distinct $x_{s_1}, \ldots, x_{s_k} \in V_n$, either $\varphi_i(x_{s_1}, \ldots, x_{s_k}) \in F_{n+1}$ or $\neg\varphi_i(x_{s_1}, \ldots, x_{s_k}) \in F_{n+1}$;
(4) if $(Q_m x)\psi \in F_n$ and all free variables of ψ are in V_n, then $\psi(x_{\langle p \rangle}) \in F_{n+1}$, where p is the least $q \in \omega$ with $x_{\langle q \rangle} \notin V_n$;

(5) $V_{n+1} = V_n \cup V_n'$, where $V_n' \subseteq V$ consists of all members of $V \setminus V_n$ introduced in (1)–(4);
(6) if $x_{s_1}, \ldots, x_{s_r}$ lists V_{n+1} in lexicographic order, then the sentence $(Qx_{s_1}) \ldots (Qx_{s_r})(\bigwedge_{\varphi \in F_{n+1}} \varphi)$ is consistent with T in K_m.

By extensive syntactic manipulation and making heavy use of the "Fubini" axiom given in (M4), it can be seen that V_{n+1} and F_{n+1} can be defined. If first-order quantifiers were present, then Skolem functions would have to be introduced before defining V_n and F_n, and (6) would have to be replaced by a modified consistency criterion. Let $F = \bigcup_{n<\omega} F_n$ and notice that $V = \bigcup_{n<\omega} V_n$.

We define an equivalence Borel structure by first defining a structure $\mathscr{M}$ and then mapping $\mathscr{M}$ suitably to a Borel subset of $\mathbb{R}$. The universe of $\mathscr{M}$ will consist of the union of

$$BT = \{f : \omega \to \omega : (\forall n > 0) f(n) \in \{0, 1\}\},$$

called the set of basic terms, and the set of all proper formal terms,

$$FT = \{t(f_1, \ldots, f_n) : t(v_1, \ldots, v_n) \text{ is an } \mathscr{L}\text{-term and } f_1, \ldots, f_n \in B\}.$$

For a relation symbol R, we define

$$\begin{aligned} \mathscr{M} \models R(t_1(f_{11}, \ldots, f_{1k_1}), \ldots, t_l(f_{l1}, \ldots, f_{lk_l})) \\ \text{iff} \quad & \text{for some finite initial segments, } s_{11}, \ldots, s_{lk_l} \\ & \text{of } f_{11}, \ldots, f_{lk_l}, \text{ respectively,} \\ & R(t_1(x_{11}, \ldots, x_{s_{1k_1}}), \ldots, t_l(x_{s_{l1}}, \ldots, x_{s_{lk_l}})) \in F. \end{aligned}$$

The equality relation is defined in exactly the same way.

Let $B \subseteq \mathbb{R}$ be a Borel set consisting of countably many disjoint perfect subsets, so that $\mathbb{R} \setminus B$ has measure 0, $\mathbb{R} \setminus B$ has power $2^{\aleph_0}$, and any basic open subset of any of the perfect sets has positive measure. The map $g : \mathscr{M} \xrightarrow{1-1} \mathbb{R}$ is then defined so that BT is mapped canonically to B, and FT is mapped in any Borel way into $\mathbb{R} \setminus B$.

Once this has been done, it can be shown by an easy induction on complexity that for any $\mathscr{L}(Q_m)$-formula $\varphi(v_1, \ldots, v_l)$ without ordinary quantifiers, we have

$$\begin{aligned} (7)\quad \mathscr{M} \models \varphi(t_1(f_{11}, \ldots, f_{1k_1}), \ldots, t_l(f_{l1}, \ldots, f_{lk_l})) \\ \text{iff} \quad & \text{for some finite initial segments } s_{11}, \ldots, s_{lk_l} \\ & \text{of } f_{11}, \ldots, f_{lk_l}, \text{ respectively,} \\ & \varphi(t_1(x_{s_{11}}, \ldots, x_{s_{1k_1}}), \ldots, t_l(x_{s_{l1}}, \ldots, x_{s_{lk_l}})) \in F, \end{aligned}$$

where the universe of $\mathscr{M}$ is now identified with its image under g. In particular, $\mathscr{M}$ is an equivalence Borel model of T.

If remains only to convert $\mathscr{M}$ into a totally Borel model. To accomplish this, it will suffice to produce $h: \mathscr{M} \to \mathscr{M}$, where again $\mathscr{M}$ is identified with its image in $\mathbb{R}$ under g, so that h takes Borel sets to Borel sets, and also if

$$\mathscr{M} \models t(f_1, \ldots, f_m) = t'(f'_1, \ldots, f'_n), \quad \text{then}$$
$$h(t(f_1, \ldots, f_m)) = h(t'(f'_1, \ldots, f'_n)).$$

Although we omit the details, that such an h can actually be defined follows from (7) above. To see that $h(\mathscr{M})$ still satisfies T, observe that (1) implies that no basic terms will be identified under h. Consequently, the property of having measure greater than 0 is preserved. ☐

3.2. Further Completeness Results

We indicate, without proof, several more completeness theorems that can be established using techniques similar to the one employed in Section 3.1. Let axioms (C0)–(C4) be the results of replacing "Q_m" everywhere in (M0)–(M4) by "Q_c", and let K_c represent the resulting proof system.

3.2.1 Theorem (Friedman [1979a]). *A set of sentences T in $\mathscr{L}(Q_c)$ has a totally Borel model iff T is consistent in K_c.* ☐

The axiom systems for logics involving Q will entail expanding the vocabularies to contain a unary predicate, $N(\cdot)$, and a binary function symbol $F(\cdot,\cdot)$. To make the axioms more comprehensible, we remark that the intended interpretation for $N(\cdot)$ is, of course, $\mathbb{N}$. Also, as x varies, $F(x, \cdot)$ is intended to represent one-to-one maps from the universe of the structure to all perfect subsets of the structure. The axioms thus are as follows.

(Q0) The usual axioms for $\mathscr{L}(Q)$ in the expanded vocabulary (see Chapter IV).
(Q1) $(\forall x)(\forall y)(\forall z)[F(x, y) = F(x, z) \to y = z]$.
(Q2) $\neg(Qx)N(x)$.
(Q3) $(Qy)\varphi \to (\exists x)(\forall y)(\forall z)(z = F(x, y) \to \varphi(z))$.
(Q4) $\neg(Qy)\varphi \to (\exists x)(\forall y)[\varphi \to (\exists z)(N(z) \wedge F(x, z) = y)]$.
(Q5) $(Qx)(x = x) \to (\exists x)(Qy)(\forall z)(F(x, z) \neq y)$.
(Q6) $Qx(x = x) \to [(\forall y_1) \ldots (\forall y_n) \exists z(N(z) \wedge \varphi(y_1, \ldots, y_n, z))$
$\to (\exists z)(\exists x_1) \ldots (\exists x_n)(\forall y_1) \ldots (\forall y_n)(N(z)$
$\wedge\, \varphi(F(x_1, y_1), \ldots, F(x_n, y_n), z))]$,
where $x_1, \ldots, x_n$ are not free in φ.

Axiom (Q5) asserts, in effect, that any perfect set contains two disjoint perfect subsets (by composition of functions, using F), and (Q6) expresses the statement that a partition of the product of perfect sets, $\prod_{i=1}^n P_i$, into countably many pieces admits a homogeneous subset of the form $\prod_{i=1}^n P'_i$, where $P'_i \subseteq P_i$ and P'_i is perfect, for each $i = 1, \ldots, n$. We let K_U represent the proof system based on axioms (A), (U0)–(U6) with the rules of inference as above.

3.2.2 Theorem (Friedman [1979a]). *A set of sentences T in $\mathscr{L}(Q)$ (in the original vocabulary) has a totally Borel model iff T is consistent in K_u.* $\square$

We will not prove this theorem, but a sketch of why K_u is sound deserves mention. Indeed, suppose that φ is a sentence in the original vocabulary that is provable in K_u. We must show that φ is true in every totally Borel $\mathscr{L}(Q)$-structure $\mathscr{M}$. To do this, we might attempt to expand $\mathscr{M}$ to obtain a structure for the new vocabulary in which the axioms of K_u are satisfied. However, for this to be possible, it might be necessary that complicated uncountable projective sets contain perfect subsets. Such a strong property cannot be guaranteed (see Jech [1978], for example, in Section 41). To circumvent this difficulty, we move up to a generic extension in which at least every uncountable projective set contains a perfect subset. Assuming that there exists an inaccessible cardinal, it is well known that this can be done (see Jech [1978], Section 42); and, by means of a more delicate construction, it can be done without the additional assumption. Let a be the Borel code for $\mathscr{M}$. By absoluteness, the interpretation of a, $\mathscr{M}^*$, in the generic extension remains a totally Borel model of $\mathscr{L}(Q)$ (recall the proof of Theorem 2.1.1). Within the generic extension, $\mathscr{M}^*$ can be expanded to a model of the axioms of K_u, and so φ is true in $\mathscr{M}^*$. Again by absoluteness—this time, however, only for the elementary statement asserting that "$\ulcorner\varphi\urcorner \in a$"—it follows that $\mathscr{M} \models \varphi$, as required.

Finally, we consider the logics $\mathscr{L}(Q, Q_m)$ and $\mathscr{L}(Q, Q_c)$. Consider

$$(\mathrm{MU})(Q_m x)(\exists y)(\varphi(x, y) \wedge N(y)) \to (\exists y)(N(y) \wedge (Q_m x)\varphi(x, y)),$$

which has the obvious interpretation that the union of countably many sets of measure 0 has measure 0. Let (CU) be the result of everywhere replacing "Q_m" by "Q_c" in (MU). Let $K_{m,u}$ be the system based on axioms (A), (M0)–(M4), (Q0)–(Q6), and (MU), and let $K_{c,u}$ be the corresponding system for $\mathscr{L}(Q, Q_c)$.

3.2.3 Theorem (Friedman [1979a]). *A set of sentences T in the original vocabulary for $\mathscr{L}(Q, Q_m)$ respectively $\mathscr{L}(Q, Q_c)$) has a totally Borel model iff T is consistent in $K_{m,u}$ (respectively $K_{c,u}$).* $\square$

We close with some pertinent remarks. Several refinements of the results above are announced in Friedman [1979a]. Of particular interest are results described there whose hypotheses are stronger than ZFC. As perhaps the soundness argument for Theorem 3.2.2 has foreshadowed, such theorems will concern Borel rather than *totally Borel* structures.

Finally, we state some problems. Pursuing a question raised in Section 2.2, we can consider

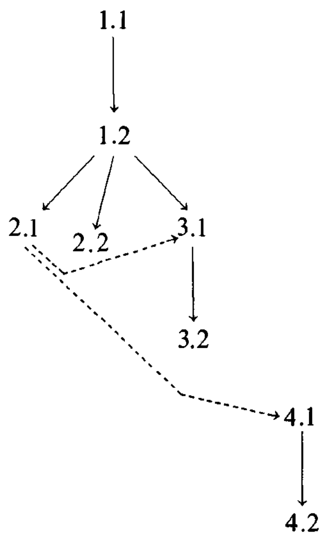
3.2.4 Problem. If the set of valid sentences of $\mathscr{L}(Q, Q_c, Q_m)$ is recursively enumerable, find a simple set of axioms for it.

3.2.5 Problem. Develop the model theory for these logics.

Dependency Chart for Chapter 16

This chapter may be read independently of others in the book.

—— indicates essential dependence.
---- indicates non-essential, but useful background.



Part F

Advanced Topics in Abstract Model Theory

Abstract model theory is the attempt to systematize the study of logics by studying the relationships between them and between various of their properties. The perspective taken in abstract model theory is discussed in Section 2 of Chapter I. The basic definitions and results of the subject were presented in Part A. Other results are scattered throughout the book. This final part of the book is devoted to more advanced topics in abstract model theory.

Chapter XVII views part of our experience with concrete logics in an abstract light. A concrete logic is presented by describing a class of structures, telling how the formulas are built up, and how formulas are interpreted in structures. Since formulas can be viewed as well-founded trees, they can be represented as set-theoretical objects. Similarly, structures are usually thought of as certain kinds of set-theoretical objects. Thus, we can think of a logic $\mathscr{L}$ as given by two predicates of sets: "x is a sentence of $\mathscr{L}$" and "the structure x satisfies the sentence y of $\mathscr{L}$." Chapter XVII deals with the following general problem: What can we say about the model-theoretic properties of $\mathscr{L}$ if we have information about how these predicates can be defined? Two forms of definitions are considered, implicit (Section 1) and explicit (the rest). The usual style of the inductive definition of truth is of the first kind, with its set-theoretical explanation being of the second kind.

When the inductive clauses for a logic $\mathscr{L}'$ can be written down in a logic $\mathscr{L}$, in a suitable precise sense, one says that $\mathscr{L}$ is *adequate to truth in* $\mathscr{L}'$. This gives a useful "effective" relation between logics which, in certain cases, agrees with the relation $\mathscr{L}' \leq_{\mathrm{RPC}} \mathscr{L}$, though not in general. Of special interest are logics which are adequate to truth in themselves.

On the explicit side, one may consider the complexity of the definition of a logic in terms of the Levy hierarchy of set-theoretic predicates, and in terms of the strength of the meta-theory T needed for the definitions. Particularly significant are the cases where the satisfaction relation for $\mathscr{L}$ is Δ_1 relative to a set theory T, which is the same as its being *absolute* relative to models of T. This insures that the meaning of a sentence is not sensitive to which universe of set theory is being considered. Absoluteness has a number of applications to the characterization of the infinitary logics $\mathscr{L}_{\infty\omega}$, $\mathscr{L}_{\infty G}$, and $\mathscr{L}_{\infty V}$ discussed in Chapters VIII and X. The discussions of the implicit and explicit approaches in this chapter are largely independent.

Chapter XVIII explores the relation between certain compactness, embedding, and definability properties. Refinements and generalizations of compactness are presented and treated at the outset. Analogues of various well-known properties from first-order model theory, such as amalgamation Robinson consistency and Beth definability are introduced and related to the various notions of compactness. Striking results emerge, such as the equivalence under certain conditions of full compactness and an abstract version of amalgamation. Also surprising is the appearance of large cardinals in both hypotheses and conclusions of many of the results in this chapter.

Chapter XIX studies the relationship between abstract equivalence relations on structures and logics. Each logic $\mathscr{L}$ determines an equivalence relation $\equiv_{\mathscr{L}}$ on $\mathscr{L}$-structures, that of being $\mathscr{L}$-equivalent. Isomorphic structures are always $\mathscr{L}$-equivalent. Many properties of $\mathscr{L}$ can be stated in terms of these equivalence relations, but it often happens that two quite different logics can give rise to the same equivalence relation.

The primary emphasis in Chapter XIX is on the relation between the equivalence relations for logics and the Robinson consistency property for logics. In Chapter I we discussed the relationship between the interpolation property and the Robinson consistency property. In Chapter XIX quite general results are obtained in an abstract setting on the relationship between compactness, interpolation and the Robinson property. There is also an extensive abstract treatment of (projective) embedding relations and the amalgamation property. Certain dualities are established between logics, equivalence relations, and embedding relations. The chapter concludes with a general study of back-and-forth systems for equivalence relations.

Chapter XVII
Set-Theoretic Definability of Logics

by J. Väänänen

Simply put, an abstract logic is determined by two predicates of set theory, "$x \in \mathscr{L}$" and "$y \models_{\mathscr{L}} x$." The general problem to be considered in this chapter is as follows: What can we say of the model-theoretic properties of $\mathscr{L}$ if we known how the predicates "$x \in \mathscr{L}$" and "$y \models_{\mathscr{L}} x$" behave as predicates of set theory?

Typical model-theoretic properties that are relevant here are Löwenheim–Skolem-type properties, various interpolation properties, completeness and compactness properties, and the conditions that are related to inductive definability of truth. Typical set-theoretic conditions that can be imposed on "$x \in \mathscr{L}$" and "$y \models_{\mathscr{L}} x$" are various forms of absoluteness. A simple example of the use of set theory in abstract model theory is the following result (see Corollary 2.2.3): *If the predicates "$y \in \mathscr{L}$" and "$x \models_{\mathscr{L}} y$" are Σ_1 in set theory, then every $\varphi \in \mathscr{L}$ such that $\varphi \in$ HC and φ has a model, has a countable model,* where HC denotes the set of hereditarily countable sets.

An important tool throughout this chapter will be the notion of adequacy to truth, a concept that is due to S. Feferman. This notion provides an analysis of implicit definability of the actual truth-definition of a logic and is, therefore, naturally connected with the explicit set-theoretical definability of "$x \in \mathscr{L}$" and "$y \models_{\mathscr{L}} x$." A study of adequacy to truth is presented in Section 1.

As opposed to the model-theoretic approach taken in Section 1, Section 2 is devoted to set-theoretic criteria. The simplest and best known example in this direction is the notion of absoluteness of a logic, due to J. Barwise. Set theoretic methods have shown themselves to be more fruitful in connection with absolute logics than anywhere else. When we pass to non-absolute logics, the various independence results of set theory blur the picture. The developments in Section 3 establish the exact relationships between model-theoretic and set-theoretic definability of truth. This is, in effect, the main part of the chapter. We will obtain set-theoretical characterizations of logics such as $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_A$ and characterize definability in the Δ-extensions of various logics.

The results of Section 4 apply the methods of the previous sections and present some new examples of the interplay between model-theoretic and set-theoretic definability. We will conclude the section by making some remarks on possible further work in the area.

1. Model-Theoretic Definability Criteria

The sole purpose of this section is to introduce the notion of adequacy to truth together with its main properties and applications. This notion was first defined by Feferman [1974a] and has its origins in generalized recursion theory. Essentially, it is part of an entire program whose aim is to bring recursion-theoretic notions to bear in abstract model theory.

1.1. Adequacy to Truth

The definitions of most logics, at least of those we would call "syntactic", are given by a recursive definition: For non-atomic φ,

(*) $\mathfrak{M} \models_{\mathscr{L}} \varphi$ if and only if $\mathfrak{M}$ and the subformulae φ_i $(i \in I)$ of φ have the property ...,

where the property ... is expressed in terms of the sequence of assertions $\mathfrak{M} \models_{\mathscr{L}} \varphi_i$ $(i \in I)$. Although (*) is usually written in plain English, it may also be formalizable in another logic, a logic which we would then call "adequate to truth" in $\mathscr{L}$. Before we examine the exact definition, we will give careful consideration to a special case.

1.1.1 Preliminary Example. Consider the logic $\mathscr{L}_{\omega\omega}$. Let us think of formulae of $\mathscr{L}_{\omega\omega}$ as elements of HF, where HF denotes the collection of hereditarily finite sets. A set $a \in \mathrm{HF}$ is an $\mathscr{L}_{\omega\omega}$-formula if it has one of the forms

(1) atomic, $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists v_n \varphi$, $\forall v_n \varphi$,

where φ and ψ are $\mathscr{L}_{\omega\omega}$-formulae and v_n is a variable symbol. We can write out an $\mathscr{L}_{\omega\omega}$-formula Form$(x)$ such that

$$\mathrm{HF} \models \mathrm{Form}(a) \quad \text{if and only if} \quad a \text{ is an } \mathscr{L}_{\omega\omega}\text{-formula.}$$

The truth-relation $\models$ of $\mathscr{L}_{\omega\omega}$ is a relation between HF and the model under consideration. Let $\mathfrak{N}$ be the structure $(M^{<\omega}, P)$, where P maps $s \in M^{<\omega}$ and $n \in \omega$ onto the nth element s_n of s. By writing out the usual clauses of the inductive truth definition, we obtain a formula η in $\mathscr{L}_{\omega\omega}$ containing a new binary predicate $S(x, y)$ such that

(2) If $(\mathfrak{M}, \mathrm{HF}, S, \mathfrak{N}) \models \eta$, then for each formula φ with free variables among $x_1, \ldots, x_n$ we have $S(\varphi, s)$ iff $\mathfrak{M} \models \varphi(s_1, \ldots, s_n)$.

Thus we can formalize the truth of $\mathscr{L}_{\omega\omega}$ in $\mathscr{L}_{\omega\omega}$ up to the definition of HF. But now comes the crucial observation: In contrast to $\mathfrak{M}$, all of HF is not needed in (2)—we can replace HF by any set-theoretical structure which is standard as far as subformulas of $\varphi(x_1, \ldots, x_n)$ are concerned.

Let $\mathfrak{B}$ be a set-theoretical structure, $\mathfrak{B} = (B, E)$. Let π_φ be a sentence in the language of $\mathfrak{B}$ which says that $\mathfrak{B}$ contains $\varphi(x_1, \ldots, x_n)$. ($\varphi$ with free variables $\subseteq \{x_1, \ldots, x_n\}$.) That is, inside $\mathfrak{B}$, regarded as a set-theoretical object, $\varphi(x_1, \ldots, x_n)$ has the same set-theoretical structure as it has in the real world. Then, of course, HF $\models \pi_\varphi$. But, moreover, for any $\mathfrak{B}$

(3) *If* $(\mathfrak{M}, \mathfrak{B}, S', \mathfrak{N}) \models \eta \wedge \pi_\varphi$, then $S'(\varphi, s)$ if and only if $\mathfrak{M} \models \varphi(s_1, \ldots, s_n)$.

Let θ be the $\mathscr{L}_{\omega\omega}$-sentence

$$\eta \wedge \forall x(\mathrm{Th}(x) \leftrightarrow \exists s\, S(x, s)),$$

where Th is a new unary predicate symbol. If we merge S into $\mathfrak{N}$, we then have, for any $\varphi \in \mathscr{L}_{\omega\omega}$:

(4) If $(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}) \models \theta \wedge \pi_\varphi$, then $\varphi \in T$ if and only if $\mathfrak{M} \models \varphi$.

We thus have an implicit definition of truth of $\mathscr{L}_{\omega\omega}$ inside $\mathscr{L}_{\omega\omega}$ using extra symbols and the infinitary sentences π_φ. This is what adequacy of $\mathscr{L}_{\omega\omega}$ to truth means in itself.

Before proceeding to the definition of adequacy to truth in general, we need some conventions concerning representation of syntax and the definition of the formulas π_φ.

For any set a, let $\mu_a(z)$ be the following (possibly) infinitary formula in the vocabulary $\tau_{\mathrm{set}} = \{\in\}$:

$$\mu_a(x) = \forall y(y \in x \leftrightarrow \bigvee_{b \in a} \mu_b(y)).$$

This recursive definition has the intuitive content $\mu_a(x) \leftrightarrow x = a$, which indeed takes place in any transitive set containing $\mathrm{TC}(\{a\})$. For example, $\mu_{\{a_1, \ldots, a_n\}}(x)$ is

$$\forall y(y \in x \leftrightarrow \mu_{a_1}(y) \vee \cdots \vee \mu_{a_n}(y)).$$

Now, let

$$\pi_a(x) = \mu_a(x) \wedge \bigwedge_{b \in \mathrm{TC}(a)} \exists y \mu_b(y).$$

If $\mathfrak{B} = (B, E)$ is a model of the axiom of extensionality, $\mathfrak{B}_0$ the well-founded part of $\mathfrak{B}$, and $\mathfrak{A}$ the transitive collapse of $\mathfrak{B}_0$ via $i\colon \mathfrak{B}_0 \to \mathfrak{A}$, then:

$$\mathfrak{B} \models \pi_a(x) \quad \text{if and only if} \quad x \in B_0, a \in A \text{ and } i(x) = a.$$

1.1.2 Convention. We have made no requirements on the way the syntax of various logics is defined. Henceforth, we will assume that associated with the logic $\mathscr{L}$ is a transitive set A such that $\mathscr{L}(\tau) \subseteq A$, for all τ considered. Moreover, it is assumed that

$$\mathrm{Mod}(\pi_a) \in \mathrm{EC}_{\mathscr{L}[\tau_{\mathrm{set}}]} \quad \text{for} \quad a \in A.$$

In other words, $\mathscr{L}$ is supposed to be strong enough to fix—or "pin down", as it were—each element of A. Finally, A is assumed to be closed under primitive recursive set functions. In this case, we say that the *syntax of $\mathscr{L}$ is represented on A*. As a standing piece of notation, $\mathscr{L}$ is represented on a set denoted by A, $\mathscr{L}'$ on A', $\mathscr{L}''$ on A'' etc. Clearly, the syntax of the logics

$$\mathscr{L}_A,\ \mathscr{L}_A(Q),\ \mathscr{L}_A^2,\ \mathscr{L}_A(\mathrm{aa}),\ \mathscr{L}_A(\mathrm{pos})$$

is represented on A, the syntax of $\mathscr{L}_{\omega_1\omega_1}$ on HC, *etc*. In this chapter, "$\mathscr{L}(Q)$" means $\mathscr{L}(Q_1)$. The logic $\Delta(\mathscr{L})$ is more problematic. However, we may identify sentences of $\Delta(\mathscr{L})$ with triples $\langle \tau, \varphi, \varphi' \rangle$ where the reductions of $\mathrm{Mod}(\varphi)$ and the complement of $\mathrm{Mod}(\varphi')$ to the vocabulary τ coincide. Understood in this way, $\Delta(\mathscr{L})$ has a canonical representation of syntax on A. We use $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, etc. to denote the set-theoretical structures $(A, \in|_A)$, $(A', \in|_{A'})$, etc.

1.1.3 Definition. We say that a logic $\mathscr{L}$ is *adequate to truth in* a logic $\mathscr{L}'$ if for every τ there is $\tau^+ = [\tau, \tau_{\mathrm{set}}, \mathrm{Th}, \tau']$ and $\theta \in \mathscr{L}[\tau^+]$ such that for every $\mathfrak{M} \in \mathrm{Str}[\tau]$, the following conditions hold:

(AT1) $(\mathfrak{M}, \mathfrak{A}', \mathrm{Th}_{\mathscr{L}'}(\mathfrak{M}), \mathfrak{N}) \models_{\mathscr{L}} \theta$ for some $\mathfrak{N}$.
(AT2) If $(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}) \models_{\mathscr{L}} \theta \wedge \pi_\varphi(b)$, then $b \in T$ if and only if $\mathfrak{M} \models_{\mathscr{L}'} \varphi$, whatever $\varphi \in A'$ and $b \in B$.

Compare (AT2) with (4) above. The role of τ' is to provide the auxiliary tools (such as the pairing function and $S(x, y)$ in Example 1.1.1) that are mainly needed for coding.

1.1.4 Example. The logic $\mathscr{L}_{\omega\omega}$ is adequate to truth in $\mathscr{L}_A$. To prove this, we need only make some additions to Example 1.1.1. There the sentence η is supposed to conjoin the different cases of the inductive truth-definition of $\mathscr{L}_{\omega\omega}$. To extend this to $\mathscr{L}_A$, we simply conjoin η with something like

$$S\left(\bigwedge_{i \in I} \varphi_i, s\right) \leftrightarrow \forall i(i \in I \to S(\varphi_i, s)),$$

$$S\left(\bigvee_{i \in I} \varphi_i, s\right) \leftrightarrow \exists i(i \in I \wedge S(\varphi_i, s)).$$

Note here that π_a will not be in $\mathscr{L}_{\omega\omega}$ unless $a \in \mathrm{HF}$.

1.1.5 Example. The logic $\mathscr{L}_{\omega\omega}(Q)$ is adequate to truth in $\mathscr{L}_A(Q)$, whatever Q. This time, we extend Example 1.1.1 by a case for Q. Suppose, for the sake of simplicity, that Q is of signature $\langle 2 \rangle$. Then we add the following case to η:

$$\begin{aligned} &S(Qx_1x_2\varphi(x_1, x_2), s) \\ &\quad \leftrightarrow Qx_1x_2 \exists s'(s'_1 = x_1 \wedge s'_2 = x_2 \wedge (s'_n = s_n \text{ for } n > 2) \\ &\quad \wedge S(\varphi(x_1, x_2), s')). \end{aligned}$$

Then η will contain Q and will no longer be a sentence of $\mathscr{L}_{\omega\omega}$ but rather of $\mathscr{L}_{\omega\omega}(Q)$.

1.1.6 Example. The logic $\mathscr{L}^2_{\omega\omega}$ is adequate to truth in $\mathscr{L}^2_A$. This case needs somewhat more changes to Example 1.1.1 than the previous ones. In Example 1.1.1 $\mathfrak{N}$ contained a new sort for sequences of elements of $\mathfrak{M}$. Now we add to $\mathfrak{N}$ a new sort N_0 for subsets of the domains of $\mathfrak{M}$ and a new sort N_1 for finite sequences of such subsets as well as the projection function for N_1. With these new sorts at hand, we can easily extend the implicit truth-definition, coded in η, to $\mathscr{L}^2_A$. At the same time, we must add the obvious axioms for N_0 and N_1 to η as well. Similarly, we see that $\mathscr{L}^2_{\omega\omega}$ is also adequate to truth in a variety of higher-order logics.

1.1.7 Example. (i) $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ is adequate to truth in $\mathscr{L}_A(\mathrm{aa})$.
(ii) $\mathscr{L}_{\lambda\lambda}$ is adequate to truth in $\mathscr{L}_{\kappa\lambda}$ for all κ.
(iii) $\mathscr{L}_{\omega_1 G}$ and $\mathscr{L}_{\omega_1 V}$ are adequate to truth in themselves. The reader should see Chapter X for the definition of these game logics.

1.1.8 Example. Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_n)_{n<\omega}$, where Q_n is the quantifier "there exists at least ω_n." This logic is not adequate to truth in itself, if represented in the canonical way on HF. The proof of this is simple enough. The Löwenheim–Skolem theorem of $\mathscr{L}$ shows that no θ in $\mathscr{L}_{\omega\omega}(Q_n)_{n<m}$ $(m < \omega)$ can capture $Q_m x(x = x)$, for example.

As we proceed, we will meet other examples of the failure of adequacy to truth. The failure of $\mathscr{L}_{\omega\omega}(Q_n)_{n<\omega}$ to be self-adequate follows intuitively from the fact that the inductive truth-definition has an infinity of genuinely different cases (in fact, one for each Q_n) and there is no way of putting them all together. A similar situation occurs in $\mathscr{L}^2_{\omega\omega}$—here, there is one case for each arity of predicate-variables—but the expressive power of $\mathscr{L}^2_{\omega\omega}$ allows us to take the long conjunction.

1.1.9 Remark. Suppose that $\mathscr{L}$ is a logic and $T\colon \mathrm{Str}[\tau] \to \mathscr{P}(A)$. Feferman [1975] calls T *#-uiid$_x$ in $\mathscr{L}$ if there is* $\tau^+ = [\tau, \tau_{\mathrm{set}}, \mathrm{Th}, \tau']$ and $\theta \in \mathscr{L}[\tau^+]$ such that if $\mathfrak{M} \in \mathrm{Str}[\tau]$, then:

$$(\#1) \qquad [\mathfrak{M}, \mathfrak{A}, T, \mathfrak{N}] \models \theta \text{ for some } \mathfrak{N};$$

and

$$(\#2) \qquad \text{If } [\mathfrak{M}, \mathfrak{B}, T', \mathfrak{N}] \models \theta \wedge \pi_a(a'), \text{ then } a \in T \quad \text{if and only if} \quad a' \in T'.$$

This is a notion which arises naturally from analogous notions in generalized recursion theory, such as the invariant implicit definability of Kunen [1968]. The ("uiid" is short for "uniformly invariantly implicitly definable," and the "x" is used to indicate the possibility of extra sorts in $\mathscr{L}'$). With this notion at hand, we could define adequacy of $\mathscr{L}$ to truth in $\mathscr{L}'$ by simply saying that the mapping $T(\mathfrak{M}) = \mathrm{Th}_{\mathscr{L}'}(\mathfrak{M})$ is $\#$-uiid$_x$ in $\mathscr{L}$. The notion $\#$-uiid$_x$ permits many variations, such as $\#$-usiid$_x$ ("s" for "semi") which replaces "if and only if" by "only if" in ($\#2$). The corresponding weaker form of adequacy to truth could be called *semi-adequacy to truth.*

The notion of adequacy to truth bears a special relation to the Δ-operation defined in Chapter II. The rest of this section is devoted to a study of this. Also recall from Chapter II the notion $\mathrm{RPC}_{\mathscr{L}}$ of relational projective class in $\mathscr{L}$.

1.1.10 Lemma. *Suppose that $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$, $\varphi \in \mathscr{L}'$ and π_φ is $\mathrm{RPC}_{\mathscr{L}}$-definable. Then* $\mathrm{Mod}(\varphi)$ *is $\Delta(\mathscr{L})$-definable.*

Proof. Suppose $\varphi \in \mathscr{L}'[\tau]$. Let $\theta \in \mathscr{L}[\tau^+]$ be as in Definition 1.1.3. The following conditions are equivalent for any $\mathfrak{M} \in \mathrm{Str}[\tau]$:

(a) $\mathfrak{M} \models_{\mathscr{L}'} \varphi$;
(b) $[\mathfrak{M}, \mathfrak{B}, \mathfrak{N}] \models \theta \wedge \pi_\varphi(b) \wedge \mathrm{Th}(b)$ for some $\mathfrak{B}$, $\mathfrak{N}$, and b;
(c) $[\mathfrak{M}, \mathfrak{B}, \mathfrak{N}] \models \theta \wedge \pi_\varphi(b) \to \mathrm{Th}(b)$ for all $\mathfrak{B}$, $\mathfrak{N}$, and b.

By substituting the $\mathrm{RPC}_{\mathscr{L}}$-definition of π_φ into (b) and (c), we obtain a $\Delta(\mathscr{L})$-definition of φ. ☐

1.1.11 Remarks. (i) If $\mathscr{L}$ is only semi-adequate to truth in $\mathscr{L}'$ as given in Lemma 1.1.10, we can still obtain a co-$\mathrm{RPC}(\mathscr{L})$-definition for φ from the proof.

(ii) Lemma 1.1.10 has interesting consequences for logics which have more power than their syntax suggests. Take, for example, $\mathscr{L}^2_{\omega\omega}$. Many set-theoretically definable $\varphi \in \mathscr{L}^2_{\infty\omega}$ satisfy the assumption that π_φ is $\mathrm{RPC}_{\mathscr{L}^2_{\omega\omega}}$. Whence, $\mathrm{Mod}(\varphi)$ is $\Delta(\mathscr{L}^2_{\omega\omega})$-definable. This shows clearly the infinitary nature of $\Delta(\mathscr{L}^2_{\omega\omega})$.

1.1.12 Corrollary. *If $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ and $A' \subseteq A$, then $\mathscr{L}' \leq_{\mathrm{RPC}} \mathscr{L}$.* ☐

1.1.13 Lemma. *If $\mathscr{L}'' \leq_{\mathrm{RPC}} \mathscr{L}$ and $\mathscr{L}''$ is adequate to truth in $\mathscr{L}'$, then $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.*

Proof. Let τ be a vocabulary and let $\theta' \in \mathscr{L}''[\tau^+]$ witness the adequacy of $\mathscr{L}''$ to truth in $\mathscr{L}'$. Let $\tau_1 \supseteq \tau^+$ and $\theta \in \mathscr{L}[\tau_1]$ such that

$$\mathfrak{M} \models \theta_1 \quad \text{if and only if} \quad (\mathfrak{M}, \mathfrak{N}) \models \theta \text{ for some } \mathfrak{N}.$$

Clearly, θ satisfies (AT1) for $\mathscr{L}$ and $\mathscr{L}'$. For (AT2), suppose that

$$(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}', \mathfrak{N}) \models \theta \wedge \pi_\varphi(b).$$

Then $(\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}') \models \theta_1 \wedge \pi_\varphi(b)$, whence

$$b \in T \quad \text{if and only if} \quad \mathfrak{M} \models \varphi,$$

as required. □

1.1.14 Proposition. *Suppose that $\mathscr{L}'$ is adequate to truth in itself and $A' \subseteq A$. Then the following are equivalent:*

(a) *$\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.*
(b) *$\mathscr{L}' \leq_{\mathrm{RPC}} \mathscr{L}$.* □

Discussion. The proposition shows that for syntacticly natural logics, adequacy to truth reduces to the familiar and much simpler concept of $\leq_{\mathrm{RPC}}$. However, this does not take place in general. Rather, we may construe the relation of adequacy to truth as an effective version of $\leq_{\mathrm{RPC}}$. This effectivity can be demonstrated by examples. Thus, unlike $\leq_{\mathrm{RPC}}$, adequacy to truth preserves Σ_1-compactness and Σ_1-definability of validity (see Section 4.3).

1.1.15 Proposition. *Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are logics such that $A' \subseteq A$. Then the following are equivalent:*

(a) *$\Delta(\mathscr{L}) \leq \mathscr{L}'$.*
(b) *Whenever $\mathscr{L}$ is adequate to truth in $\mathscr{L}''$, with $A'' \subseteq A$, then $\mathscr{L}'' \leq \mathscr{L}'$.*

Proof. The argument for (a) implies (b) follows from Corollary 1.1.12. To prove the converse, suppose that $\mathscr{K}$ is a $\Delta(\mathscr{L})$-definable model class. Let Q be the generalized quantifier associated with $\mathscr{K}$. By Proposition 1.1.14, we have that $\mathscr{L}$ is adequate to truth in $\mathscr{L}_A(Q)$. Thus, by letting $\mathscr{L}'' = \mathscr{L}_A(Q)$ in (b), we get $\mathscr{L}_A(Q) \leq \mathscr{L}'$. Whence, Q is $\mathscr{L}'$-definable. □

1.1.16 Definition. A logic $\mathscr{L}$ is *truth maximal* if $\mathscr{L}' \leq \mathscr{L}$ whenever $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ with $A' \subseteq A$. If, in addition, $\mathscr{L}$ is adequate to truth in itself, we say that $\mathscr{L}$ is *truth complete.*

1.1.17 Corollary. *$\mathscr{L}$ has the Δ-interpolation property if and only if $\mathscr{L}$ is truth maximal.* □

1.1.18 Examples. If $A \subseteq \mathrm{HC}$ is admissible, then $\mathscr{L}_A$ is truth complete. If A is the union of countable admissible sets, then $\mathscr{L}_A$ is truth maximal but not necessarily truth complete, in the case where A is not admissible. The Δ-extension of any logic is truth maximal.

The concepts of truth maximality and truth completeness were introduced by Feferman [1974a] and Corollary 1.1.17 was also proven there. Feferman's paper was among the first to discuss Souslin–Kleene interpolation in an abstract setting, and it provided strong support for further work on the Δ-operation.

1.2. Definability of Syntax Set

We have assumed that there is associated with every logic $\mathscr{L}$ a syntax set A on which the syntax of $\mathscr{L}$ is represented. Part of this convention is that every element of A is definable in $\mathscr{L}$. For some $\mathscr{L}$, it happens that A itself is in one form or another definable in $\mathscr{L}$. The results below suggest that such $\mathscr{L}$ have been defined without proper concern to the balance between syntax and semantics. Recall that we use $\mathfrak{A}$ for $(A, \varepsilon|_A)$. Along these same lines, let us use $\mathscr{I}(A)$ for the isomorphism class of $\mathfrak{A}$, $\mathscr{E}(A)$ for the class of structures isomorphic to an end-extension of $\mathfrak{A}$. That is,

$$\mathscr{I}(A) = \{\mathfrak{B} \in \mathrm{Str}[\tau_{\mathrm{set}}] \mid \mathfrak{B} \cong \mathfrak{A}\}$$

$$\mathscr{E}(A) = \{\mathfrak{B} \in \mathrm{Str}[\tau_{\mathrm{set}}] \mid \mathscr{C} \subseteq_{\mathrm{e}} \mathfrak{B}, \text{ for some } \mathscr{C} \in \mathscr{I}(A)\}.$$

We will now consider definability of $\mathscr{I}(A)$ and $\mathscr{E}(A)$.

1.2.1 Examples. (i) $\mathscr{E}(\mathrm{HF}) \in \mathrm{EC}_{\mathscr{L}_{\omega\omega}}$.
(ii) $\mathscr{I}(\mathrm{HF}) \in \mathrm{EC}_{\mathscr{L}_{\omega\omega}(Q_0)}$.
(iii) $\mathscr{E}(A) \in \mathrm{PC}_{\mathscr{L}_A}$ if $A = B^+$, B admissible (see Barwise [1975, V. 3.9]).
(iv) $\mathscr{I}(\mathrm{HC}) \in \mathrm{EC}_{\mathscr{L}_{\omega_1\omega_1}}$.

1.2.2 Proposition. *Suppose $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ and $\Phi \subset \mathscr{L}'$. Then* $\mathrm{Mod}(\Phi)$ *is* $\mathrm{RPC}_{\mathscr{L}}$ *if either of the following conditions holds*:

(a) *$\mathscr{E}(A')$ is $\mathrm{RPC}_{\mathscr{L}}$ and Φ is a Σ_1 subset of A.*
(b) *$\mathscr{I}(A')$ is $\mathrm{RPC}_{\mathscr{L}}$ and Φ is a Π^1_1 subset of A.* □

Remarks. The method of proving this proposition is similar to that used in the proof of Lemma 1.1.10. In (b) we only need to know that Φ is definable by a co-$\mathrm{RPC}_{\mathscr{L}}$-formula over A'. If $A' \subseteq A$, then Σ_1 can be replaced by $\boldsymbol{\Sigma}_1$ and Π^1_1 by $\boldsymbol{\Pi}^1_1$. If $\mathscr{I}(A')$ is $\Delta(\mathscr{L})$-definable in (b), and if Φ is Δ^1_1, then $\mathrm{Mod}(\Phi)$ is $\Delta(\mathscr{L})$-definable.

Applications. The Kleene–Craig–Vaught theorem says that recursively axiomatizable theories in $\mathscr{L}_{\omega\omega}$ can be finitely axiomatized using extra predicates (see Craig–Vaught [1958]). This is exactly what Proposition 1.2.2(a) says if $\mathscr{L} = \mathscr{L}' = \mathscr{L}_{\omega\omega}$ and $A = A' = \mathrm{HF}$. By letting $\mathscr{L} = \mathscr{L}' = \mathscr{L}_A(Q)$, we get the same theorem for $\mathscr{L}_A(Q)$.

An element of paradox is always near when we speak about definability of truth. The following application of this paradoxical element has a long history:

1.2.3 Proposition. *Suppose that $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$ and $\mathscr{I}(A')$ is $\Delta(\mathscr{L})$-definable. Then $\Delta(\mathscr{L}) \not\leq \mathscr{L}'$.*

Proof. Let τ be a vocabulary, and let $\theta \in \mathscr{L}[\tau^+]$ witness the adequacy of $\mathscr{L}$ to truth in $\mathscr{L}'$. Also, let θ' be the conjunction of θ and the $\mathrm{RPC}_{\mathscr{L}}$-definition of $\mathscr{I}(A)$, and let

$$\mathscr{K} = \{(\mathfrak{M}, \mathfrak{B}, b) \mid \exists \mathfrak{N}(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta' \wedge \mathrm{Th}(b))\}.$$

By its definition, $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$-definable. On the other hand, we claim that

$$(*) \qquad (\mathfrak{M}, \mathfrak{B}, b) \notin \mathscr{K} \quad \text{if and only if} \quad \mathfrak{B} \notin \mathscr{I}(A') \quad \text{or} \\ \exists \mathfrak{N}((\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta' \wedge \neg \mathrm{Th}(b)).$$

Suppose first that $(\mathfrak{M}, \mathfrak{B}, b) \notin \mathscr{K}$ but $\mathfrak{B} \in \mathscr{I}(A')$. By (AT1), $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta'$ holds, for some $\mathfrak{N}$. By the definition of $\mathscr{K}$, $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \neg\mathrm{Th}(b)$. Now, for the converse, we suppose that $(\mathfrak{M}, \mathfrak{B}, b) \in \mathscr{K}$. By the definition of $\mathscr{K}$, $\mathfrak{B} \in \mathscr{I}(A')$ and $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}) \models \theta' \wedge \mathrm{Th}(b)$, for some $\mathfrak{N}$. Now if $(\mathfrak{M}, \mathfrak{B}, \mathfrak{N}') \models \theta' \wedge \neg\mathrm{Th}(b)$, for some $\mathfrak{N}'$, then by (AT2), $\mathfrak{M} \models_{\mathscr{L}'} b$ and $\mathfrak{M} \not\models_{\mathscr{L}'} b$, which is absurd. This ends the proof of (*). It follows that $\mathscr{K}$ is $\Delta(\mathscr{L})$-definable. To conclude the proof we show that $\mathscr{K}$ is not definable in $\mathscr{L}'$. To this end, suppose that $\mathscr{K} = \mathrm{Mod}(\varphi)$, for some

$$\varphi \in \mathscr{L}'[\tau \cup \tau_{\mathrm{set}} \cup \{c\}].$$

For any $\mathfrak{M} \in \mathrm{Str}[\tau]$ and $\psi \in \mathscr{L}'[\tau]$, we thus have:

$$(\#) \qquad \mathfrak{M} \models_{\mathscr{L}'} \psi \quad \text{if and only if} \quad (\mathfrak{M}, \mathfrak{A}', \psi) \models_{\mathscr{L}'} \varphi.$$

Now we choose $\tau = \tau_{\mathrm{set}} \cup \{c\}$, $\psi = \neg\varphi$ and $\mathfrak{M} = (\mathfrak{A}', \psi)$. Then (#) gives

$$\begin{aligned} \mathfrak{M} \models_{\mathscr{L}'} \psi \quad &\text{if and only if} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi \\ &\text{if and only if} \quad \mathfrak{M} \not\models_{\mathscr{L}'} \psi. \end{aligned}$$

This contradiction completes the proof. □

1.2.4 Corollary. *If $\mathscr{I}(A)$ is $\Delta(\mathscr{L})$-definable, then $\Delta(\mathscr{L})$ is not adequate to truth in itself.* □

Applications. $\Delta(\mathscr{L})$ is not selfadequate if $\mathscr{L}$ happens to be one of $\mathscr{L}_{\omega\omega}(Q_0)$, $\mathscr{L}^2_{\omega\omega}$, $\mathscr{L}_{\omega_1\omega_1}$ among many others.

Remark. We will later prove that $\Delta(\mathscr{L}_{\omega\omega}(Q_0)) \equiv \mathscr{L}_A$ for $A = (\mathrm{HF})^+$, the smallest admissible set containing HF. Thus, $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$ is selfadequate if represented on $(\mathrm{HF})^+$ rather than on HF. This provides an example of the importance of the exact manner in which the syntax is defined.

Some Refinements. The notion of adequacy to truth is based very heavily on the use of extra predicates. Our prime example (see Example 1.1.1) uses the extra symbols

$M^{<\omega}$—finite sequences of elements of the model,

$P(s, n)$—the nth element of the sequence s,

$S(x, y)$—the sequence y satisfies the formula x.

The use of the new sort $M^{<\omega}$ is actually unnecessary if the model $\mathfrak{M}$ is infinite. Let us say that $\mathscr{L}$ is *simply adequate to truth* in $\mathscr{L}'$ if Definition 1.1.3 can be satisfied (for infinite models) with no new sorts of τ' over those of $\tau \cup \tau_{\text{set}}$. The following are examples of simply selfadequate logics:

$$\mathscr{L}_A, \mathscr{L}_A(Q), L_{AG}, \mathscr{L}_{\kappa\omega}.$$

The above results connecting adequacy to truth and RPC carry over to simple adequacy to truth if RPC is replaced by PC and all models are infinite. As $\text{PC}_{\mathscr{L}} = \text{EC}_{\mathscr{L}}$, for $\mathscr{L} = \mathscr{L}^2_{\omega\omega}$, we see from the analogue of Proposition 1.2.3 that $\mathscr{L}^2_{\omega\omega}$ is not simply adequate to truth in itself.

Another refinement arises in the following way. Looking again at Example 1.1.1, we notice that the only really new symbol one needs is S. That is, we can allow $\mathfrak{M}$ in Example 1.1.1(4) to contain the symbols $M^{<\omega}$ and P. On the other hand, S is implicitly defined by θ as soon as there are no non-standard formulae. This observation motivates the following definition. We say that $\mathscr{L}$ is *uniquely adequate to truth* in $\mathscr{L}'$ if there is a vocabulary τ_{code} such that Lemma 1.1.13 can be satisfied with $\mathfrak{M} \in \text{Str}[\tau \cup \tau_{\text{code}}]$; and, moreover, the relations of $\tau' - \tau_{\text{code}}$ are implicitly defined by θ. The following are examples of uniquely selfadequate logics:

$$\mathscr{L}_{\omega\omega}(Q_0), \mathscr{L}^{2,w}_{\omega\omega}, \mathscr{L}^2_{\omega\omega}, \mathscr{L}_{\omega_1 G}, \mathscr{L}_{\omega_1\omega_1}.$$

If $\mathscr{L}$ is uniquely adequate to truth in $\mathscr{L}'$ and $\mathscr{I}(A')$ is $\text{WB}(\mathscr{L})$-definable, then it can be proven as in Proposition 1.2.3 that $\text{WB}(\mathscr{L}) \not\leq \mathscr{L}'$. Thus, $\text{WB}(\mathscr{L})$ is not uniquely adequate to truth in itself for $\mathscr{L}$ as above.

Another uniform feature in the examples we have is the following: The new type τ^+ is obtained from τ effectively. This gives rise to the following refinement. $\mathscr{L}$ is *effectively adequate to truth in* $\mathscr{L}'$ if 1.1.3 can be satisfied in such a way that τ^+ is obtained from τ via a Σ_1 operation on A.

Historical and Bibliographical Remarks. The notion of adequacy to truth was introduced in Feferman [1974a] and has been further developed in Feferman [1975]. Indeed, Corollary 1.1.17 is from Feferman [1974a]. Definability of syntax set is discussed in Paulos [1976] where Proposition 1.2.3 is (essentially) proven. The roots of Proposition 1.2.3 go back to Craig [1965] and Kreisel [1967]. While Craig only considered higher-order logics, it was Mostowski [1968] who first explicitly proved the failure of interpolation and Beth-definability (see Section 4.1 for this) for logics which can define their own syntax set. On the other hand, we may trace the roots of the application of self-reference and the Liar Paradox in logic back to K. Gödel and A. Tarski. The Kleene–Craig–Vaught theorem is proven in Craig–Vaught [1958] and its generalization to $\mathscr{L}_{\omega\omega}(Q)$ was used in Lindström [1969]. Its generalization (see Proposition 1.2.2(a)) to abstract model theory was remarked in Feferman [1974a]. The reader is referred to Barwise [1975] for a proof of Example 1.2.1(iii).

2. *Set-Theoretic Definability Criteria*

Suppose that we are given a logic $\mathscr{L}$. The predicates $\varphi \in \mathscr{L}$ and $\mathfrak{M} \models_{\mathscr{L}} \varphi$ of φ and $\mathfrak{M}$ are certain set-theoretic predicates, and we may raise the following question: What is the set-theoretic complexity of these predicates? In this section we will study logics with a fixed upper bound for these complexities. Moreover, we will be particularly interested in those definitions of the predicates whose meaning does not depend on the particular interpretation given to set-theoretical axioms.

2.1. Absolute Logics

The idea of absoluteness of a logic is that the truth or falsity of the predicate $\mathfrak{M} \models_{\mathscr{L}} \varphi$ should not depend on the entire set-theoretical universe but rather should depend on the sets that are required to exist (in addition to $\mathfrak{M}$ and φ) by the axioms of a fixed set theory T only. An important candidate for such a set theory T is the Kripke–Platek axioms KP (with the axiom of infinity included). For details on KP and the set-theoretic notion of absoluteness the reader is referred to Barwise [1975], where the following crucial characterization (due to Feferman and Kreisel) can also be found on page 35: For any T, a predicate is absolute in models of T if and only if it is Δ_1 with respect to T (see Feferman [1974a] for a proof of this result). Absolute logics were first studied systematically by Barwise [1972a].

2.1.1 Definition. Let $\mathscr{L}$ be a logic and T a set theory. We say that $\mathscr{L}$ is *absolute relative to* T if there is a predicate $S(x, y)$, Δ_1 with respect to T, such that for $\varphi \in A$ and for any $\mathfrak{M}$

$$\text{(A)} \qquad S(\mathfrak{M}, \varphi) \leftrightarrow \varphi \in \mathscr{L} \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi,$$

and the syntactic operations of $\mathscr{L}$ are Δ_1 with respect to T. The logic $\mathscr{L}$ is (*strictly*) *absolute* if it is absolute relative to some T (relative to KP) which is true in the real world.

Explanations. By "syntactic operations" we mean finitary conjunction, disjunction, permutation, $\pi_a(x)$, *etc.*, which are built into the definition of an abstract logic. By "true in the real world" we mean that T is a consequence of the axioms of our meta-set-theory. It would make little sense to allow T to be, for instance, inconsistent! The most important consequence of T being a true set theory is that if $(A, \ )$ is a transitive model of T and $\mathscr{L}$ is absolute relative to T, and if $\varphi \in \mathscr{L}$ and $\mathfrak{M}, \varphi \in A$, then we have

$$(A,\in) \models \text{“}\mathfrak{M} \models_{\mathscr{L}} \varphi\text{”} \quad \text{if and only if} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi.$$

2.1.2 Example. The infinitary logic $\mathscr{L}_A$ is strictly absolute. The fact that the satisfaction relation of $\mathscr{L}_A$ is Δ_1 in KP-Inf-(= Axiom of Infinity) is proven in Barwise [1975]. The crucial property of KP-Inf is that it allows the definition of Δ-predicates by recursion. All the syntactic and semantic notions of $\mathscr{L}_A$ can be defined in KP-Inf by set-recursion using Δ-predicates.

2.1.3 Example. The logics $\mathscr{L}_{\omega\omega}(Q_0)$ and $\mathscr{L}_A^{2,w}$ are strictly absolute. This can be seen by reducing these logics to $\mathscr{L}_A$ or by considering the proof of the selfadequacy of these logics. The point to note here is that the predicate "x is finite" is Δ_1 in KP. However, the predicate "x is countable" is not Δ_1 in any first-order set-theory; and, indeed, the logic $\mathscr{L}_{\omega\omega}(Q_1)$ turns out to be non-absolute.

2.1.4 Example. The game logics $\mathscr{L}_{AG}$, $\mathscr{L}_{AV}$, and $\mathscr{L}_{AS}$ are absolute relative to KP + Σ_1-separation + DC (= Axiom of Dependent Choices) (see Chapter X). Burgess [1977] introduced the Borel game logic $\mathscr{L}_{\infty B}$ which extends $\mathscr{L}_{\infty V}$ by allowing the operation

$$\bigwedge_{i_0 \in I} \forall v_0 \bigvee_{i_1 \in I} \exists v_1 \ldots \{n \mid \varphi_{i_0 \ldots i_n}(v_0, \ldots, v_{2n+1}) \text{ true}\} \in B,$$

where B is any Borel set of sets of natural numbers and I is a set. It follows from Martin's Borel Determinacy Theorem that $\mathscr{L}_{AB}$ is absolute relative to ZFC.

2.2. *Some Properties of Absolute Logics*

The two principal properties of absolute logics are the downward Löwenheim–Skolem theorem (see Theorem 2.2.2) and the approximation theorem (see Theorem 2.2.8). Many useful implications can be drawn from these two results. Interestingly enough, both have the form of an approximation result, although the two notions of approximation are unrelated. The notion of countable approximation, to be studied first, is due to Kueker [1972, 1977]. It leads to a very strong formulation of the downward Löwenheim–Skolem theorem. The second notion of approximation is the result of gradual development starting from Moschovakis.

The Löwenheim–Skolem Theorem. In order to obtain a particularly simple formulation of the Löwenheim–Skolem theorem, we will assume for a moment that there is a proper class of urelements and that the elements of all models are urelements. This is not an essential restriction, because every model is isomorphic to one consisting of urelements. Moreover, urelements could be avoided by using a more cumbersome notation.

For any sets a and s, let

$$a^s = a \text{ if } a \text{ urelement}, \quad a^s = \{x^s \mid x \in s \cap a\} \text{ otherwise.}$$

If s is countable, then a^s is called a *countable approximation of* a. Note that if $\alpha \in On$, then $\alpha^s < \omega_1$, for all countable s. If $\mathfrak{M}$ is a (relational) structure, then $\mathfrak{M}^s$ is a countable substructure of $\mathfrak{M}$.

If $P(x_1, \ldots, x_n)$ is a predicate of set theory, then we say that

$P(a_1^s, \ldots, a_n^s)$ holds *almost everywhere* (abbreviated by a.e.)

if $P(a_1^s, \ldots, a_n^s)$ holds for all s in a closed unbounded (cub) set of countable subsets of $\mathrm{TC}(\{a_1, \ldots, a_n\})$. See Chapter II, and Chapter IV, Section 4 for more on "almost all countable sets".

2.2.1 Lemma. *If $P(x_1, \ldots, x_n)$ is a Σ_1-predicate and $P(a_1, \ldots, a_n)$ holds, then $P(a_1^s, \ldots, a_n^s)$ holds almost everywhere.* □

2.2.2 Downward Löwenheim–Skolem Theorem. *Suppose that $\mathscr{L}$ is an absolute logic, $\varphi \in \mathscr{L}$ and $\mathfrak{M}$ is a model. Then $\varphi^s \in \mathscr{L}$ almost everywhere and*

$$\mathfrak{M} \models_{\mathscr{L}} \varphi \quad \textit{if and only if} \quad \mathfrak{M}^s \models_{\mathscr{L}} \varphi^s \textit{ almost everywhere.}$$

Proof. The predicate $\varphi \in \mathscr{L}$ is Σ_1, and hence $\varphi^s \in \mathscr{L}$ a.e., by Lemma 2.2.1. Similarly, the predicate $\mathfrak{M} \models_{\mathscr{L}} \varphi$ is Σ_1 and we get $\mathfrak{M}^s \models_{\mathscr{L}} \varphi^s$ a.e. from Lemma 2.2.1. □

If $a \in \mathrm{HC}$, then $a^s = a$ a.e., since the set $\mathrm{TC}(\{a\})$ is countable. Hence, we have

2.2.3 Corollary. *Suppose $\mathscr{L}$ is an absolute logic, $\varphi \in \mathscr{L}$ and $\varphi \in \mathrm{HC}$. If φ has a model, then φ has a countable model.* □

Application. $\mathscr{L}_A(Q_1)$ and $\mathscr{L}_A^2$ are not absolute as they do not satisfy Corollary 2.2.3.

For a sharper application of Theorem 2.2.2, we need a sharper cub set calculation. The proof of the following lemma is not hard.

2.2.4 Lemma. *Suppose that X is a* cub *set of countable subsets of A. Suppose further that $I \subseteq A$ and κ is an infinite cardinal such that $|I| \leq \kappa < |A|$. Then there is a $B \subset A$ such that $I \subseteq B$, $|B| = \kappa$ and the set of countable subsets of B in X form a* cub *set on B.* □

2.2.5 Corollary. *Suppose $\mathscr{L}$ is an absolute logic, $\varphi \in \mathscr{L}$, $\mathfrak{M} \models_{\mathscr{L}} \varphi$, $N_0 \subset M$ has cardinality at most κ and $\varphi \in H_{\kappa^+}$. Then there is an $\mathfrak{N} \subseteq \mathfrak{M}$ of cardinality κ such that $N_0 \subset N$ and $\mathfrak{N} \models_{\mathscr{L}} \varphi$.* □

Proof. Let $A = \mathrm{TC}(\{\varphi, \mathfrak{M}, \kappa^+\})$, and let X be a cub set of countable $s \subset A$ such that $\mathfrak{M}^s \models_{\mathscr{L}} \varphi^s$. Furthermore, let $I = \mathrm{TC}(\{N_0, \varphi\})$. Finally, let $B \subset A$ be given by Lemma 2.2.4 and define $\mathfrak{N}$ to be the restriction of $\mathfrak{M}$ to $B \cap M$. If s is in the cub set of countable subsets of B that are in X, then $\mathfrak{M}^s \models_{\mathscr{L}} \varphi^s$. But $\mathfrak{M}^s = \mathfrak{N}$ a.e. on B and $\varphi^s = \varphi$ a.e. on B. Hence, $\mathfrak{N} \models_{\mathscr{L}} \varphi$. □

The Approximation Theorem

The countable approximations φ^s that we studied in the above discussions were defined from a set-theoretical point of view. We will now associate with every formula φ of an absolute logic approximations $A(\varphi, \alpha)$ ($\alpha \in \mathrm{On}$) which are formulae of $\mathscr{L}_{\infty\omega}$ and which are logically related to φ. It is instructive to first examine the approximations of game formulae. This is the historical order of events: The approximations were developed by Moschovakis and others for game formulae, and it was only later that Burgess [1977] presented the general case (Theorem 2.2.8).

Let us consider a disjunctive game formula

$$(*) \qquad \forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \cdots \bigvee_{n<\omega} \varphi^{i_0 j_0 \ldots i_n j_n}(x_0, y_0, \ldots, x_n, y_n).$$

In order to better understand the idea of approximation, it is useful to write $(*)$ in a new form. Recall that the truth of $(*)$ is determined according to whether player I or II has a winning strategy in the (determinate) infinite game in which each play consists of running through $(*)$ from left to right, with $\exists x_n$ and $\bigvee_{i_n \in I}$ moves of I (pick an x_n or an i_n) and $\forall y_n$ and $\bigwedge_{j_n \in I}$ moves of II (pick a y_n or a j_n) and I wins if one of $\varphi^{i_0 \cdots j_n}(x_0, \ldots, y_n)$ is true in the end. As the truth of $\varphi^{i_0 \cdots j_n}(x_0, \ldots, y_n)$ does not depend on the moves number $n + 1$, $n + 2$, etc., we may as well construe the sense of $(*)$ as

$$(**) \qquad \forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \left(\varphi^{i_0 j_0}(x_0, y_0) \vee \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} (\varphi^{i_0 \cdots j_1}(x_0, \ldots, y_1) \vee \ldots)\right).$$

Let us now define approximations $A(\alpha, \varphi)$ for formulae φ obtained from atomic formulae using $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$, $\bigvee$, $\bigwedge$ and $(**)$:

$$A(0, \varphi) = \perp (= \text{false}),$$

$$A(\alpha + 1, \varphi) = \varphi \text{ if } \varphi \text{ is atomic},$$

$$A(\alpha + 1, \neg\varphi) = \neg A(\alpha, \varphi),$$

$$A(\alpha + 1, \varphi \wedge \psi) = A(\alpha, \varphi) \wedge A(\alpha, \psi),$$

$$A(\alpha + 1, \varphi \vee \psi) = A(\alpha, \varphi) \vee A(\alpha, \psi),$$

$$A(\alpha + 1, \exists x\varphi(x)) = \exists x A(\alpha, \varphi(x)),$$

$$A(\alpha + 1, \forall x\varphi(x)) = \forall x A(\alpha, \varphi(x)),$$

$$A\left(\alpha + 1, \bigvee_{i \in I} \varphi_i\right) = \bigvee_{i \in I} A(\alpha, \varphi_i),$$

$$A\left(\alpha + 1, \bigwedge_{i \in I} \varphi_i\right) = \bigwedge_{i \in I} A(\alpha, \varphi_i),$$

$$A(\nu, \varphi) = \bigvee_{\alpha < \nu} A(\alpha, \varphi) \text{ for limit } \nu.$$

In order to see what happens to $A(\alpha, \varphi)$ for various α and for φ as in $(**)$ above, we will assume that $\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \varphi^{i_0 j_0}(x_0, y_0)$ is true and every $\varphi^{i_0 j_0}(x_0, y_0)$ is atomic. Then $A(6, \varphi)$ is true. If the formulae $\varphi^{i_0 j_0}(x_0, y_0)$ are not atomic but are still in $\mathscr{L}_{\infty\omega}$, then $A(\omega + 5, \varphi)$ is true. We can now prove that φ is true in a model $\mathfrak{M}$ if and only if $A(\alpha, \varphi)$ is true in $\mathfrak{M}$, for some $\alpha \in \mathrm{On}$. Observe that this would not be true if the syntax $(*)$ were used as no approximation would have reached to the long disjunction at the end. If we start with a conjuctive game formula φ, we can define $A(\alpha, \varphi) = \neg A(\alpha, \varphi\neg)$, where $\varphi\neg$ is the dual of φ obtained by everywhere interchanging $\wedge$ and $\vee$, $\exists$ and $\forall$, $\bigvee$ and $\bigwedge$, and an atomic formula and its negation.

A trivial induction on α shows that if $A(\alpha, \varphi)$ is true, then $A(\beta, \varphi)$ is true for all $\beta \geq \alpha$. On the other hand, one need not study very large α: the first α as above is below $|\mathfrak{M}|^+$. The reader should see Chapter X for more on approximations.

2.2.6 Definition. An *approximation function* for a logic $\mathscr{L}$ is a mapping $A\colon \mathrm{On} \times \mathscr{L} \to \mathscr{L}_{\infty\omega}$ such that for all $\varphi \in \mathscr{L}$ and $\mathfrak{M}$:

$$\mathfrak{M} \models_{\mathscr{L}} \varphi \quad \text{if and only if} \quad \mathfrak{M} \models A(\alpha, \varphi) \text{ for some } \alpha \in \mathrm{On}.$$

2.2.7 Example. $\mathscr{L}_{AV}$ has a Δ_1 approximation function. This function was defined above. It is easily proven by induction on α that $A(\alpha, \varphi) \in \mathscr{L}_{\infty\omega}$, for all $\alpha \in \mathrm{On}$ and $\varphi \in \mathscr{L}_{AV}$.

2.2.8 Approximation Theorem. *Every absolute logic has a Δ_1 approximation function.*

Idea of Proof. It will be shown how to define a Δ_1 approximation function on countable $\mathfrak{M}$. The general case is based on forcing and is omitted here see Burgess [1977] for the details). As $\mathscr{L}$ is absolute, there is a Σ_1 predicate $S(x, y)$ such that for $x, y \subset \omega$ we have

$S(x, y)$ if and only if x codes a model $\mathfrak{M}$, y codes a $\varphi \in \mathscr{L}$ and $\mathfrak{M} \models_{\mathscr{L}} \varphi$.

But Σ_1-properties of reals are Σ_2^1 over ω. By using the standard tree representation of Π_2^1 sets, we find a recursive F such that

$S(x, y)$ if and only if $F(x, y, z)$ wellorders ω in some type $< \omega_1$, for some $z \subset \omega$.

Thus, we have a Σ_1^1 property $\psi(x, y, u)$ of reals such that

$$S(x, y) \leftrightarrow \exists\alpha < \omega_1\, \exists u((\omega, u) \cong (\alpha, \in|_\alpha) \wedge \psi(x, y, u)).$$

Recall that a Σ_1^1 property of reals can be defined by a game formula. Let $A'(\alpha, \varphi)$ be a formula of $\mathscr{L}_{\infty V}$ which says that for some $u \subset \omega$, $(\omega, u) \cong (\alpha, \in|_\alpha)$ and $\psi(x, y, n)$ holds for the code y of φ and for the code x of the model we are considering. It thus follows that

$$S(\mathfrak{M}, \varphi) \leftrightarrow \exists\alpha < \omega_1 A'(\alpha, \varphi).$$

To get an approximation $A(\alpha, \varphi) \in \mathscr{L}_{\infty\omega}$, we use the fact that $\mathscr{L}_{\infty V}$ permits approximation (see Example 2.2.7). For more details, consult Burgess [1977]. □

Remark. A logic with a Δ_1 approximation function is, in fact, absolute if its syntactic operations are Δ_1. In particular, every logic with a Δ_1 approximation function has the downward Löwenheim–Skolem property of Theorem 2.2.2. □

2.2.9 Corollary. *Every absolute logic has the Karp property.*

Proof. This is such a basic property of absolute logics that we indicate two proofs here, one using countable approximations and the other approximations in $\mathscr{L}_{\infty\omega}$. Let $\mathscr{L}$ be an absolute logic.

First Proof. Suppose that $\mathfrak{M} \cong_p \mathfrak{N}$ but not $\mathfrak{M} \equiv_{\mathscr{L}} \mathfrak{N}$. The previous sentence is a Σ_1-property of $\mathfrak{M}$ and $\mathfrak{N}$. By Lemma 2.2.1 there are countable approximations

$\mathfrak{M}^s$ and $\mathfrak{N}^s$ of $\mathfrak{M}$ and $\mathfrak{N}$ with the same property. But then $\mathfrak{M}^s \cong \mathfrak{N}^s$, which implies that $\mathfrak{M}^s \equiv_{\mathscr{L}} \mathfrak{N}^s$.

Second Proof. If $\mathfrak{M} \equiv_{\infty\omega} \mathfrak{N}$, then $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same approximations of $\mathscr{L}$-sentences. Hence, by the approximation theorem (see Theorem 2.2.8), $\mathfrak{M} \equiv_{\mathscr{L}} \mathfrak{N}$. $\square$

2.2.10 Corollary. *If $\mathscr{L}$ is an absolute logic, then $\Delta(\mathscr{L}_{\omega_2\omega}) \not\leq \mathscr{L}$.*

Proof. Consider the structures $\mathfrak{M}$ and $\mathfrak{N}$ over the empty vocabulary such that $|\mathfrak{M}| = \aleph_0$ and $|\mathfrak{N}| = \aleph_1$. Now, $\mathfrak{M} \cong_p \mathfrak{N}$, and the model classes $\{\mathfrak{A} \mid \mathfrak{A} \cong \mathfrak{M}\}$, $\{\mathfrak{A} \mid \mathfrak{A} \cong \mathfrak{N}\}$ are $\Delta(\mathscr{L}_{\omega_2\omega})$-definable. If $\Delta(\mathscr{L}_{\omega_2\omega}) \leq \mathscr{L}$, then $\mathfrak{M} \not\equiv_{\mathscr{L}} \mathfrak{N}$, which is contrary to Karp property (see Theorem 2.2.8). $\square$

The rather simple observation given in Corollary 2.2.10 has the following immediate but important consequence: There is no way of extending $\mathscr{L}_{\infty\omega}$ to a logic which obeys the Craig interpolation theorem and which would still be absolute.

2.2.11 Corollary. (i) *Let $\mathscr{L}$ be an absolute logic and $\varphi \in \mathscr{L}$ such that $\varphi \in H_\kappa$ $(\kappa > \omega)$. There are $\varphi_\alpha \in \mathscr{L}_{\kappa\omega}$ $(\alpha < \kappa)$ such that for any $\mathfrak{M}$ of cardinality $< \kappa$:*

$$\mathfrak{M} \models \varphi \leftrightarrow \bigvee_{\alpha < \kappa} \varphi_\alpha.$$

(ii) *If $\mathscr{L}$ is absolute and $\varphi \in \mathscr{L}$ such that $\varphi \in \mathrm{HC}$, then the number of non-isomorphic countable models of φ is either $\leq \aleph_1$ or $2^{\aleph_0}$.*

Proof. The proof of (i) follows from Theorem 2.2.8 and Levy's reflection principle. We take $\varphi_\alpha = A(\alpha, \varphi)$. The proof of (ii) follows from (i) and the similar result for $\mathscr{L}_{\omega_1\omega}$. $\square$

Definability of Well-Order

2.2.12 Definition. A sentence $\varphi(M, <, \ldots)$ *pins down* an ordinal α if $(M, <)$ is well-ordered in every model of $\varphi(M, <, \ldots)$ and $\varphi(M, <, \ldots)$ has at least one model with $(M, <)$ of order type $\geq \alpha$. A logic $\mathscr{L}$ *pins down* α if some $\varphi \in \mathscr{L}$ does. A logic $\mathscr{L}$ is *strong* if some $\varphi \in \mathscr{L}$ pins down every countable ordinal. A logic $\mathscr{L}$ is *bounded* if no $\varphi \in \mathscr{L}$ is able to pin down every ordinal, otherwise it is *unbounded.*

2.2.13 Examples. (i) $\mathscr{L}_{\omega\omega}(Q_0)$ and $\mathscr{L}_{(\mathrm{HF})^+}$ pin down every $\alpha < \omega_1^{\mathrm{CK}}$.

(ii) $\mathscr{L}_{\omega_1\omega}$ pins down every $\alpha < \omega_1$.

(iii) If $\mathrm{cf}(\kappa) > \omega$, then $\mathscr{L}_{\kappa^+\omega}$ pins down κ^+.

(iv) $\mathscr{L}_{\omega_1 S}$ is strong.

(v) $\mathscr{L}_{\omega_1 G}$ is unbounded.

Remarks on Proofs. Recall that ω_1^{CK} (the Church–Kleene ω_1) is the smallest ordinal which is not the order type of a recursive well-ordering of ω. We have used $(\mathrm{HF})^+$ for the set $L_{\omega_1^{\mathrm{CK}}}$. If $\alpha < \omega_1^{\mathrm{CK}}$, then π_α (as defined in Section 1.1) is in $\mathscr{L}_{(\mathrm{HF})^+}$

and pins down α. It will be proven in Chapter VIII that $\mathscr{L}_{(\mathrm{HF})^+}$ cannot pin down ω_1^{CK}. Similarly, π_α will pin down any $\alpha < \omega_1$ in $\mathscr{L}_{\omega_1\omega}$. To pin down an $\alpha < \omega_1^{\mathrm{CK}}$ in $\mathscr{L}_{\omega\omega}(Q_0)$, we simply write down the standard definition of $(\mathbb{N}, +, \cdot, 0, 1, <)$ in $\mathscr{L}_{\omega\omega}(Q_0)$ and then use the recursive definition of α to define α. Since $\mathbb{N}$ is standard, this will really define α. For a proof of (iii) see Barwise–Kunen [1971]. The example (iv) is based on the observation that a linear ordering $<$ of ω is a well-ordering if and only if

$$\bigvee_{i_0} \bigvee_{i_1} \cdots \bigwedge_{n<\omega} i_{n+1} < i_n.$$

It is known that $\mathscr{L}_{\omega_1 S}$ does not pin down ω_2 but pins down every $\alpha < \omega_2$ if $MA + 2^\omega > \omega_1 + \omega_1 = \omega_1^L$. $\square$

In Chapter III it was proven (Theorem III.3.6) that every bounded logic with the downward Löwenheim–Skolem property as in Theorem 2.2.2, is a sublogic of $\mathscr{L}_{\infty\omega}$. The result is interesting enough to be rephrased here as

2.2.14 Theorem. *Suppose that $\mathscr{L}$ is a regular, absolute, and bounded logic. Then $\mathscr{L} \leq \mathscr{L}_{\infty\omega}$.* $\square$

2.3. *Relative Absoluteness and Generalized Quantifiers*

We have observed that $\mathscr{L}_{\omega\omega}(Q_1)$ is not absolute, the reason being that the predicate "x is countable" is itself not absolute. However, even $\mathscr{L}_{\omega\omega}(Q_1^{\mathrm{E}})$ is absolute if $\aleph_1$ is preserved. More generally, by suitably relativizing the notion of absoluteness, we will be able to examine non-absolute logics.

In the following definition, R is an arbitrary predicate of set theory. Recall the characterization of absoluteness as stated earlier in Definition 2.1.1. This characterization is valid in extended languages as well.

2.3.1 Definition. Let $\mathscr{L}$ be a logic, R a predicate of set theory, and T a set theory. We say that $\mathscr{L}$ is *absolute relative to R (and T)* if it is absolute (relative to T) in the extended language $\{\in, R\}$.

2.3.2 Examples. (i) The logic $\mathscr{L}_A(Q)$ is absolute relative to Q and KP(Q) (= KP in the language $\{\in, Q\}$).

(ii) The logic $\mathscr{L}_A^2$ is absolute relative to Pw and KP(Pw) + axiom of power set, where $Pw(x, y) \leftrightarrow y = \mathscr{P}(x)$.

There is a difficulty in proceeding with relative absoluteness in the same way as with absoluteness. The basic method in the theory of absolute logics is to appeal to transitive models of set theory. In the case of relative absoluteness, however, the analogue of transitivity of a model of set theory is the property of being of the form $(M, \in, R \cap M^n)$, where n is the arity of the predicate R. Very little is known of models of this form; and, accordingly, there are very few general results about relatively absolute logics.

In special cases, more specific results obtain. Let Cbl(x) be the predicate of set theory expressing that x is countable (that is, mappable one to one into ω). Clearly, $\mathscr{L}_{\omega\omega}(Q_1)$ is absolute relative to Cbl. But so is $\mathscr{L}_{\omega\omega}(Q_1^{\mathrm{E}})$, where Q_1^{E} says that an equivalence relation has $\aleph_1$ classes. Furthermore, combinations such as $\mathscr{L}_{AG}(Q_1^{\mathrm{E}})$ are absolute relative to Cbl.

2.3.3 Proposition. *Suppose that $\mathscr{L}$ is absolute relative to* Cbl *and $\varphi \in \mathscr{L}$* such that $\varphi \in H_{\omega_2}$. *If φ has a model, then φ has a model of power at most $\aleph_1$.*

Idea of Proof. Let $S(x, y)$ be a predicate Σ_1 relative to Cbl, which defines the truth of $\mathscr{L}$. Then there is a Σ_1-predicate $S'(x, y, z)$ such that $S(x, y) \leftrightarrow S'(x, y, \aleph_1)$ holds in ZFC$^-$ ($=$ ZFC-power set axiom). If $\exists x S(x, \varphi)$, then by Levy's reflection principle $H_{\omega_2} \models \exists x S'(x, \varphi, \aleph_1)$. Whence,

$$S(\mathfrak{M}, \varphi) \quad \text{for some} \quad \mathfrak{M} \in H_{\omega_2}. \quad \square$$

We can improve Proposition 2.3.3 in the direction of Theorem 2.2.2 by studying Kueker's uncountable approximations (see Kueker [1977]).

Hutchinson [1976] showed that the axiomatizability and countable compactness of $\mathscr{L}_{\omega\omega}(Q_1)$ follow from properties of countable models of set theory. Although we will not go into the details, these set-theoretical methods extend naturally to logics that are absolute relative to Cbl.

2.3.4 Example. Let us define

$$\mathrm{Cd}(x) \leftrightarrow \text{“}x \text{ is a cardinal.}$$

Then $\mathscr{L}_A(I)$ is absolute relative to Cd, where I is the equicardinality quantifier $IxyA(x)B(y) \leftrightarrow |A(\cdot)| = |B(\cdot)|$.

To get a result analogous to Proposition 2.3.3 for $\mathscr{L}_A(I)$ we would have to start with an H_κ having the following rather strong reflection property: *If $a \in H_\kappa$, $\varphi(x)$ is Σ_1 relative to* Cd *and $\varphi(a)$ holds, then $H_\kappa \models \varphi(a)$.* Such κ exist, but how large are they? From a standpoint of consistency κ could be 2^ω or κ could be bigger than a measurable cardinal (see Väänänen [1978]).

2.3.5 Proposition. *If $V = L$, then $\mathscr{L}_A^2$ is absolute relative to* Cd.

Proof. If $V = L$, then Pw is Σ_1 relative to Cd:

$$Pw(x, y) \leftrightarrow \exists\kappa\, \exists\alpha \in \kappa(x \in L_\alpha \wedge \mathrm{Cd}(\kappa) \wedge L_\kappa \models Pw(x, y)). \quad \square$$

2.4. *Absoluteness and Boolean Extensions*

In this discussion we will assume that the reader is familiar with Boolean-valued models of set theory and forcing.

2.4.1 Definition. Let $\mathbb{B}$ be a complete Boolean algebra. A logic $\mathscr{L}$ is *absolute for* $\mathbb{B}$, if for all $\mathfrak{M}$:

$$\mathfrak{M} \models_{\mathscr{L}} \varphi \quad \text{if and only if} \quad [\check{\mathfrak{M}} \models_{\mathscr{L}} \check{\varphi}]^{\mathbb{B}} = 1. \quad \square$$

Remarks. It may be that $[\check{\varphi} \in \mathscr{L}]^{\mathbb{B}} = 1$ although $\varphi \notin \mathscr{L}$, for instance, if $\mathscr{L} = \mathscr{L}_{\omega_1\omega}$ and $\mathbb{B}$ collapses φ to a countable set. It may also happen that $[\check{\mathfrak{M}} \models_{\mathscr{L}} \check{\varphi}]^{\mathbb{B}}$ is neither 0 nor 1. However, for homogenous $\mathbb{B}$ this never happens.

2.4.2 Example. If $\mathscr{L}$ is absolute relative to T and $V^{\mathbb{B}} \models T$, then $\mathscr{L}$ is absolute for $\mathbb{B}$. In particular, $\mathscr{L}_A$, $\mathscr{L}_{AG}$, $\mathscr{L}_{AB}$ are all absolute for any $\mathbb{B}$.

2.4.3 Example. $\mathscr{L}_A(I)$ is absolute relative to all Boolean algebras with c.c.c. This is because every c.c.c. algebra preserves the predicate Cd.

2.4.4 Example. $\mathscr{L}_A(Q_1)$ and $\mathscr{L}_{\omega_1\omega_1}$ are absolute for countably closed forcing, since such extensions preserve the predicate Cbl and do not add new countable subsets.

2.4.5 Example. $\mathscr{L}_A(\mathrm{aa})$ is absolute for proper forcing (a notion of forcing is *proper* if it does not destroy stationary subset of ω_1; this condition is, of course, weaker than both countable closure and c.c.c.).

2.4.6 Proposition. *There is no extension of $\mathscr{L}_{\omega\omega}(Q_1)$ which provably in* ZFC *satisfies the Craig interpolation property and is provably absolute for* c.c.c. *forcing.*

Idea of Proof. We shall consider tree-like partially ordered structures as defined in Baumgartner *et al.* [1970]. Let $\mathscr{K}_1$ be the class of tree-like structures with an uncountable branch and $\mathscr{K}_2$ the class of tree-like structures homomorphic to the ordering of the rationals. Then $\mathscr{K}_1$ and $\mathscr{K}_2$ are disjoint PC-classes of $\mathscr{L}_{\omega\omega}(Q_1)$. Suppose that θ is a sentence in a logic absolute for c.c.c. forcing, such that $\mathscr{K}_1 \subseteq \mathrm{Mod}(\theta)$ and $\mathrm{Mod}(\theta) \cap \mathscr{K}_2 = \varnothing$. Let T be a Souslin tree (if there is none, we can obtain one by c.c.c. forcing). Suppose that $T \models \theta$. Let $\mathbb{B}$ be a c.c.c. algebra which embeds T homomorphically into the rationals (see *ibid.*). Then $[T \models \theta]^{\mathbb{B}} = 1$—a contradiction. If $T \not\models \theta$, let $\mathbb{B}$ be a c.c.c. algebra which produces a long branch through T. Then $[T \in \mathscr{K}_1]^{\mathbb{B}} = 1$—a contradiction again. $\square$

Considering the great interest in extensions of $\mathscr{L}_{\omega\omega}(Q_1)$—especially those satisfying Craig—the above result is most useful. It shows that such an extension has to be based on something more complicated than cardinality, cofinality, or stationary sets. In this sense, Proposition 2.4.6 is analogous to Corollary 2.2.9.

The following result shows another direction in the applications of forcing to absolute logics. It is but one— and a simple one, at that—in the range of independence results concerning strong abstract logics.

2.4.7 Proposition. *If* CON(ZF), *then it is consistent that every logic* $\mathscr{L}_{\omega\omega}(Q)$, *provably absolute for* c.c.c. *forcing, has Löwenheim number* $< 2^{\omega}$.

Proof. We shall construct a c.c.c. algebra $\mathbb{B}$ such that $[\mathscr{L}_{\omega\omega}(I)$ has Löwenheim number $< 2^{\omega}]^{\mathbb{B}} = 1$. The more general result will then follow by compactness.

Let $\mathscr{L}_{\omega\omega}(I)[\tau] = \{\varphi_n | n < \omega\}$, where τ is a vocabulary general enough to give the right Löwenheim number. Let $\mathbb{B}_0 = \{0, 1\}$. If $\mathbb{B}_n$ is defined, let $\mathbb{B}_{n+1} \supseteq \mathbb{B}_n$ be a c.c.c. algebra such that if $[\check{\varphi}$ has a model$]^{\mathbb{B}} > 0$ for some c.c.c. $\mathbb{B} \supseteq \mathbb{B}_n$, then $[\check{\varphi}_n$ has a model of power $< 2^{\omega}]^{\mathbb{B}_{n+1}} = 1$. This is possible in view of the unlimited size of 2^{ω} in c.c.c. extensions. If $\mathbb{B}_0 \subseteq \cdots \subseteq \mathbb{B}_n \cdots$ is defined for $n < \omega$, let $\mathbb{B}$ be the direct limit of $(\mathbb{B}_n)_{n<\omega}$. Then $\mathbb{B}$ has c.c.c. Suppose now that $[\check{\varphi}_n$ has a model$]^{\mathbb{B}} > 0$. Then also $[\check{\varphi}_n$ has a model of power $< 2^{\omega}]^{\mathbb{B}_{n+1}} = 1$, by construction. Hence, $[\check{\varphi}_n$ has a model of power $< 2^{\omega}]^{\mathbb{B}} = 1$, by the absoluteness of $\mathscr{L}_{\omega\omega}(I)$ for $\mathbb{B}_{n+1}$ and $\mathbb{B}$. ☐

Historical and Bibliographical Remarks. The definition of an absolute logic goes back to Barwise [1972a]. The Löwenheim–Skolem theorem for absolute logics was first proven in the weaker form (see Corollary 2.2.3) in Barwise [1972a], and the full result Theorem 2.2.2 appeared in Barwise [1974b] using ideas from Kueker [1972]. The results Lemma 2.2.4 and Corollary 2.2.5 are from Kueker [1977]. The notion of approximation was developed for game and Vaught formulae by Vaught [1973b] and has been since generalized for all absolute logics in Burgess [1977], where Theorem 2.2.8 and its corollary is proved. Corollaries 2.2.9 and 2.2.10 were already proven by Barwise [1972a]. The reader is referred to Ellentuck [1975] and Burgess [1978] for results on Souslin logic $\mathscr{L}_{AS}$. The characterization given in Theorem 2.2.13 of $\mathscr{L}_{\infty\omega}$ is due to Barwise [1972a]. Proposition 2.4.6 is due to S. Shelah, while Proposition 2.4.7 is from Väänänen [1980b], where other related results are also proven.

3. *Characterizations of Abstract Logics*

In this section we shall relate the model-theoretic notion of adequacy to truth and the set-theoretic notion of relative absoluteness. In rough terms, we show that if a logic $\mathscr{L}$ is sufficiently strong and is sufficiently absolute, then $\mathscr{L}$ is adequate to truth in $\mathscr{L}$. As applications, we get rather strong results on Δ-extensions of various logics—the main results of this chapter.

3.1. A General Framework

In order that a logic be adequate to truth in a given logic, it must have enough expressive power to enable it to capture the truth definition of the logic. In our set-theoretical approach, this leads to the following new notion:

3.1.1 Definition. Let $\mathscr{L}$ be a logic and R a predicate of set theory. We say that $\mathscr{L}$ *captures* R if there is an $\mathrm{RPC}_{\mathscr{L}}$-class $\mathscr{K}$ of set-theoretical structures such that

(C1) For any set a there is a transitive set M such that $a \in M$, and $(M, \in|_M) \in \mathscr{K}$.
(C2) If $\mathfrak{M} \in \mathscr{K}$ and $\mathfrak{M} \models \pi_{a_i}(m_i)$ $(i = 1 \dots n)$, then $R(a_1, \dots, a_n)$ if and only if $\mathfrak{M} \models R(m_1, \dots, m_n)$.

Explanations. Intuitively, in the above $\mathscr{K}$ is a class of transitive models of set theory. Condition (C1) says only that $\mathscr{K}$ is non-trivial. (C2) is the critical condition and asserts that models of $\mathscr{K}$ preserve R upwards and downwards.

3.1.2 Example. Let $R(x)$ be a predicate which is Δ_1 in KP-Inf. Then $\mathscr{L}_{\omega\omega}$ captures R. To see this, let $\mathscr{K}$ be the $\mathrm{EC}_{\mathscr{L}_{\omega\omega}}$-class of models of a large finite part of KP-Inf. Condition (C1) is then true, since $H_\kappa \in \mathscr{K}$ for all κ. In order to verify (C2), we let $\mathfrak{M} \in \mathscr{K}$ and $\mathfrak{M} \models \pi_a(m)$. We may assume that the well-founded part $\mathfrak{N}$ of $\mathfrak{M}$ is a standard $\in$-structure and thus that $m = a$ also. By the truncation lemma (see Barwise [1975], p. 73), $\mathfrak{N} \in \mathscr{K}$ holds. And, by the absoluteness of Δ_1-predicates we have

$$\begin{aligned} R(a) \quad &\text{if and only if} \quad \mathfrak{N} \models R(a) \\ &\text{if and only if} \quad \mathfrak{M} \models R(a). \end{aligned}$$

Before we examine more examples of capture, let us prove the main result of this section. In this theorem, we will again assume that the elements of all models considered are urelements. In fact, we do not need this convention before Theorem 3.4.15 except in the proof of

3.1.3 Theorem. *Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are logics. Suppose also that $\mathscr{L}$ captures the predicate $S(x, y)$ such that*

$$S(\mathfrak{M}, \varphi) \quad \textit{if and only if} \quad \varphi \in \mathscr{L}' \quad \textit{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi,$$

for all φ and all $\mathfrak{M}$. Then $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.

Proof. Suppose that $\mathscr{K}$ witnesses the capture of $S(x, y)$. Let τ'_{set} (disjoint form τ_{set}) be the vocabulary of $\mathscr{K}$. In order to demonstrate the adequacy of $\mathscr{L}$ to truth in $\mathscr{L}'$, we shall begin with an arbitrary type τ. As a means of simplifying notation, we will assume that τ contains only one binary predicate symbol R and one sort s. We let $\tau^+ = [\tau, \tau_{\mathrm{set}}, T, \tau']$, where τ' contains τ'_{set} and three constant symbols m, n, and r of the sort of τ'_{set}. Let $S'(x, y)$ be the predicate $S(x, y)$ in vocabulary τ'_{set}, and let $\mathscr{K}'$ be the class of τ^+-structures $\mathfrak{M}' = [\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}, m, n, r, f]$ such that all the following hold:

(a) $\mathfrak{N} \in \mathscr{K}$.
(b) $\mathfrak{B} \subseteq_{\mathbf{end}} \mathfrak{N}$.
(c) $\mathfrak{N} \models$ "m is a structure (n, r) of type $\langle 2 \rangle$ and n is a set of urelements."
(d) $\forall x(x \in M \leftrightarrow \mathfrak{N} \models f(x) \in n)$ & "f is 1–1 on M."
(e) $\forall x, y \in M(R(x, y) \leftrightarrow \mathfrak{N} \models (f(x), f(y)) \in r$.
(f) $\forall x \in B(T(x) \leftrightarrow S'(m, x))$.

Intuitively, the following idea is behind $\mathscr{K}'$. $\mathfrak{B}$ is the syntax set of $\mathscr{L}'$, and $\mathfrak{N}$ is a larger set-theoretical universe within which $S(x, y)$ is captured by $\mathscr{K}$. In view of the choice of $S(x, y)$, this essentially entails that $\models_{\mathscr{L}'}$ be captured within $\mathfrak{N}$. Inside the universe $\mathfrak{N}$ m is a structure (n, r) of the same type as the structure $\mathfrak{M}$, the true sentences of which we try to define. Conditions (d) and (e) assert that m looks exactly like $\mathfrak{M}$. Finally, condition (f) defines the truth-predicate T in the obvious way.

Clearly, $\mathscr{K}'$ is an $\mathrm{RPC}_{\mathscr{L}}$-class, so that it is RPC-defined by some $\theta \in \mathscr{L}$. Now, in order to prove (AT1), we let $\mathfrak{M} \in \mathrm{Str}[\tau]$ be given. By (C1), there is a transitive set N such that A', $\mathfrak{M} \in N$ and $\mathfrak{N} = (N, \in|_N) \in \mathscr{K}$. Let us examine the structure

$$\mathfrak{M}' = [\mathfrak{M}, \mathfrak{A}, \mathrm{Th}_{\mathscr{L}'}(\mathfrak{M}), \mathfrak{N}, n, m, r, f],$$

where n, m, r, and f are defined so as to make conditions (a) through (e) true. Now, also condition (f) holds, since by (C2) we have

$$\begin{aligned} \mathfrak{N} \models S'(\mathfrak{M}, \varphi) \quad &\text{if and only if} \quad S(\mathfrak{M}, \varphi) \\ &\text{if and only if} \quad \varphi \in \mathrm{Th}_{\mathscr{L}'}(\mathfrak{M}). \end{aligned}$$

Thus, $\mathfrak{M}' \in \mathscr{K}'$ and therefore expands to a model of θ. This ends the proof of (AT1).

As to the proof of (AT2), we suppose that

$$[\mathfrak{M}, \mathfrak{B}, T, \mathfrak{N}, m, n, r, f, \ldots] \models \theta \wedge \pi_\varphi(b),$$

where $\varphi \in A'$ and $b \in B$. Furthermore, let $\mathfrak{N}'$ be the well-founded part of $\mathfrak{N}$ and i a transitive collapse of $\mathfrak{N}$, $i\colon \mathfrak{N} \to (N, \in)$. As $\mathfrak{B} \models \pi_\varphi(b)$, we have $b \in N'$ and $i(b) = \varphi$. Since n is a set of urelements in $\mathfrak{N}$, $m \in N'$ and $i(m)$ is a structure $\mathfrak{M}'$ isomorphic to $\mathfrak{M}$. Now we may reason as follows:

$$\begin{aligned} b \in T \quad &\text{if and only if} \quad \mathfrak{N} \models S'(m, b), \\ &\text{if and only if} \quad S(\mathfrak{M}', \varphi), \\ &\text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M}' \models_{\mathscr{L}'} \varphi, \\ &\text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi. \end{aligned}$$

This ends the proof of (AT2). $\square$

3.1.4 Corollary. *Suppose $\mathscr{L}$ captures the predicate $x \in \mathscr{K}$, where $\mathscr{K}$ is a model class. Then $\mathscr{K}$ is $\Delta(\mathscr{L})$-definable.*

3.1.5 Corollary (Characterization of $\mathscr{L}_{\omega\omega}$). *$\mathscr{L}_{\omega\omega}$ is the only logic which is represented on* HF *and is absolute relative to* KP-Inf.

Proof. $\mathscr{L}_{\omega\omega}$ certainly has the stated property by Example 2.1.2. On the other hand, if $\mathscr{L}$ is represented on HF and absolute relative to KP-Inf, then by Example 3.1.2 and Theorem 3.1.3, we have that $\mathscr{L}_{\omega\omega}$ is adequate to truth in $\mathscr{L}$. Whence, by Corollary 1.1.12, it follows that $\mathscr{L} \leq \Delta(\mathscr{L}_{\omega\omega}) = \mathscr{L}_{\omega\omega}$. □

Observe that the proof of Corollary 3.1.5 actually gives the stronger conclusion that $\mathscr{L}_{\omega\omega}$ is adequate to truth in any logic that is absolute relative to KP-Inf.

3.1.6 Remarks. (i) The proof of Theorem 3.1.3 also gives a sufficient condition for simple adequacy to truth. All we have to know in addition about capture is that the transitive set M in (C1) can be chosen to be of cardinality at most $|a| \cdot \aleph_0$.

(ii) If $\mathscr{L}$ captures S in the weaker sense that "if" holds in (C2) rather than "if and only if," then we still get $\mathscr{L}$ semi-adequate to the truth in $\mathscr{L}'$ in the sense of Remark 1.1.9, and we still obtain $\mathrm{RPC}_{\mathscr{L}}$-definability of $\mathscr{K}$ in Corollary 3.1.4.

(iii) Some logics may be defined with respect to a parameter (for example α is the parameter for $\mathscr{L}_{\omega\omega}(Q_\alpha)$). There is no essential difficulty in having a parameter p in Theorem 3.1.3 and Corollary 3.1.4, but in Definition 3.1.1(C2), we must assume, of course, that every $\mathfrak{M} \in \mathscr{K}$ contains a set q such that $\mathfrak{M} \models \pi_p(q)$.

(iv) To prove Corollary 3.1.4 we do not need the full strength of "capture." Thus, in (C1), we can restrict to $a = (\mathfrak{M}, p)$, where $\mathfrak{M}$ is an arbitrary model of the type of $\mathscr{K}$ and p is a parameter in the definition of $\mathscr{K}$ (if any).

(v) There is a certain uniformity in the way τ^+ is obtained from τ above. More precisely, the conclusion of Theorem 3.1.3 can be improved to: *$\mathscr{L}$ is effectively adequate to truth in $\mathscr{L}'$.*

3.2. Absolute Logics Revisited

The notion of capture is used to prove the following theorem concerning strict absoluteness (Definition 2.1.1).

3.2.1 Theorem. *$\mathscr{L}_{\omega\omega}(Q_0)$ is adequate to truth in any strictly absolute logic.*

Proof. Suppose that $\mathscr{L}$ is strictly absolute. Thus, *there is a predicate $S(x, y)$ Δ_1 in* KP, *such that*

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L} \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi,$$

for all $\mathfrak{M}$ and φ. An argument similar to the one used in Example 3.1.2 shows that $\mathscr{L}_{\omega\omega}(Q_0)$ captures $S(x, y)$. By Theorem 3.1.3, $\mathscr{L}_{\omega\omega}(Q_0)$ is adequate to truth in $\mathscr{L}$. $\square$

Observe that Theorem 3.2.1 does not allow us to conclude that $\mathscr{L} \leq \Delta(\mathscr{L}_{\omega\omega}(Q_0))$ if $\mathscr{L}$ is strictly absolute, since $\mathscr{L}$ may not be represented on the same syntax set. However, we do have the following important result:

3.2.2 Theorem. $\Delta(\mathscr{L}_{\omega\omega}(Q_0)) \equiv \mathscr{L}_{(\mathrm{HF})^+}$.

Proof. It suffices to prove that $\mathscr{L}_{(\mathrm{HF})^+} \leq \Delta(\mathscr{L}_{\omega\omega}(Q_0))$, as it is well known that $\mathscr{L}_{(\mathrm{HF})^+}$ satisfies the Craig interpolation theorem. As $\mathscr{L}_{(\mathrm{HF})^+}$ is strictly absolute, there is a predicate $S(x, y)$, Δ_1 in KP, such that

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L} \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} \varphi.$$

Let $\varphi \in \mathscr{L}_{(\mathrm{HF})^+}$. As an element of $\mathscr{L}_{\omega_1^{\mathrm{CK}}}$, the set φ is definable by a predicate which is Δ_1 in KP (see, for example, Barwise [1975], Section II.5.14). Thus, the model class $\mathrm{Mod}(\varphi)$ is definable by a predicate, Δ_1 in KP. By Example 3.1.2, $\mathscr{L}_{\omega\omega}(Q_0)$ captures this predicate; and, by Corollary 3.1.4, $\mathrm{Mod}(\varphi)$ is $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$-definable, as desired. $\square$

The above theorem can be relativized to a parameter in the following way. Recall what was said about logics defined with respect to a parameter in Remark 3.1.6(iii).

If $X \subseteq \omega$, let Q_X be the generalized quantifier associated with

$$\{\mathfrak{A} \mid \mathfrak{A} \cong (\omega, <, X)\}$$

and $\mathrm{HYP}(X)$ the smallest admissible set containing X as an element.

3.2.3 Theorem. *If* $X \subseteq \omega$, *then* $\Delta(\mathscr{L}_{\omega\omega}(Q_X)) \equiv \mathscr{L}_{\mathrm{HYP}(X)}$. *Hence,*

$$\mathscr{L}_{\omega_1\omega} \equiv \Delta(\mathscr{L}_{\omega\omega}(Q_X)_{X \subseteq \omega}).$$

Proof. As $\mathscr{L}_{\mathrm{HYP}(X)}$ satisfies Craig, it is again enough to pick $\varphi \in \mathscr{L}_{\mathrm{HYP}(X)}$ and show that $\mathscr{L}_{\omega\omega}(Q_X)$ captures the predicate $x \in \mathrm{Mod}(\varphi)$, as we did in the proof of Theorem 3.2.2. Note that $\mathscr{L}_{\mathrm{HYP}(X)}$ is strictly absolute and $\mathrm{Mod}(\varphi)$ is therefore definable by a predicate, Δ_1 in KP, with φ as a parameter. As an element of $\mathrm{HYP}(X)$, φ is itself definable by a predicate Δ_1 in KP, with X as a parameter (see Barwise [1975], Section IV.1.6). Using $\mathscr{L}_{\omega\omega}(Q_X)$, it is now easy to capture the predicate $x \in \mathrm{Mod}(\varphi)$: We simply proceed as in Example 3.1.2 and use Q_X to capture the parameter X. $\square$

We shall apply Theorem 3.2.1 now to prove the main result of Barwise [1972a]:

3.2.4 Theorem. *Let A be an admissible set containing ω, and let $\mathscr{L}$ be a strictly absolute logic the syntax of which is represented on A. Then $\mathscr{L} \leq \mathscr{L}_A$.*

Proof. Let $\mathscr{L}$ be a strictly absolute logic represented on A, and let $\varphi \in \mathscr{L}$. Suppose, for a *reductio ad absurdum*, that $\mathrm{Mod}(\varphi)$ is not definable in $\mathscr{L}_A$. Then the following holds:

$$\exists A_0\ \exists\varphi_0 \in A_0((A_0, \in|_{A_0}) \models KP \ \wedge\ \varphi_0 \in \mathscr{L}\ \wedge\ \forall\psi \in \mathscr{L}_{A_0} \neg(\varphi_0 \leftrightarrow \psi)).$$

This can be written in Σ-form. Whence, by Levy's reflection lemma, it holds in HC. Thus, we have a countable admissible set A_0 such that some $\varphi_0 \in \mathscr{L}$ is in A_0 but is not definable in $\mathscr{L}_{A_0}$. Let $\mathscr{L}'$ be the strictly absolute sublogic of $\mathscr{L}$ containing those sentences of $\mathscr{L}$ which are in A_0. By Theorem 3.2.1, $\mathscr{L}_{\omega\omega}(Q_0)$, and hence also $\mathscr{L}_{A_0}$, is adequate to truth in $\mathscr{L}'$. By Corollary 1.1.12, $\mathscr{L}' \leq \Delta(\mathscr{L}_{A_0})$. As A_0 is a countable admissible set, $\Delta(\mathscr{L}_{A_0}) \equiv \mathscr{L}_{A_0}$, and hence, $\mathscr{L}' \leq \mathscr{L}_{A_0}$ also. But this contradicts the assumption that $\mathrm{Mod}(\varphi_0)$ is not definable in $\mathscr{L}_{A_0}$. □

Application. The logics $\mathscr{L}_{AG}$, $\mathscr{L}_{AV}$, $\mathscr{L}_{AB}$ and other unbounded absolute logics are not strictly absolute.

Theorem 3.2.4 is an important characterization of admissible languages $\mathscr{L}_A$. It uses essentially the Souslin–Kleene property of $\mathscr{L}_A$ for countable A. The lack of this property is the main obstacle to proofs of a similar result for other logics. The following is a local version:

3.2.5 Corollary. *Let A be an admissible set containing ω. A model class is definable in $\mathscr{L}_A$ if and only if it is definable by a predicate Δ_1 in* KP, *with parameters in A.* □

3.2.6 Corollary (Characterization of $\mathscr{L}_A$). *If $\omega \in A$, then $\mathscr{L}_A$ is the strongest strictly absolute logic represented on A.* □

3.3. Unbounded Logics

Recall that a logic L is unbounded if L contains a sentence which pins down every ordinal or, equivalently, if the notion of well-ordering is RPC in L.

3.3.1 Lemma. *An unbounded logic captures every Δ_1 predicate.*

Proof. The capturing RPC-class $\mathscr{K}$ asserts that $\mathfrak{M}$ is a well-founded model of the sentence expressing the Δ_1-definability of the predicate in question. Condition (C2) then follows from the absoluteness of Δ_1 predicates in transitive domains. □

3.3.2 Theorem. *Any unbounded logic is adequate to truth in any absolute logic.*

Proof. The claim follows from the definition of absolute logics (see Lemma 3.3.1 and Theorem 3.1.3). □

It would now be in order to search for the simplest possible unbounded logic. Unfortunately, there is no natural choice. The simplest logic in which the notion of well-ordering is EC-definable (rather than RPC-definable) is the logic $\mathscr{L}_{\omega\omega}(W)$, where

$$WxyA(x, y) \leftrightarrow A(\cdot, \cdot)\text{well-orders its field.}$$

But, for example, the unbounded logic $\mathscr{L}_{\omega\omega}(I)$ does not contain $\mathscr{L}_{\omega\omega}(W)$ (see Lindström [1966]).

3.3.3 Corollary. (i) *If $\mathscr{L}$ is absolute, then $\mathscr{L} \leq \Delta(\mathscr{L}_A(W))$.*

(ii) *The logics $\mathscr{L}_A(W)$, $\mathscr{L}_{AG}$, $\mathscr{L}_{AV}$, $\mathscr{L}_{AB}$ and all unbounded absolute logics represented on A are Δ-equivalent.*

(iii) *A model class is definable in $\Delta(\mathscr{L}_A(W))$ if and only if it is definable by a Δ_1 predicate with parameters in A.* □

Remark. We can replace "unbounded" by "strong" in Theorem 3.3.2 and Corollary 3.3.3(ii) if the syntax set A is assumed to be contained in HC. Respectively, if $A \subseteq \text{HC}$, $\mathscr{L}_{AS}$ can be added to the list of logics in Corollary 3.3.3(ii).

It is interesting to observe that there is no strongest absolute logic (this follows from Theorem 3.4.7 below). The family of absolute logics divides into two categories: The first consists of sublogics of $\mathscr{L}_{\infty\omega}$, and the second of Δ-equivalent logics (up to difference of syntax set).

There is an important relation between descriptive set theory and infinitary logic. In order to see this, let us restrict ourselves to countable structures and logics represented on HC for a moment. A class $\mathscr{K}$ of countable models can be viewed as a set of reals, and it thus is meaningful to ask, for example, whether $\mathscr{K}$ is Borel or not. If $\mathscr{K}$ is invariant (that is, closed under isomorphisms), then Diagram 1 shows the equivalence of $\mathscr{K}$ being definable on a level in topology and $\mathscr{K}$ being definable in an infinitary logic. The reader is referred to Vaught [1973] for details on these equivalences. Observe, however, that on the last row, we can replace $\mathscr{L}_{\omega_1 V}$ by any unbounded absolute logic (by Corollary 3.3.3(ii)). Thus, every $\mathscr{K}$ definable in an absolute logic is Δ^1_2. Burgess [1977] showed that the question (posed by Vaught) of whether the converse holds, that is, of whether every Δ^1_2 $\mathscr{K}$ is definable in an absolute logic, is independent of ZFC.

Topology	Infinitary Logic
Borel	$\mathscr{L}_{\omega_1\omega}$
Analytic	PC in $\mathscr{L}_{\omega_1\omega}$
C-set	$\mathscr{L}_{\omega_1 V}$
Σ^1_2	PC in $\mathscr{L}_{\omega_1 V}$

Diagram 1

3.4. Relatively Absolute Logics Revisited

As we remarked earlier, the role that transitive models of set theory play in the theory of absolute logics is taken up by models of the form $(M, \in \cap M^2, R \cap M^n)$ in the theory of relatively absolute logics (see Section 2.3). Getting a hold on $R \cap M^n$ is no easier than making sure that $\in$ is the true $\in$. While unboundedness is a good means for $\in$, we need the following relativized version of pinning down for $R \cap M^n$:

3.4.1 Definition. Let $\mathscr{L}$ be a logic and $R(x_1, \ldots, x_n)$ a predicate. We say that $\mathscr{L}$ *pins down* $R(x_1, \ldots, x_n)$ if there is an $\mathrm{RPC}_{\mathscr{L}}$-class $\mathscr{K}$ such that

$$\mathfrak{N} \in \mathscr{K} \quad \text{if and only if} \quad \mathfrak{N} \cong (N, \in \cap N^2, R \cap N^n), \text{ for some transitive set } N.$$

3.4.2 Examples. (i) $\mathscr{L}_{\omega\omega}(I)$ pins down Cd.
(ii) $\mathscr{L}^2_{\omega\omega}$ pins down Pw.
(iii) $\mathscr{L}_{\omega\omega}(H)$ pins down Pw.
(iv) If $V = L$, then $\mathscr{L}_{\omega\omega}(I)$ pins down Pw.
(v) $\mathscr{L}_{\omega\omega}(Q)$ pins down Q, if unbounded.

3.4.3 Lemma. *If a logic $\mathscr{L}$ pins down a predicate R, then $\mathscr{L}$ is unbounded and captures R. Moreover, $\mathscr{L}$ captures every Δ_1 predicate in the extended language $\{\in, R\}$.* □

Proof. The claim concerning unboundedness and capture is trivial. For the second claim, let $S(x)$ be Δ_1 in the language $\{\in, R\}$. Let $\mathscr{K}$ witness the pinning down of R and let $\mathscr{K}'$ be $\mathscr{K}$ intersected with a statement witnessing the Δ_1 nature of $S(x)$. By reflection, $\mathscr{K}'$ satisfies (C1). Condition (C2) follows from the absoluteness of Δ_1 predicates in end extensions. □

3.4.4 Theorem. *If $\mathscr{L}$ pins down R and $\mathscr{L}'$ is absolute relative to R, then $\mathscr{L}$ is adequate to truth in $\mathscr{L}'$.*

Proof. Let $S(x, y)$ be a predicate, Δ_1 in the extended language $\{\in, R\}$, such that

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi,$$

for all $\mathfrak{M}$ and φ. Now, $\mathscr{L}$ captures $S(x, y)$ by Lemma 3.4.3. Thus, Theorem 3.1.3 gives the desired result. □

3.4.5 Corollary. (i) *If $\mathscr{L}$ pins down R, $\mathscr{L}'$ is absolute relative to R and $A' \subseteq A$, then $\mathscr{L}' \leq \Delta(\mathscr{L})$.*
(ii) *If $\mathscr{L}$ is absolute relative to R and pins down R, then a model class is definable in $\Delta(\mathscr{L})$ ($\mathrm{RPC}_{\mathscr{L}}$) if and only if it is Δ_1 (Σ_1) definable in the extended language $\{\in, R\}$, with parameters in A.* □

3.4.6 Examples. (i) The logic $\mathscr{L}_A(I)$ is absolute relative to Cd and pins down Cd. Therefore:
If $\mathscr{L}$ pins down Cd, then $\mathscr{L}_A(I) \leq \Delta(\mathscr{L})$.
If $\mathscr{L}$ is absolute relative to Cd, then $\mathscr{L} \leq \Delta(\mathscr{L}_A(\mathrm{I}))$.
(ii) The logic $\mathscr{L}_A^2$ is absolute relative to Pw and pins down Pw. Then, using the fact that $\Delta_2 = \Delta_1(Pw)$, we get: A model class is definable in $\Delta(\mathscr{L}_A^2)$ if and only if it is Δ_2 with parameters in A.

The above results lead naturally to the following question: When is $\Delta(\mathscr{L})$ absolute? We can answer this for many unbounded $\mathscr{L}$, but the problem remains unsettled for most bounded $\mathscr{L}$.

3.4.7 Theorem. *If $\mathscr{L}$ pins down R and $\mathscr{L}'$ is absolute relative to R, then $\Delta(\mathscr{L}) \nleq \mathscr{L}'$.*

Proof. Let $S(x, y)$ be a Δ_1 predicate in the extended language $\{\in, R\}$ such that

$$S(\mathfrak{M}, \varphi) \quad \text{if and only if} \quad \varphi \in \mathscr{L}' \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}'} \varphi,$$

for all $\mathfrak{M}$ and φ. Let $\mathscr{K}$ be the class of models $\mathfrak{B}$ such that

$$\mathfrak{B} \cong (B, \in \cap B^2), \text{ where } B = \mathrm{TC}(\{a\}) \text{ for some } a \text{ such that } \neg S(\mathfrak{B}, a).$$

$\mathscr{K}$ is clearly, Δ_1 in the language $\{\in, R\}$. By Corollary 3.4.5(ii), $\mathscr{K}$ is definable in $\Delta(\mathscr{L})$. Suppose that $\mathscr{K}$ were definable by some $\varphi \in \mathscr{L}'$ and let $\mathfrak{N} = (N, \in \cap N^2)$, where $N = \mathrm{TC}(\{\varphi\})$. Then

$$\begin{aligned} S(\mathfrak{N}, \varphi) \quad &\text{if and only if} \quad \mathfrak{N} \in \mathscr{K} \\ &\text{if and only if} \quad \neg S(\mathfrak{N}, \varphi). \end{aligned}$$

This contradiction shows that $\mathscr{K}$ is not $\mathrm{EC}_{\mathscr{L}'}$ and the proof is thus completed. ☐

3.4.8 Examples. (i) $\Delta(\mathscr{L}_{\omega\omega}(W)) \nleq \mathscr{L}_{\infty B}$.
(ii) $\Delta(\mathscr{L}_{\omega\omega}(I)) \nleq \mathscr{L}_{\infty\omega}(I)$.
(iii) $\Delta(\mathscr{L}_{\omega\omega}(H)) \nleq \mathscr{L}^2_{\infty\omega}$.
(iv) $\Delta(\mathscr{L}_{\omega\omega}(Q)) \nleq \mathscr{L}_{\infty\omega}(Q)$, if $\mathscr{L}_{\omega\omega}(Q)$ is unbounded.

Theorem 3.4.7 shows that if $\mathscr{L}$ pins down R, then $\Delta(\mathscr{L})$ cannot be extended to a logic absolute relative to R; even less is $\Delta(\mathscr{L})$ itself absolute relative to R.

3.4.9 Examples. The logic $\Delta(\mathscr{L}_{\omega\omega}(W))$ is not absolute, and neither is $\Delta(\mathscr{L}_{\omega_1 G})$ nor $\Delta(\mathscr{L}_{\omega_1 V})$. Moreover, the logic $\Delta(\mathscr{L}_{\omega\omega}(I))$ is not absolute relative to Cd, nor is the logic $\Delta(\mathscr{L}^2_{\omega\omega})$ absolute relative to Pw.

We observed in Corollary 1.2.4 that $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$ is not selfadequate but is equivalent to one on a larger syntax set. The results we have here are stronger. For example $\Delta(\mathscr{L}_{\omega\omega}(W))$ is not absolute even if represented on a larger syntax set.

Application. If $\mathscr{L}$ is unbounded, then there are no generalized quantifiers $Q_1, \ldots, Q_n$ and no p.r. closed set A such that

$$\Delta(\mathscr{L}) \equiv \mathscr{L}_A(Q_1, \ldots, Q_n).$$

For otherwise $\mathscr{L}_A(Q_1, \ldots, Q_n)$ would be a Δ-logic which pins down and is absolute relative to $Q_1, \ldots, Q_n$.

Iterated Δ-extensions

3.4.10 Definition. Let $\mathscr{L}$ be a logic and let $\Sigma_0(\mathscr{L})$ and $\Pi_0(\mathscr{L})$ mean the same as $EC_{\mathscr{L}}$. Moreover, $\Sigma_{n+1}(\mathscr{L})$ means $RPC_{\Pi_n(\mathscr{L})}$ and $\Pi_{n+1}(\mathscr{L})$ means $RPC_{\Sigma_n(\mathscr{L})}$. Finally,

$$\Delta_{n+1}(\mathscr{L}) \text{ means } \Sigma_{n+1}(\mathscr{L}) \cap \Pi_{n+1}(\mathscr{L}).$$

Explanation. Here we have a hierarchy of RPC-definability defined very much like the hierarchy of Σ_n predicates of set theory or the hierarchy of Σ_n^1-sets in recursion theory. We treat $\Sigma_n(\mathscr{L})$, $\Pi_n(\mathscr{L})$, and $\Delta_n(\mathscr{L})$ as if they were logics, which, in fact, they actually are, as one can easily see. Of course, $\Sigma_n(\mathscr{L})$ and $\Pi_n(\mathscr{L})$ are not closed under negation. However, $\Delta_n(\mathscr{L})$ is closed, if $\mathscr{L}$ is. Moreover, it is easy to see that each $\Delta_n(\mathscr{L})$ is Δ-closed.

3.4.11 Theorem. *Let $n > 1$. A model class is definable in $\Delta_n(\mathscr{L}_A)$ if and only if it is Δ_n-definable in set theory, with parameters in A.* $\square$

Remark. If $\mathscr{L}_A$ is replaced by $\mathscr{L}_{AG}$, the result also holds for $n = 1$.

Proof of Theorem 3.4.11. We use induction on n. For $n = 2$, the claim is true since $\mathscr{L}_A^2 \leq \Sigma_2(\mathscr{L}_A)$ implies that $\Delta(\mathscr{L}_A^2) \leq \Delta_2(\mathscr{L}_A)$ holds, and $\Sigma_1(\mathscr{L}_A) \leq \Delta(\mathscr{L}_A^2)$ implies that $\Delta_2(\mathscr{L}_A) \leq \Delta(\mathscr{L}_A^2)$ holds. Assume, then, that the claim holds for n. Let $\mathscr{K}$ be a Σ_{n+1}-definable model class, and let R be a Π_n predicate such that $\mathscr{K}$ is Σ_1 in the extended language $\{\in, R\}$. Moreover, let $\mathscr{L}$ be the logic $\mathscr{L}_A(Q)$, where Q is the quantifier associated with the model class

$$\{\mathfrak{N} \mid \mathfrak{N} \cong (N, \in \cap N^2, R \cap N^m) \text{ for some transitive set } N\}.$$

Then $\mathscr{L}$ is absolute relative to R and pins down R. By Corollary 3.4.5(ii), $\mathscr{K}$ is $RPC_{\mathscr{L}}$-definable. As a Π_n-definable model class, Q is $\Sigma_n(\mathscr{L})$-definable. The converse is similar. $\square$

3.4.12 Corollary. $\Delta_{n+1}(\mathscr{L}_A) \equiv \Delta_n(\mathscr{L}_A^2)$, *for $n > 0$.*

The logics $\Delta_n(\mathscr{L}_A)$ are extremely powerful and gradually exhaust all logics definable in set theory. In fact, $\Delta_3(\mathscr{L}_A)$ already contains most familiar logics.

Second-Order Logic

We can construe second-order logic $\mathscr{L}_A^2$ as the result of iteratively closing $\mathscr{L}_A$ under the PC-operation. Therefore, let us examine the extent to which the above results hold for PC in place of RPC. To this purpose, we now consider

3.4.13 Definition. Let $\varphi(x_0, \ldots, x_n)$ be a formula of set theory. The expressions

$$\exists x_0(\mathrm{HC}(x_0) \leq \mathrm{HC}(x_1 \cup \cdots \cup x_n) \wedge \varphi(x_0, \ldots, x_n))$$

and

$$\forall x_0(\mathrm{HC}(x_0) \leq \mathrm{HC}(x_1 \cup \cdots \cup x_n) \rightarrow \varphi(x_0, \ldots, x_n)),$$

where $\mathrm{HC}(x) = \max(\aleph_0, |\mathrm{TC}(x)|)$, are called *flat quantifiers*. The class of *flat formulae* of set theory is the smallest class of formulae which contains Σ_0-formulae and which is closed under $\wedge$, $\vee$, $\neg$ and flat quantification.

The following characterization of second-order logic can be proven by slightly modifying the proof of Theorems 3.3.2 and 3.4.11.

3.4.14 Theorem. (i) *Second-order logic is simply adequate to truth in any logic definable by a flat formula of set theory.*

(ii) *A model class is definable in second order logic $\mathscr{L}_A^2$ if and only if it is definable by a flat formula of set theory with parameters in A.* □

Likewise, we may characterize $\mathrm{PC}_{\mathscr{L}}$-definability for a variety of $\mathscr{L}$ by modifying Corollary 3.4.5(ii).

The Logic $\mathscr{L}_{\omega\omega}(Q)$

Let Q be any quantifier. For reasons which will become apparent in the sequel, no characterization of $\mathscr{L}_{\omega\omega}(Q)$ can be proven along the above lines. However, we can say something about $\Delta(\mathscr{L}_{\omega\omega}(Q))$. In particular, we can assert

3.4.15 Theorem. *$\mathscr{L}_{\omega\omega}(Q)$ is adequate to truth in any logic that is absolute relative to Q and* KP(Q)-Inf.

Proof. Suppose that $\mathscr{L}$ is absolute relative to Q and KP(Q)-Inf. Then the predicate "$\varphi \in \mathscr{L} \wedge \mathfrak{M} \models_{\mathscr{L}} \varphi$" is Δ_1 in Q and KP(Q)-Inf. It is easy to show that $\mathscr{L}_{\omega\omega}(Q)$ captures such predicates. Thus, the claim follows from Theorem 3.1.3. □

3.4.16 Corollary. *If K is a model class, then* (a) $\rightarrow$ (b) $\rightarrow$ (c) *as below holds*:

(a) *K is definable in $\mathscr{L}_A(Q)$.*
(b) *K is Δ_1 in* KP-Inf *in the extended language $\{\in, Q\}$ with parameters in A.*
(c) *K is definable in $\Delta(\mathscr{L}_A(Q))$.* □

The main obstacle to improving Corollary 3.4.16 to (b) $\leftrightarrow$ (c) lies in the fact that certain $\mathscr{K}$ are $\Delta(\mathscr{L}_{\omega\omega}(Q))$-definable in some models of set theory but not in others. For example, if $\mathscr{K}$ is the class of tree-like structures with an uncountable branch, then $MA + \neg\mathrm{CH}$ implies that $\mathscr{K}$ is $\Delta(\mathscr{L}_{\omega\omega}(Q_1))$-definable, but ZFC alone is not enough for this, let alone ZFC-Inf. On the other hand, if the axiom of infinity is added to the picture, much more than $\Delta(\mathscr{L}_{\omega\omega}(Q_1))$ will be Δ_1, for instance, $\mathscr{L}_{\omega\omega}(W)$. These situations manifest the difficulties inherent in trying to prove general set-theoretical characterizations for logics of the form $\mathscr{L}_{\omega\omega}(Q)$.

Historical and Bibliographical Remarks. The first result proven in the direction of this section is the characterization Theorem 3.2.4 of strictly absolute logics, due to Barwise [1972a]. The observation that absolute logics and $\mathscr{L}_A(W)$ are related as in Corollary 3.3.3(i) was made by Swett [1974]. Corollary 3.3.3 was rediscovered independently by Oikkonen [1978]. The relativization to an arbitrary predicate R (see Corollary 3.4.5) was carried out in Oikkonen [1978] and Väänänen [1978]. Finally, the iteration in Theorem 3.4.11 is due to Oikkonen [1978]. The computation of $\Delta(\mathscr{L}_{\omega\omega}(Q_0))$ in Theorem 3.2.2 and its generalization Theorem 3.2.3 are due independently to Barwise [1974a] and Makowsky [1975b]. Burgess [1977] is a good reference to absolute logics. Essentially, it contains Theorem 3.4.7, among other things. Theorem 3.4.14 on second-order logic is from Väänänen [1979a]. The results on first-order logic are due to Manders [1980] and G. Wilmers. In Väänänen [1979a], a logic $\mathscr{L}$ was called symbiotic with a predicate R if $\Delta(\mathscr{L})$-definability coincides with Δ_1-definability in $\{\in, R\}$. The present terminology, centered around absoluteness, capture, and pinning down seems more useful and emphasizes the relation to adequacy to truth. Theorem 3.1.3 is formally new but in fact is really only the codification of the underlying ideas of the above characterization results. The general approach was chosen in an attempt to shed light on these ideas.

4. Other Topics

4.1. The Weak Beth Property Revisited

Recall the definition of weak Beth property: if a formula $\varphi(R)$ defines the predicate R implicitly (that is, $\varphi(R) \wedge \varphi(R') \models \forall x_1 \cdots x_n(R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n))$ and if every model can be expanded to a model of $\varphi(R)$, then some formula $\eta(x_1, \ldots, x_n)$ defines R explicitly (that is, $\varphi(R) \models \forall x_1 \cdots x_n(R(x_1, \ldots, x_n) \leftrightarrow \eta(x_1, \ldots, x_n))$. With every logic $\mathscr{L}$ can be associated the smallest extension of $\mathscr{L}$ to a logic $\mathrm{WB}(\mathscr{L})$ with the weak Beth property.

We have already mentioned the following result in discussing some refinements at the end of Section 1.

4.1.1 Theorem. *If $\mathscr{L}$ is uniquely adequate to truth in $\mathscr{L}'$, $\mathscr{L}$ is closed under negation and $\mathscr{I}(A')$ is definable in* $\mathrm{WB}(\mathscr{L})$, *then* $\mathrm{WB}(\mathscr{L}) \not\leq \mathscr{L}'$. □

Applications. The logics $\mathscr{L}_{\omega\omega}(Q_0)$, $\mathscr{L}^{2,w}_{\omega\omega}$, $\mathscr{L}^2_{\omega\omega}$, $\mathscr{L}_{\omega\omega}(H)$, and $\mathscr{L}_{\omega_1\omega_1}$ do not have the weak Beth property.

For unbounded logics (see Corollary 2.2.11) we have the following result of Burgess. The proof uses methods from descriptive set theory, notably the Π^1_1-uniformization property, combined with Theorem 3.4.7.

4.1.2 Theorem. *Suppose that $\mathscr{L}$ is a strong absolute logic closed under countable disjunctions and negations, and $A \subseteq \mathrm{HC}$. Then $\mathscr{L}$ fails to have the weak Beth property.*

Applications. If $\mathscr{L}$ is any of $\mathscr{L}_{\omega\omega}(W)$, $\mathscr{L}_{\omega_1 G}$, $\mathscr{L}_{\omega_1 S}$, $\mathscr{L}_{\omega_1 V}$, or $\mathscr{L}_{\omega_1 B}$, then $\mathscr{L}$ does not have the weak Beth property and $\mathrm{WB}(\mathscr{L})$ is not absolute.

Particularly strong results on weak Beth closure come from the following theorem of Gostanian–Hrbacek [1976].

4.1.3 Theorem. $\mathrm{WB}(\mathscr{L}_{\omega\omega}(W)) \nleq \mathscr{L}_{\infty\infty}$.

Proof. Let $\mathscr{K}$ be the class of models (A, E, R) such that either (A, E) is non-well-founded and $R = \varnothing$ or (A, E) is well-founded and R is the set of pairs (φ, f) where (A, E) satisfies [$\varphi \in \mathscr{L}_{\infty\infty}$ and f is a function such that the inductive clauses for satisfaction of $\mathscr{L}_{\infty\infty}$-formulae hold]; an example of the inductive clauses here is:

$$(\exists(x_\alpha)_{\alpha<\kappa}\varphi, f) \in R \quad \text{if and only if} \quad \exists g \in A \text{ such that } g(x) = f(x) \text{ for variables } x \neq x_\alpha\ (\alpha < \kappa) \text{ and } (\varphi, g) \in R.$$

If (A, E, R) and (A, E, R') are in $\mathscr{K}$, then we can use induction to prove that $R = R'$. Thus, $\mathscr{K}$ defines R implicitly. Moreover, for all (A, E), there is an R such that $(A, E, R) \in \mathscr{K}$. Suppose that there were a formula $\eta(x, y)$ in $\mathscr{L}_{\infty\infty}$ which defines R explicitly in models of $\mathscr{K}$. Let κ be a regular cardinal such that $\eta \in \mathscr{L}_{\kappa\kappa}$. We shall consider the model $\mathfrak{M} = (H_\kappa, \in \cap H^2)$. The point to notice here is that if $\exists(x_\alpha)_{\alpha<\beta}\psi$ is in H_κ, then $\beta < \kappa$ and every sequence $(x_\alpha)_{\alpha<\beta}$ of elements of H_κ that one might need to satisfy ψ already exists as an element of H_κ. Thus, if R is chosen such that $(H_\kappa, \in \cap H^2, R) \in \mathscr{K}$, then

$$R = \{(\varphi, f) \mid \varphi \in \mathscr{L}_{\kappa\kappa} \text{ and } f \text{ satisfies } \varphi \text{ in } \mathfrak{M}\}.$$

Combining this with the choice of $\eta(x, y)$ yields

$$\mathfrak{M} \models \eta(\varphi, f) \quad \text{if and only if} \quad f \text{ satisfies } \varphi \text{ in } \mathfrak{M}.$$

The standard diagonal argument ends the proof. Hence, let ξ $(= \xi(x))$ be the formula $\neg\eta(x, f)$, where f is a term denoting the function which maps the variable x to x ($f = \{(x, x)\}$). We now have $\xi \in H_\kappa$ and

$$\mathfrak{M} \models \xi(\xi) \leftrightarrow \eta(\xi, \{(x, x)\}) \leftrightarrow \neg\xi(\xi).$$

The contradiction shows that $\mathscr{K}$ does not define R explicitly in $\mathscr{L}_{\infty\infty}$; and this implies the claim, as $\mathscr{K}$ itself is $\mathscr{L}_{\omega\omega}(W)$-definable. □

The above theorem permits several improvements. An immediate observation is that $\mathscr{K}$ need not assert that all its models (with $R \neq \varnothing$) are well-founded. In fact, it is enough that $\mathscr{K}$ pins down the κ such that $\eta \in \mathscr{L}_{\kappa\kappa}$. Thus we have

4.1.4 Theorem. *If $\mathscr{L}$ pins down the regular κ, then* $\mathrm{WB}(\mathscr{L}) \not\leq \mathscr{L}_{\kappa\kappa}$. □

4.1.5 Corollary. (i) *If* $\mathrm{cf}(\kappa) > \omega$, *then* $\mathrm{WB}(\mathscr{L}_{\kappa^+\omega}) \not\leq \mathscr{L}_{\kappa^+\kappa^+}$.

(ii) $\mathrm{WB}(\mathscr{L}_{\omega_1\omega_1}) \not\leq \mathscr{L}_{\infty\infty}$. □

Our second improvement concerns generalized quantifiers. We can make the R in the above proof work for formulae of $\mathscr{L}_{\infty\infty}(Q)$, where Q is an arbitrary generalized quantifier. However, $\mathscr{K}$ is then definable in $\mathscr{L}_{\omega\omega}(W, Q)$ and not in $\mathscr{L}_{\omega\omega}(W)$.

4.1.6 Theorem. *If $\mathscr{L}_{\omega\omega}(Q)$ pins down the regular κ, then* $\mathrm{WB}(\mathscr{L}_{\omega\omega}(Q)) \not\leq \mathscr{L}_{\kappa\kappa}(Q)$. □

4.1.7 Corollary. (i) $\mathrm{WB}(\mathscr{L}_{\omega\omega}(I)) \not\leq \mathscr{L}_{\infty\infty}(I)$.

(ii) $\mathrm{WB}(\mathscr{L}_{\omega_1 G}) \not\leq \mathscr{L}_{\infty\infty}(G)$.

(iii) $\mathrm{WB}(\mathscr{L}_{\omega\omega}(H)) \not\leq \mathscr{L}_{\infty\infty}(H)$.

(iv) $\mathrm{WB}(\mathscr{L}_{\kappa^+\omega}(Q)) \not\leq \mathscr{L}_{\kappa^+\kappa}(Q)$, if $\mathrm{cf}(\kappa) > \omega$.

What about logics which pin down ordinals but no interesting regular cardinals? Here, we may notice that the only role of regularity of κ in the proof of Theorem 4.1.3 (or of Theorem 4.1.4) is that it gives the correct interpretation for R as the satisfaction relation of $\mathscr{L}_{\kappa\kappa}$. However, if we replace $\mathscr{L}_{\kappa\kappa}$ by $\mathscr{L}_{\kappa\omega}$, any admissible A with $o(A) = \kappa$ can replace H_κ, and we have

4.1.8 Theorem. *If A is an admissible set and $\mathscr{L}'$ pins down $o(A)$, then* $\mathrm{WB}(\mathscr{L}') \not\leq \mathscr{L}_A$. □

Again, we can add an arbitrary generalized quantifier Q to this result. In fact, we have

4.1.9 Theorem. *If $\mathscr{L}_{\omega\omega}(Q)$ pins down $o(A)$, where A is an admissible set, then* $\mathrm{WB}(\mathscr{L}_{\omega\omega}(Q)) \not\leq \mathscr{L}_A(Q)$. □

4.2. Σ_1-*Compactness*

Recall that a logic $\mathscr{L}$, represented on A, is called Σ_1-*compact* if every $T \subset \mathscr{L}$, Σ_1 over A, which has no models, has an A-finite subset with no models. The following result is thus straightforward.

4.2.1 Proposition. *If $\mathscr{L}$ is effectively adequate to truth in $\mathscr{L}'$ (as explained in the refinement at the end of Section 1), and if $A' = A$ are admissible sets and $\mathscr{L}$ is Σ_1-compact, then $\mathscr{L}'$ is Σ_1-compact.* □

This result, when combined with Theorem 3.3.2, this gives

4.2.2 Corollary. *If $\mathscr{L}_A(W)$, where A admissible, is Σ_1-compact, then so is every absolute logic represented on A.* ∎

Similarly, for stronger logics we have

4.2.3 Corollary. *If $\mathscr{L}$ is Σ_1-compact and pins down R, then every logic, absolute relative to R and represented on A, is Σ_1-compact.* ∎

A third kind of consequence of Proposition 4.2.1 is given in

4.2.4 Corollary. *If $\mathscr{L}$ is adequate to truth in itself and Σ_1-compact and Q is $\Delta(\mathscr{L})$-definable, then $\mathscr{L}_A(Q)$ is Σ_1-compact.* ∎

It is well-known that $\mathscr{L}_A$ is Σ_1-compact if A is a countable admissible set. More generally,

(*) $\qquad \mathscr{L}_A$ is Σ_1-compact if and only if A satisfies s-Π_1^1-reflection.

The reader is referred to Chapter VIII for more on this and other results on $\mathscr{L}_A$. The result given in (*) above has been generalized to all absolute logics by Cutland–Kaufmann [1980]. In this development, use is made of the notion of a s-Π_1^1-Souslin formula. These formulae are (in their normal form) of the form

$$\forall V_1 \ldots \forall V_m Q_s x_1 \ldots Q_s x_n \exists y_1 \ldots \exists y_2 \psi,$$

where ψ is Σ_0 and Q_s is the Souslin quantifier

$$Q_s x\varphi(x) \leftrightarrow \exists x_0 \exists x_1 \ldots \bigwedge_{n<\omega} \varphi(\langle x_0, \ldots, x_n \rangle).$$

4.2.5 Theorem. *An admissible set A satisfies s-Π_1^1-Souslin reflection if and only if every absolute logic represented on A is Σ_1-compact.* ∎

4.2.6 Corollary. *If $\mathscr{L}_{\kappa\omega}$ is Σ_1-compact and* $\mathrm{cf}(\kappa) > \omega$, *then $\mathscr{L}_{\kappa\omega}(W)$ is Σ_1-compact.* ∎

Recall that an admissible set A is *resolvable* if $A = \cup_{\alpha < o(A)} F(\alpha)$ for some A-recursive function F.

4.2.7 Theorem. *If A is a countable resolvable admissible set, then A satisfies Σ_2^1-reflection if and only if every absolute logic represented on A is Σ_1-compact.* ∎

4.2.8 Theorem. *If A is a resolvable admissible set, then $\mathscr{L}_A(Q)$ is Σ_1-compact if and only if $\mathscr{I}(A)$ is not* RPC-*defined by a Σ_1-theory of $\mathscr{L}_A(Q)$.*

Proof. The argument is similar to Barwise [1975, VIII.4.8]. ∎

Remark. Let A be a resolvable admissible set and assume that $\mathscr{E}(A)$ is RPC in $\mathscr{L}_A(Q)$. By Proposition 1.2.2, the conjunction of a Σ_1-theory of $\mathscr{L}_A(Q)$ is RPC in $\mathscr{L}_A(Q)$. Thus, if $\mathscr{L}_A(Q)$ is not Σ_1-compact, then $\mathscr{I}(A)$ is RPC in $\mathscr{L}_A(Q)$, which means that $\mathscr{L}_A(A)$ pins down $o(A)$. By Theorem 4.1.9, $\mathscr{L}_A(Q)$ fails to satisfy the weak Beth property. We have, in effect, a proof of

4.2.9 Theorem. *Suppose A is a resolvable admissible set. If $\mathscr{L}_A(Q)$ satisfies the weak Beth property, then $\mathscr{L}_A(Q)$ is Σ_1-compact.* ▯

Σ_1-compactness is somewhat related to weak compactness. A logic $\mathscr{L}$ is *weakly compact* if it is Σ_1-compact with any $R \subseteq A$ as a parameter. For $A = H_\kappa$, this assumes the more familiar form: If a theory $T \subset \mathscr{L}$ (and $T \subset H_\kappa$) has no models, then some subtheory of power $< \kappa$ has no models. It is well-known that $\mathscr{L}_{\kappa\omega}$ is weakly compact if and only if $\mathscr{L}_{\kappa\kappa}$ is weakly compact if and only if $\kappa \to (\kappa)^2_2$.

4.2.10 Theorem. *Let $\mathscr{L}$ be any logic and κ a measurable cardinal. There is a stationary set of cardinals $\lambda < \kappa$ such that $\mathscr{L}$ restricted to H_λ is weakly compact.*

Proof. Let U be a normal ultrafilter on κ and $i\colon V \to M$ the associated embedding (see, for example, Jech [1978, p. 305]). The fundamental property of i is that if $\varphi(x, y)$ is any formula of set theory, then

$$(*) \qquad M \models \varphi(\kappa, i(x)) \quad \text{if and only if} \quad \{\lambda < \kappa \mid \varphi(\lambda, x)\} \in U.$$

We let $\varphi(\lambda, x)$ be the formula "If $T \subset \mathscr{L}$, $T \subseteq H_\lambda \cap x$ and T has no models, then some subset $T_0 \in H_\lambda$ of T has no models". In view of $(*)$ it suffices to prove that $M \models \varphi(\kappa, i(x))$ holds, for $x = \{\varphi \in \mathscr{L} \mid \varphi \in H_\kappa\}$. Suppose we have $T = i(T')$, for some T'. By $(*)$ the set

$$A = \{\lambda < \kappa \mid T' \subset \mathscr{L}, T' \subseteq H_\lambda \cap x \text{ and every subset } T_0 \in H_\lambda \text{ of } T' \text{ has a model}\}$$

is in U. Let $\lambda, \mu \in A$ such that $\lambda < \mu$. Then $T' \in H_\mu$, as $\lambda \in A$; and, hence, T' has a model, as $\mu \in A$. Therefore, $M \models T$ has a model, using $(*)$ again. ▯

4.3. The Problem of Validity

Recall that if $\mathscr{L}$ is a logic represented on A, we say that *validity in $\mathscr{L}$ is Σ_1* if the set

$$\mathrm{Val}_{\mathscr{L}} = \{\varphi \in \mathscr{L} \mid \models_{\mathscr{L}} \varphi\}$$

is Σ_1 over A.

4.3.1 Proposition. *If $\mathscr{L}$ is effectively adequate to truth in $\mathscr{L}'$, $A' = A$ is an admissible set and validity in $\mathscr{L}$ is Σ_1, then the same holds for $\mathscr{L}'$.* ▯

4.3.2 Corollary. *If validity in $\mathcal{L}_A(W)$, where A admissible, is Σ_1, then the same holds for every absolute logic represented on A.* $\square$

Similarly, for stronger logics, we have

4.3.3 Corollary. *If the validity in $\mathcal{L}$ is Σ_1 and $\mathcal{L}$ pins down R, then validity in any logic absolute relative to R and represented on A is Σ_1.* $\square$

A third kind of consequence of Proposition 4.3.1 is given in

4.3.4 Corollary. *If $\mathcal{L}$ is effectively adequate to truth in itself, and validity in $\mathcal{L}$ is Σ_1 and Q is $\Delta(\mathcal{L})$-definable, then validity in $\mathcal{L}_A(Q)$ is Σ_1.*

Application. $\mathcal{L}_{\omega\omega}(Q_1^E)$ is axiomatizable, because Q_1^E is $\Delta(\mathcal{L}_{\omega\omega}(Q_1))$-definable.

As it actually turns out, validity in an unbounded logic is hardly ever Σ_1. In order to see this let us first make two simple remarks. In the following, a subset X of A is said to be Π_1 if it has the form $\{x \in A \mid \varphi(x)\}$, where $\varphi(x)$ is Π_1. Observe that Π_1 over A refers to sets of the form $\{x \in A \mid A \models \varphi(x)\}$, $\varphi(x) \in \Pi_1$. A set $X \subseteq A$ is *complete for* Π_1 *on* A if for every Π_1 subset Y of A there is a Σ_1-function f of A such that for $a \in A$

$$a \in Y \leftrightarrow f(a) \in X.$$

4.3.5 Lemma. (i) *If $\mathcal{L}$ is absolute relative to R, then* $\mathrm{Val}_{\mathcal{L}}$ *is Π_1 in the extended language $\{\in, R\}$.*

(ii) *If $\mathcal{L}$ pins down R, then* $\mathrm{Val}_{\mathcal{L}}$ *is complete for Π_1 on A in the extended language $\{\in, R\}$.*

Proof. In order to prove (i), we use absoluteness of $\mathcal{L}$ to write the definition

$$a \in \mathrm{Val}_{\mathcal{L}} \leftrightarrow a \in \mathcal{L} \wedge \forall \mathfrak{A}(\mathfrak{A} \models_{\mathcal{L}} a)$$

in Π_1-form. In order to prove (ii), we suppose that Y is a subset of A defined by the Π_1-formula $\varphi(x)$. For $a \in A$, let $g(a)$ be an $\mathcal{L}$-sentence equivalent to

$$\forall x(\pi_a(x) \rightarrow \varphi(x)).$$

If $\mathcal{K}$ is the class of models $(M, \in \cap M^2, R \cap M^n)$, M transitive, then for all $a \in A$

$$a \in Y \leftrightarrow \mathcal{K} \subseteq \mathrm{Mod}(g(a)).$$

Using the fact that $\mathcal{K}$ is $\mathrm{RPC}_{\mathcal{L}}$, we find a Σ_1-function f on A such that for all $a \in A$, we have

$$a \in Y \leftrightarrow f(a) \in \mathrm{Val}_{\mathcal{L}}. \quad \square$$

In order to be able to apply Lemma 4.3.5 we would like to know that Π_1 coincides with Π_1 over A for subsets of A. In general, this is not true. An equivalent condition is that $A \prec_1 V$ and this is known to hold for $A = H_\kappa$, $\kappa > \omega$, (Levy reflection principle) and for $A = L_\alpha$, where $\alpha \leq \omega_1^L$ is stable, at least (Schoenfield absoluteness lemma). In such a case, Π_1 plus complete for Π_1 coincide with the ordinary notion of complete Π_1, which is never Σ_1 over A (if A is admissible). Thus, we have the proof of

4.3.6 Theorem. *If $\mathscr{L}$ is an unbounded absolute logic represented on $A \prec_1 V$, then* $\mathrm{Val}_{\mathscr{L}}$ *is complete Π_1 over A and validity in $\mathscr{L}$ is not Σ_1.* □

4.3.7 Corollary. *Validity in an unbounded absolute logic represented on $H_\kappa, \kappa > \omega$, is not Σ_1.* □

Cutland–Kaufman [1980] obtained the following improvement of Theorem 4.3.6 in the case of $A = L_\alpha$.

4.3.8 Theorem. *Validity is not Σ_1 in any unbounded absolute logic represented on an admissible set of the form L_α.* □

Corollary. *If $V = L$, then validity in an unbounded absolute logic represented on an admissible set is never Σ_1.* □

Considering these negative results, one might raise the question of whether some more general completeness property would be more tractable. In this direction Cutland and Kaufman proved

4.3.9 Theorem. *If $\mathscr{L}$ is an absolute logic represented on an admissible set A, then* $\mathrm{Val}_{\mathscr{L}}$ *is s-Π_1^1-Souslin over A.*

Feferman [1975] proves a more general completeness theorem. Recall the notion of $\#$-siid$_x$ from Remark 1.1.9. In the following theorem $\mathscr{L}$ has to satisfy a property called "join property," a property which most logics do indeed satisfy and which essentially says that $\mathscr{L}$ permits the construction of disjoint unions of structures.

4.3.10 Theorem. *Let $\mathscr{L}$ be adequate to truth in itself. Then for each τ, the set $\{\varphi \in \mathscr{L}[\tau] \mid \models_{\mathscr{L}} \varphi\}$ is $\#$-siid$_x$ in $\mathscr{L}$. Moreover, if $S \subset \mathscr{L}[\tau]$ is siid$_x$ in $\mathscr{L}$, then the same holds for $\{\varphi \in \mathscr{L}[\tau] \mid S \models_{\mathscr{L}} \varphi\}$.* □

This theorem shows that validity and even consequence is "r.e." in any self-adequate logic once we use an appropriate notion of "r.e.". The notion $\#$-siid$_x$ does indeed have many of the characteristics of r.e. on ω and Σ_1 on an admissible set (see Feferman [1975] and Kunen [1968]).

4.4. Löwenheim Numbers and Spectra

The Löwenheim number of a logic is related to the more general problem of spectra. The *spectrum* of a sentence φ of a logic $\mathscr{L}$ is the class of cardinals of models of φ. That is, in symbols the spectrum of φ is

$$\mathrm{Sp}(\varphi) = \{|\mathfrak{A}| \mid \mathfrak{A} \models_{\mathscr{L}} \varphi\}.$$

The problem of spectra of such strong logics as $\mathscr{L}^2_{\omega\omega}$ or $\mathscr{L}_{\omega\omega}(I)$ is a difficult subject and remains mostly unsettled. However, even the spectra of $\mathscr{L}_{\omega\omega}$ present open problems. Well-known is the *Finite Spectrum Problem*: *Is the complement of a spectrum of $\mathscr{L}_{\omega\omega}$ also a spectrum of $\mathscr{L}_{\omega\omega}$, if only finite models are considered*? On the other hand, the infinite part of a spectrum of $\mathscr{L}_{\omega\omega}$ is trivial: It is either empty or contains every infinite cardinal. The spectra of $\mathscr{L}_{\omega_1\omega}$ are more complex: Every set of natural numbers is one, as are also $\{\kappa \mid \kappa < 2^{\omega_\alpha}\}$, for $\alpha < \omega_1$. Even more complex, however, are spectra of $\mathscr{L}^2_{\omega\omega}$. The strength of $\mathscr{L}^2_{\omega\omega}$ makes it possible to represent every spectrum as the spectrum of an identity sentence. Thus, the spectra of $\mathscr{L}^2_{\omega\omega}$ form a boolean algebra with respect to complementation, union and intersection. In fact, the spectra of $\mathscr{L}^2_{\omega\omega}$ permit the following general characterization, a consequence of Theorem 3.4.14(ii).

4.4.1 Theorem. *A class C of cardinals is a spectrum of $\mathscr{L}^2_A$ if and only if C is defined by a flat formula of set theory with parameters in A.* □

For logics such as $\mathscr{L}_{\omega_1\omega}$, $\mathscr{L}_{\omega\omega}(Q_1)$, and $\mathscr{L}_{\omega\omega}(W)$ the complexity of spectra is limited by a strong downward Löwenheim–Skolem theorem. Another limiting factor is the upward Löwenheim–Skolem theorem.

Recall that the *Lowenheim number* of $\mathscr{L}$ is the cardinal

$$\ell(L) = \sup\{\min C \mid C \text{ is a spectrum of } \mathscr{L}\}.$$

Despite our occasional reference to logics such as $\mathscr{L}_{\infty\omega}$ and $\mathscr{L}_{\infty\infty}$, every logic is represented on a set and therefore has a Löwenheim number. The explicit computations $\ell(\mathscr{L}_{\kappa^+\omega}) = \kappa$ and $\ell(\mathscr{L}_{\omega\omega}(Q_\alpha)) = \omega_\alpha$ are immediate. Following are two easy preservation results.

4.4.2 Proposition. (i) *If $\mathscr{L} \leq_{\mathrm{RPC}} \mathscr{L}'$, then $\ell(\mathscr{L}) \leq \ell(\mathscr{L}')$.*

(ii) *If $\mathscr{L}$ is absolute relative to R, $A \subseteq A'$ and $\mathscr{L}'$ pins down R, then $\ell(\mathscr{L}) \leq \ell(\mathscr{L}')$.*

In the following theorem we shall estimate Löwenheim numbers in purely set-theoretical terms

4.4.3 Theorem. *Let $\mathscr{L}$ be a logic, R a predicate and*

$\delta = \sup\{\kappa \mid \kappa$ *in Π_1-definable in the extended language $\{\in, R\}$ with parameters in A*$\}$.

(i) *If $\mathscr{L}$ is absolute relative to R, then $\ell(\mathscr{L}) \leq \delta$.*

(ii) *If $\mathscr{L}$ pins down R, then $\delta \leq \ell(\mathscr{L})$.*

Proof. For (i) we suppose that $\lambda = \min C$, where $C = \mathrm{Sp}(\varphi)$ is spectrum of $\mathscr{L}$. The cardinal λ has the following definition:

$$\alpha \in \lambda \leftrightarrow \forall \beta(\beta \leq \alpha \to \beta \notin C).$$

Using absoluteness of $\mathscr{L}$, we can write this in Π_1-form with φ as a parameter.

(ii) We suppose κ that is Π_1-definable with parameters in A. Let $\mathscr{K}$ be the class of well-ordered structures of type $\geq \kappa$. $\mathscr{K}$ is clearly Σ_1-definable with parameters in A. By Theorem 3.4.4 (letting $\mathscr{L}' = \mathscr{L}_{\omega\omega}(Q)$ such that $\mathscr{K}$ is $\mathrm{EC}_{\mathscr{L}'}$ and $\mathscr{L}'$ is absolute relative to R), $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$. Let

$$\lambda = \min\{|\mathfrak{A}| \mid \mathfrak{A} \in \mathscr{K}\}.$$

Now $\kappa \leq \lambda \leq \ell(\mathscr{L})$. $\square$

Remark. If $\ell(\mathscr{L})$ is a limit cardinal, we can replace κ by α in Theorem 4.4.3.

4.4.4 Corollary. *Suppose that $\mathscr{L}$ is absolute relative to R and pins down R. Then*

$$\ell(\mathscr{L}) = \sup\{\kappa \mid \kappa \textit{ is } \Pi_1\textit{-definable in the extended language } \{\in, R\} \textit{ with parameters in } A\}. \quad \square$$

4.4.5 Examples. (i) $\ell(\mathscr{L}_A(I)) = \sup\{\alpha \mid \alpha$ is Π_1-definable in $\{\in, \mathrm{Cd}\}$ with parameters in $A\}$.

(ii) $\ell(\mathscr{L}_A^2) = \sup\{\alpha \mid \alpha$ is Π_2-definable with parameters in $A\}$.

An inductive argument based on Theorem 4.4.3 can be used to prove:

4.4.6 Theorem. $\ell(\Delta_n(\mathscr{L}_A)) = \sup\{\alpha \mid \alpha$ *is Π_n-definable with parameters in* $A\}$ $(n > 1)$. $\square$

We can actually replace Π by Δ in the above results. When this is done, we then have

4.4.7 Theorem. (i) $\ell(\mathscr{L}_A^2) = \sup\{\alpha \mid \alpha$ *is Δ_2-definable with parameters in* $A\}$.

(ii) $\ell(\Delta_n(\mathscr{L}_A)) = \sup\{\alpha \mid \alpha$ *is Δ_n-definable with parameters in* $A\}$, $(n > 1)$. $\square$

Following is a third characterization of $\ell(\Delta_n(\mathscr{L}_A))$ in set-theoretical terms.

4.4.8 Theorem. *If $n > 1$, then $\ell(\Delta_n(\mathscr{L}_{\kappa\omega})) = \kappa$ if and only if $R_\kappa \prec_n V$.* $\square$

Combined with the facts that $R_\kappa \prec_2 V$ for κ supercompact and $R_\kappa \prec_3 V$ for κ extendible, this yields

4.4.9 Corollary. (i) *If κ supercompact, then $\ell(\mathscr{L}_{\kappa\omega}^2) = \kappa$.*

(ii) *If κ is extendible, then $\ell(\Delta_3(\mathscr{L}_{\kappa\omega})) = \kappa$.* $\square$

Remark. Magidor [1971] proves a downward Löwenheim–Skolem theorem for $\mathscr{L}^2_{\omega\omega}$ on a supercompact cardinal, a result which is stronger than that given by Corollary 4.4.9(i).

No upper bound to $\ell(\mathscr{L}^2_{\omega\omega})$ or even to $\ell(\mathscr{L}_{\omega\omega}(I))$ is known in terms of large cardinals below supercompact cardinals. However, observe the following

4.4.10 Theorem. (i) *If a spectrum C of $\mathscr{L}^2_{\omega\omega}$ contains a measurable cardinal κ, then $C \cap \kappa$ is stationary on κ.*

(ii) *If a spectrum C of $\mathscr{L}_{\omega\omega}(I)$ contains a weakly inaccessible cardinal κ, then $C \cap \kappa$ is* cub *on κ.* □

4.5. Hanf Numbers

Recall that the Hanf number of a logic $\mathscr{L}$ is the cardinal

$$\hbar(\mathscr{L}) = \sup\{\sup C \mid C \text{ is a bounded spectrum of } \mathscr{L}\}.$$

There are a few explicit Hanf number computations, such as $\hbar(\mathscr{L}_A) = \beth_\alpha$ for countable admissible A of ordinal α and $\hbar(\mathscr{L}_{\omega\omega}(Q_1)) = \beth_\omega$. But for many $\mathscr{L}$, for more than with $\ell(\mathscr{L})$, $\hbar(\mathscr{L})$ is simply unknown. The following estimates are the best known ones.

4.5.1 Examples. (i) $\hbar(\mathscr{L}_{\omega\omega}(W))$ exceeds the first κ such that $\kappa \to (\omega)^{<\omega}$.

(ii) If $\kappa \to (\omega_1)^{<\omega}$, then $\hbar(\mathscr{L}_{\omega\omega}(W)) < \kappa$.

(iii) $\hbar(\mathscr{L}_{\omega_1 S}) \leq \beth_{\omega_2}$.

Remark. As to (iii), Burgess [1978] shows that $\hbar(\mathscr{L}_{\omega_1 S}) = \beth_{\omega_2}$ under $MA + \neg\mathrm{CH} + \omega_1^L = \omega_1$.

When we proceed to set-theoretical characterization or estimation of Hanf numbers, the first point to notice is the failure of Hanf numbers to be preserved—in general—under Δ-operation. Thus, our general results will mostly concern $\hbar(\mathrm{RPC}_{\mathscr{L}})$ rather than $\hbar(\mathscr{L})$. (Note that $\ell(\mathrm{RPC}_{\mathscr{L}}) = \ell(\mathscr{L})$). In particular examples, on the other hand, $\hbar(\mathrm{RPC}_{\mathscr{L}}) = \hbar(\mathscr{L})$ usually holds, as we shall see.

A typical $\mathrm{RPC}_{\mathscr{L}}$-definition has the form

$$(*) \qquad \mathfrak{M} \in \mathscr{K} \leftrightarrow \exists\mathfrak{N}([\mathfrak{M}, \mathfrak{N}] \models_{\mathscr{L}} \varphi).$$

Problems with $\hbar(\mathscr{L})$ arise because there is no upper bound on the size of $\mathfrak{N}$. But suppose that the following holds in addition to (*) above:

$$\forall\mathfrak{M}\, \exists\kappa\, \forall\mathfrak{N}([\mathfrak{M}, \mathfrak{N}] \models_{\mathscr{L}} \varphi \to |\mathfrak{N}| \leq \kappa).$$

In this case, we say that $\mathscr{K}$ is *bounded* $\mathrm{RPC}_{\mathscr{L}}$. This notion clear, we have

4.5.2 Lemma. $\hbar(\text{bounded RPC}_{\mathscr{L}}) = \hbar(\mathscr{L})$.

Proof. The argument for this result is easy. ◻

4.5.3 Examples. $\text{RPC}_{\mathscr{L}} = \text{bounded RPC}_{\mathscr{L}}$ if $\mathscr{L}$ is one of $\mathscr{L}_{\kappa^+\omega}$, $\mathscr{L}_{\omega\omega}(Q_\alpha)$, $\mathscr{L}_{\omega\omega}(Q_\alpha^{<\omega})$, $\mathscr{L}_{\kappa G}$ or if Bounded $\text{RPC}_{\mathscr{L}}$ contains $\mathscr{L}^2_{\omega\omega}$. In the first four cases this follows from the strong Löwenheim–Skolem theorems of these logics. In the fifth case, we may use the strength of $\mathscr{L}^2_{\omega\omega}$ to make sure that the set-theoretical rank of $\mathfrak{N}$ in (*) is minimal.

4.5.4 Theorem. *If* Con(ZF), *then* $\text{Con}[\text{ZFC} + \hbar(\mathscr{L}_{\omega\omega}(I)) < \hbar(\text{RPC}_{\mathscr{L}_{\omega\omega}(I)})]$.

Proof. Let $A(\alpha)$ be the statement "there is a sequence $(\omega_{\gamma+\beta+1})_{\beta<\alpha}$ of cardinals κ such that $2^\kappa \geq \kappa^{++}$" and let $B(\alpha)$ be the statement $A(\alpha) \wedge \forall\beta(A(\beta) \to \beta \leq \alpha)$. It can be seen without too much trouble that $B(\alpha)$ implies that $\alpha < \hbar(\text{RPC}_{\mathscr{L}_{\omega\omega}(I)})$. Thus, it remains to construct a boolean extension in which $\hbar(\mathscr{L}_{\omega\omega}(I)) \leq \alpha \wedge B(\alpha)$. The idea here is the following. Construct notions of forcing (proper classes) $F_{\alpha\beta}$ such that $F_{\alpha\beta} \Vdash B(\alpha)$, $F_{\alpha\beta}$ is ω_β-closed, $F_{\alpha\beta} \supseteq F_{\alpha\gamma}$ if $\beta < \gamma$, and $F_{\alpha\beta}$ preserves cardinals. Call $F_{\alpha\beta}$ *a failure* if it fails to force $\hbar(\mathscr{L}_{\omega\omega}(I)) \leq \alpha$. Construct a sequence $(\varphi_\alpha)_{\alpha<\omega_1}$ of sentences of $\mathscr{L}_{\omega\omega}(I)$ and sequences $(\lambda_\alpha)_{\alpha<\omega_1}$ and $(\kappa_\alpha)_{\alpha<\omega_1}$ of cardinals such that for $\lambda = \hbar(\mathscr{L}_{\omega\omega}(I))$, $\lambda_\alpha = \sup(\kappa_\beta)_{\beta<\alpha}$, $F_{\lambda\lambda_\alpha}$ is a failure, because it forces φ_α to have a model of power $\geq\lambda$ but none $\geq\kappa_\alpha$ $(>\lambda_\alpha)$. Take $\alpha < \beta < \omega_1$ such that $\varphi_\alpha = \varphi_\beta$. Then $F_{\lambda\lambda_\alpha}$ forces φ_α to have a model $\mathfrak{M}$ of power $\geq\lambda$ such that $|\mathfrak{M}| < \kappa_\alpha$. As $F_{\lambda\lambda_\beta} \subseteq F_{\lambda\lambda_\alpha}$, $F_{\lambda\lambda_\beta}$ forces the same thing. But since $F_{\lambda\lambda_\beta}$ is λ_β-closed and $\kappa_\alpha \leq \lambda_\beta$, we may assume that $\mathfrak{M} \in V$, whence φ_α already has a model of power $\geq\kappa_\alpha$ in V. This is a contradiction of the definition of φ_α. ◻

We can establish a similar relation between bounded RPC and "bounded Σ_1" as holds between RPC and Σ_1 (Corollary 3.4.5(ii)). A Σ_1-formula $\exists x\varphi(x, y)$ is called "bounded" if for all y, the class $\{x \mid \varphi(x, y)\}$ is a set. Using such "bounded" formulae, we could actually characterize $\hbar(\mathscr{L})$ set-theoretically for a variety of $\mathscr{L}$. As $\hbar(\mathscr{L}) = \hbar(\text{RPC}_{\mathscr{L}})$ in so many practical cases, we confine ourselves to characterizing $\hbar(\text{RPC}_{\mathscr{L}})$. However, we will first make the simple observation given in

4.5.5 Proposition. *If $\mathscr{L}$ is absolute relative to R, $A \subseteq A'$ and $\mathscr{L}'$ pins down R, then* $\hbar(\text{RPC}_{\mathscr{L}}) \leq \hbar(\text{RPC}_{\mathscr{L}'})$.

Proof. See Corollary 3.4.5(i). ◻

4.5.6 Corollary. *If $\mathscr{L}$ is absolute, then* $\hbar(\mathscr{L}) \leq \hbar(\mathscr{L}_A(W))$. ◻

4.5.7 Theorem. *Let $\mathscr{L}$ be a logic, R a predicate and*

> $\delta = \sup\{\alpha \mid \alpha$ *is Σ_1-definable in the extended language* $\{\in, R\}$ *with parameters in* $A\}$.

(i) *If $\mathscr{L}$ is absolute relative to R, then* $\hbar(\text{RPC}_{\mathscr{L}}) \leq \delta$.
(ii) *If $\mathscr{L}$ pins down R, then* $\delta \leq \hbar(\text{RPC}_{\mathscr{L}})$.

Proof. As to (i) suppose that $\lambda = \sup(C)$, for a bounded spectrum C of $\mathrm{RPC}_{\mathscr{L}}$. The cardinal λ has the following definition:

$$\alpha \in \lambda \leftrightarrow \exists\beta(\alpha \leq \beta \wedge \beta \in C).$$

Using absoluteness of $\mathscr{L}$, we can write this in Σ_1-form with a parameter in A. As to (ii), suppose that α is Σ_1-definable with parameters in A. Let $\mathscr{K}$ be the class of well-ordered structures of type $<\alpha$. $\mathscr{K}$ is clearly Σ_1-definable, with parameters in A. By Theorem 3.4.4, $\mathscr{K}$ is $\mathrm{RPC}_{\mathscr{L}}$-definable. Moreover, $\mathrm{Sp}(\mathscr{K})$ is bounded by $|\alpha|^+$. Let $\lambda = \sup \mathrm{Sp}(\mathscr{K})$. Well-known properties of all Hanf numbers imply that $\alpha^+ < \hbar(\mathscr{L})$. Thus, we have that $\lambda < \hbar(\mathscr{L})$. $\square$

4.5.8 Corollary. *If $\mathscr{L}$ is absolute relative to R and pins down R, then*

$$\hbar(\mathrm{RPC}_{\mathscr{L}}) = \sup\{\alpha \mid \alpha \textit{ is } \Sigma_1\textit{-definable in } \{\varepsilon, R\} \textit{ with parameters in } A\}. \quad \square$$

4.5.9 Examples. (i) $\hbar(\mathscr{L}_A(W)) = \sup\{\alpha \mid \alpha$ is Σ_1-definable with parameters in $A\}$.
(ii) $\hbar(\mathscr{L}_A^2) = \sup\{\alpha \mid \alpha$ is Σ_2-definable with parameters in $A\}$.

Theorem 4.5.7 permits an iteration similar to that of Theorem 4.4.3.

4.5.10 Theorem. $\hbar(\Delta_n(\mathscr{L}_A)) = \sup\{\alpha \mid \alpha$ *is* Σ_n*-definable with parameters in* $A\}$ $(n > 1)$.

Up until now, we have characterized the suprema of Σ_n-, Π_n-, and Δ_n-definable ordinals in terms of Hanf and Löwenheim numbers. The logics in question—$\Delta_n(\mathscr{L}_A)$—are so strong that there is little hope of deciding any questions concerning them in ZFC alone. However, one rather curious relation between the different Hanf and Löwenheim numbers is not hard to prove.

4.5.11 Theorem. $\ell(\Delta_n(\mathscr{L}_A)) < \hbar(\Delta_n(\mathscr{L}_A)) = \ell(\Delta_{n+1}(\mathscr{L}_A))$ $(n > 1)$.

4.5.12 Corollary. *If the required large cardinals exist, then*

1st measurable $< \ell(\Delta_2(\mathscr{L}_A)) <$
1st supercompact $< \hbar(\Delta_2(\mathscr{L}_A)) = \ell(\Delta_3(\mathscr{L}_A)) <$
1st extendible $< \hbar(\Delta_3(\mathscr{L}_A))$.

Remark. If we consider Theorem 4.5.10 for $n = 1$, we have to add the quantifier W to $\mathscr{L}_A$ to make the situation non-trivial (for $n > 1$, this would make no difference). The inequality-part remains true then. The equality-part fails, for if κ is the last κ such that $\kappa \to (\omega_1)^{<\omega}$, then $\hbar(\mathscr{L}_{\omega\omega}(W)) < \kappa < \ell(\Delta_2(\mathscr{L}_{\omega\omega}))$. But, of course, there need not exist such a large κ. Indeed, the theorem does hold also for $n = 1$ in L. And we have

4.5.13 Theorem. (i) $\hbar(\mathscr{L}_A(W)) \leq \ell(\mathscr{L}_A^2)$.
(ii) *If* $V = L$, *then* $\hbar(\mathscr{L}_A(W)) = \ell(\mathscr{L}_A^2)$.

Remarks. A number "about the size" of $h(\mathscr{L}_{\omega\omega}(W))$ and $\ell(\mathscr{L}^2_{\omega\omega})$ is $h(\mathscr{L}_{\omega_1\omega_1})$. If $V = L$, then $\ell(\mathscr{L}^2_{\omega\omega}) < h(\mathscr{L}_{\omega_1\omega_1})$, because the former equals $h(\mathscr{L}_{\omega\omega}(W)) < h(\mathscr{L}_{\omega_1\omega_1})$. Observe that $\mathrm{cf}(h(\mathscr{L}_{\omega_1\omega_1})) > \omega$, so that $h(\mathscr{L}_{\omega_1\omega_1})$ can never really equal either $h(\mathscr{L}_{\omega\omega}(W))$ or $\ell(\mathscr{L}^2_{\omega\omega})$. Kunen [1970] showed that if $V = L^\mu$, then $h(\mathscr{L}_{\omega_1\omega_1})$ exceeds the 1st measurable cardinal. On the other hand, there is a model in which $h(\mathscr{L}_{\omega_1\omega_1})$ is below the first weakly compact (Väänänen [1980c]) and, hence, also below $\ell(\mathscr{L}^2_{\omega\omega})$. Another curiosity in this field is that although $\ell(\mathscr{L}) < h(\mathscr{L})$ is true of almost all logics, it is not a rule: The statement $\ell(\mathscr{L}_{\omega\omega}(I)) < h(\mathscr{L}_{\omega\omega}(I))$ is independent of ZFC. Also, all non-trivial claims of relation between $\ell(\mathscr{L}_{\omega\omega}(I))$, $h(\mathscr{L}_{\omega\omega}(I))$ and large cardinals turn out to be independent (Väänänen [1982a]). The numbers can be as small or as large as conceivably possible, if measured by large cardinals. The interrelations of the Hanf and Löwenheim numbers discussed can be visualized in the form of Diagram 2, where an arrow means "less or equal to".

$$\begin{tikzcd}
& & h(\mathscr{L}^2_{\omega\omega}) & & \\
\mathscr{L}(\mathscr{L}_{\omega\omega}(I)) \arrow[urr] & & \ell(\mathscr{L}^2_{\omega\omega}) \arrow[u] & & h(\mathscr{L}_{\omega_1\omega_1}) \arrow[ull] \\
& \ell(\mathscr{L}_{\omega\omega}(I)) \arrow[ur] & & & \\
& & h(\mathscr{L}_{\omega\omega}(W)) \arrow[uull] \arrow[uu] \arrow[uurr] & &
\end{tikzcd}$$

Diagram 2

It is not known whether $h(\mathscr{L}_{\omega_1\omega_1}) < h(\mathscr{L}_{\omega\omega}(I))$ holds absolutely or not, but no arrows are otherwise missing. If $V = L$, the picture collapses (Diagram 3).

$$\begin{tikzcd}
h(\mathscr{L}_{\omega\omega}(I)) = h(\mathscr{L}^2_{\omega\omega}) \\
h(\mathscr{L}_{\omega_1\omega_1}) \arrow[u] \\
\ell(\mathscr{L}_{\omega\omega}(I)) = \ell(\mathscr{L}^2_{\omega\omega}) = h(\mathscr{L}_{\omega\omega}(W)) \arrow[u]
\end{tikzcd}$$

Diagram 3

Historical and Bibliographical Remarks. The main results on the failure of the weak Beth property, Theorems 4.1.2 and 4.1.3 are respectively due to Burgess [1977] and Gostanian–Hrbacek [1976]. They have many precedents in the literature, Mostowski [1968] being perhaps the most notable. Also Theorem 4.1.4, 5, 7(ii) and 8 are from Gostanian–Hrbacek [1976].

The results given in Sections 4.2.5–7 are from Cutland–Kaufman [1980]. Theorem 4.2.10 is from Stavi [1978] which also contains refinements of Theorem 4.2.10. The incompleteness results, Corollary 4.3.7 and Theorem 4.3.8, are respectively due to Barwise [1972a] and Cutland–Kaufman [1980]. The latter is also the best reference to Theorem 4.3.9 and many other results on Σ_1-compactness and validity questions for unbounded absolute logics. The motivation behind Theorem 4.3.10 as well as its proof are given in all details in Feferman [1975].

Theorems 4.4.3–9 are from Väänänen [1979a], while Example 4.4.5(ii) is independently due to Krawczyk–Marek [1977]. The relation between supercompactness, extendibility and $R_\kappa \prec_n V$ are from Solovay *et al.* [1978]. Magidor [1971] establishes important relations between supercompactness, extendibility, and second-order logic. Theorem 4.4.10(i) is proven in a way similar to Theorem 4.2.10. Part (ii) of this result is due to Pinus [1978].

Examples 4.5.1(i) and (ii) are due to Silver [1971]; while (iii) is due to Burgess [1978]. The results given in sections 4.5.2–4 are from Väänänen [1983] where Theorem 4.5.4 is also proven in the stronger form, namely

$$\mathrm{Con}[\hbar(\mathscr{L}_{\omega\omega}(I)) < \hbar(\Delta(\mathscr{L}_{\omega\omega}(I)))].$$

Corollary 4.5.6 is due to Barwise [1972a], and the results in Sections 4.5.7–13 are from Väänänen [1979a]. Example 4.5.9(ii) is independently due to Krawczyk and Marek [1977]. Theorem 4.5.13 is proven in Väänänen [1979b].

Suggestions for Further Work in the Area. It seems likely that further progress can be made in the following parts of this chapter:

1. *The Program Presented in Feferman* [1974b, 1975]. The analysis of adequacy to truth presented here, as well as Theorem 4.3.10 are parts of the program. However, the entire program is much more ambitious.
2. *Relative Absoluteness.* The set-theoretical method is at its best in the context of absolute logics and there are but few results on relatively absolute logics. In view of Hutchinson [1976], it seems possible to develop set-theoretical proofs for compactness theorems. Although, in general we have tended to ignore compact logics in this chapter, it would nevertheless be interesting to extend the scope in their direction.
3. *Canonical Failure of Interpolation.* We have undefinability of truth style proofs for the failure of different forms of interpolation in various logics. These proofs do not apply directly to $\mathscr{L}_{\omega\omega}(Q_1)$ or to $\mathscr{L}_{\omega\omega}(\mathrm{aa})$, for example. Is there a canonical anti-interpolation theorem which applies to these countably compact logics?
4. *Can Validity in an Unbounded Absolute Logic be* Σ_1? The validity problem seems to provide a fruitful framework for further work in abstract model theory.
5. *Löwenheim–Skolem and Hanf Prospects.* One may formulate downward or upward Löwenheim–Skolem theorems which are stronger than those related to Löwenheim and Hanf numbers. Magidor [1971] is an example. The proofs of such theorems tend to depend on large cardinal or combinatorial axioms of set theory.

Chapter XVIII
Compactness, Embeddings and Definability

by J. A. MAKOWSKY

This chapter presents an overview of the author's joint work with S. Shelah in abstract model theory, which had started as early as 1972. It is mainly based on our papers (Makowsky–Shelah–Stavi [1976]; Makowsky–Shelah [1979, 1981, 1983]) and on an unpublished manuscript of S. Shelah (Shelah [198?e]) which he wrote while this chapter came into being. The present exposition, however, tries to give a more coherent picture by putting all our results into a single perspective together with results of M. Magidor, H. Mannila, D. Mundici, and J. Stavi.

The main theme of this chapter is abstract model theory proper, especially the relationship between various compactness, embedding, and definability properties which do not characterize first-order logic. More precisely, we look at various classes of logics defined axiomatically, such as compact logics, logics satisfying certain model existence or definability properties. The classes of logics are sometimes further specified by set-theoretic parameters, such as finitely generated, absolute, set presentable, bounds on the size function, or by set-theoretic assumptions such as large cardinal axioms. Within such classes of logics we want to explore which other properties of logics follow from the axiomatic description of the class. In Chapter III first-order logic was characterized in this way. In Chapter XVII the class of absolute logics was studied. Most of the other chapters (with the exception of Chapters XIX and XX) study families of logics which bear some inherent similarity which stems from the way they evolved, such as infinitary logics or logics based on cardinality quantifiers, and establish particular model-theoretic results for those logics. In this chapter we want to clarify the conceptual and metalogical relationship between these model theoretic properties. Success in this program can be achieved in three ways: by establishing non-trivial connections between these properties; by applying the former to gain new insight about particular logics previously studied; and by using this insight to construct new examples of logics, and ultimately, by showing, that our list of examples is, in some reasonable sense, exhaustive.

The chapter consists of four sections, in each of which one aspect of abstract model theory is developed to a certain depth.

Section 1 is devoted to compactness properties and is almost self-contained. Its main results are the abstract compactness theorem and the description of the compactness spectrum. Here a thorough understanding of various compactness phenomena is obtained and the theory is provided with new examples. Especially, the examples described in Section 1.6 play an important role in the successive sections as well.

Section 2 is devoted to the study of the dependence number. Its main result is the finite dependence theorem, the proof of which is given completely on the basis of three lemmas, which are only stated. The complete proof may be found in Makowsky–Shelah [1983]. The finite dependence theorem clarifies how little compactness is needed to ensure that a logic is equivalent to a logic which has the finite dependence property. In fact, assuming there are no uncountable measurable cardinals, $[\omega]$-compactness suffices. Finally, the dependence structure is introduced, a concept which appears here for the first time. It is the appropriate generalization of the dependence number, as the examples and the finite dependence structure theorem show.

Section 3 is devoted to various aspects of embeddings, whose existence is implied by the compactness theorem, such as proper extensions, amalgamation, and joint embeddings. Joint embeddings are also discussed in Chapter XIX and amalgamations in Chapter XX. The main result here is the connection between $[\omega]$-compactness and proper extensions and the abstract amalgamation theorem. Again, this section is rather self-contained. The abstract amalgamation theorem also leads to the discovery that various logics with cardinality quantifiers do not satisfy the amalgamation property. This solves a problem which had been stated explicitly in Malitz–Reinhardt [1972b].

Section 4 is devoted to definability properties, as introduced already in Section II.7, and to preservation properties. Preservation properties for sum-like operations already played an important role in Chapters XII and XIII. A common generalization of these two properties, the uniform reduction property, was introduced in Feferman [1974b]. The first two subsections are devoted to an exposition of those properties and their interrelations. The main results here are the equivalence of the uniform reduction property UR_1 with the interpolation property and the equivalence, for compact logics, of the pair preservation property and the uniform reduction property for pairs. The Robinson property and especially its weaker versions, the finite Robinson property and the weak finite Robinson property are the topic of the next three subsections. In Chapter XIX the Robinson property is studied further.

Our main results here are: The finite Robinson property together with the pair preservation property implies that a logic is ultimately compact, and therefore has the finite dependence property, provided that there are no uncountable measurable cardinals. The Beth property together with the tree preservation property implies the weak finite Robinson property and the Robinson property together with the pair preservation property implies the existence of models with arbitrarily large automorphism groups. The last subsection discusses more examples, in particular a compact logic which satisfies the Beth property, the pair preservation property, but not the interpolation property.

Measurable cardinals play an important role in our presentation. They are in some sense $\mathscr{L}$-compact cardinals, which is to say, if such a cardinal μ exists then every finitely generated logic is, stationary often, weakly compact below μ. The first cardinal for which a logic is $[\kappa]$-compact is always measurable (or ω). But measurable cardinals, of which the first could conceivably be as big as the first strongly compact cardinal, also appear frequently in the hypotheses of various of our theorems. They also appear in various examples and counterexamples and sometimes their existence turns out to be equivalent to certain assumptions in abstract model theory.

In the same sense, it turns out, Vopenka's principle is a compactness axiom: It is equivalent to the statement that every finitely generated logic is ultimately compact or, alternatively, that every finitely generated logic has a global Hanf number. We have not centered our presentation around this theme, but the reader will easily discern it throughout the chapter.

Finally, a word on future research. Some of the possible directions of future research in abstract model theory are outlined in Chapters XIX and XX. The purpose there is to get *away* from the syntactic aspects of logic completely and to study classes of structures more in the spirit of universal algebra. If we want to stay in the framework of abstract model theory and logics I can see three directions in which to pursue further research.

The first direction is to study, what we have rather neglected in this chapter, the impact of various axiomatizability and dependence properties of logics on their respective model theory. We know that axiomatizability implies recursive compactness. But we do not know, for instance, if there are any model-theoretic properties distinguishing axiomatizable logics from logics axiomatizable by a finite set of axiom schemas. Only recently, in Shelah–Steinhorn [1982], it is shown that the logic $\mathscr{L}_{\omega\omega}(Q_{\beth_\omega})$ is an axiomatizable logic which cannot be axiomatized by schemas. This was the first example of its kind. Similarly, we know that $[\omega]$-compactness implies the finite dependence property (assuming there are no uncountable measurable cardinals), but we have not investigated if other model-theoretic properties, such as Lowenheim or Hanf numbers, have similar effects. The same holds for the finite dependence structure and dependence filters, as discussed in Section 2.4.

The second direction is the search for more model-theoretic properties which fit into the abstract framework. In Section 4.5 an attempt in this direction is presented: the existence of models with large automorphism groups. Incidentally, this also gives us a new proof for the case of first-order logic. In Shelah [198?e] a host of new notions occur in his study of Beth closures of logics preserving compactness and preservation properties. There is a danger here of proving theorems which apply only to first-order logic, such as compactness and chain properties imply the Robinson property. Since it is open whether there are logics satisfying both the Robinson property and the pair preservation property, the results in Section 4.5 should be taken with a grain of salt.

The third direction consists in incorporating the theory of second-order quantifiers, as presented in Chapter XII, into the study of the model-theoretic properties as presented in this chapter. What are the compact second-order quantifiers, what are the second-order quantifiers satisfying preservation and definability properties, etc? I am convinced that abstract model theory will remain a fruitful area of active research for many years to come.

We have not included detailed historical notes. Most of the results presented in this chapter are taken from my joint papers with S. Shelah and from his unpublished manuscript mentioned above. Some of the theorems and corollaries are stated here for the first time as a result of reflection upon the material presented. Results which appear here for the first time in print are marked with an asterisk. Whenever possible, we refer to the other chapters in the book rather than to original papers.

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1. Compact Logics

1.1. $[\kappa, \lambda]$-compactness

In this section we will study compactness properties of abstract logics. Traditionally, one looks at a set Σ of sentences of cardinality κ such that every subset $\Sigma_0 \subset \Sigma$ of cardinality less than λ has a model and concludes that Σ has a model. This is called (κ, λ)-compactness. By abuse of notation we write (∞, κ)-compactness instead of $(<\infty, \kappa)$-compactness. We call $(<\infty, \omega)$-compactness just compactness.

In contrast to this we look at two different situations:

(*) Given two sets of sentences Δ and Σ with $\text{card}(\Sigma) = \kappa$, $\text{card}(\Delta)$ arbitrary and such that for every subset $\Sigma_0 \subset \Sigma$ of cardinality less than λ $\Sigma_0 \cup \Delta$ has model. Then $\Sigma \cup \Delta$ has a model.

(**) Given a family Γ_α $(\alpha < \kappa)$ of sets of sentences such that for every set $X \subset \kappa$ of cardinality less than λ the union $\bigcup_{\alpha \in X} \Gamma_\alpha$ has a model. Then $\bigcup_{\alpha < \kappa} \Gamma_\alpha$ has a model.

1.1.1 Proposition. *For a regular logic $\mathscr{L}$ properties* (*) *and* (**) *are equivalent.*

Proof. (*) $\to$ (**) Let P_α ($\alpha < \mu$) be unary predicates not in $\bigcup_{\alpha<\kappa} \Gamma_\alpha$ and let ψ_α be the formula $\exists x\, P_\alpha(x)$. Now we put

$$\Delta = \{\psi_\alpha \to \varphi : \alpha < \kappa, \varphi \in \Gamma_\alpha\},$$

and

$$\Sigma = \{\psi_\alpha : \alpha < \kappa\}.$$

Clearly $\Delta \cup \Sigma_0$ has a model iff $\bigcup_{\psi_\alpha \in \Sigma_0} \Gamma_\alpha$ has a model.

(**) $\to$ (*) Let $\{\psi_\alpha : \alpha < \kappa\}$ be an enumeration of the formulas of Σ and put

$$\Gamma_\alpha = \Delta \cup \{\psi_\alpha\}. \ \Box$$

1.1.2 Remark. (*) was first systematically studied in Makowsky–Shelah [1979b] and in Makowsky–Shelah [1983]. (**) was introduced for topological spaces in Alexandroff–Urysohn [1929], as was pointed out to us by H. Mannila. (*) was called first *relative* (κ, λ)-*compact* and then $(\kappa, \lambda)^*$-compact. (**) is called in the topological literature $[\kappa, \lambda]$-*compact*.

The motivation behind (*) stems from working with elementary extensions and with diagrams. Δ usually plays the role of a diagram, and Σ describes the properties the extension should have. A similar situation occurs in Chang–Keisler [1973, Exercise 4.3.22].

1.1.3 Definition. A regular logic $\mathscr{L}$ with property (*) or (**) is called $[\kappa, \lambda]$-*compact*. If $\kappa = \lambda$ we simply write $[\kappa]$-compact.

1.1.4 Examples. (i) $\mathscr{L}(Q_{\omega_1})$ is (ω, ω)-compact but not $[\omega]$-compact.

(ii) (Bell–Slomson [1969, Theorem 2.2, p. 263]). If κ is small for λ, then $\mathscr{L}(Q_\lambda)$ is $[\kappa, \omega]$-compact. In particular, ω is small for $(2^\omega)^+$.

Recall that κ is small for λ if for every family μ_i ($i < \kappa$) such that $\mu_i < \lambda$ $\prod_i \mu_i < \lambda$.

1.1.5 Definition. We write $[\kappa, \lambda] \to [\mu, \nu]$ whenever $[\kappa, \lambda]$-compactness implies $[\mu, \nu]$-compactness. Similarly for conjunctions of compactness properties implying other such properties.

The following lemma collects some simple but useful facts:

1.1.6 Lemma. (i) $[\kappa, \lambda] \to [\mu, \lambda]$ *for* $\mu < \kappa$.

(ii) $[\kappa, \lambda] \to [\kappa, \nu]$ *for* $\nu > \lambda$.

(iii) $[\mu] \wedge [\kappa, \mu^+] \to [\kappa, \mu]$.

(iv) $[\kappa^+] \wedge [\kappa, \mu] \to [\kappa^+, \mu]$.

(v) *If* $[\beta]$ *and for every* $\alpha < \beta$, $[\kappa_\alpha]$ *and* $[\kappa_\alpha, \mu]$ *then* $[\sum_{\alpha<\beta} \kappa_\alpha, \mu]$.

(vi) $[\mathrm{cf}(\kappa)] \to [\kappa]$.

Proof. Trivial for (i) and (ii).

(iii), (iv) and (v) follow from definition (*).

(vi) follows from definition (**). $\Box$

1.1.7 Proposition. (i) *A logic $\mathscr{L}$ is $[\kappa, \lambda]$-compact iff $\mathscr{L}$ is $[\mu]$-compact for every μ, $\lambda \leq \mu \leq \kappa$.*

(ii) *A logic $\mathscr{L}$ is $[\infty, \kappa]$-compact iff $\mathscr{L}$ is (∞, κ)-comact.*

Proof. For (i) we use Lemma 1.1.6 and (ii) follows from the definition. □

Mannila [1982, 1983] has investigated what results from topology give us refinements of Theorem 1.1.7. He showed that results from Alexandroff–Urysohn [1929] and Vaughan [1975] can be translated into our framework and we obtain

1.1.8 Proposition. (i) *A logic $\mathscr{L}$ is $[\kappa, \omega]$-compact iff $\mathscr{L}$ is $[\mu]$-compact for every regular μ, $\omega \leq \mu \leq \kappa$.*

(ii) *Assume* $\mathrm{cf}(\kappa) \geq \lambda$. *A logic $\mathscr{L}$ is $[\kappa, \lambda]$-compact iff $\mathscr{L}$ is $[\mu, \lambda]$-compact for every regular μ, $\lambda \leq \mu \leq \kappa$.*

Proposition 1.1.8 was first stated in Makowsky–Shelah [1983], where it was derived from Lemma 1.1.6.

Using the methods developed in Sections 1.3 and 1.4 this can be sharpened to:

1.1.9 Theorem. *Let λ be a cardinal and $\mathscr{L}$ a logic. The following are equivalent:*

(i) *$\mathscr{L}$ is $[\mu]$-compact for every regular $\mu \geq \lambda$.*
(ii) *$\mathscr{L}$ is $[\mu]$-compact for every $\mu \geq \lambda$.*
(iii) *$\mathscr{L}$ is $[\infty, \lambda]$-compact.*
(iv) *$\mathscr{L}$ is (∞, λ)-compact.*

Proof. (ii) implies (iii) by Proposition 1.1.7(i); (iii) is equivalent to (iv) by Proposition 1.1.7(ii) and (iii) implies (i) by Lemma 1.1.6(i) and (ii). So we have to prove that (i) implies (ii). Assume (i) and that λ is singular. So $\mathscr{L}$ is $[\lambda^+]$-compact. Now we use the abstract compactness theorem (1.3.9(ii)) which gives us a uniform ultrafilter F on λ^+. By Lemma 1.3.11(i) F is $[\lambda^+, \lambda]$-regular, so by Theorem 1.3.9(i) $\mathscr{L}$ is $[\lambda^+, \lambda]$-compact, and therefore $[\lambda]$-compact. □

We have put this proof here, though it uses material from Section 1.3, to illustrate the power of the abstract compactness theorem, which gives rise to various transfer results. We shall see more transfer results in Section 1.5.

We shall call logics $\mathscr{L}$ satisfying any of equivalent properties above *ultimately compact.*

1.2. Cofinal Extensions

One useful tool for the study of $[\kappa]$-compactness is its characterization via the non-characterizability of certain ordered structures. In Chapter II, Proposition 5.2.4 we have seen the paradigm of this procedure: A logic $\mathscr{L}$ is (∞, ω)-compact iff its well-ordering number is ω. Here the well-ordering is replaced by the cofinality of some linear order.

1.2.1 Definition. (i) Let $\mathfrak{A}$ be an expansion (possibly with new sorts) of the structure $\langle \kappa, < \rangle$ and $\mathfrak{B}$ and $\mathscr{L}$-extension of A. $\mathfrak{B}$ *extends* $\mathfrak{A}$ *beyond* κ if there is an element $b \in B \cap \mathrm{dom}(<^B)$ such that for every $a \in A \cap \mathrm{dom}(<^A)$ $B \models a < b$. If there is no such element, we call $\mathfrak{B}$ a *confinal extension of* $\mathfrak{A}$.

(ii) Let $\mathscr{L}$ be a logic and κ a regular cardinal. $\mathscr{L}$ *confinally characterizes* κ or κ is *cofinally characterizable in* $\mathscr{L}$ if there exists an expansion $\mathfrak{A}$ (possibly many-sorted with additional sorts) of the structure $\langle \kappa, < \rangle$ such that every $\mathscr{L}$-extension $\mathfrak{B}$ of $\mathfrak{A}$ is a cofinal extension of $\mathfrak{A}$. In this case we also say that $\mathscr{L}$ *cofinally characterizes* κ *via* $\mathfrak{A}$.

1.2.2 Theorem. *Let κ be a regular cardinal. A logic $\mathscr{L}$ is $[\kappa]$-compact iff κ is not cofinally characterizable in $\mathscr{L}$.*

Proof. Like in Chapter II, Proposition 5.2.4. ☐

Theorem 1.2.2 gives a quick proof of Lemma 1.1.6(vi). It can be used, together with a classical result due to Rabin and Keisler (Keisler [1964]) (cf. also Chang–Keisler [1973, Theorem 6.4.5]), to study the existence of $\mathscr{L}$-maximal structures.

Recall that a *complete structure* $\mathfrak{A}$ is a one-sorted structure where every subset $X \subset A^n$ is the interpretation of some relation symbol R_X. In the case of many-sorted structures we have to allow also relations with mixed arities.

1.2.3 Theorem (Rabin–Keisler). *Let $\mathfrak{A}$ be a complete structure of cardinality $\lambda <$ first uncountable measurable cardinal, P^A be a countable infinite predicate of $\mathfrak{A}$ and $\mathfrak{B}$ be a proper $\mathscr{L}_{\omega\omega}$-extension of $\mathfrak{A}$. Then $P^A \subsetneq P^B$.*

One can now easily prove from Theorems 1.2.2 and 1.2.3 a generalization of a result of Malitz–Reinhardt [1972b] and independently (Shelah [1967]):

1.2.4 Proposition. *If a logic $\mathscr{L}$ is not $[\omega]$-compact then there are arbitrarily large $\mathscr{L}$-maximal structures of cardinality less than the first uncountable measurable cardinal.*

Recall that a structure is $\mathscr{L}$-maximal if it has no proper $\mathscr{L}$-extensions. $\mathscr{L}$-extensions are further studied in Section 3.

The following observations will be useful later:

1.2.5 Lemma (Mundici). *Let κ, λ be regular cardinals and $\mathscr{L}$ a logic. Let $\mathfrak{A}_\kappa, \mathfrak{A}_\lambda$ be expansions of $\langle \kappa, < \rangle, \langle \lambda, < \rangle$ to τ-structures such that $\mathscr{L}$ cofinally characterizes $\kappa, (\lambda)$ via $\mathfrak{A}_\kappa, \mathfrak{A}_\lambda$, respectively. Then there exists no $\mathscr{L}$-embedding of $\mathfrak{A}_\kappa$ into $\mathfrak{A}_\lambda$.*

The proof is left to the reader.

1.2.6 Proposition*. *Let $\mathscr{L}$ be a logic which is not ultimately compact. Then there is a proper class of τ-structures $\mathfrak{C}$ such that for no two $\mathfrak{A}, \mathfrak{B} \in \mathfrak{C}$ there is an $\mathscr{L}$-embedding from $\mathfrak{A}$ into $\mathfrak{B}$.*

Proof. If $\mathscr{L}$ is not ultimately compact, there is a proper class $\mathfrak{C}_0$ of regular cardinals κ for which $\mathscr{L}$ is not $[\kappa]$-compact (use Lemma 1.1.6(vi)). So by Theorem 1.2.2 each $\kappa \in \mathfrak{C}_0$ is cofinally characterizable in $\mathscr{L}$ via some $\mathfrak{A}_\kappa$. We can arrange it that each $\mathfrak{A}_\kappa$ is a τ-structure for some countable τ. For this we code many n-ary relation symbol by one $(n + 1)$-ary relation symbol and the use of constants. Now put $\mathfrak{C} = \{\mathfrak{A}_\kappa : \kappa \in \mathfrak{C}_0\}$. By Lemma 1.2.5 $\mathfrak{C}$ has the required property. □

1.3. Ultrafilters, Ultrapowers and Compactness

In first-order logic compactness is intimately related to the ultrapower construction. One can turn this observation easily into a characterization theorem for $\mathscr{L}_{\omega\omega}$.

1.3.1 Definition. Let $\mathscr{L}$ be a logic. $\mathscr{L}$ is said to have the *Los property* if for every τ-structure $\mathfrak{A}$ and every ultrafilter F and every formula $\varphi \in \mathscr{L}[\tau]$ the ultrapower $\prod \mathfrak{A}_i/F \models \varphi$ iff $\{i \in I : \mathfrak{A}_i \models \varphi\} \in F$.

1.3.2 Theorem. *Let $\mathscr{L}$ be a regular logic which has the Los property. Then* $L \equiv \mathscr{L}_{\omega\omega}$.

Proof. By coding a family of structures in one structure and using the Keisler–Shelah theorem, that elementarily equivalent structures have isomorphic ultrapowers, the proof is straightforward. □

1.3.3 Remark. Theorem 1.3.2 was folklore already around 1972. A detailed version may be found in Sgro [1977] and Monk [1976, Exercise 25.53]. Sgro [1977] contains interesting additional material concerning maximal logics.

To study compactness for abstract logics we need a generalization of the Los property.

1.3.4 Definitions. (i) Let $\mathscr{L}$ be a logic and F be an ultrafilter over I. We say that F *relates to* $\mathscr{L}$ if for every τ and for every τ-structure $\mathfrak{A}$ there exists a τ-structure $\mathfrak{B}$ extending $\prod_I \mathfrak{A}/F$ such that for every formula $\varphi \in \mathscr{L}[\tau]$, $\varphi = \varphi(x_1, x_2, \ldots, x_i, \ldots)$, $i < \alpha$ with α many free variables and every $f_i \in A^I$, $i < \alpha$ we have:

$$\mathfrak{B} \models \varphi(f_1/F, f_2/F, \ldots, f_i/F, \ldots)$$

iff

$$\{j \in I : \mathfrak{A} \models \varphi(f_1(j), f_2(j), \ldots, f_i(j), \ldots)\} \in F.$$

(ii) We define UF($\mathscr{L}$) to be the class of ultrafilters F which are related to $\mathscr{L}$.

1.3.5 Remark. Note that $\mathfrak{B}$ is always an elementary extension of $\prod_I \mathfrak{A}/F$.

1.3.6 Examples. (i) Every ultrafilter is in $\mathrm{UF}(\mathscr{L}_{\omega\omega})$.

(ii) Let $\mathscr{L}$ be $\mathscr{L}_{\omega\omega}(Q_\kappa)$, i.e., first-order logic with the additional quantifier "there exist at least κ many." Then every ultrafilter on ω is related to $\mathscr{L}$, provided ω is small for κ.

1.3.7 Proposition. *$\mathscr{L}$ is compact iff every ultrafilter is related to $\mathscr{L}$.*

Proof. Let $\mathfrak{M}$ be a τ-structure and F an ultrafilter on a set I. For every $f \in M^I$ let c_f be a new constant symbol not in τ. Put

$$T = \{\varphi(c_{f_1}, c_{f_2}, \ldots) : \varphi \in \mathscr{L}[\tau] \quad \text{and} \quad \{t \in I : \mathfrak{M} \models \varphi(f_1(t), f_2(t), \ldots)\} \in F\}.$$

If $\mathscr{L}(\tau)$ is a set, so is T and obviously every finite subset of T has a model: We just expand $\mathfrak{M}$ appropriately. So let $\mathfrak{N}$ be a model of T. Clearly

$$\prod_I \mathfrak{M}^I/F \subset \mathfrak{N},$$

and by the definition of T, $\mathfrak{N}$ satisfies the requirements for $F \in \mathrm{UF}(\mathscr{L})$. In the case $\mathscr{L}(\tau)$ is a proper class, we have to take a subclass $T_0 \subset T$ which is a set and still guarantees that

$$\prod_I \mathfrak{M}^I/F \subset \mathfrak{N},$$

and that $\mathfrak{N}$ satisfies the requirements for $F \in \mathrm{UF}(\mathscr{L})$. For this we observe that over the structure $\mathfrak{M}^I$ there are only set many inequivalent formulas with less than $\mathrm{card}(\mathfrak{M}^I)^+$-many free variables.

The converse is trivial. ☐

The next theorem connects the compactness spectrum $\mathrm{Comp}(\mathscr{L})$ with the filters in $\mathrm{UF}(\mathscr{L})$. To be more explicit, we need some more definitions.

1.3.8 Definitions. Let F be an ultrafilter on I, and λ, μ be cardinals with $\lambda \geq \mu$.

(i) F is said to be (λ, μ)-regular if there is a family $\{X_\alpha : \alpha < \lambda\}$, $X_\alpha \in F$ such that if $\{\alpha_i < \lambda : i < \mu\}$ is any enumeration of μ ordinals less than λ, then $\bigcap_{i<\mu} X_{\alpha_i} = \varnothing$. The family $\{X_\alpha, \alpha < \lambda\}$ is called a (λ, μ)-regular family.

(ii) A (λ, ω)-regular ultrafilter on λ is called regular.

(iii) F is λ-descendingly incomplete if there exists a family $\{X_\alpha : \alpha < \lambda\}$, $X_\alpha \in F$ with $X_\beta \subset X_\alpha$ for $\alpha < \beta < \lambda$ such that $\bigcap_{\alpha<\lambda} X_\alpha = \varnothing$.

(iv) F is uniform on λ if every $X \in F$ has cardinality λ.

1.3.9 Theorem (Abstract Compactness Theorem). *Let λ, μ be cardinals, $\lambda \geq \mu$, and let $\mathscr{L}$ be a logic.*

(i) *$\mathscr{L}$ is $[\lambda, \mu]$-compact iff there is a (λ, μ)-regular ultrafilter F on $I = P_{<\mu}(\lambda)$ in* $\mathrm{UF}(\mathscr{L})$.

(ii) *If $\lambda = \mu$ and μ regular, then $\mathscr{L}$ is $[\lambda]$-compact iff there is a uniform ultrafilter F on λ in* $\mathrm{UF}(\mathscr{L})$.

The proof of this theorem is delayed to Section 1.4.

Theorem 1.3.9 allows us to use known results from the theory of ultrafilters to understand $[\lambda, \mu]$-compactness. The following lemma collects some simple results from (but not due to) Comfort–Negrepontis [1974].

1.3.10 Lemma. (i) *If F is (λ, μ)-regular and $\mu \leq \mu_1 \leq \lambda_1 \leq \lambda$ then F is (λ_1, μ_1)-regular.*

(ii) *If λ is a regular cardinal and F is λ-descendingly incomplete, then F is (λ, λ)-regular.*

(iii) *If E is uniform on λ then F is (λ, λ)-regular.*

(iv) *If F is* $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$*-regular then F is (λ, λ)-regular.*

The abstract compactness theorem and Lemma 1.3.10 give us immediately the corresponding statements in Lemma 1.1.6.

The next lemma collects some more sophisticated theorems from the literature on ultrafilters. For Lemma 1.3.11(ii) one may also consult Comfort–Negrepontis [1974, Theorem 8.36].

1.3.11 Lemma. (i) (Kanamori [1976]). *If F is uniform on λ^+ and λ is singular, then F is (λ^+, λ)-regular.*

(ii) (Kunen–Prikry [1971]; Cudnovskii–Cudnovskii [1971]). *If F is uniform on λ^+ and λ is regular, then F is λ-descendingly incomplete, and hence (λ, λ)-regular.*

This lemma, together with the abstract compactness theorem, is the key to the study of the compactness spectrum in Sections 1.5 and 1.6. It is also used in the proof of Theorem 1.1.9.

1.4. Proof of the Abstract Compactness Theorem

Before we prove the abstract compactness theorem we shall give a model-theoretic characterization of (λ, μ)-regular ultrafilters which will give us the link between $[\lambda, \mu]$-compactness and the existence of (λ, μ)-regular ultrafilters. This is implicitly in Keisler [1967b] (cf. also Comfort–Negrepontis [1974, Theorem 13.6]).

Let $H(\lambda)$ denote the set of sets hereditarily of cardinality $<\lambda$ and let $\mathfrak{H}(\lambda)$ be the structure $\langle H(\lambda), \in\rangle$ where $\in$ is the natural membership relation on $H(\lambda)$.

1.4.1 Lemma (Keisler). *For an ultrafilter F on a set I the following are equivalent:*

(i) *F is (λ, μ)-regular.*

(ii) *In the structure $\mathfrak{N} = \prod_I \mathfrak{H}(\lambda^+)/F$ there is an element $\mathfrak{b} = b/F$ where $b\colon I \to H(\lambda^+)$ is a function, such that $\mathfrak{N} \models \mathfrak{b} \subset \lambda^N$ and $\mathfrak{N} \models \mathrm{card}(\mathfrak{b}) < \mu^N$ but for every $\alpha < \lambda$ $\mathfrak{N} \models \alpha^N \in \mathfrak{b}$.*

Recall that for an ordinal $\alpha \leq \lambda$, α^N denotes the image of α under the natural embedding into $\mathfrak{N}$.

Proof. (i) → (ii) Define $b: I \to H(\lambda^+)$ by $b(t) = \{\alpha \in \lambda: t \in X_\alpha\}$ for $t \in I$ and $\{X_\alpha: \alpha \in \lambda\}$ a (λ, μ)-regular family. Now $X_\alpha = \{t \in I: \alpha \in b(t)\}$ so $\mathfrak{N} \models \alpha^N \in \mathfrak{b}$, since for each $\alpha \in \lambda$, $X_\alpha \in F$. But clearly, $b(t)$ has cardinality $<\mu$ for each $t \in I$, since $\{X_\alpha: \alpha \in \lambda\}$ is a (λ, μ)-regular family, so $\mathfrak{N} \models \operatorname{card}(b) < \mu$. Trivially, we have also $\mathfrak{N} \models b \subset \lambda^N$.

(ii) → (i) Let $\mathfrak{b} = b/F$ be the required element in $\mathfrak{N}$. Define $\mathfrak{b}'$ by $b'(t) = b(t)$ if $b(t) \subset \lambda$ and $\operatorname{card}(b(t)) < \mu$ and $b'(t) = \varnothing$ otherwise.

Obviously $\mathfrak{b}/F = \mathfrak{b}'/F$ since $\mathfrak{N} \models \mathfrak{b} \subset \lambda^N$. We want to construct a (λ, μ)-regular family. Put $X_\alpha = \{t \in I: \alpha \in b'(t)\}$ for each $\alpha \in \lambda$. Now suppose that for some $\{\alpha_i: i \in \mu\}$ the intersection $\bigcap_{i \in \mu} X_{\alpha_i} \neq \varnothing$. So there is a $t \in I$ such that for each $i \in \mu$, $\alpha_i \in b'(t)$, which contradicts the fact that $\operatorname{card}(b'(t) < \mu$. □

1.4.2 Definition. Let F_i be ultrafilters on I_i $(i = 1, 2)$. *F_2 is a projection of F_1* if there is a map $f: I_1 \to I_2$ which is onto and such that $F_1 = \{f^{-1}(X): X \in F_2\}$.

Projections are closely related to the *Rudin–Keisler order* on ultrafilters over a fixed set I, cf. Comfort–Negrepontis [1974]. We use now Lemma 1.4.1 together with *complete expansions* (i.e., complete structures over their original universe, cf. Section 1.2), to get:

1.4.3 Lemma. *If λ is regular and F_1 is (λ, λ)-regular ultrafilter on I then there is a uniform ultrafilter F_2 on λ which is a projection of F_1.*

Proof. Let $\mathfrak{N}^\#$ be the complete expansion of $\mathfrak{N} = \prod_I \mathfrak{H}(\lambda^+)$ and $b: I \to H(\lambda^+)$ as in Lemma 1.4.1 and without loss of generality $b(t) \subset \lambda$ for all $t \in I$. Now put $c(t) = \sup(b(t))$ so $c(t) \in \lambda$ since λ is regular, and $\mathfrak{N} \models \mathfrak{b} \subset \mathfrak{c}$. Clearly $c: I \to \lambda$. We define now F_2 by $F_2 = \{S \subset \lambda: \mathfrak{N}^\# = \mathfrak{c} \in \mathbf{S}\}$ where $\mathbf{S}$ is the name of S in $\mathfrak{N}^\#$. It is now easy to verify that F_2 is a uniform ultrafilter on λ which is a projection of F_1. □

To prove the abstract compactness theorem we shall prove a slightly more elaborate statement:

1.4.4 Theorem (Abstract Compactness Theorem). *Let $\mathscr{L}$ be a logic, λ, μ be cardinals and $\lambda \geq \mu$.*

(i) *The following are equivalent*:

  (a) *There is (λ, μ)-regular ultrafilter F on $I = P_{<\mu}(\lambda)$ which is in* $\mathrm{UF}(\mathscr{L})$.
  (b) *For every (relativized) expansion $\mathfrak{A}$ of $\mathfrak{H}(\lambda^+)$ there is an $\mathscr{L}$-extension $\mathfrak{B}$ and an element $b \in B$ such that $\mathfrak{B} \models \operatorname{card}(b) < \mu^B$ but for every $\alpha < \lambda$ we have $\mathfrak{B} \models \alpha^B \in b$.*
  (c) *$\mathscr{L}$ is $[\lambda, \mu]$-compact.*

(ii) *Furthermore, if λ is regular then the following are equivalent*:

  (d) *There is a uniform ultrafilter F on λ which is in* $\mathrm{UF}(\mathscr{L})$.
  (e) *$\mathscr{L}$ is $[\lambda]$-compact.*

(iii) In particular, we have:

  (f) If there is a (λ, μ)-regular ultrafilter F on any set I which is in $\mathrm{UF}(\mathscr{L})$, then $\mathscr{L}$ is $[\lambda, \mu]$-compact.

Proof. (a) → (b) Let F be a (λ, μ)-regular ultrafilter in $\mathrm{UF}(\mathscr{L})$ and let $\mathfrak{M}$ be any expansion of $\langle H(\lambda^+), \in\rangle$. Put $\mathfrak{N}_0$ to be the ultrapower $\prod_I \mathfrak{M}/F$ and $\mathfrak{N}_1$ the extension of $\mathfrak{N}_0$ as required for $F \in \mathrm{UF}(\mathscr{L})$. First we observe that $\mathfrak{N}_0 < \mathfrak{N}_1$ $(\mathscr{L}_{\omega\omega})$ and, by Lemma 1.4.1 there is an element b in $\mathfrak{N}_0$ with the required properties. But then the same element b has the same properties also in $\mathfrak{N}_1$ since $\mathfrak{N}_0 < \mathfrak{N}_1(\mathscr{L}_{\omega\omega})$. But by the definition of $\mathfrak{N}_1$, $\mathfrak{M} < \mathfrak{N}_1(\mathscr{L})$, so we are done.

(b) → (c) Let Δ, Σ be sets of $\mathscr{L}[\tau]$-sentences satisfying the hypothesis of $[\lambda, \mu]$-compactness. We define an expansion $\mathfrak{M}(\Delta, \Sigma)$ of $\langle H(\lambda^+), \in\rangle$ to apply (b). For this purpose let $\{S_\alpha: \alpha < \lambda^{<\mu}\}$ be an enumeration of all the subsets of Σ of cardinality less than μ, $\mathfrak{A}_\alpha$ be a model of $\Delta \cup S_\alpha$ and $\{c_\alpha: \alpha < P_{<\mu}(\lambda)\}$ an enumeration of all the subsets of λ of cardinality less than μ. Finally we put $\nu = (\sup_\alpha(\mathrm{card}(\mathfrak{A}_\alpha)) + \lambda^+)$, and define $\lambda_\alpha = \mathrm{card}(\mathfrak{A}_\alpha)$. We now define $\mathfrak{M}(\Delta, \Sigma)$ to be $\langle H(\nu), d_\alpha, \in, R, P\rangle_{\alpha<\lambda^+, P\in\tau}$ such that d_α is the name of $\alpha < \lambda^+$, R is a binary predicate not in τ and the domain of R is λ. We arrange it such that for each $\alpha < \lambda$ the set $R_\alpha = \{x \in H(\nu): (\alpha, x) \in R\}$ has cardinality λ_α and such that $\langle R_\alpha, P\rangle_{P\in\tau} \cong \mathfrak{A}_\alpha$. In other words we put all the models $\mathfrak{A}_\alpha$ into $\mathfrak{M}(\Delta, \Sigma)$ in way, that when we now apply (b) we shall get a model for $\Delta \cup \Sigma$. More precisely, we observe that for each formula $\phi \in \Delta$:

(1) $$\mathfrak{M}(\Delta, \Sigma) \models \mathrm{card}(b) < d_\mu \to \varphi^{R_b}$$

and for each $\beta < \lambda$ and for $\Sigma = \{\varphi_i: i < \lambda\}$ an enumeration of Σ we have

(2) $$\mathfrak{M}(\Delta, \Sigma) \models (d_\beta \in c \wedge \mathrm{card}(c) < d_\mu) \to \varphi_\beta^{R_b}.$$

Now let $\mathfrak{B}$, $b \in B$ be as in the conclusion of (b) for $\mathfrak{A} = \prod \mathfrak{M}(\Delta, \Sigma)/F$.

Claim. $\langle R_b, P\rangle_{P\in\tau} \models \Delta \cup \Sigma$.

This follows from the definition and from (1) and (2).

(c) → (a): So assume $\mathscr{L}$ is $[\lambda, \mu]$-compact but no (λ, μ)-regular ultrafilter F on $P_{<\mu}(\lambda)$ is related to $\mathscr{L}$. So for every such F there is an $\mathscr{L}_F$-structure $\mathfrak{A}_F$ exemplifying this.

We now proceed to construct an ultrafilter F_0 on λ which contradicts the choice of the $\mathfrak{A}_F$'s. For this we construct first a rich enough structure $\mathfrak{M}$ such that:

(1) for each $\mathfrak{A}_F$ there is a unary predicate P_F in $\mathfrak{M}$ with $\langle P_F, P\rangle_{P\in\tau} \cong \mathfrak{A}_F$;
(2) $\mathfrak{M}$ is a model of enough set theory to carry out the argument; and
(3) $\mathfrak{M}$ is an extension and expansion of $\langle H(\lambda^+), \in\rangle$ (or equivalently $\langle H(\lambda^+), \in\rangle$ is a relativized reduct of $\mathfrak{M}$).

Let $\mathfrak{M}^\#$ be the complete expansion of $\mathfrak{M}$ and put $\Delta = \mathrm{Th}_{\mathscr{L}}$ $(\mathfrak{M}^\#)$, the first-order theory of $\mathfrak{M}^\#$ where $\mathscr{L}^\#$ is the vocabulary of $\mathfrak{M}^\#$. Furthermore, put

$$\Sigma = \{b \subset d_\lambda \wedge \mathrm{card}(b) < d_\mu \wedge d \ \in b: \alpha < \lambda\}.$$

Clearly Δ and Σ satisfy the hypothesis of $[\lambda, \mu]$-compactness using the model $\mathfrak{M}^\#$. So $\Delta \cup \Sigma$ has a model $\mathfrak{N}$. We want to use $\mathfrak{N}$ to construct our filter F_0. First we observe that $\mathfrak{M}^\# <_{\mathscr{L}} \mathfrak{N}$. Let a_b be the interpretation of b in $\mathfrak{N}$. We define F_0 on

$P_{<\mu}(\lambda)$ by $F_0 = \{R \in P_{<\mu}(\lambda): \mathfrak{N} \models a_b \in R^{\mathfrak{N}}\}$. This makes sense, since $\mathfrak{M}^{\#}$ is a complete expansion and hence every subset of λ of cardinality $<\mu$ corresponds to a predicate in $\mathfrak{M}^{\#}$ (remember $\langle H(\lambda^+), \in\rangle$ is present in $\mathfrak{M}^{\#}$).

To complete the proof we have to verify several claims:

Claim 1. *F_0 is ultrafilter.*

Obvious.

Claim 2. *F_0 is (λ, μ)-regular.*

Let $X_\alpha = \{t \in P_{<\mu}(\lambda): \alpha \in t\}_{\alpha<\lambda}$. Now $X_\alpha \in F_0$, for say X_α corresponds to R_α then $\mathfrak{N} \models a_b \in R_\alpha$ iff $\mathfrak{N} \models d_\alpha \in a_b$, which is true for all $\alpha < \lambda$ be definition of a_b. Now $\{X_{\alpha_i}: i < \mu\}$ be a subfamily of the X_α's. Clearly, $\bigcap_{i<\mu} X_{\alpha_i} = \varnothing$, since each t in some X_α has cardinality $<\mu$.

Now consider the ultraproduct $\prod \mathfrak{M}^{\#}/F_0 = \mathfrak{N}_0$. If g is an element of $\mathfrak{N}_0$ then g is an F_0-equivalence class of functions $g: P_{<\mu}(\lambda) \to \mathfrak{M}^{\#}$ so g corresponds to a function $\mathrm{g}^{\mathfrak{M}}$ in $\mathfrak{M}^{\#}$ with name g (since $\mathfrak{M}^{\#}$ is the complete expansion) and $a_b \in \mathrm{Dom}(\mathrm{g}^{\mathfrak{N}})$. So we define an embedding $f: \mathfrak{N}_0 \to \mathfrak{N}$ by $f(g/F_0) = \mathrm{g}^{\mathfrak{N}}(a_b)$.

Claim 3. *f is well defined and* 1–1.

Let $g/F_0 = g'/F_0$. We want to show that this is equivalent to $\mathfrak{N} \models \mathrm{g}(a_c) = \mathrm{g}'(a_c)$ iff $Y = \{t \in P_{<\mu}: g(t) = g'(t)\} \in F_0$. But the latter is true iff $a_b \in Y^{\mathfrak{N}}$ which is equivalent to $\mathrm{g}(a_b) = \mathrm{g}'(a_b)$.

So we have shown that f is an embedding of $\mathfrak{N}_0$ into $\mathfrak{N}$.

Now let $\bar{g} = \{g_i/F_0: i < \alpha\}$ be in $\mathfrak{N}_0$.

Claim 4. *For every $\mathscr{L}$-formula φ we have*

$$\mathfrak{N} \models \varphi(\bar{g}) \quad \textit{iff} \quad Y = \{t \in P_{<\mu}: \mathfrak{M} \models \varphi(g_1(t), g_2(t), \ldots)\} \in F_0.$$

Clear, since $Y \in F_0$ iff $Y^{\mathfrak{N}}$ contains a_c iff $\mathfrak{N} \models \phi(\mathrm{g}_1(a_c), \mathrm{g}_2(a_c), \ldots)$.

Now look at $\mathfrak{A}_{F_0}$. By assumption there is no $\mathfrak{N}'$ extending $\prod \mathfrak{A}_{F_0}/F_0$ satisfying Claim 4. But $\langle P^{\mathfrak{N}}_{F_0}, P\rangle_{P \in \tau_{F_0}}$ is such an $\mathfrak{N}'$ by construction. This completes the proof of (i).

(d) $\to$ (e) This follows from the above, since uniform ultrafilters on λ are $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$-regular and λ is a regular cardinal by our hypothesis.

(e) $\to$ (d) Here we use Lemma 1.4.3 and (a) $\to$ (c). This completes the proof of (ii).

To prove (f) we just observe that in the proof of (a) $\to$ (c) we did not use that $I = P_{<\mu}(\lambda)$. This completes the proof of Theorem 1.3.9. $\square$

1.5. The Compactness Spectrum

In this section we study the structure of the *compactness spectrum* Comp($\mathscr{L}$) and the *regular compactness spectrum* RComp($\mathscr{L}$) defined below.

1.5.1 Definition. For a logic $\mathscr{L}$ we define Comp($\mathscr{L}$), (RComp($\mathscr{L}$)) to be the class of all (regular) cardinals such that $\mathscr{L}$ is $[\kappa]$-compact.

1.5.2 Theorem. *The first cardinal λ_0 in* Comp($\mathscr{L}$) *is measurable* (*or* ω).

Proof. By Theorem 1.2.2(i) each regular $\lambda < \lambda_0$ is cofinally characterizable in $\mathscr{L}$ via a structure $\mathfrak{B}(\lambda)$ with κ_λ the cardinality of $\mathfrak{B}(\lambda)$. Let μ be defined by

$$\mu = (\sup\{\kappa_\lambda : \lambda < \lambda_0\}) + \lambda_0^+$$

and let $\mathfrak{B}$ be the complete expansion of the structure $\langle \mu, \varepsilon \rangle$. Therefore (*) in every $\mathscr{L}$-extension of $\mathfrak{B}$ all the ordinals smaller than λ_0 are standard. By $[\lambda_0]$-compactness $\mathfrak{B}$ has an $\mathscr{L}$-elementary extension $\mathfrak{C}$ with some $c \in C - B$ and such that $\mathfrak{C} \models c \in \lambda_0^B$. Since λ_0 is minimal we have for no $\lambda < \lambda_0$ that $\mathfrak{C} \models c \in \lambda^C$. We now define an ultrafilter F on λ_0 by

$$F = \{X \subset \lambda_0 : \mathfrak{C} \models c \in \mathbf{X}\},$$

where $\mathbf{X}$ is the name of the set X in $\mathfrak{B}$. Clearly F is an ultrafilter. We propose to show that F is λ_0-complete.

Let $\{X_\alpha : \alpha < \mu < \lambda_0\}$ be any family in F. The function f with $f(\alpha) = X_\alpha$ is a function in $\mathfrak{B}$ with name, say, $\mathbf{f}$. Put now $X = \bigcap_{\alpha<\mu} X_\alpha$. So $\mathfrak{B} \models \mathbf{X} = \bigcap_{\alpha<\mu} \mathbf{X}_\alpha$ and therefore

$$\mathfrak{B} \models \forall x(\forall i(i < \alpha \to x \in \mathbf{f}(i)) \to x \in \bigcap_{i<\alpha} \mathbf{f}(i)).$$

But by (*) the ordinals $\alpha < \lambda_0$ in $\mathfrak{B}$ are the same as in $\mathfrak{C}$. So $\mathfrak{C} \models c \in \mathbf{X}$ since $\mathbf{f}$ is a function of $\mathfrak{C}$ with $\mathbf{f}^C \upharpoonright B = \mathbf{f}^B$. So $X \in F$ and therefore λ_0 is measurable. $\square$

1.5.3 Example. If κ is a strongly compact cardinal, the logic $\mathscr{L}_{\kappa\kappa}$ is (∞, κ)-compact and therefore $[\kappa]$-compact. But the logic $\mathscr{L}_{\kappa\kappa}$ is not $[\lambda]$-compact for any $\lambda < \kappa$.

Note that, as a corollary, we get that strongly compact cardinals are measurable. By Magidor [1976] it is consistent that the first measurable and the first strongly compact cardinal coincide.

Our next aim is to study the structure of Comp($\mathscr{L}$). The main theorem here is

1.5.4 Theorem. *For every cardinal λ and every logic $\mathscr{L}$, $\lambda^+ \in$ Comp($\mathscr{L}$) implies $\lambda \in$ Comp($\mathscr{L}$).*

Proof. Use the abstract compactness theorem 1.4.4 and Lemma 1.3.11. $\square$

For λ regular this was first proved in Makowsky–Shelah [1979] giving a direct proof by relating $[\lambda]$-compactness to descendingly incomplete ultrafilters. The general result was proved in Makowsky–Shelah [1983]. There the connection with ultrafilters was first recognized, on which the presentation here is based.

The next result concerns the structure of Comp($\mathscr{L}$). The following was proven in Makowsky–Shelah [1979, Lemma 6.4(ii)] by an extension of the argument for Theorem 1.5.2.

1.5.5 Lemma. *Let $\lambda > \mu$ be two regular cardinals and $\mathscr{L}$ be a logic such that $\lambda \in \mathrm{Comp}(\mathscr{L})$ but $\mu \notin \mathrm{Comp}(\mathscr{L})$. Then there is a uniform μ-descendingly complete ultrafilter on λ.*

Consider the following assumption $A(\lambda)$, where λ is an uncountable cardinal.

> $A(\lambda)$: "if $\mathfrak{F}$ is a uniform ultrafilter on λ, then $\mathfrak{F}$ is μ-descendingly incomplete for every $\mu \leq \lambda$."
> We denote by $A(\infty)$ the statement "for every infinite cardinal λ, $A(\lambda)$ holds."

Donder–Jensen–Koppelberg [1981] and Magidor [198?] have studied this assumption. The following theorem summarizes their results (with part (v) being Theorem 8.36 in Comfort–Negrepontis [1974], see also Lemma 1.3.11).

1.5.6 Theorem. (i) (Jensen–Koppelberg). *Assume $\neg O^{\#}$. Then for every regular cardinal λ we have $A(\lambda)$.*

(ii) (Donder). *Assume there is no inner model of* ZFC *with an uncountable measurable cardinal. Then $A(\infty)$ holds.*

(iii) *If $A(\infty)$ holds then there are no uncountable measurable cardinals.*

(iv)* (Woodin). *Assume there are uncountable measurable cardinals. Then it is consistent with* ZFC *that $A(\omega_\omega)$ fails.*
However, in ZFC *we already have:*

(v) (Kunen–Prikry and Cudnovskii–Cudnovskii). *For every $n \in \omega$, $A(\omega_n)$ holds.*

Magidor has informed us of the yet unpublished result of Theorem 1.5.6(iv) of Woodin. He had previously proved a similar result, where one has to replace the existence of an uncountable measurable cardinal in the hypothesis by the existence of a supercompact cardinal.

The assumption $A(\infty)$ is intimately connected with compactness properties: It implies that Comp($\mathscr{L}$) has no gaps. On the other hand, the existence of strongly compact cardinals allows us to construct logics where Comp($\mathscr{L}$) does have gaps. More precisely:

1.5.7 Theorem. (i) *Assume $A(\infty)$ holds. Then* Comp($\mathscr{L}$) *is an initial segment of the cardinals, i.e., $\lambda \in \mathrm{Comp}(\mathscr{L})$ and $\mu < \lambda$ implies that $\mu \in \mathrm{Comp}(\mathscr{L})$.*

(ii)* (Shelah). *Let $\mu_1 < \mu_2$ be two uncountable strongly compact cardinals. Then there is a logic $\mathscr{L}$ which is $[\kappa]$-compact iff $\kappa < \mu_1$ or $\kappa \geq \mu_2$.*

Proof. (i) Assume Comp(L) $\neq \varnothing$. Since $A(\infty)$ implies that there are no uncountable measurable cardinals, by Theorem 1.5.6(iii), the first cardinal in Comp($\mathscr{L}$) is

ω, by Theorem 1.5.2. Now, if $\omega < \lambda \in \mathrm{Comp}(\mathscr{L})$ and $\omega < \mu < \lambda$, $\mu \notin \mathrm{Comp}(\mathscr{L})$, μ regular, we apply Lemma 1.5.5 and get a contradiction to $A(\infty)$. If μ is singular, we apply Lemma 1.5.5 to $\mathrm{cf}(\mu)$ and then use Lemma 1.1.6(v).

(ii) will follow from Proposition 1.6.7. ▯

The question which remains, is whether $\mathrm{Comp}(\mathscr{L})$ is empty or not. Now clearly the logic $\mathscr{L}_{\infty\omega}$ is not compact in any sense, so $\mathrm{Comp}(\mathscr{L}_{\infty\omega})$ is empty. But if we assume that the logic $\mathscr{L}$ is bounded in some sense and have some very strong assumption on the existence of large cardinals we can get more specific results. For terminology and results on large cardinals we refer to Jech [1978].

1.5.8 Definition. A logic is *set presentable* if:

(i) there is a cardinal κ such that whenever a vocabulary $\tau \in H(\kappa)$ and $\Sigma \subset \mathscr{L}[\tau]$ has cardinality $<\kappa$ then $\Sigma \subset H(\kappa)$; and

(ii) for every $\varphi \in \mathscr{L}[\tau]$ $\mathrm{Mod}(\varphi)$ is a set-theoretically definable class of τ-structures.

(Recall that $H(\kappa)$ is the family of sets hereditarily of cardinality $<\kappa$.)

1.5.9 Example. Let $\mathscr{L} = \mathscr{L}_\kappa^n$ be like nth-order logic except that we allow conjunctions and disjunctions of less than κ many formulas. Clearly $\mathscr{L}$ is set presentable and so is every sublogic of it.

1.5.10 Definition. Let $\mathrm{SComp}(\mathscr{L})$ be the class of cardinals κ such that $\mathscr{L}$ is (∞, κ)-compact and $\mathrm{WComp}(\mathscr{L})$ be the class of cardinals κ such that $\mathscr{L}$ is (κ, κ)-compact. Clearly we have $\mathrm{SComp}(\mathscr{L}) \subset \mathrm{Comp}(\mathscr{L}) \subset \mathrm{WComp}(\mathscr{L})$.

1.5.11 Proposition (Magidor [1971]). *If κ is an extendible cardinal then $\kappa \in \mathrm{SComp}(\mathscr{L}_\kappa^n)$.*

1.5.12 Definition. The following statement is called *Vopenka's principle*:
Let $\mathbf{C}$ be a proper class of τ-structures for some finite vocabulary τ. Then there are two structures $\mathfrak{A}, \mathfrak{B} \in \mathbf{C}$ such that $\mathfrak{A}$ is (first-order) elementary embeddable into $\mathfrak{B}$.

Now Magidor [1971] also shows

1.5.13 Proposition. *If Vopenka's principle holds then the class of all extendible cardinals is closed unbounded.*

So Propositions 1.5.11 and 1.5.13 give us immediately:

1.5.14 Theorem (Magidor–Stavi). *Assume Vopenka's principle holds and that $\mathscr{L}$ is a set presentable logic. Then* $\mathrm{SComp}(\mathscr{L})$ *is a non-empty final segment of the cardinals (in other words, $\mathscr{L}$ is ultimately compact).*

For $\mathrm{WComp}(\mathscr{L})$ we do not need Vopenka's principle to prove an analogue of Theorem 1.5.14.

1.5.15 Theorem (Stavi [1978]). *Let μ be a uncountable measurable cardinal and F be a normal ultrafilter on μ and $\mathscr{L}$ be a sublogic of $\mathscr{L}^n_\mu$. Then* $\mathrm{WComp}(\mathscr{L}) \cap \mu \in F$.

Theorem 1.5.15 holds under much weaker assumptions (cf. Stavi [1978, Section 5]) and is also discussed and proved in Chapter XVII, Section 4.2.

The structure of $\mathrm{Comp}(\mathscr{L})$ definitely deserves further investigation. We combine the content of Lemma 1.1.6(v), and Theorems 1.5.2, 1.5.4, and 1.5.14 into the statement:

1.5.16 Theorem. *For a logic $\mathscr{L}$ we have:*

(i) $\mathrm{cf}(\kappa) \in \mathrm{Comp}(\mathscr{L}) \to \kappa \in \mathrm{Comp}(\mathscr{L})$.
(ii) $\kappa^+ \in \mathrm{Comp}(\mathscr{L}) \to \kappa \in \mathrm{Comp}(\mathscr{L})$.
(iii) *The first cardinal in* $\mathrm{Comp}(\mathscr{L})$ *is measurable (or ω).*
(iv) *If $\mathscr{L}$ is set presentable and Vopenka's principle holds, then* $\mathrm{Comp}(\mathscr{L})$ *contains a final segment of the class of all cardinals.*

Our last theorem illustrates that Vopenka's principle is the right large cardinal assumption in this context.

1.5.17 Theorem* (Makowsky). *The following are equivalent:*

(i) *Vopenka's principle.*
(ii) *For every logic $\mathscr{L}$* $\mathrm{SComp}(\mathscr{L}) \neq \varnothing$.
(iii) *For every finitely generated logic $\mathscr{L}$* $\mathrm{SComp}(\mathscr{L}) \neq \varnothing$.
(iv) *For every finitely generated logic $\mathscr{L}$* $\mathrm{Comp}(\mathscr{L}) \neq \varnothing$.

Proof. (i) $\to$ (ii) follows from Proposition 1.2.6. So we only have to prove (iv) $\to$ (i). Let $\mathbf{C}$ be a proper class of τ-structures and let Q_C be the Lindstrom quantifier defined by $\mathbf{C}$ and $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_C)$. Clearly $\mathbf{C}$ contains a proper subclass $\mathbf{C}_0$ of the form $\mathbf{C}_0 = \mathrm{Mod}(T)$ where T is a complete $\mathscr{L}[\tau]$-theory. Assume that $\kappa \in \mathrm{Comp}(\mathscr{L})$ and let $\mathfrak{A} \in \mathbf{C}_0$ be of cardinality $\geq \kappa$. Using $[\kappa]$-compactness we now find $\mathfrak{B} \models T$ which is an (first-order) elementary extension of $\mathfrak{A}$ and clearly $\mathfrak{B} \in \mathbf{C}$. ▯

1.6. Gaps in the Compactness Spectrum

In this section we want to study a family of examples of logics with gaps in the compactness spectrum. These examples will also be used in the subsequent sections to illustrate various phenomena concerning dependence numbers and amalgamation properties (see Example 2.2.5 and Section 3.5).

1.6.1 Example. Let κ be a cardinal and F be an ultrafilter on κ. We define a logic $\mathscr{L} = \mathscr{L}_{F\omega}$ by adding to first-order logic $\mathscr{L}_{\omega\omega}$ the following formation rule: If $\{\varphi_i : i < \kappa\}$ is an indexed family of $\mathscr{L}$-sentences, then $\bigcap_F \{\varphi_i : i \leq \kappa\}$ is an $\mathscr{L}$-sentence. We additionally assume that $\mathscr{L}$-formulas have $<\omega$ many free variables. Satisfaction for $\mathscr{L}$ is defined by the additional clause: If $\mathfrak{A}$ is a τ-structure then $\mathfrak{A} \models \bigcap_F \{\varphi_i : i \leq \kappa\}$ iff $\{i \leq \kappa : \mathfrak{A} \models \varphi_i\} \in F$.

1.6.2 Proposition. *Let μ be a measurable cardinal and F be a μ-complete non-principal ultrafilter on μ.*

(i) $\mathscr{L}_{F\omega} < \mathscr{L}_{(2^\mu)^+\omega}$.
(ii) *$\mathscr{L}_{F\omega}$ is not $[\mu]$-compact.*
(iii) *$\mathscr{L}_{F\omega}$ is $[\lambda]$-compact for every $\lambda < \mu$.*

Proof. (i) and (ii) are left to the reader. To prove (iii) we make use of the abstract compactness theorem (1.3.9) and we show that every ultrafilter D on λ is in $\mathrm{UF}(\mathscr{L})$. Let us spell this out precisely:

1.6.3 Lemma. *Let $\mathscr{L} = \mathscr{L}_{F\omega}$ and D be any ultrafilter on $\lambda < \mu$. Furthermore let $\{\mathfrak{A}_i: i < \lambda\}$ be a family of τ-structures, $\varphi \in \mathscr{L}[\tau]$ and $\{\mathbf{f}_j: j \leq \nu < \mu\}$ be a family of functions in $\prod_{i\in\lambda}\mathfrak{A}_i$. Then the following are equivalent:*

(i) $\prod_{i\in\lambda}\mathfrak{A}_i/D \models \varphi(\mathbf{f}_1\mathbf{f}_2, \ldots, \mathbf{f}_j, \ldots)_{j\leq\nu}$.
(ii) $X_\varphi = \{i \in \lambda: \mathfrak{A}_i \models \varphi(\mathbf{f}_1(i), \mathbf{f}_2(i), \ldots, \mathbf{f}_j(i), \ldots)_{j\leq\nu}\} \in D$.

Proof. Like Los' theorem for first-order logic. □

Example 1.6.1 can be still further extended:

1.6.4 Example*. Let $\mu_1 < \mu_2$ with μ_1 measurable and μ_2 strongly compact. Let $\mathfrak{F}$ be a μ_1-complete non-principal ultrafilter on μ_1. We define the logic $\mathscr{L}_{F,\mu_2}$ as above, but we allow existential quantification over sequences of variables $\{x_j: j < \alpha < \mu_2\}$.

1.6.5 Proposition* (Shelah). (i) $\mathscr{L}_{F,\mu_2} < \mathscr{L}_{\mu_2,\mu_2}$.
(ii) *The logic $\mathscr{L}_{F,\mu_2}$ is $[\kappa]$-compact for every $\kappa < \mu_1$ and $\kappa \geq \mu_2$.*

Proof. (i) Clearly, the operation $\bigcap_F$ can be expressed by conjunctions and disjunctions in $\mathscr{L}_{\mu_2,\mu_2}$, since μ_2 is a strong limit cardinal and $\mu_1 < \mu_2$.

(ii) For $\kappa < \mu_1$ this is similar to Lemma 1.6.3 and for $\kappa \geq \mu_2$ this follows from (i) and the fact that μ_2 is strongly compact. □

Clearly, in Proposition 1.6.5, $[\mu_1]$-compactness fails. But it is not clear, whether for any κ with $\mu_1 < \kappa < \mu_2$, we have $[\kappa]$-compactness. However, we can construct a more refined example:

1.6.6 Example* (Shelah). Let $D(\mu_1, \mu_2)$ be the set of μ_1-complete ultrafilter F on some set $I \subset \mu_2$ such that $\mu_1 \leq \mathrm{card}(I) < \mu_2$. Instead of allowing $\bigcap_F$ for one ultrafilter we can now form a logic $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2}$ as follows: We close first-order logic $\mathscr{L}_{\omega_1\omega}$ under all the operations $\bigcap_F$ for $F \in D(\mu_1, \mu_2)$ as in the previous example. Additionally we close under existential quantification over strictly less than μ_2 many individual variables.

The next proposition is proved exactly as Proposition 1.6.2.

1.6.7 Proposition* (Shelah). *Let μ_1 be measurable and μ_2 be a strongly compact cardinal bigger than μ_1. Then:*

(i) $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2} < \mathscr{L}_{\mu_2,\mu_2}$,
(ii) $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2}$ *is $[\kappa]$-compact for every $\kappa < \mu_1$ and $\kappa \geq \mu_2$; and*
(iii) $\mathscr{L}_{D(\mu_1,\mu_2),\mu_2}$ *is not $[\kappa]$-compact for any κ with $\mu_1 \leq \kappa < \mu_2$.*

This also establishes Theorem 1.5.7(ii). Using the same type of examples we can actually find logics with a compactness spectrum containing various gaps. How far we can go with this, is described in the following theorem:

1.6.8 Theorem*. (i) *Assume there are arbitrarily large measurable cardinals. Then there is a $[\omega]$-compact logic $\mathscr{L}$ such that both* Comp($\mathscr{L}$) *and its complement are confinal in the class of all cardinals.*
(ii) *Assume there are arbitrarily large strongly compact cardinals. Then there is a $[\omega]$-compact logic $\mathscr{L}$ such that both* Comp($\mathscr{L}$) *and its complement are cofinal in the class of all cardinals and consist of intervals whose length is a strongly compact cardinal.*

Proof. Combine Examples 1.6.1 and 1.6.4, respectively. □

Note however, that for set-presentable logics $\mathscr{L}$, Vopenka's principle (Theorem 1.5.14) implies that Comp($\mathscr{L}$) is a final segment of all cardinals.

2. *The Dependence Number*

2.1. *Introduction*

In this section we develop further an idea mentioned briefly in Chapter II, Section 5.1, namely the meaning of the assertion that a formula $\varphi \in \mathscr{L}[\tau]$ depends only on a subset $\sigma \subset \tau$. We present the material of this section for one-sorted logics only. We leave it to the reader to adopt the definitions and results to the many-sorted case. Let us recall a definition:

2.1.1 Proposition. Let $\mathscr{L}$ be a logic and $\varphi \in \mathscr{L}[\tau]$.

(i) φ *depends* (*only*) on (the symbols in) σ, $\sigma \subset \tau$ if for all τ-structures $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A} \upharpoonright \sigma \cong \mathfrak{B} \upharpoonright \sigma$ we have $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.
(ii) A logic $\mathscr{L}$ is *weakly regular, if $\mathscr{L}$ satisfies the basic closure properties* (1.2.1) and the relativization property (1.2.2) of Chapter II.

The difference between weakly regular and regular is the absence of the substitution property (1.2.3) of Chapter II.

If $\varphi \in \mathscr{L}[\tau]$ does only depend on $\sigma \subset \tau$, one would generally expect, that there is a $\psi \in \mathscr{L}[\sigma]$ which is equivalent to φ. If this is the case, we say that the logic

$\mathscr{L}$ is *occurrence normal*. However, our definition of a weakly regular logic does not imply this. Nevertheless we have:

2.1.2 Proposition*. *For every weakly regular logic $\mathscr{L}$ there is a logic $\mathscr{L}_1$ such that:*

(i) *$\mathscr{L} \equiv \mathscr{L}_1$; and*

(ii) *if $\varphi \in \mathscr{L}_1[\tau]$, $\sigma \subset \tau$ and φ depends only on σ, then there is a $\psi \in \mathscr{L}_1[\sigma]$ such that for every σ-structure $\mathfrak{A}$, $\mathfrak{A} \models \psi$ iff every expansion of $\mathfrak{A}$ to a τ-structure $\mathfrak{A}_1$, $\mathfrak{A}_1 \models \varphi$.*

Proof. We just add new atomic formulas and consider them as being of the required vocabulary. □

Regular logics are closed under substitutions of formulas for atomic predicate letters. For one-sorted logics there is no problem in stating this directly, for many-sorted logics we have to be a bit careful about the sorts. Gaifman pointed out that the definition of a regular logic ensures that $\mathscr{L}_1$ actually is $\mathscr{L}$.

2.1.3 Proposition*. *Every regular logic is occurrence normal.*

Proof. One-sorted case: Assume φ, σ, and τ as in the definition of occurrence normal above. To construct ψ we first make use of the eliminability of function symbols (which follows from the substitution property, Definition II.1.2.3) and assume that $\tau - \sigma$ contains only relation symbols. Next we construct for every predicate symbol $R \in \tau - \sigma$ a formula of first-order logic ϑ_R with equality only and with free variables according to the specifications of the one-sorted arity of R. We now obtain ψ by substituting ϑ_R for every occurrence of R in φ, using the substitution property again. Note that we do not need the relativization property here.

In the case of many-sorted logics, the definition of the substitution property (1.2.3) from Chapter II has to be modified. There is no difficulty in doing this so that it implies occurrence normality. We leave this as an exercise to the reader. □

In the light of Propositions 2.1.2 and 2.1.3 we can restrict ourselves for the rest of this chapter to occurrence normal or regular logics. For such logics we can define the concept of a *dependence number* in a semantical way. In Chapter II (after Definition 1.2.3) a syntactic concept of occurrence property was introduced.

2.1.4 Definition. (i) Given a regular logic $\mathscr{L}$, we define a cardinal $o(\mathscr{L}) = \kappa$ to be the smallest cardinal such that every formula $\varphi \in \mathscr{L}[\tau]$ depends only on some subset $\tau_0 \subset \tau$ with $\operatorname{card}(\tau_0) < \kappa$. If no such κ exists we write $o(\mathscr{L}) = \infty$. If $o(\mathscr{L}) = \omega$ we also say that $\mathscr{L}$ has *finite dependence* or has the *finite dependence property*.

(ii) Given a regular logic $\mathscr{L}$, we define a cardinal $OC(\mathscr{L}) = \kappa$ to be the smallest cardinal such that for every formula $\varphi \in \mathscr{L}[\tau]$ there is $\sigma \subset \tau$ with $\operatorname{card}(\sigma) < OC(\mathscr{L})$ and $\varphi \in L(\sigma)$.

In Chapter II (Definition 6.1.3) the *finite* occurrence property was introduced, which is the syntactic counterpart of our finite dependence property. In our terminology the finite occurrence property is equivalent to $\mathrm{OC}(\mathscr{L}) = \omega$. Using Proposition 2.1.3 one easily sees that every logic $\mathscr{L}$ which has the finite dependence property, contains a sublogic $\mathscr{L}_0$ equivalent to it which has the occurrence property in the syntactic sense. In fact, more generally we have:

2.1.5 Proposition*. *Let $\mathscr{L}$ be a regular logic with dependence number* $\mathrm{o}(\mathscr{L})$. *Then there is a regular logic $\mathscr{L}_1$ with* $\mathrm{OC}(\mathscr{L}) = \mathrm{o}(\mathscr{L})$ *which is equivalent to $\mathscr{L}$.*

Proof. Similar to Proposition 2.1.2. □

The above proposition shows that up to equivalence of logics, the occurrence number and the dependence number coincide. In Makowsky–Shelah [1983] the dependence number is, indeed, called occurrence number. The change in terminology was motivated by the requirements of Chapter II and by the notion of the *dependence structure*, introduced in Section 2.4.

2.1.6 Examples. (i) In Chapter II, Proposition 5.1.3 shows that for a (κ, λ)-compact logic with $\mathrm{o}(\mathscr{L}) \leq \kappa$ we actually have $\mathrm{o}(\mathscr{L}) \leq \lambda$. This fact was first pointed out in H. Friedman [1970].
(ii) Let us look at the logic $\mathscr{L}_{F\omega}$ defined in Example 1.6.1. Obviously $\mathrm{o}(\mathscr{L}) \leq \kappa^+$. But if $\varphi \in \mathscr{L}[\tau]$, $\mathrm{card}(\tau) = \kappa$ then there is no smallest $\tau_0 \subset \tau$ such that φ depends exactly on the symbols in τ_0.

2.1.7 Substitutes for the Dependence Number. The dependence number is a concept which keeps the size of a logic limited. Other assumptions in this direction are:

(i) For every vocabulary τ with τ *a set* $\mathscr{L}[\tau]$ is also a set. We call such logics *small*. In Section 4.3 this concept will be used.
(ii) For every vocabulary τ, if τ is a set, $\mathrm{card}(\mathscr{L}[\tau]) = \mathrm{card}(\tau) + \kappa$ for some fixed cardinal κ. This gives us a special case of a *size function*, as defined in Section 4.3. There we also look at *tiny* logics, i.e., logics $\mathscr{L}$ such that whenever $\mathrm{card}(\tau)$ is smaller than the first uncountable measurable cardinal μ_0, then $\mathrm{card}(\mathscr{L}[\tau])$ is also smaller than μ_0.
(iii) The presence of a Lowenheim number $l_\kappa(\mathscr{L})$, as introduced in Section II.6.2.

For various theorems in abstract model theory such limiting assumptions are needed, as we shall see in the further course of this and the next chapter. Note that from the above properties (ii) → (i) and in the presence of an dependence number (iii) → (ii), up to equivalence of logics. In fact, we have the following:

2.1.8 Proposition. *Let $\mathscr{L}$ be a logic with* $\mathrm{o}(\mathscr{L}) = \mu$ *and* $l_1(\mathscr{L}) = \kappa$ *and τ be a vocabulary with* $\mathrm{card}(\tau) = \lambda$ *and* $\mu \leq \lambda \leq \kappa$. *Then there are, up to logical equivalence, only* $\leq 2^{2^\kappa}$ *many τ-sentences.*

The proof consists of a crude counting argument. Note that we do not get the stronger conclusion $\operatorname{card}(\mathscr{L}[\tau]) \leq 2^{2^\kappa}$, since there may be many equivalent formulas.

One would actually expect that if $l_1(\mathscr{L}) = \kappa$ then $o(\mathscr{L}) \leq \kappa^+$ and one might add this to the definition of the Lowenheim number, but it is an open field to determine which model-theoretic properties have what impact on the size of the dependence numbers. The only exception is compactness and the rest of Section 2 is devoted to this.

2.2. *Compactness and Dependence Numbers*

This section is devoted to the statement of the finite dependence theorem and the discussion of several examples. The proof of the finite dependence theorem is discussed in the following section but for a technically complete exposition of the proof we refer the reader to Makowsky–Shelah [1983].

To simplify the statements of the following theorem and its corollaries, we denote by $\bar{\mu}$ the first uncountable measurable cardinal, if there is one, and ∞ otherwise. We stipulate further that if $\bar{\mu} = \infty$, then $\bar{\mu}^+ = \infty$.

2.2.1 Theorem (Finite Dependence Theorem). (i) (Global version). *Let $\mathscr{L}$ be a regular, $[\omega]$-compact logic with dependence number $o(\mathscr{L}) < \bar{\mu}$. Then $\mathscr{L}$ has the finite dependence property, i.e., $o(\mathscr{L}) = \omega$.*

(ii) (Local version). *Let $\mathscr{L}$ be a regular, $[\omega]$-compact logic, τ a vocabulary and $\varphi \in \mathscr{L}[\tau]$ a formula which depends only on some $\tau_0 \subset \tau$ with $\operatorname{card}(\tau_0)$ less than the first uncountable measurable cardinal. Then there is a finite $\tau_1 \subset \tau_0$ such that φ depends only on τ_1.*

Clearly, (ii) implies (i). The proof of (ii) is presented in Section 2.3.

2.2.2 Corollary. *Let $\mathscr{L}$ be a regular, $[\kappa]$-compact logic, $\kappa < \bar{\mu}$ and $o(\mathscr{L}) \leq \bar{\mu}^+$. Then $\mathscr{L}$ has the finite dependence property.*

Proof of Corollary. By Theorem 1.5.2 $\mathscr{L}$ is $[\omega]$-compact, so we can apply the finite dependence theorem. ▯

As a second corollary we get a representation theorem of some compact logics via Lindstrom quantifiers (cf. Section II.4). Let us recall a definition:

2.2.3 Definition. A logic $\mathscr{L}$ is a *Lindstrom logic* if $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_i)_{i \in I}$ for some indexed set of Lindstrom quantifiers Q_i $(i \in I)$. $\mathscr{L}$ is *finitely generated* if $\mathscr{L}$ is a Lindstrom logic and $\operatorname{card}(I) < \omega$.

Note that by Theorem 4.1.3 of Chapter II every regular logic $\mathscr{L}$ which has the (syntactic) finite occurrence property is a Lindstrom logic.

2.2.4 Proposition*. (i) *Let $\mathscr{L}$ be a regular logic with* $\mathrm{o}(\mathscr{L}) = \omega$. *Then $\mathscr{L}$ is equivalent to a Lindstrom logic.*

(ii) *If a regular logic $\mathscr{L}$ is small, $[\kappa]$-compact and $\mathrm{o}(\mathscr{L}) \leq \kappa < \bar{\mu}^+$ then $\mathscr{L}$ is equivalent to a Lindstrom logic.*

Proof. Using Corollary 2.2.2 we can reduce (ii) to (i). So assume that $\mathscr{L}$ has finite dependence. Let τ be a finite vocabulary. We want to replace every $\varphi \in \mathscr{L}[\tau]$, which is not equivalent to a first-order formula, by a formula consisting of a new quantifier Q_φ applied to a sequence of atomic formulas. The problem is to keep the number of quantifiers so introduced small. But the type of the quantifier does not really depend on the vocabulary τ, but only on the similarity type, i.e., on the number and arities of the symbols τ. Now there is a countable universal vocabulary τ_∞ such that for every finite τ there is $\tau' \subset \tau_\infty$ which is of the same similarity type as τ. Therefore, every $\varphi \in \mathscr{L}[\tau]$ can be obtained from some $\psi \in \mathscr{L}[\tau_\infty]$ by an application of substitution. By our assumption, $\mathscr{L}[\tau_\infty]$ is a set. So writing every formula in $\mathscr{L}[\tau_\infty]$ as a Lindstrom quantifier, we complete the proof. □

Both the theorem and the corollaries have assumptions involving measurable cardinals. In the sequel we shall discuss examples which show, that these assumptions are necessary.

2.2.5 Examples. (i) Let μ be a strongly compact cardinal. So $\mathscr{L} = \mathscr{L}_{\mu\mu}$ is $[\mu]$-compact and $\mathrm{o}(\mathscr{L}) = \mu$. As noted before, it is consistent that the first strongly compact and the first measurable cardinal coincide, by Magidor [1976]. This shows that the assumption on measurable cardinals cannot be dropped in the corollaries.

(ii) Let μ be a measurable cardinal and F be a μ-complete non-principal ultrafilter on μ. We look again at the logic $\mathscr{L} = \mathscr{L}_{F\omega}$ from Example 1.6.1. By Proposition 1.6.2 this logic is $[\omega]$-compact, but clearly its dependence number is μ^+. This shows that the assumption on the measurable cardinal cannot be dropped in the finite dependence theorem.

2.3. *Proof of the Finite Dependence Theorem*

The proof of the finite dependence theorem uses three lemmas (Lemmas A, B, C). We do not prove these lemmas here and refer the reader to [Makowsky–Shelah [1983]. Instead, we present the three lemmas without proofs and show how the finite dependence theorem is proved from them. The reader will gain a rather transparent picture of the structure of the proof.

Let us fix a $[\lambda]$-compact logic $\mathscr{L}$, a vocabulary τ and a sentence $\varphi \in \mathscr{L}[\tau]$. We want to study subsets of τ on which φ does not depend. Each lemma introduces a new aspect of the notions involved: Lemma A uses compactness to construct a dummy subset of τ. Lemma B builds a function on the power set of τ which is used to apply Lemma C, which makes us conclude that card(τ) was measurable.

Lemma A′ is an improvement of Theorem 5.1.2 in Chapter II, and its proof is very similar.

2.3.1 Lemma A′. (i) *For every* $\tau_1 \subset \tau$ *with* $\mathrm{card}(\tau_1) \leq \lambda$ *there is a* $\tau_0 \subset \tau_1$ *with* $\mathrm{card}(\tau_0) < \lambda$ *such that* φ *does not depend on* $\tau_1 - \tau_0$.

(ii) *There is a* $\mu < \lambda$ *such that for every* $\tau_1 \subset \tau$ *with* $\mathrm{card}(\tau_1) \leq \lambda$ *there is a* $\tau_0 \subset \tau_1$ *with* $\mathrm{card}(\tau_0) \leq \mu$ *such that* φ *does not depend on* $\tau_1 - \tau_0$.

Now Lemma A′ can be used to prove Lemma A.

2.3.2 Lemma A. There is a $\tau_1 \subset \tau$ *with* $\mathrm{card}(\tau_1) < \lambda$ *such that for every* $\tau_0 \subset \tau - \tau_1$ *with* $\mathrm{card}(\tau_0) \leq \lambda$ *does not depend on* τ_0.

The second lemma used in the proof of the finite dependence theorem gives us the connection to ultrafilters. Here we use some material from Section 1.3, in particular, the definition of $\mathrm{UF}(\mathscr{L})$.

2.3.3 Lemma B. *Let* μ *be a cardinal,* $\mathscr{L}$ *be a logic and* φ *a* $\mathscr{L}[\tau]$*-sentence. If* $\tau_2 \subset \tau$ *but for each* $\tau_1 \subset \tau_2$ *with* $\mathrm{card}(\tau_1) \leq \lambda$, φ *does not depend on* τ_1, *then there is a function* $f: P(\tau_2) \to \{0, 1\}$ *such that*:

(i) f *is non-constant.*
(ii) *For every* $\sigma_1, \sigma_2 \subset \tau_1$ *with* $\mathrm{card}(\sigma_1 \Delta \sigma_2) \leq \lambda$ *we have* $f(\sigma_1) = f(\sigma_2)$.
(iii) *For every ultrafilter* $F \in \mathrm{UF}(\mathscr{L})$ *(on* μ*)* f *is* F*-continuous.*

Recall that if F is an ultrafilter on μ, $\{\sigma_i : i < \mu\}$, σ are subsets of τ_2 then $\lim_F \sigma_i = \sigma$ iff for every $P \in \tau_2$ the set $I_P = \{i \in \mu : P \in \sigma_i \leftrightarrow P \in \sigma\} \in F$ and f is F-continuous iff $\sigma = \lim_F \sigma_i$ implies that $f(\sigma) = \lim_F f(\sigma_i)$.

The third lemma, used in the proof of the finite dependence theorem, gives us the connection to measurable cardinals:

2.3.4 Lemma C. *If* F *is a uniform ultrafilter on* ω *and* $f: P(k) \to \{0, 1\}$ *satisfies* (i)–(iii) *of the previous lemma, then there is a measurable cardinal* μ_0 *such that* $\omega < \mu_0 \leq \kappa$.

We are now in a position to prove the finite dependence theorem.

Proof of the Finite Dependence Theorem. Assume $\mathscr{L}$ is $[\omega]$-compact and $\mathrm{o}(\mathscr{L}) > \omega$. Then there is an $\mathscr{L}[\tau]$-sentence φ which does not depend only on a finite subset of τ. So $\mathrm{card}(\tau) \geq \omega$, and if $\mathrm{card}(\tau) = \omega$ we are done by Theorem 5.1.2 of Chapter II. So $\mathrm{card}(\tau) > \omega$. By Lemma A (for $\lambda = \omega$) we can assume that φ does not depend on any countable subset of τ. Now we apply Lemma B to construct the function f and by the abstract compactness theorem (1.3.9) and Lemma A we know that f is F-continuous for some uniform ultrafilter on ω. So by Lemma C we know that $\mathrm{card}(\tau) \geq \mu_0$, the first uncountable measurable cardinal. But this shows that $\mathrm{o}(\mathscr{L}) \geq \mu_0$, a contradiction. □

2.4. Dependence Filters

So far we have studied the concept of a formula depending on some subset of a vocabulary τ, and our main result was the finite dependence theorem. However, as the examples in Section 1.6 and their discussion in Examples 2.1.6 show, this need not be the appropriate notion. We are facing here a similar problem as in the analysis of compactness properties. There it turned out that the more appropriate tool to study compactness is the class of ultrafilters UF($\mathscr{L}$). Similarly here, we have to look at *dependence filters.*

2.4.1 Definition. Let τ be an infinite vocabulary and assume, for notational simplicity, that $\tau = \{R_i: i < \lambda\}$, where R_i are relation symbols. Let $\varphi \in \mathscr{L}[\tau]$ be a formula of some logic $\mathscr{L}$. If $X \subset \lambda$ we write τ_X for $\{R_i: i \in X\}$.

(i) Let F be an ultrafilter on λ. We say that φ *depends on F only*, if, given two τ-structures $\mathfrak{A} = \langle A, R_i^A\rangle_{i<\lambda}$ and $\mathfrak{B} = \langle B, R_i^B\rangle_{i<\lambda}$, and a set $X \in F$ such that $\mathfrak{A} \upharpoonright \tau_X \cong \mathfrak{B} \upharpoonright \tau_X$ then $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. We call F an *dependence filter for* φ.

(ii) Let $Y_0 \cup Y_1 \cup \cdots \cup Y_n$ be a finite partition of λ and F_k $(k = 0, 1, \ldots, n)$ be ultrafilters on Y_k, respectively. We say that φ *depends on* $F_0, F_1, \ldots, F_n$ *only*, if, given two τ-structures $\mathfrak{A} = \langle A, R_i^A\rangle_{i<\lambda}$ and $\mathfrak{B} = \langle B, R_i^B\rangle_{i<\lambda}$, and sets $X_k \in F_k$ such that $\mathfrak{A} \upharpoonright \tau_X \cong \mathfrak{B} \upharpoonright \tau_X$, where $X = \bigcup_0^n X_i$, then $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. We call $F_0, F_1, \ldots, F_n$ a *finite dependence structure for* φ.

(iii) We can modify (ii) to allow infinite partitions. In this case we speak of *dependence structures for* φ.

2.4.2 Examples. (i) If a logic $\mathscr{L}$ has finite dependence, $\varphi \in \mathscr{L}[\tau]$, then φ has a principal dependence filter generated by the finite set $\tau_0 \subset \tau$ on which φ only depends.

(ii) Let us return to the logic $\mathscr{L}_{F\omega}$ from Example 2.1.6(ii), introduced in Example 1.6.1 Recall that F is an ultrafilter on some set I. Let R_i, $i \in I$ be relation symbols. The formula $\bigcap_F \{R_i: i \in I\}$ has among its dependence filters also the ultrafilter F. However, if $\tau = \{R_i: i \in I\} \cup \{S_i: i \in I\}$ then the dependences of the formula $\bigcap_F \{R_i: i \in I\} \wedge \bigcap_F \{R_i: i \in I\}$ has to be described by a finite partition of τ and a filter on each of the components, which in this case is F.

(iii) If we look at Example 1.6.5 it is easy to construct examples of sentences whose dependence is described by more complicated partitions and more complicated ultrafilters.

That those examples are more than accidental is shown by the following theorem from the treasure box (Shelah [198?e]).

2.4.3 Theorem* (Shelah's Finite Dependence Structure Theorem). *Let $\mathscr{L}$ be a $[\omega]$-compact logic, $\tau = \{R_i: i < \lambda\}$ a vocabulary and $\varphi \in \mathscr{L}[\tau]$. Then there is a finite partition $Y_0 \cup Y_1 \cup \cdots \cup Y_n$ of λ and countably complete ultrafilters F_k $(k = 0, 1, \ldots, n)$ on Y_k, respectively, such that φ only depends on $F_0, F_1, \ldots, F_n$. In other words, every $\varphi \in \mathscr{L}[\tau]$ has finite dependence structure.*

The proof of the finite dependence structure theorem consists of elaborations of the Lemmas A, B, and C in Section 2.3. The finite dependence structure theorem opens new perspectives in the study of dependence phenomena for compact logics for the case that there are uncountable measurable cardinals.

3. $\mathscr{L}$-Extensions and Amalgamation

3.1. Basics

Given a logic $\mathscr{L}$, it is clear how to define the analogue of elementary equivalence of two structures of the same language τ: They have to satisfy the same τ-sentences. It is more problematic to generalize the notion of elementary embeddings, because already in the first-order case either free variables or new constant symbols are used in the definition and various definitions are equivalent only because of the finite occurrence (finite dependence) or even because of compactness. In the general case it is convenient to introduce a cardinal parameter.

Let us recall that the $\mathscr{L}$-diagram of an τ-structure $\mathfrak{A}$ is the set of $\mathscr{L}$ sentences true in the structure $\langle \mathfrak{A}, A\rangle$, i.e., the structure $\mathfrak{A}$ augmented with names for all its elements. We denote the $\mathscr{L}$-diagram of $\mathfrak{A}$ by $D_{\mathscr{L}}(\mathfrak{A})$.

3.1.1 Definitions. (i) A τ-structure $\mathfrak{B}$ is an $\mathscr{L}$-extension of a τ-structure $\mathfrak{A}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the two structures $\langle \mathfrak{A}, A\rangle$ and $\langle \mathfrak{B}, A\rangle$ satisfy the same $\mathscr{L}$-sentences. In this case we write $\mathfrak{A} <_{\mathscr{L}} \mathfrak{B}$.

(ii) A τ-structure $\mathfrak{B}$ is a $(\kappa, \mathscr{L})$-extension of a τ-structure $\mathfrak{A}$, if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for every subset $A_0 \models A$ with $\mathrm{card}(A_0) < \kappa$ the two structures $\langle \mathfrak{A}, A_0\rangle$ and $\langle \mathfrak{B}, A_0\rangle$ are $\mathscr{L}$-equivalent. In this case we write $\mathfrak{A} <_{\mathscr{L}}^{\kappa} \mathfrak{B}$.

3.1.2 Examples. (i) For $\mathscr{L} = \mathscr{L}_{\infty\omega}$ without occurrence restrictions we have clearly $\mathfrak{A} <_{\mathscr{L}} \mathfrak{B}$ iff $\mathfrak{A} = \mathfrak{B}$. Using indiscernibles, it is easy to construct $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} <_{\mathscr{L}_{\infty\omega}}^{\kappa} \mathfrak{B}$ for a given κ.

(ii) If $\mathrm{o}(\mathscr{L}) = \kappa$ then clearly every $(\kappa, \mathscr{L})$-extension is an $\mathscr{L}$-extension.

(iii) If $\mathscr{L}$ is a compact logic, then we have, by the finite dependence theorem of the previous section, that $\mathscr{L}$-extensions and $(\kappa, \mathscr{L})$-extensions coincide for every κ.

In model-theory extensions are studied extensively and the following three situations are characteristic:

(i) Do models have $(\kappa, \mathscr{L})$-extensions?

(ii) Given a chain of extensions, is the union an extension of each member of the chain?

(iii) Given three τ-structures $\mathfrak{A}_i$, $i = 0, 1, 2$ such that $\mathfrak{A}_0$ is an $\mathscr{L}$-substructure of both $\mathfrak{A}_1$ and $\mathfrak{A}_2$, does there exists an amalgamating extension $\mathfrak{A}_3$?

In fact, in Chapter XX we shall describe an approach to abstract model theory, which is entirely based on those aspects and not on the notion of formulas and logics. Here, however, we shall study logics which allow these constructions.

In this chapter we shall deal with logics which allow one of the above constructions (i)–(iii) universally.

3.1.3 Definition. (i) A logic $\mathscr{L}$ satisfies $\mathrm{EXT}(\mathscr{L})$ or has the *extension property*, if every infinite τ-structure $\mathfrak{A}$ has an $\mathscr{L}$-extension $\mathfrak{B}$.

(ii) A logic $\mathscr{L}$ satisfies REXT($\mathscr{L}$) or has the *relativized extension property*, if for every infinite definable set X in some τ-structure $\mathfrak{A}$ there is a τ-structure $\mathfrak{B}$ which is a $\mathscr{L}$-extension of $\mathfrak{A}$ which extends X properly.

Clearly, REXT($\mathscr{L}$) implies EXT($\mathscr{L}$) for every logic $\mathscr{L}$.

3.1.4 Example. Every compact logic $\mathscr{L}$ satisfies REXT($\mathscr{L}$).

In fact, the following proposition is easily proved by the reader:

3.1.5 Proposition. *If a logic $\mathscr{L}$ is $[\omega]$-compact then $\mathscr{L}$ satisfies* REXT($\mathscr{L}$).

We shall return to the study of EXT and REXT in Section 3.2.

3.1.6 Definitions. (i) A family of τ-structures $\mathfrak{A}_i$, $i < \kappa$ is an $\mathscr{L}$-chain if $\mathfrak{A}_i$ is an $\mathscr{L}$-extension of $\mathfrak{A}_j$ for every $j < i < \kappa$.

(ii) A logic $\mathscr{L}$ satisfies CHAIN(κ, $\mathscr{L}$) or *respects* chains of length κ, if given a $\mathscr{L}$-chain $\mathfrak{A}_i$, $i < \kappa$ then $\bigcup_{i<\kappa} \mathfrak{A}_i$ is an $\mathscr{L}$-extension of each of the $\mathfrak{A}_i$'s.

(iii) A logic $\mathscr{L}$ satisfies CHAIN($\mathscr{L}$) or has the *chain property*, if it satisfies CHAIN(κ, $\mathscr{L}$) for every κ.

3.1.7 Remark. CHAIN(ω, $\mathscr{L}$) was called in Chapter III the Tarski-union-property.

3.1.8 Examples. (i) $\mathscr{L}_{\kappa\omega}$ has the chain property.

(ii) If κ is regular than $\mathscr{L}_{\kappa\kappa}$ respects chains of length λ, $\mathrm{cf}(\lambda) > \kappa$.

In Chapter III (Theorem 2.2.2) the following result of Lindström [1973] was proved:

3.1.9 Theorem (Lindström). *If a logic $\mathscr{L}$ is compact and respects chains of length ω then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*

There are no logics known which are $[\omega]$-compact and satisfy CHAIN($\mathscr{L}$). It is open whether this is due to a theorem or simple ignorance of more examples. It would be interesting to explore more consequences of CHAIN-properties. In Tharp [1974] and Makowsky [1975] "continuous" or "securable" quantifiers are studied, which, if added to first-order logic, give us logics which do satisfy CHAIN($\mathscr{L}$). In Lindström [1973a, 1983] a variation of Theorem 3.1.9 is studied involving only (λ, ω)-compactness and a modification of the Tarski-union-property.

3.1.10 Definitions. (i) A logic $\mathscr{L}$ satisfies Am(κ, $\mathscr{L}$) or has the *κ-amalgamation property* if, given three τ-structures $\mathfrak{A}_i$, $i = 0, 1, 2$ such that $\mathfrak{A}_0 <^{\kappa}_{\mathscr{L}} \mathfrak{A}_j$, $j = 1, 2$ there is a τ-structure $\mathfrak{B}$ such that $\mathfrak{A}_i <^{\kappa}_{\mathscr{L}} \mathfrak{B}$, $i = 0, 1, 2$ and the diagram commutes.

(ii) A logic $\mathscr{L}$ satisfies Am($\mathscr{L}$) or has the *amalgamation property*, if Am(κ, $\mathscr{L}$) holds for every κ.

(iii) A logic $\mathscr{L}$ satisfies JEP($\mathscr{L}$) or has the *joint embedding property* if any two $\mathscr{L}$-equivalent τ-structures $\mathfrak{A}_i$, $i = 1, 2$ have a common $\mathscr{L}$-extension $\mathfrak{B}$.

One can also introduce cardinal parameters for $\mathscr{L}$-equivalence and the joint embedding property, but we shall not need this in our exposition.

3.1.11 Theorem. (i) *Every compact logic $\mathscr{L}$ has the joint embedding property.*
(ii) *If a logic $\mathscr{L}$ satisfies* JEP($\mathscr{L}$) *then it has the amalgamation property.*

Proof. (i) Since $\mathscr{L}$ is compact, $\mathscr{L}$ has finite dependence, by the finite dependence theorem. So we can use compactness again to show that $D_L(\mathfrak{A}_1) \cup D_L(\mathfrak{A}_2)$ has a model $\mathfrak{B}$ which is a $(\kappa, \mathscr{L})$-extension of both the $\mathfrak{A}_i$, $i = 1, 2$.

(ii) Let $\mathfrak{A}_i$, $i = 0, 1, 2$ be as in the hypothesis of the amalgamation property. Clearly the two structures $\langle \mathfrak{A}_1, A_0 \rangle$, $\langle \mathfrak{A}_2, A_0 \rangle$ are $\mathscr{L}$-equivalent, so let $\mathfrak{B}$ be an $\mathscr{L}$-extension of both of them. Clearly this $\mathfrak{B}$ satisfies the requirements of the amalgamation property. $\square$

3.1.12 Examples. (i) If κ is a strongly compact cardinal, then $\mathscr{L}_{\kappa\kappa}$ satisfies the joint embedding property.
(ii) Let $\mathscr{L} = \mathscr{L}_{\infty\omega}$, but with finite occurrence. It is easy to see that $\mathscr{L}$ does not satisfy the amalgamation property.

3.1.13 Definition. A logic $\mathscr{L}$ has the *Robinson property* if whenever $\Sigma_i \subset \mathscr{L}[\tau_i]$, $i = 0, 1, 2$ are such that $\tau_0 = \tau_1 \cap \tau_2$ and Σ_0 is complete and $\Sigma_0 \cup \Sigma_j, j = 1, 2$ has a model, then $\bigcup_{i=0}^{i=2} \Sigma_i$ has a model. Recall that a set of sentences Σ is *complete* if any two models of Σ are $\mathscr{L}$-equivalent.

D. Mundici has studied various aspects of the Robinson property, cf. Mundici [1981d, 1981c]. The Robinson property is extensively discussed in Chapter XIX. Here we only note the following theorem:

3.1.14 Theorem. *Every logic $\mathscr{L}$, which has the Robinson property also has the amalgamation property.*

Proof. Let $\Sigma_i = D_{\mathscr{L}}(\mathfrak{A}_i)$ where the $\mathfrak{A}_i$ are as in the hypothesis of the amalgamation property. $\square$

The amalgamation property is further studied in Sections 3.3 and 3.4.

Let us summarize here some rather unexpected consequences of the amalgamation property, as they follow from Theorem 3.2.1 and the abstract amalgamation theorem (3.3.1).

3.1.15 Theorem. *Let $\mathscr{L}$ be a regular logic with occurrence (dependence) number less than the first uncountable measurable cardinal.*

(i) *If $\mathscr{L}$ has the amalgamation property, then* REXT($\mathscr{L}$) *holds.*
(ii) *If* CHAIN(ω, $\mathscr{L}$) *holds and $\mathscr{L}$ has the amalgamation property then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*

This theorem stresses the connections between the more "algebraic" properties of logics, as they are at the core of Chapter XX. In our context the theorem is trivial. But then, the reader may try to prove (i) directly. The same challenge applies to Corollary 3.3.4.

3.2. $\mathscr{L}$-Extensions

In this section we prove a converse of Proposition 3.1.5 and explore further variations of extension properties.

3.2.1 Theorem. *A regular logic $\mathscr{L}$ satisfies* REXT($\mathscr{L}$) *iff $\mathscr{L}$ is $[\omega]$-compact.*

Proof. Assume REXT($\mathscr{L}$) and that $\mathscr{L}$ is not $[\omega]$-compact. So by Theorem 1.2.2 (or Chapter II, Proposition 5.2.4) ω is cofinally characterizable in $\mathscr{L}$ by some expansion $\mathfrak{A}$ of $\langle \kappa, < \rangle$. But clearly ω^A is a maximal definable subset of $\mathfrak{A}$, a contradiction. The other direction was Proposition 3.1.5. $\square$

We next introduce a cardinal parameter into our extension properties:

3.2.2 Definition. A logic $\mathscr{L}$ satisfies EXT(κ, $\mathscr{L}$) if, whenever a τ-structure $\mathfrak{A}$ has no proper $\mathscr{L}$-extension then card($\mathfrak{A}$) $< \kappa$.

3.2.3 Proposition. *If a logic $\mathscr{L}$ is $[\lambda]$-compact then $\mathscr{L}$ satisfies* EXT(λ, $\mathscr{L}$).

The proof is left to the reader.

The next theorem is one of the least constructive theorems in logic: Its proof uses the replacement axiom very heavily. To test our assertion the reader should try to prove Theorem 3.2.4 below in ZC rather than in ZFC. (This problem was suggested by A. Dodd.)

3.2.4 Theorem. *Let λ_0 be an infinite cardinal and $\mathscr{L}$ satisfies* EXT(λ_0, $\mathscr{L}$) *then there is a cardinal κ such that $\mathscr{L}$ is $[\kappa]$-compact.*

Proof. We prove the contraposition: If $\mathscr{L}$ is not $[\kappa]$-compact for any cardinal κ then for every cardinal λ_0 there is a maximal structure $\mathfrak{B}$ with card($\mathfrak{B}$) $\geq \lambda_0$. (Recall that a structure is maximal for $\mathscr{L}$ if it has no proper $\mathscr{L}$-extensions.)

By Theorem 1.2.2 every regular cardinal λ is cofinally characterizable via some expansion $\mathfrak{B}_\lambda$ which we assume without loss of generality of minimal cardinality $g(\lambda)$.

Now let μ be the first cardinal such that:

(i) If $\nu < \mu$ then $g(\nu) \leq \mu$.
(ii) $\lambda_0 \leq \mu$.
(iii) $\mathrm{cf}(\mu) = \omega$.

Clearly such a cardinal exist, e.g., the ω-limit of the first fixed points of the function $g(\nu)$. (This is where the replacement axiom is used without control over the complexity of the set-theoretic formula involved.)

Let $\mathfrak{B}$ be the complete expansion of the structure $\langle \mu, \in \rangle$. We claim that $\mathfrak{B}$ is maximal. For otherwise, let $\mathfrak{C}$ be an $\mathscr{L}$-extension of $\mathfrak{B}$. If $\mathfrak{C}$ is proper there is a $c \in C - B$. Remember $\mathrm{cf}(\mu) = \omega$ and let $\{b_n : n \in \omega\}$ be a cofinal sequence in $\mathfrak{B}$. Since ω is cofinally characterizable in $\mathscr{L}$ via $\mathfrak{B}$, $g(\omega) \leq \mu$ and $\mathfrak{B}$ is a complete structure, $\{b_n : n \in \omega\}$ is also cofinal in $\mathfrak{C}$. So clearly, $\mathfrak{C} \models c \in b_k$ for some $k \in \omega$. Now let $d \in B$ be the smallest (with respect to $\in$) element in $\mathfrak{B}$ such that $\mathfrak{C} \models c \in d$. We note that d is an ordinal. Let $\delta = \mathrm{cf}(d)$ and $\{d_i : i < \delta\}$ be a sequence cofinal to d in $\mathfrak{B}$. Again, since $g(\delta) \leq \mu$ and δ is cofinally characterizable in $\mathscr{L}$ via $\mathfrak{B}$ $\{d_i : i < \delta\}$ is cofinal to d in $\mathfrak{C}$. So there is a $j < \delta$ with $\mathfrak{C} \models c \in d_j$, which contradicts the minimality of d. This establishes that $\mathfrak{B}$ is maximal. Clearly, $\mathrm{card}(\mathfrak{B}) > \lambda_0$ by our construction, which completes the proof. □

If there are no uncontable measurable cardinals, we get the following situation:

3.2.5 Theorem. *Assume there are no uncountable measurable cardinals and $\mathscr{L}$ is a regular logic. Then the following are equivalent:*

(i) *$\mathscr{L}$ is $[\omega]$-compact.*
(ii) *$\mathscr{L}$ satisfies* $\mathrm{EXT}(\mathscr{L})$.
(iii) *$\mathscr{L}$ satisfies* $\mathrm{REXT}(\mathscr{L})$.

Proof. (i) → (iii) was Proposition 3.1.5 and (iii) → (ii) follows from the definitions. To prove (ii) → (i) we apply Theorem 3.2.4 and then Theorem 1.5.2. □

Also the existence of uncountable measurable cardinals is closely related to our extension properties. Let us look at the following example:

3.2.6 Example. A logic $\mathscr{L}$ for which $\mathrm{EXT}(\mathscr{L})$ and $\mathrm{REXT}(\mathscr{L})$ do not coincide. Let $Q_{\lambda\kappa}$ be a quantifier of type $\langle 1, 1 \rangle$ with satisfaction defined by

$$\mathfrak{A} \models Q_{\lambda\kappa} xy(\varphi(x), \psi(y)) \text{ iff } \mathrm{card}(\varphi^A) < \lambda \quad \text{and} \quad \mathrm{card}(\psi^A) > \kappa$$

3.2.7 Lemma. *Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\omega 2^{\mu_0}})$ where μ_0 is the first uncountable measurable cardinal.*

(i) *$\mathscr{L}$ is $[\mu_0]$-compact.*
(ii) *$\mathscr{L}$ satisfies* $\mathrm{EXT}(\mathscr{L})$.
(iii) *$\mathscr{L}$ does not satisfy* $\mathrm{REXT}(\mathscr{L})$ *and therefore is not $[\omega]$-compact.*

Proof. We prove (iii) first. For this we look at the structure $\mathfrak{A} = \langle (2^{\mu_0})^+, \in \rangle$. It is straightforward to find an expansion $\mathfrak{A}_1$ of $\mathfrak{A}$ in which $\langle \omega, \in \rangle$ is cofinally characterized in $\mathscr{L}$, so we apply Theorem 3.2.1 together with Theorem 1.2.2.

To prove (ii) we distinguish two cases: On structures $\mathfrak{A}$ with $\mathrm{card}(\mathfrak{A}) \leq 2^{\mu_0}$ $\mathscr{L}$ is equivalent to first-order logic, since the quantifier Q acts trivially, being always false, so first-order extensions will do. On structures $\mathfrak{A}$ with $\mathrm{card}(\mathfrak{A}) > 2^{\mu_0}$ we apply (i).

To prove (i) we use the abstract compactness theorem (1.3.9) and show that every μ_0-complete ultrafilter F on μ_0 is in UF($\mathscr{L}$). We need μ_0-completeness to see that finiteness is preserved under ultrapowers over F and we need that μ_0 is small for $(2^{\mu_0})^+$ to see that the other cardinality restriction is preserved under such ultrapowers. □

This example together with Theorem 3.2.5 gives us immediately the following characterization of the existence of uncountable measurable cardinals.

3.2.8 Theorem. *The following are equivalent:*

(i) *For every logic $\mathscr{L}$* EXT($\mathscr{L}$) *holds iff* REXT($\mathscr{L}$) *holds.*
(ii) *There are no uncountable measurable cardinals.*

Finally, let us have a look at Hanf numbers. We shall draw some corollaries from results in the previous sections, giving links between existence of some new type of Hanf numbers and various forms of compactness. The existence of this new Hanf number for every finitely generated logic is, as it turns out, equivalent to Vopenka's principle. Let us first recall some definitions from Section II.6:

3.2.9 Definitions. Let $\mathscr{L}$ be a logic.

(i) Let $\Phi \subset \mathscr{L}[\tau]$ be a set of sentences and λ be a cardinal. Φ *pins down the cardinal* λ, iff Φ has a model of cardinality λ, but Φ has no models of arbitrary large cardinalities.
(ii) We define a function $h_\kappa(\mathscr{L})$ to be the supremum of all cardinals that can be pinned down by a set of $\mathscr{L}$-sentences of power $\leq \kappa$. $h_1(\mathscr{L}) = h(\mathscr{L})$ from Section II.6.
(iii) We define $h_\infty(\mathscr{L})$ to be the supremum of all $h_\kappa(\mathscr{L})$ if it exists, and otherwise we write $h_\infty(\mathscr{L}) = \infty$. We say that $\mathscr{L}$ *has a global Hanf number*, if $h_\infty(\mathscr{L}) < \infty$.

Global Hanf numbers do not necessarily exist, even for finitely generated logics. Clearly compact logics do have global Hanf number ω. The following clarifies the relationship between compactness and global Hanf numbers:

3.2.10 Proposition* (Makowsky). *Let $\mathscr{L}$ be a logic.*

(i) *If $\mathscr{L}$ is (∞, λ)-compact then $h_\infty(\mathscr{L}) \leq \lambda$.*
(ii) *If $\mathscr{L}$ is $[\omega]$-compact and* CHAIN($\mathscr{L}$) *holds, then $h_\infty(\mathscr{L}) = \omega$.*
(iii) *If $\mathscr{L}$ has a global Hanf number, then* Comp($\mathscr{L}$) $\neq \varnothing$.

Proof. (i) This is a standard application of the method of diagrams.

(ii) Using Proposition 3.1.5 we construct an $\mathscr{L}$-chain of proper $\mathscr{L}$-extensions. Now CHAIN($\mathscr{L}$) allows us to go as far as we want.

(iii) Let λ_0 be the global Hanf number of $\mathscr{L}$. Clearly, every structure of cardinality $\geq \lambda_0$ has a proper $\mathscr{L}$-extension, i.e., EXT(λ_0, $\mathscr{L}$) holds, so the result follows from Theorem 3.2.4. □

3.2.11 Corollary* (Makowsky). *Assume there are no uncountable measurable cardinals. If $\mathscr{L}$ is a logic which has a global Hanf number then $\mathscr{L}$ has finite dependence.*

Proof. By Proposition 3.2.10, $\mathrm{Comp}(\mathscr{L}) \neq \emptyset$, so by Theorem 1.5.2 and the assumption on measurable cardinals, $\mathscr{L}$ is $[\omega]$-compact. Now we apply the finite dependence theorem (2.2.1). ☐

The following is an improvement of Theorem 1.5.17.

3.2.12 Theorem* (Makowsky). *The following statements are equivalent:*

(i) *For every finitely generated logic $\mathscr{L}$* $\mathrm{SComp}(\mathscr{L}) \neq \emptyset$.
(ii) *Every finitely generated logic $\mathscr{L}$ has a global Hanf number.*
(iii) *For every finitely generated logic $\mathscr{L}$* $\mathrm{Comp}(\mathscr{L}) \neq \emptyset$.
(iv) *Vopenka's principle.*

Proof. (i) → (ii) This follows from Proposition 3.2.10(i) above.
(ii) → (iii) This follows from Proposition 3.2.10(iii) above.
(iii) → (iv) and (iv) → (i) both follow from Theorem 1.5.17. ☐

Theorem 3.2.12 tells us that there are logics which have no global Hanf number provided Vopenka's principle is false. Let us end this section with some examples:

3.2.13 Examples. (i) Let $\mathscr{L}$ be $\mathscr{L}_{\omega_1\omega}$. Let $\mathfrak{A}$ be a complete expansion of a structure of cardinality λ. If there are no uncountable measurable cardinals, $\mathfrak{A}$ has no proper $\mathscr{L}$-extensions (see Theorem 1.2.3), so the complete $\mathscr{L}$-theory of $\mathfrak{A}$ pins down λ. Hence, assuming there are no uncountable measurable cardinals, $\mathscr{L}$ has no global Hanf number.

(ii) Let $\mathscr{L}_0$ be the logic $\mathscr{L}_{\omega\omega}(Q_0)$ and $\mathscr{L}_1$ be $\mathscr{L}_{\omega\omega}(Q_1)$. In Malitz–Reinhardt [1972b] it is shown that $h_\omega(\mathscr{L}_i)$ $(i = 0, 1)$ is bigger than the first uncountable measurable cardinal.

(iii) Let $\mathscr{L}$ be $\mathscr{L}^2_{\omega\omega}$, i.e., second-order logic. By Magidor [1971] $h_\infty(\mathscr{L})$ is smaller than the first extendible cardinal.

(iv) In Corollary XVII.4.5.12 it is shown that $h_1(\Delta_3(\mathscr{L}_A))$ is bigger than the first extendible cardinal.

3.3. The Amalgamation Property

In this section we present our main theorem in the analysis of the amalgamation properties:

3.3.1 Theorem (Abstract Amalgamation Theorem). *Let $\mathscr{L}$ be a logic with dependence number $\mathrm{o}(\mathscr{L}) = \lambda$ and with the amalgamation property. Then $\mathscr{L}$ is ultimately compact. In fact it is $[\infty, \lambda]$-compact.*

The proof of this theorem will be outlined in Section 3.5. Here we mainly illustrate various consequences of this theorem and discuss examples and limitations.

For logics with finite dependence we immediately get:

3.3.2 Theorem. *For a logic $\mathscr{L}$ with finite dependence the following are equivalent:*

(i) *$\mathscr{L}$ is compact.*
(ii) *$\mathscr{L}$ has the amalgamation property.*
(iii) *$\mathscr{L}$ has the joint embedding property.*

Proof. We have seen in Theorem 3.1.11 that (i) implies (ii) and (iii), and that (iii) implies (ii). So let us assume (ii). From Theorem 3.3.1 we get immediately that $\mathscr{L}$ is $[\lambda]$-compact for every regular λ and therefore compact by Theorem 1.1.8 □

D. Mundici has studied the joint embedding property extensively, cf. Mundici [1982b, 1983a]. In general the joint embedding property is not known to be equivalent to the *amalgamation property.* In Chapter XIX some consequences of the joint embedding property are studied. Using more of the set-theoretic machinery we get

3.3.3 Theorem. *If $\mathscr{L}$ is a logic with $o(\mathscr{L}) < \mu_0$, where μ_0 is the first uncountable measurable cardinal, then the following are equivalent:*

(i) *$\mathscr{L}$ is compact.*
(ii) *$\mathscr{L}$ has the amalgamation property.*
(iii) *$\mathscr{L}$ has the joint embedding property.*

Proof. We only have to prove (ii) $\rightarrow$ (i): Using Theorem 3.3.1 we get $[\kappa]$-compactness for some $\kappa < \mu_0$, so by Theorem 1.5.2 we get $[\omega]$-compactness and therefore by Theorem 2.2.1, finite dependence. So now the results follows by another application of Theorem 3.3.1. □

3.3.4 Corollary. *Let $\mathscr{L}$ be a logic with $o(\mathscr{L}) < \mu_0$, where μ_0 is the first uncountable measurable cardinal.*

(i) *If $\mathscr{L}$ has the amalgamation property (joint embedding property) then every sublogic $\mathscr{L}_0 < \mathscr{L}$ has the amalgamation property (joint embedding property).*
(ii) *If $\mathscr{L}$ has the amalgamation property (joint embedding property) then $\Delta(\mathscr{L})$ also has the amalgamation property (joint embedding property).*

Proof. (i) This is clearly true for compactness, so by Theorem 3.3.3 also for the amalgamation property.

(ii) It is easy to see, that the Δ-closure of logics preserves compactness and finite dependence. □

The reader may try to prove this without using Theorem 3.3.3.

3.3.5 Corollary. *Let $\mathscr{L}$ be a logic with $o(\mathscr{L}) < \mu_0$, where μ_0 is the first uncountable measurable cardinal. If $\mathscr{L}$ has the Robinson property, then $\mathscr{L}$ is compact.*

Proof. Use Theorem 3.1.14 and Theorem 3.3.3. □

For logics with finite dependence we shall see in Chapter XIX another proof of Corollary 3.3.5 without using Theorem 3.3.1.

The rest of this section is devoted to examples and applications of the above theorems. The first example gives a real application of Theorem 3.3.2 for the following result was originally derived from it:

3.3.6 Example. Let $\mathscr{L}_{\omega\omega}(Q_\kappa)$ be the first-order logic with the additional quantifier "there exist at least κ many." Theorem 3.3.2 gives us immediately that this logic does not satisfy the amalgamation property for any cardinal κ. For $\kappa = \omega$ or ω_1 this was shown by Malitz–Reinhardt [1972b], the other cases were open till Theorem 3.3.2 was proven.

The next examples all show that the assumption on large cardinals cannot be dropped in any of the above statements.

3.3.7 Examples. (i) The logic $\mathscr{L}_{\infty\infty}$ has no occurrence number. Since this logic can describe any structure up to isomorphism, one easily verifies that the Robinson property and the amalgamation property hold trivially, but $\mathscr{L}_{\infty\infty}$ has no compactness whatsoever.

(ii) In Makowsky–Shelah [1983] it is shown that if κ is an extendible cardinal, then $\mathscr{L}^2_{\kappa\kappa}$, i.e., second-order logic with conjunctions, first-order and second-order quantification over $<\kappa$ many formulas or variables, satisfies the Robinson property, and hence the Amalgamation property and is $[\infty, \kappa]$-compact. Clearly, $\mathrm{o}(\mathscr{L}^2_{\kappa\kappa}) = \kappa$ and $\mathscr{L}^2_{\kappa\kappa}$ is not $[\lambda]$-compact for any $\lambda < \kappa$.

(iii) Now let us look at $\mathscr{L}_{\lambda\omega}$ with additionally the finite dependence property. It is easy to see, that for $\lambda > \omega$ the amalgamation property fails. But $\mathscr{L}_{\lambda\omega} < \mathscr{L}^2_{\kappa\kappa}$ for $\lambda < \kappa$, so Corollary 3.3.4 cannot be improved.

(iv) The logic $\mathscr{L}_{\infty\omega}$ satisfies the amalgamation property trivially, but does not satisfy the Robinson property, as pointed out in Makowsky–Shelah [1979].

(v) In Section 3.5 we present a $[\omega]$-compact logic $\mathscr{L}$ which has the amalgamation property, but for which $\mathrm{Comp}(\mathscr{L})$ has a large gap. This example presupposes the existence of strongly compact cardinals.

3.4. *Proof of the Abstract Amalgamation Theorem*

3.4.1 Synopsis. We first observe that by Theorem 1.1.9 it suffices to prove the following weaker theorem:

3.4.2 Theorem. *Let λ be a regular cardinal and $\mathscr{L}$ be a logic with dependence number $\mathrm{o}(\mathscr{L}) \le \lambda$ and with the amalgamation property. Then $\mathscr{L}$ is $[\lambda]$-compact.*

We give first an outline of the proof, to help the reader. We assume for contradiction that λ is regular and $\mathscr{L}$ is not $[\lambda]$-compact. Using Theorem 1.2.2 we construct a class K of linear orderings with additional predicates in which points of confinality λ are absolute. Inside K we show the existence of some sufficiently homogeneous structure $\mathfrak{N}$. In $\mathfrak{N}$ we shall find $\mathfrak{M}_i$ $(i = 0, 1, 2)$ being a counterexample to the amalgamation property for $\mathscr{L}$. The dependence number and the isomorphism axiom will be needed to show that $\mathfrak{M}_0 <_L \mathfrak{M}_i$ $(i = 1, 2)$ and the absoluteness of "cofinality λ" to show that there is no amalgamating structure.

The counterexample to amalgamation is patterned after the following example: Let K be the class of dense linear orderings with an additional unary predicate Red such that both Red and its complement are dense. Let $\mathfrak{A} <_K \mathfrak{B}$ hold if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ and the universe of $\mathfrak{A}$ is a dense subset of the universe of $\mathfrak{B}$. We shall show that K with this notion of substructure $<_K$ does not allow amalgamation: For this let $\mathfrak{A}_0$ be the rationals properly coloured, and let $\mathfrak{A}_i$ $(i = 1, 2)$ the rationals augmented by one element (say π) coloured Red in $\mathfrak{A}_1$ and not coloured in $\mathfrak{A}_2$. Clearly, $\mathfrak{A}_0 <_K \mathfrak{A}_i$ $(i = 1, 2)$, but no amalgamating structure exists, since otherwise π is simultaneously coloured and not coloured.

3.4.3 The Structure $\mathfrak{M}$. Now, let $\lambda \geq \mathrm{OC}(\mathscr{L})$ be regular and $\mathscr{L}$ not $[\lambda]$-compact. By Theorem 1.2.2, λ is cofinally characterizable in $\mathscr{L}$ in a structure $\mathfrak{M}$. We need some more information on $\mathfrak{M}$:

Let Δ, $\Sigma_1 = \{\varphi_\alpha : \alpha < \lambda\}$ be the counterexample to $[\lambda]$-compactness. Put $\Sigma^{\alpha_1} = \{\varphi_\beta : \beta < \alpha\}$ and $\mathfrak{M}_\alpha \models \Delta \cup \Sigma^{\alpha_1}$. Without loss of generality the $\mathfrak{M}_\alpha$'s are structures of some countable vocabulary τ (coding more predicates with parameters), and have the same power $\mu \geq \lambda$, $\mathfrak{M}_\alpha = \langle M_\alpha, Q_n\ (n \in \omega)\rangle$.

We want to code all the $\mathfrak{M}_\alpha$'s into one structure. So we let $\mathfrak{M}$ be such that:

(1) $\mathfrak{M} = \langle M, <, \bar{Q}_n, c_j\ (n \in \omega, j \in \lambda)\rangle$.
(2) $\langle M, <\rangle$ is a linear order of cofinality λ such that every initial segment has power μ (of order type $\mu^* + \lambda$, for example).
(3) $\{c_j : j < \lambda\} \subset M$ is increasing and unbounded.
(4) If $x \leq c_j$ but $x > c_i$ for every $i < j$ then

$$\langle \{y \in M : y < x\}, \bar{Q}_n(x, -, -, \ldots, -) \rangle \cong \mathfrak{M}_\alpha.$$

Let $T = Th_{\mathscr{L}}(\mathfrak{M})$ for some fixed $\mathfrak{M}$ as described above.

Claim. Then T cofinally characterizes λ.

This is proved like Theorem 1.2.2.

3.4.4 The Class $K(\mathfrak{M})$. For the rest of this section $\mathfrak{M}$ is fixed. We now define a class of structures $K(\mathfrak{M})$:

The vocabulary of $K(\mathfrak{M})$ is that of $\mathfrak{M}$ without the constant symbols for c_j but with two additional unary predicate symbols P and R and one additional binary predicate symbol I. Actually our main focus is on the order together with P, R, and I is used to code copies of $\mathfrak{M}$, which we need to guarantee the absoluteness of cofinality λ.

A model in $K(\mathfrak{M})$ is of the form $\mathfrak{A} = \langle A, <, \bar{Q}_i, P, R, I\rangle$ with the requirements:

(K1) If $x \in P$ then the cofinality of x in $\langle A, <\rangle$ is λ with a witnessing sequence $\{c_j(x): j < \lambda\}$.
(K2) $(a, x) \in I$ implies that $a < x$.
(K3) $(a, x) \in I$ implies that $x \in P$ and $a \notin P$.
(K4) $P(x)$ implies that $I(c_j(x), x)$ for every $j \in \lambda$.

Put $J_A^x = \{a \in A: (a, x) \in I\}$ and $\mathfrak{J}_A^x$ be the substructure of $\langle A, <, \bar{Q}_i\rangle$ induced by J_A^x.

(K5) The structure $\langle \mathfrak{J}_A^x, c_j(x)\rangle$ is isomorphic to $\mathfrak{M}$.
(K6) $R \subset P$.

We call a structure in $K(\mathfrak{M})$ *pure* if additionally

(K7) $\bar{Q}_i$ is false where not defined by the previous requirements.

3.4.5 Comments. Note that if $\mathfrak{A} \in K(\mathfrak{M})$ is pure and P in $\mathfrak{A}$ is empty, then $\mathfrak{A}$ is just a linear ordering, i.e., all the other relations are empty, too, by (K7). If we add to $\mathfrak{M}$ one point at the end, say x and let $P = \{x\}$, we get a structure in $K(\mathfrak{M})$. We denote this structure by $\mathfrak{M}^{+1}$.

In general the structures in $K(\mathfrak{M})$ are linearly ordered structures where every point in P has a copy of $\mathfrak{M}$ attached to it in such a way that different points have almost disjoint copies of $\mathfrak{M}$, and $\mathfrak{M}$ cofinally reaches its point in P. The choice of R can be any subset of P. More precisely:

Fact 1. For every $\mathfrak{A} \in K(\mathfrak{M})$ and every $a, a' \in A$, $J_A^a \cap J_A^{a}$ is bounded below both a, a'.

This is proved using the fact that $\mathfrak{M}$ is of order type $\mu^* + \lambda$. Note that this is first-order expressible and could have been stated also as an axiom among (K1–K7).

Fact 2. If $\mathfrak{A} \in K(\mathfrak{M})$ and $a \in P^A$ and we form $\mathfrak{A}'$ by changing the truth value of $a \in R^A$, but leaving everything else fixed, then $\mathfrak{A}' \in K(\mathfrak{M})$.

Next we define the notion of K-substructure, $\mathfrak{A} \subset_K \mathfrak{B}$ for, $\mathfrak{A}, \mathfrak{B} \in K(\mathfrak{M})$ by:

(K8) $\mathfrak{A} \subset \mathfrak{B}$.
(K9) If $x \in P^A$ then $J_B^x \subset A$.
(K10) If $x \in P^B - P^A$ then $\{a \in A: a < x\}$ is bounded below x in $\mathfrak{B}$, i.e., there is $b_x \in B$ such that $b_x < x$ and for each $a \in A$ with $a < x$ we have $a < b_x$.

The idea behind this is that in $\mathfrak{B}$ new points in P_B are added to P^A in a way that they are not limits of points from $\mathfrak{A}$, and that points in $\mathfrak{A}$ which are of cofinality λ, are also of cofinality λ in $\mathfrak{B}$ with the same copy of $\mathfrak{M}$ ensuring this as in $\mathfrak{A}$.

This ends the definition of $K(\mathfrak{M})$ and of K-substructures.

3.4.6 Some More Facts About $K(\mathfrak{M})$. Before we proceed with the proof of the theorem we collect some more facts:

Definition. If $\mathfrak{A}_1$. $\mathfrak{A}_2 \in K(\mathfrak{M})$ we define $\mathfrak{A}_1 + \mathfrak{A}_2$ to be the disjoint union of $\mathfrak{A}_1, \mathfrak{A}_2$ with the linear ordering of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ for their elements and $a_1 < a_2$ for every $a_1 \in A_1, a_2 \in A_2$. For the other relations we just take their unions.

Fact 3. If $\mathfrak{A}_1, \mathfrak{A}_2 \in K(\mathfrak{M})$ so $\mathfrak{A}_1 + \mathfrak{A}_2 \in K(\mathfrak{M})$ and $\mathfrak{A}_i \subset_K \mathfrak{A}_1 + \mathfrak{A}_2$ $(i = 1, 2)$.

This is clear from the definitions.

Definition. Denote by $L_A^x = \{a \in A : a < x\}$ and by $\mathfrak{L}_A^x$ the structure $\mathfrak{A} \upharpoonright L_A^x$. If $\mathfrak{B} \in K(\mathfrak{M})$ and $A \subset B$ we define a substructure $\mathfrak{C}(A)$ of $\mathfrak{B}$ by

$$\mathfrak{C}(A) = \mathfrak{B} \upharpoonright \bigcup_{a \in A} J_B^a \cup A.$$

This makes sense by Fact 1 and ensures that:

Fact 4. For every $\mathfrak{B} \in K(\mathfrak{M})$, $A \subset B$, $\mathfrak{C}(A) \subset_K \mathfrak{B}$, but in general $\mathfrak{C}(A)$ is not pure. Furthermore, if A is bounded in $\mathfrak{B}$ by b, i.e., there is $b \subset B$ with $A \subset L_B^b$, so $\mathfrak{C}(A) \subset \mathfrak{L}_B^b$ and $\mathfrak{C}(L_B^b) = \mathfrak{L}_B^b$.

Fact 5. If $\mathfrak{A} \in K(\mathfrak{M})$ and $d \in P^A$ then $\mathfrak{A} \upharpoonright L_A^d \subset_K \mathfrak{A}$.

Fact 6. If $\{\mathfrak{A}_i : i < \alpha\}$ is a sequence of structures in $K(\mathfrak{M})$ such that $\mathfrak{A}_i \subset_K \mathfrak{A}_{i+1}$ then $\mathfrak{A} = \bigcup_{i < \bar{a}} \mathfrak{A}_i \in K(\mathfrak{M})$ and $\mathfrak{A}_i \subset_K \mathfrak{A}$ for each $i < \alpha$.

Definition. If $\mathfrak{A}_1, \mathfrak{A}_2 \in K(\mathfrak{M})$, $\mathfrak{B}_i \subset_K \mathfrak{A}_i$ $(i = 1, 2)$ and $f: \mathfrak{B}_1 \cong \mathfrak{B}_2$ is an isomorphism, we define $\mathfrak{A}_1 +_f \mathfrak{A}_2$ in the following way: Form the disjoint union of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ modulo f (i.e., identify elements only via f). This makes it into a partially ordered structure where $a_i \in A_i$ $(i = 1, 2)$ are comparable only if one of them is in the range or domain of f, or there is b between a_1, a_2 which has been identified. For incomparable a_1, a_2 we extend the order on $\mathfrak{A}_1 +_f \mathfrak{A}_2$ setting $a_1 < a_2$.

Fact 7. If $\mathfrak{A}_1, \mathfrak{A}_2 \in K(\mathfrak{M})$ and $f: \mathfrak{B}_i \cong \mathfrak{B}_2$, $\mathfrak{B}_i \subset_K \mathfrak{A}_i$ $(i = 1, 2)$ then $\mathfrak{A}_1 +_f \mathfrak{A}_2 \in K(\mathfrak{M})$ and $\mathfrak{A}_i \subset_K \mathfrak{A}_1 +_f \mathfrak{A}_2$.

The proofs of the facts are left to the reader.

3.4.7 Two Lemmas. The next lemma is crucial for our construction:

Lemma 1. *If $\mathfrak{A} \in K(\mathfrak{M})$ and $\mathfrak{B}$ is an $\mathscr{L}$-extension of $\mathfrak{A}$ and $\{d_j : j < \lambda\}$ is confinal in J_A^a for $a \in P^A$, then $\{d_j : j < \lambda\}$ cofinal in J_B^a.*

Proof. Let $a \in P^A$, so $\mathfrak{J}_A^a \cong \mathfrak{M}$ by (K5) and by our assumption on $\mathscr{L}$ and $\mathfrak{M}$, $\mathscr{L}$ cofinally characterizes λ in $\mathfrak{M}$. Using relativization of $\mathscr{L}$ the structure $\mathfrak{J}_B^a$ is an $\mathscr{L}$-extension of $\mathfrak{M}$ so $\mathfrak{M}$ is confinal in $\mathfrak{J}_B^a$, hence $\{d_j : j < \lambda\}$ is cofinal in $\mathfrak{J}_B^a$ which proves the lemma. □

The next lemma is proved in a similar way as one usually proves the existence of homogeneous structures for Jonsson classes (cf. Chapter XX). We omit the proof here and show how one can now complete the proof of the theorem. A detailed proof of the lemma may be found in Makowsky–Shelah [1983].

Lemma 2. *There is a structure $\mathfrak{N}$ in $K(\mathfrak{M})$ and $d_1 < d_2 < d_3$ in $\mathfrak{N}$ with $d_i \in P^N$ $(i = 1, 2, 3)$, $d_1 \in R^N$, $d_2 \notin R^N$ such that:*

(i) $\mathfrak{N} \upharpoonright L_N^{d_1} \cong \mathfrak{N} \upharpoonright L_N^{d_2} \cong \mathfrak{N} \upharpoonright L_N^{d_3}$; *and*
(ii) *If* $\mathfrak{A} \subset_K \mathfrak{N} \upharpoonright L_N^{d_i}$ $(i = 1, 2)$ *is bounded in* $\mathfrak{N} \upharpoonright L_N^{d_i}$ *then* $\mathfrak{N} \upharpoonright L_N^{d_i} \cong \mathfrak{N} \upharpoonright L_N^{d_3}$ *over* $\mathfrak{A}$ $(i = 1, 2)$.

3.4.8 Proof of the Abstract Amalgamation Theorem. Put $\mathfrak{M}_i = \mathfrak{N} \upharpoonright L_N^{d_i}$ $(i = 1, 2, 3)$. We have to verify some claims:

Claim 1. $\mathfrak{M}_i <_{\mathscr{L}} \mathfrak{M}_3$ $(1 = 1, 2)$.

Proof. Let φ be an $L[\tau(\mathfrak{M}_i)]$-sentence. Since the dependence number $\mathrm{o}(\mathscr{L}) \leq \lambda$, φ depends on $< \lambda$ many constants, hence there is $a \in M_i$ and all the constants of φ are in $L_{M_i}^a$. So by Fact 4, $\mathfrak{M}_i \upharpoonright L_{M_i}^a$ is a bounded K-substructure of both $\mathfrak{M}_i$ and $\mathfrak{M}_3$. So, by Lemma 2(ii) above, $\langle \mathfrak{M}_i, L_{M_i}^a \rangle$ is isomorphic to $\langle \mathfrak{M}_3, L_{M_3}^a \rangle$ hence by the basic isomorphism axiom,

$$\langle \mathfrak{M}_i, L_{M_i}^a \rangle \models \varphi \quad \text{iff} \quad \langle \mathfrak{M}_3, L_{M_3}^a \rangle \models \varphi.$$

Now let $f: \mathfrak{M}_1 \cong \mathfrak{M}_2$ be the isomorphism from Lemma 2(i) above, and $g_i: \mathfrak{M}_i \to \mathfrak{M}_3$ $(i = 1, 2)$ the $\mathscr{L}$-embeddings from Claim 1.

Since $\mathscr{L}$ has AP, let $\mathfrak{A}$ be the amalgamation for $g_1: \mathfrak{M}_1 \to \mathfrak{M}_3$, $g_2 f: \mathfrak{M}_1 \to \mathfrak{M}_3$.

Claim 2. $\mathfrak{A} \models d_1 = d_2$.

Proof. $d_i \in P^{M_3}$ $(i = 1, 2)$ are both of cofinality λ and $g_i(M_1)$ is cofinal in $\mathfrak{M}_3 \upharpoonright L_{M_3}^{d_1}$, and $g_2 f(M_1)$ is cofinal in $\mathfrak{M}_3 \upharpoonright L_{M_3}^{d_2}$, so by Lemma 1 above also in $\mathfrak{A} \upharpoonright L_A^{d_1}$ and $\mathfrak{A} \upharpoonright L_A^{d_2}$, hence $\mathfrak{A} \models d_1 = d_2$.

But Claim 2 contradicts our assumption of Lemma 2 above that $d_1 \in R^{\mathfrak{A}}$ and $d_2 \notin R^{\mathfrak{A}}$. This completes the proof of the abstract amalgamation theorem. □

In fact the same proof gives also the following versions of the abstract amalgamation theorem:

3.4.9 Theorem*. *Let κ be a regular cardinal and $\mathscr{L}$ be a logic such that:*

(i) *The Lowenheim number $l_\kappa(\mathscr{L})$ of $\mathscr{L}$ is κ.*
(ii) $\mathrm{Am}(\kappa, \mathscr{L})$ *holds.*

Then $\mathscr{L}$ is (κ, κ)-compact.

3.4.10 Theorem*. *Let $\mathscr{L}$ be a logic with dependence number* $\mathrm{o}(\mathscr{L}) \leq \lambda$. *If* $\mathrm{Am}(\kappa, \mathscr{L})$ *holds for every $\kappa \geq \lambda$ then $\mathscr{L}$ is $[\infty, \lambda]$-compact.*

It is open whether the converse of Theorem 3.4.10 also holds. Note however that for λ smaller than the first uncountable measurable cardinal the converse does hold.

3.5. An Intriguing Example

Let us now look at logics which do have the amalgamation property, but have a large occurrence number. One naturally wonders if such a logic has to be an extension of $\mathcal{L}_{\kappa\kappa}$ for some uncountable κ, possibly bigger than the occurrence number. This is clearly not the case, provided the logic $\mathcal{L}$ is $[\omega]$-compact. The purpose of this section is to present an example of a logic $\mathcal{L}$ with occurrence number $\mathrm{OC}(\mathcal{L})$ bigger than the first uncountable measurable cardinal μ_0, which is still $[\lambda]$-compact for every $\lambda < \mu_0$, satisfies the amalgamation property, but is not compact. If, however, a logic $\mathcal{L}$ satisfies the amalgamation property but is not $[\omega]$-compact, then we know that its occurrence number is bigger than μ_0, and therefore, by Proposition 1.2.4, every τ-structure $\mathfrak{A}$ with $\mathrm{card}(\mathfrak{A}) < \mu_0$ has an $\mathcal{L}$-maximal expansion. This can be used to show that for every $\varphi \in \mathcal{L}_{\mu_0\omega}[\tau]$ there is τ', $\tau \subset \tau'$ and a set $\Sigma \subset \mathcal{L}[\tau']$ such that $\mathrm{Mod}_{\mathcal{L}_{\mu_0\omega}}(\varphi) = \mathrm{Mod}_{\mathcal{L}}(\Sigma) \upharpoonright \tau$. In the presence of the Robinson property τ' can be assumed to be τ. We develop this idea further in Chapter XIX, Theorem 1.12.

3.5.1 Definitions. Let μ be a cardinal and $E \subset P(\mu)$ a family of subsets of μ.

(i) We say that E is $(<\kappa)$-closed, κ a cardinal, if for every $\lambda < \kappa$ and every ultrafilter F on λ the following holds: Given $\{A_i \subset \mu : i < \lambda\}$, then $\{i \in \mu : A_i \in E\} \in F$ implies that $\lim_F A_i = \{\alpha \in \mu : \{i \in \lambda : \alpha \in A_i\} \in F\} \in E$. We say that E is $(<\kappa)$-bi-closed if both E and $P(\mu) - E$ are $(<\kappa)$-closed.

(ii) If $\{\psi_i : i \in \mu\}$ is a family of $\mathcal{L}$-formulas, we define a connective $\bigwedge^E_{i\in\mu} \psi_i$ by $\bigvee_{A\in E}(\bigwedge_{i\in A} \psi_i \vee \bigwedge_{i\in\mu-A} \neg\, \psi_i)$.

3.5.2 Remarks. (i) If E is a κ-complete ultrafilter on μ then both E and $P(\mu) - E$ are $(<\kappa)$-closed.

(ii) The connective $\bigwedge^E_{i\in\mu} \psi_i$ is a generalization of the connective $\bigcap_F$ where F is some ultrafilter.

3.5.3 Definitions. (i) Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals. We denote by $E(\kappa_1, \kappa_2)$ the set of $(<\kappa_1)$-bi-closed families $E \subset P(\mu)$ with $\mu < \kappa_2$.

(ii) Let $\mathcal{L}_{E(\kappa_1,\kappa_2),\kappa_2}$ be the closure of first-order logic under all the infinitary operations $\bigwedge^E_{i\in\mu}$ for $E \in E(\kappa_1, \kappa_2)$.

(iii) Recall that $\mathcal{L} = \mathcal{L}_{D(\kappa_1,\kappa_2),\kappa_2}$ was defined in Example 1.6.6 in a similar way as (ii) above, but instead of $(<\kappa_1)$-bi-closed sets we only used κ_1-complete ultrafilters.

3.5.4 Proposition* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals and $\mathcal{L} = \mathcal{L}_{E(\kappa_1,\kappa_2),\kappa_2}$.*

(i) $\mathcal{L}_{D(\kappa_1,\kappa_2),\kappa_2} < \mathcal{L}$.

(ii) $\mathcal{L} < \mathcal{L}_{\kappa_2,\kappa_2}$.

(iii) $\mathcal{L}$ *is* $[\infty, \kappa_2]$-*compact.*

(iv) *Every ultrafilter F on $\mu < \kappa_1$ is in* UF($\mathscr{L}$), *i.e., is related to $\mathscr{L}$.*
(v) *For every cardinal $\mu < \kappa_1$ is the logic $\mathscr{L}$ $[\mu]$-compact.*
(vi) *For no cardinal μ, $\kappa_1 \leq \mu < \kappa_2$ is $\mathscr{L}$ $[\mu]$-compact.*

Proof. Essentially the same as in Section 1.6. □

3.5.5 Theorem* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals and $\mathscr{L} = \mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$. Then $\mathscr{L}$ satisfies the joint embedding property, and therefore the amalgamation property.*

Outline of Proof. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be two disjoint τ-structures such that $\mathfrak{M}_1 \equiv_{\mathscr{L}} \mathfrak{M}_2$ and let $D_{\mathscr{L}}(\mathfrak{M}_i)$ $(i = 1, 2)$ be their $\mathscr{L}$-diagrams. We want to show that $D^* = D_{\mathscr{L}}(\mathfrak{M}_1) \cup D_{\mathscr{L}}(\mathfrak{M}_2)$ has a model. Since $\mathscr{L} < \mathscr{L}_{\kappa_2,\kappa_2}$ and κ_2 is strongly compact, it suffices to show that for every subset $\Gamma_1 \subset D_{\mathscr{L}}(\mathfrak{M}_1)$ and $\Gamma_2 \subset D_{\mathscr{L}}(\mathfrak{M}_2)$ with $\mathrm{card}(\Gamma_i) < \kappa_2$, $\Gamma_1 \cup \Gamma_2$ has a model.

Let Γ_1, Γ_2 be given and assume $\Gamma_2 = \{\varphi_i(\bar{a}) : i < \mu < \kappa_2\}$. Put

$$E_0 = \{A \subset \mu : \Gamma_1 \cup \{\varphi_i : i \in A\} \textit{ has a model}\}.$$

If $\mu \in E_0$ we are done. So assume, for contradiction that $\mu \notin E_0$. Clearly, $\varnothing \in E_0$, since $\mathfrak{M}_1$ can be expanded to a model of Γ_1.

Claim 1. *E_0 is $(<\kappa_1)$-closed.*

This can be established using Proposition 3.5.4(iv).

Claim 2. *If $E \subset P(\mu)$, $\mu < \kappa_2$ is $(<\kappa_1)$-closed and $\mu \notin E$, then there is $E_1 \subset P(\mu) - E$ with $\mu \in E_1$ such that E_1 is $(<\kappa_1)$-bi-closed.*

This is proved using a reduction to infinitary propositional calculus with conjunctions of length less than κ_1 and the fact that κ_1 is strongly compact.

Clearly, $\mathfrak{M}_2 \models \bigwedge_{i\in\mu} \varphi_i(\bar{a})$, and therefore, $\mathfrak{M}_2 \models \bigwedge^{E_1}_{i\in\mu} \varphi_i(\bar{a})$. Since $\mathscr{L}$ is closed under existential quantification of length less than κ_2, $\exists\bar{x} \bigwedge^{E_2}_{i\in\mu} \varphi_i(\bar{x})$ is an $\mathscr{L}$-sentence and $\mathfrak{M}_2 \models \exists\bar{x} \bigwedge^{E_1}_{i\in\mu} \varphi_i(\bar{x})$. So also $\mathfrak{M}_1 \models \exists\bar{x} \bigwedge^{E_1}_{i\in\mu} \varphi_i(\bar{x})$. Therefore there is $\bar{b}$ from $\mathfrak{M}_1$ and $A \in E_1$ such that $\mathfrak{M}_1 \models \bigwedge_{i\in A} \varphi_1(\bar{b})$ which shows that $\Gamma_1 \cup \{\varphi_i(a) : i \in A\}$ has a model. From this we conclude that $A \in E_0$, contradicting $E_1 \subset P(\mu) - E_0$. □

Using the finite dependence structure theorem and the fact that $\mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$ is $[\omega]$-compact, we get now

3.5.6 Proposition* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals. Then the two logics $\mathscr{L}_{E(\kappa_1,\kappa_2),\kappa_2}$ and $\mathscr{L}_{D(\kappa_1,\kappa_2),\kappa_2}$ are equivalent.*

3.5.7 Corollary* (Shelah). *Let $\kappa_1 < \kappa_2$ be two strongly compact cardinals. Then the logic $\mathscr{L}_{D(\kappa_1,\kappa_2),\kappa_2}$ has the joint embedding property, and therefore the amalgamation property.*

3.5.8 Remark. In Chapter XIX, Theorem 1.1 states that, if $\mathscr{L}$ is a small logic with $s(\omega) = \lambda$ (s the size function of $\mathscr{L}$) which satisfies the joint embedding property, then there are atmost 2^λ many regular cardinals μ such that $\mathscr{L}$ not $[\mu]$-compact. Theorem 3.5.4 shows that this is best possible.

4. Definability

4.1. Preservation Theorems for Sum-like Operations

In model theory one frequently builds new models from a set of given models and it is often very useful to know that the theory of the so-constructed model only depends on the theories of the models it was built from. Examples are the ultraproduct construction and various other product-like constructions, which mostly go back to the seminal papers (Mostowski [1952], Łos–Suszko [1957], Feferman–Vaught [1959], and Frayne–Morel–Scott [1962]). The possibilities of generalizations of the Łos lemma to logics in general are rather limited, as we have shown in Section 1. For simpler constructions, such as disjoint unions or ordered sums, the preservation properties are usually proved with the use of back-and-forth arguments, as they are generalized in Chapter XIX. The first to consider such properties in the context of abstract model theory was S. Feferman in his papers (Feferman [1972, 1974a, b, 1975]. The theme was then pursued in Shelah [1975], Makowsky [1978], and Makowsky–Shelah [1979].

In the context of abstract model theory, in contrast to specific examples of logics, only sum-like operations have played an independent role. They are also used heavily in Chapters XII and XIII. For this reason we restrict our exposition here to the description of sum-like operations as they are used in the following subsections, and as we think they are of interest for future research. Recent trends in theoretical computer science have shown that abstract model theory offers the appropriate framework to state problems and theorems dealing with specification of abstract data types (Goguen–Burstall [1983] and Mahr–Makowsky [1983a, b, 1984]), correctness of programs (Harel [1979, 1983], Makowsky [1980], and Manders–Daley [1982]) and data base theory (Makowsky [1984]). Especially sum-like operations on abstract data types have been recently investigated by Bergstra–Tucker [1984] to show that some of the concepts in program correctness are probably not stable enough to be transferred from one formalization to another.

4.1.1 Definitions. (i) (Pair of Two Structures). Let τ_1, τ_2 be two disjoint one-sorted vocabularies and $\mathfrak{A}_1$, $\mathfrak{A}_2$ be τ_i-structures, respectively. We define the *pair* $[\mathfrak{A}_1, \mathfrak{A}_2]$ to be the two-sorted $\tau_1 \cup \tau_2$-structure with universes A_1, A_2 and their respective relations, functions, and constants. If the vocabularies τ_1, τ_2 are not disjoint, we make them disjoint by a name changer and write nevertheless $[\tau_1, \tau_2]$.

(ii) (Pair Preservation Property). If $\mathscr{L}$ is a logic, we say that $\mathscr{L}$ satisfies the *pair preservation property* and write PPP($\mathscr{L}$), if whenever $\mathfrak{A}_{11}$, $\mathfrak{A}_{12}$, $\mathfrak{A}_{21}$, $\mathfrak{A}_{22}$ are structures such that $\mathfrak{A}_{i1} \equiv_{\mathscr{L}} \mathfrak{A}_{i2}$ then $[\mathfrak{A}_{11}, \mathfrak{A}_{21}] \equiv_{\mathscr{L}} [\mathfrak{A}_{12}, \mathfrak{A}_{22}]$.

To verify that a given logic satisfies PPP($\mathscr{L}$) it is often useful to use back-and-forth type arguments, as described in Chapter II and more generally in Chapter XIX. It should be possible to state a general theorem to the effect of when a back-and-forth property implies the pair preservation property, but this does not seem to be a very rewarding line of thought. For the traditional back-and-forth arguments for infinitary logics this analysis has been carried out in Feferman [1972].

4.1.2 Examples. (i) Both $\mathscr{L}_{\omega\omega}$ and $\mathscr{L}_{\infty\omega}$ satisfy the pair preservation property.

(ii) $\mathscr{L}_{\omega_1\omega}$ does not satisfy the pair preservation property (Malitz [1971]). $\mathscr{L}_{\kappa\lambda}$ does satisfy the pair preservation property iff κ is strongly inaccessible (Malitz [1971]).

(iii) $\mathscr{L}_{\omega\omega}(Q_\kappa)$ satisfies the pair preservation property by Wojciechowska [1969].

(iv) For logics with second-order quantification, such as stationary logic $\mathscr{L}_{\omega\omega}(aa)$ we have to distinguish between the possibility that subsets range over the union of the universes, or that we have also two sorts of set variables. In the former case $\mathscr{L}_{\omega\omega}(aa)$ does not satisfy the pair preservation property (cf. Example IV.6.1.2), in the latter case it does (Makowsky–Shelah [1981]).

4.1.3 Definitions. (i) (Algebraic Operations). Let $n \in \omega$ and $\tau_1, \tau_2, \ldots, \tau_n, \sigma$ be vocabularies. Let $\mathbf{F}$: $\mathrm{Str}(\tau_1) \times \cdots \times \mathrm{Str}(\tau_n) \to \mathrm{Str}(\sigma)$ be a function. We say that $\mathbf{F}$ is an *n-ary algebraic operation of type* $\tau = [\tau_1, \tau_2, \ldots, \tau_n, \sigma]$, if $\mathfrak{A}_i$, $\mathfrak{B}_i$ are τ_i-structures and $\mathfrak{A}_i \cong \mathfrak{B}_i$ $(i = 1, \ldots, n)$ then

$$\mathbf{F}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n) \cong \mathbf{F}(\mathfrak{B}_1, \ldots, \mathfrak{B}_n).$$

(ii) ($\mathscr{L}$-Projective Operations). Let $\mathscr{L}$ be a logic. An algebraic operation $\mathbf{F}$ of type τ as above is an *$\mathscr{L}$-projective operation* if the graph of $\mathbf{F}$ is an $\mathscr{L}$-projective class.

(iii) (Preservation Property for Projective Operations). We say, a logic $\mathscr{L}$ has the *preservation property for projective operations* and write PPPO($\mathscr{L}$), if for every $\mathscr{L}$-projective operation $\mathbf{F}$ of type τ, if $\mathfrak{A}_i$, $\mathfrak{B}_i$ are τ_i-structures and $\mathfrak{A}_i \equiv_{\mathscr{L}} \mathfrak{B}_i$ $(i = 1, \ldots, n)$ then $\mathbf{F}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n) \equiv_{\mathscr{L}} \mathbf{F}(\mathfrak{B}_1, \ldots, \mathfrak{B}_n)$.

4.1.4 Examples. (i) First-order logic satisfies the PPPO by Feferman [1974].

(ii) The PPPO follows from the uniform reduction property UR_2 defined in Section 4.2.

(iii) The pair construction in Definition 4.1.1 is a first-order projective operation. Therefore PPP follows from PPPO for any regular logic.

(iv) Various other algebraic operations are studied in Gaifman [1967, 1974], Isbell [1973], Hodges [1974, 1975, 1980], and H. Friedman [1979c].

The preservation property for projective operations seems to be very rare. In fact, it is only known to hold for first-order logic, or for logics with uniform reduction (see Section 4.2). For many applications, however, we need much less. A construction somewhere between disjoint unions and general projective operations is enough to obtain interesting theorems in abstract model theory. In the spirit of this section, dealing with definability properties in logics, we give both an implicit and an explicit definition.

4.1.5 Definitions. (i) (Tree-like Structure). Let τ_{tree} be one-sorted and consist of one unary function symbol $\mathbf{f}$ and one constant symbol $\mathbf{c}$. A τ_{tree}-structure $\mathfrak{T} = \langle T, f, c\rangle$ is a *tree-like* structure, if the following hold.

(a) For every $x \in T, f(x) = x$ iff $x = c$, i.e., f is cycle-free but for its only fixed point c, the root of f.
(b) f is onto.
(c) For every $x \in T$ there is an $n \in \omega$ with $f^n(x) = c$.

For $x \in T$ we denote by T_x the set $f^{-1}(x) - \{x\}$.

(ii) (Augmented Tree-like Structure). Let τ_{aug} be $\tau_{\text{tree}} \cup \{\mathbf{P}\}$, where $\mathbf{P}$ is a unary predicate symbol. A τ_{aug}-structure $\mathfrak{T} = \langle T, f, c, P\rangle$ is an *augmented tree-like structure*, if $\mathfrak{T} \restriction \tau_{\text{tree}}$ is a tree-like structure.

(iii) (Tree-like Sum, Implicit Version). Let τ be a vocabulary with a distinguished predicate symbol $\mathbf{P}$ and let $\mathfrak{A}$, $\mathfrak{B}$ two τ-structures. We now define two structures over the vocabulary $\tau \cup_{\text{disjoint}} \tau_{\text{tree}}$, $\mathfrak{N}^i = \text{Tree}^i_P(\mathfrak{A}, \mathfrak{B})$, $i = 0, 1$, the *tree-like sum over* $\mathbf{P}$, in the following way:

(a) $\mathfrak{N}^i \restriction (\tau_{\text{tree}} \cup \{\mathbf{P}\})$ is an augmented tree-like structure. We write now N^{ix} for T_x above.
(b) For every $x \in N^i$ there is bijection $\varepsilon_x: \mathfrak{C} \to N^{ix}$ where $\mathfrak{C}$ is either $\mathfrak{A}$ or $\mathfrak{B}$. This bijection makes N^i_x naturally into a τ-structure which we denote by $\mathfrak{N}^i_x$.
(c) For each symbol $\mathbf{R} \in \tau$ let R_x be its interpretation in $\mathfrak{N}^i_x$. We require now that $R = \mathbf{R}^N = \bigcup_{x \in N} R_x$.
(d) We require further that $P = \mathbf{P}^{Ni}$ be defined by: If $x \in P$ then $\mathfrak{N}^i_x \cong \mathfrak{A}$ and $x \notin P$ then $\mathfrak{N}^i_x \cong \mathfrak{B}$.
(e) If $i = 1$ then $c \in P$ and if $i = 0$ then $c \notin P$.

(iv) (Tree-like Sum, Explicit Version). To make the definition of the tree-like sum $\mathfrak{N}^i = \text{Tree}^i_P(\mathfrak{A}, \mathfrak{B})$ explicit we proceed as follows: We let the universe of $\mathfrak{N}^i$ consist of the set of finite sequences $\langle a_k : k < n\rangle$ such that:

(a) $a_k \in A \cup B$;
(b) if $i = 0$ then $a_0 \in A$, but if $i = 1$ then $a_0 \in B$;
(c) $a_k \in P^A \cup P^B$ iff $a_{k+1} \in A$;

Next we define f, the interpretation of $\mathbf{f}$:

(d) For the empty sequence $\langle\ \rangle$ we put $f(\langle\ \rangle) = \langle\ \rangle$;
(e) $f(\langle a_k : k \leq n\rangle) = \langle a_k : k < n\rangle$.

Finally, for every relation symbol $\mathbf{R} \in \tau$ we define its interpretation R by

(f) $(\langle a_k : k \leq n\rangle, \langle b_k : k \leq n\rangle) \in R$ iff $a_k = b_k$ for every $k < n$ and

$$(a_n, b_n) \in R^A \cup R^B.$$

(v) (Tree Preservation Property). Let $\mathscr{L}$ be a logic. We say that $\mathscr{L}$ has the *tree preservation property* and write TPP($\mathscr{L}$), if whenever $\mathfrak{A}$, $\mathfrak{B}$ are as above, $\tau = \tau_0 \cup \{\mathbf{P}\}$ and additionally $\mathfrak{A} \restriction \tau_0 \equiv_{\mathscr{L}} \mathfrak{B} \restriction \tau_0$ then

$$\mathrm{Tree}_P^0(\mathfrak{A}, \mathfrak{B}) \restriction \tau_0 \cup \tau_{\mathrm{tree}} \equiv_{\mathscr{L}} \mathrm{Tree}_P^1(\mathfrak{A}, \mathfrak{B}) \restriction \tau_0 \cup \tau_{\mathrm{tree}}.$$

4.1.6 Remarks. (i) The tree-like sum is not, in general, a projective operation, since Definition 4.1.5(c) is not first-order definable. However, if the logic $\mathscr{L}$ is such, that the structure $\langle \omega, < \rangle$ is $\mathrm{PC}_{\mathscr{L}}$-characterizable, then the tree-like sum is an $\mathscr{L}$-projective operation.

(ii) For regular logics $\mathscr{L}$ the tree preservation property implies the pair preservation property, since the pair can be constructed as a relativized reduct of the tree sum.

(iii) If the distinguished predicate $\mathbf{P}$ in the tree-like sum is not unary, we can still define a tree-like sum over $\mathbf{P}$. We just replace f by a function $s: T \to T^n$ and define s_1 to be s followed by a projection to the first coordinate. Then we express Definition 4.1.5(i)(a) and (b) with s and (c) with s_1.

The construction of the tree-like sum over a predicate P can sometimes be used to define the predicate P implicitly. The precise situation where this is possible is given in the following lemma from Makowsky–Shelah [1979b]. The idea goes back to S. Shelah.

4.1.7 Lemma. *Let $\mathscr{L}$ be a logic, $\tau_i = \tau_0 \cup_{\mathrm{disjoint}} \{\mathbf{P}_i\}$ $(i = 1, 2)$ vocabularies, and $\varphi_i \in \mathscr{L}[\tau_i]$ be sentences having a model, but such that $\varphi_1 \wedge \varphi_2$ has no model. Then there is a sentence $\psi \in \mathscr{L}[\tau_0 \cup_{\mathrm{disjoint}} \tau_{\mathrm{aug}}]$ such that:*

(i) *Every $\tau_0 \cup_{\mathrm{disjoint}} \tau_{\mathrm{tree}}$-structure $\mathfrak{A}$ has at most one expansion $\mathfrak{A}^* \models \psi$.*

(ii) *If $\mathfrak{A}_i (i = 1, 2)$ are τ_i-structures and $\mathfrak{A}_i \models \varphi_i$ then* $\mathrm{Tree}_P^i(\mathfrak{A}_1, \mathfrak{A}_2) \models \psi$ provided we substitute $\mathbf{P}$ for $\mathbf{P}_1$, $\mathbf{P}_2$, respectively.

Proof. Let $\psi = \psi_0 \wedge \psi_1 \wedge \psi_2$ with:

ψ_0 expresses Definition 4.1.5(i)(a) and (b);
ψ_1 is the $\mathscr{L}$-formalization of "If $x \in P$ then $\mathfrak{N}_x \models \varphi_1$";
ψ_2 is the first-order formalization of "If $x \notin P$ then $\mathfrak{N}_x \models \varphi_2$."

The latter two involve the appropriate substitutions and relativizations. Clearly (ii) too, holds, by our construction of $\mathrm{Tree}_P^i(\mathfrak{A}_1, \mathfrak{A}_2)$. And (i) holds because $\varphi_1 \wedge \varphi_2$ has no model. $\square$

We shall use Lemma 4.1.7 in Section 4.4 to prove some abstract theorems.

4.2. Definability, Interpolation and Uniform Reduction

We first recall some definitions from Chapter II, Section 7.

4.2.1 Definitions. (i) A logic $\mathscr{L}$ has the *interpolation property*, and we write INT($\mathscr{L}$), if any two disjoint classes of τ-structures, which are RPC in $\mathscr{L}$, can be separated by some EC-class of $\mathscr{L}$.

(ii) A logic $\mathscr{L}$ has the *Δ-interpolation property*, and we write Δ-INT($\mathscr{L}$), if any class K of τ-structures, such that K and its complement are RPC in $\mathscr{L}$, then K is an EC-class of $\mathscr{L}$.

(iii) A logic $\mathscr{L}$ has the *weak Beth property*, and we write WBETH($\mathscr{L}$), if every *strong implicit definition* can be replaced by some explicit definition in $\mathscr{L}$.

(iv) A logic $\mathscr{L}$ has the *Beth property*, and we write BETH($\mathscr{L}$), if every *implicit definition* can be replaced by some explicit definition in $\mathscr{L}$.

(v) A logic $\mathscr{L}$ has the *projective weak Beth property*, and we write PWBETH($\mathscr{L}$), if every implicit definition which is RPC in $\mathscr{L}$, can be replaced by some explicit definition in $\mathscr{L}$.

The following summarizes the relationship between these properties.

4.2.2 Theorem. (i) *A logic $\mathscr{L}$ has the weak projective Beth property iff it has the Δ-interpolation property.*

(ii) *For a logic $\mathscr{L}$ the interpolation property implies, but is strictly stronger than, the Δ-interpolation property (and therefore the projective weak Beth property). This is true even for compact logics.*

(iii) *For a logic $\mathscr{L}$ the interpolation property implies, but is strictly stronger than, the Beth property. This is true even for compact logics.*

(iv) *For a logic $\mathscr{L}$ the Δ-interpolation property implies, but is strictly stronger than, the weak Beth property. In fact, the Δ-interpolation property does not imply the Beth property. This is true even for compact logics.*

(v) *For a logic $\mathscr{L}$ the Beth property implies, but is strictly stronger than, the weak Beth property, in fact the Beth property does not imply the Δ-interpolation property. This is even true for compact logics.*

Proof. The implications are all straightforward. (i) is Proposition 7.3.3 and (ii) is 7.2.7 in Chapter II. (iii) follows from (v). (iv) is Theorem 2.5 in Makowsky–Shelah [1979b] and (v) is proven in Makowsky–Shelah [1976] and will appear in Makowsky–Shelah [198?]. For compact logics (ii)–(v) follow from Theorems 4.6.12 and 4.6.13. □

4.2.3 Remark. For sublogics of $\mathscr{L}_{\omega_1\omega}$ of the form $\mathscr{L}_A$ with A primitive recursive closed, the Δ-interpolation property implies the interpolation property and therefore the Beth property. This is due to H. Friedman and proved in Makowsky–Shelah–Stavi [1976]. See also Chapter VIII, Theorem 6.3.1.

Next we investigate the relationship between the weak Beth property and recursive compactness. Of special interest here is that we need an additional assumption, namely either that the logic is finitely generated or the pair preservation property.

4.2.4 Definitions. (i) A logic $\mathscr{L}$ is *finitely generated*, if it is a Lindström logic over a finite set of new quantifier symbols.

(ii) A logic $\mathscr{L}$ is *recursively generated*, if it is a Lindström logic over a recursive set of new quantifier symbols.

(iii) A logic $\mathscr{L}$ is *recursively compact*, if $\mathscr{L}$ is recursively generated and if Σ is any recursive set of $\mathscr{L}$-sentences such that every finite subset of Σ has a model, so Σ has a model.

4.2.5 Remarks. (i) By Theorem 5.2.5 in Chapter II every logic, for which validity is recursively enumerable, is recursively compact.

(ii) A logic $\mathscr{L}$ is recursively compact iff no single sentence $\varphi \in \mathscr{L}[\tau]$, with τ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega, < \rangle$ up to isomorphism among (relativized) reducts of models of φ. Cf. also Chapter II, Section 5.2.

4.2.6 Theorem. (i) (Lindström). *Assume a logic $\mathscr{L}$ is finitely generated and has the weak Beth property, then $\mathscr{L}$ is recursively compact.*

(ii) *Assume a logic $\mathscr{L}$ is recursively generated and satisfies the weak Beth property and the pair preservation property. Then $\mathscr{L}$ is recursively compact.*

Proof. The proof of (i) is similar to the proof of Theorem 5.2.5 in Chapter II, cf. also Chapter III, Remark 2.1.5 or Chapter XVII, Section 4.

To prove (ii) we assume for contradiction that there is a $\varphi \in \mathscr{L}[\tau]$ as in the remark (ii) above. Since $\mathscr{L}$ is recursively generated we have at most 2^ω many theories over a countable vocabulary. Now consider the τ-structure

$$\mathfrak{A} = \langle A, P^n, Q, \in \rangle$$

where $A = \bigcup_{n \in \omega} P^n(\omega)$, P^n is the nth iteration of the power set operation, P^n are unary predicates with $P^n = P^n(\omega)$, $\in$ is the natural membership relation, and $Q \subset P^k$ where k is fixed and such that $\beth(k)$ is bigger than the number κ of inequivalent theories in $\mathscr{L}[\tau]$. Now consider the structure $[\mathfrak{A}, \mathfrak{A}]$ with universe of the first sort A_1 and universe of the second sort A_2 and let ψ be the formula in $\mathscr{L}$ which expresses:

(i) P^0 is standard ω. (Here we use φ.)
(ii) F is a partial map from A_1 to A_2, where F is a new function symbol.
(iii) F and F^{-1} preserve $\in$.
(iv) F is hereditary, i.e., if F is defined for x and $y \in x$ so F is defined for y.
(v) The domain of F is maximal with respect to (i)–(iv).

Clearly, ψ defines F strongly implicitly. Since there are at most $\kappa = 2^\omega$ many theories over τ, we can find two structures $\mathfrak{A}_1 = \langle A, P^n, Q_1, \in \rangle$, $\mathfrak{A}_2 = \langle A, P^n, Q_2, \in \rangle$, such that $\mathfrak{A}_1 \equiv_{\mathscr{L}} \mathfrak{A}_2$ but $Q_1 \neq Q_2$.

Let $\mathfrak{B}_1 = [\mathfrak{A}_1, \mathfrak{A}_2]$ and $\mathfrak{B}_2 = [\mathfrak{A}_1, \mathfrak{A}_1]$. Now we use PPP($\mathscr{L}$) to conclude that $\mathfrak{B}_1 \equiv_{\mathscr{L}} \mathfrak{B}_2$. Using the weak Beth property, let $\vartheta \in \mathscr{L}[\tau]$ define F explicitly. So ϑ defines on $\mathfrak{B}_i$ a partial map F_i with domain D_i. Clearly $Q_1 \subset D_1$, and since $\mathfrak{B}_1 \equiv_{\mathscr{L}} \mathfrak{B}_2$, also $Q_1 \subset D_2$. But then we can show by induction on $l \leq k$ that $Q_1 = Q_2$, contrary to our assumption. Note that, in this proof, we have only used a finite subset of the vocabulary τ. ☐

The same proof actually only requires that the number of theories for a countably vocabulary is smaller than $\beth(\omega_1^{\mathrm{CK}})$. This can be achieved by assuming either that the Löwenheim number is smaller than $\beth(\omega_1^{\mathrm{CK}})$ or directly, by assuming that there are not too many different formulas for a given countable vocabulary. One can vary the prove further for logics $\mathscr{L}$ such that card($\mathscr{L}[\tau]$) $< \beth(\alpha)$ for countable vocabulary τ. We state the corresponding results without proof:

4.2.7 Theorem. (i) *Assume a logic $\mathscr{L}$ satisfies the weak Beth property and the pair preservation property, and has a Lowenheim number $l(\mathscr{L}) < \beth(\omega_1^{\mathrm{CK}})$. Then no single sentence $\varphi \in \mathscr{L}[\tau]$, with τ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega, < \rangle$ up to isomorphism among reducts of models of φ. In other words, the well-ordering number $w_1(\mathscr{L})$ for single sentences of $\mathscr{L}$ is ω.*

(ii) *Assume a logic $\mathscr{L}$ satisfies the weak Beth property and the pair preservation property, and* card($\mathscr{L}[\tau]$) $< \beth(\alpha)$ *for countable vocabulary τ. Then no single sentence $\varphi \in \mathscr{L}[\tau]$, with τ containing a binary relation symbol denoted by $<$, characterizes the structure $\langle \omega + \alpha, < \rangle$ up to isomorphism among reducts of models of φ. In other words, the well-ordering number $w_1(\mathscr{L})$ for single sentences of $\mathscr{L}$ is $\omega + \alpha$.*

4.2.8 Corollary. *Let A be a countable admissible set with $\omega \in A$, or $A = \omega_1$. Then $\mathscr{L}_A$ does not satisfy the pair preservation property.*

Proof. Clearly $\langle \omega, < \rangle$ is characterizable in $\mathscr{L}_A$ and the interpolation property holds. ☐

We now want to look at a property introduced in Feferman [1974b] and further studied in Makowsky [1978], which is a generalization of both the interpolation property and some of the preservation properties.

4.2.9 Definition. Let $\mathscr{L}$ be a logic and $\mathfrak{A}_i$ be τ_i-structures ($i = 1, 2$) with τ the vocabulary for $[\mathfrak{A}_1, \mathfrak{A}_2]$. We say that $\mathscr{L}$ allows *uniform reduction for pairs*, or has the uniform reduction property for pairs, anc write URP($\mathscr{L}$), if for every $\varphi \in \mathscr{L}[\tau]$ there exists a pair of finite sequences of formulas $\psi_1^1, \ldots, \psi_{n_1}^1$ and $\psi_1^2, \ldots, \psi_{n_2}^2$ with $\psi_k^i \in \mathscr{L}[\tau_i]$ and a boolean function $B \in 2^{n_1+n_2}$ such that for every τ_i-structures $\mathfrak{A}_i$ $(i = 1, 2)$ $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$ iff $B(a_1^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2) = 1$, where a_k^i is the truth value of $\mathfrak{A}_i \models \psi_k^i$.

4.2.10 Examples. (i) URP($\mathscr{L}$) holds for

$\mathscr{L} = \mathscr{L}_{\omega\omega}$ by Feferman–Vaught [1959].
$\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_\kappa)$ by Wojciechowska [1969].
$\mathscr{L} = \mathscr{L}_{\infty\infty}$ by Malitz [1971].

(ii) URP($\mathscr{L}$) does hold for $\mathscr{L} = \mathscr{L}_{\kappa\lambda}$ iff κ is strongly inaccessible, by Malitz [1971].

We want to generalize URP to constructions different from the simple pair.

4.2.11 Definitions. (i) Let $\tau_0, \tau_1, \ldots, \tau_n$ be disjoint vocabularies and let

$$R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1) \times \cdots \times \mathrm{Str}(\tau_n)$$

be an n-ary relation on structures. A sentence $\varphi \in \mathscr{L}[\tau_n]$ is said to be *invariant on the range of* R, if for all $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n, \mathfrak{A}'_n$ such that $R(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ and $R(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}'_n)$ $\mathfrak{A}_n \models \varphi$ iff $\mathfrak{A}'_n \models \varphi$.

(ii) An n-tuple of sequences of sentences $\bar{\psi}_0, \bar{\psi}_1, \ldots, \bar{\psi}_{n-1}$ with

$$\bar{\psi}_k = (\psi^k_1, \ldots, \psi^k_{m_k})$$

and $\psi^k_i \in \mathscr{L}[\tau_k]$ together with a boolean function

$$B \in 2^{m_1 + \cdots + m_{n-1}}$$

is called an UR *n-tuple for φ on the domain of R* if for all $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n$ we have that $R(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ implies that $\mathfrak{A}_n \models \varphi$ iff $B(a^1_1, \ldots, a^1_{m_1}, a^2_1, \ldots, a^{n-1}_{m_{n-1}}) = 1$ where a^i_j is defined as in Definition 4.2.9.

(iii) We say a logic $\mathscr{L}$ satisfies the *uniform reduction property for $(n + 1)$-ary relations*, and we write UR$_n$($\mathscr{L}$), if for every relation $R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1) \times \cdots \times \mathrm{Str}(\tau_n)$ which is PC in $\mathscr{L}$ and for every $\varphi \in \mathscr{L}[\tau_n]$ which is invariant in the range of R, there is an UR tuple for φ on the domain of R.

4.2.12 Remarks. (i) Clearly UR$_2$($\mathscr{L}$) implies URP($\mathscr{L}$), since the construction of the pair $[\mathfrak{A}_1, \mathfrak{A}_2]$ is a PC$_{\mathscr{L}}$-operation, i.e., its graph is a PC$_{\mathscr{L}}$ relation.

(ii) Instead of the pair construct we could consider the cartesian product of a fixed finite number n of structures $\mathfrak{A}_i$ and define similarly *uniform reduction for n-fold cartesian products* (URProd$_n$($\mathscr{L}$)). Again UR$_n$($\mathscr{L}$) implies URProd$_n$($\mathscr{L}$).

The following clarifies the relationship between PPP and various uniform reduction properties.

4.2.13 Theorem. *Let $\mathscr{L}$ be a logic. Then*

(i) URP($\mathscr{L}$) *implies* PPP($\mathscr{L}$),
(ii) UR$_2$($\mathscr{L}$) *implies* PPPO($\mathscr{L}$).

If additionally $\mathscr{L}$ *has an dependence number* $o(\mathscr{L}) = \kappa$ *and is* (μ, ω)*-compact, with* $\mu = \sup\{\mathrm{card}(\mathscr{L}[\tau]): \mathrm{card}(\tau) < \kappa\}$, *then:*

(iii) (Shelah [198?e]). $\mathrm{PPP}(\mathscr{L})$ *implies* $\mathrm{URP}(\mathscr{L})$; *and*
(iv) (Shelah [198?e]). $\mathrm{PPPO}(\mathscr{L})$ *implies* $\mathrm{UR}_n(\mathscr{L})$ *for every* $n \in \omega$.

Proof. (i) and (ii) are straightforward. To prove (iii) assume φ is a counterexample to URP. So for every pair of sequences of formulas $\bar{\psi}_1 = (\psi_1^1, \ldots, \psi_{n_1}^1)$ and $\bar{\psi}_2 = (\psi_1^2, \ldots, \psi_{n_2}^2)$ with $\psi_k^i \in \mathscr{L}[\tau_i]$ and every boolean function $B \in 2^{n_1+n_2}$ there are τ_i-structures $\mathfrak{A}_i^j$ such that $[\mathfrak{A}_1^1, \mathfrak{A}_2^1] \models \varphi$, $[\mathfrak{A}_1^2, \mathfrak{A}_2^2] \models \neg\varphi$, but

$$B(a(j)_1^1, \ldots, a(j)_{n_1}^1, a(j)_1^2, \ldots, a(j)_{n_2}^2) = 1,$$

where $a(j)_k^i$ is the truth value of $\mathfrak{A}_i^j \models \psi_k^i$.

Claim 1. *For every such pair of sequences of formulas* $\bar{\psi}_1$, $\bar{\psi}_2$ *there is a function* $h: \bar{\psi}_1 \cup \bar{\psi}_2 \to 2$ *such that*

$$\Sigma_h^1 = \{\varphi\} \cup \{\vartheta \leftrightarrow h(\vartheta): \vartheta \in \bar{\psi}_1 \cup \bar{\psi}_2\},$$

and

$$\Sigma_h^0 = \{\neg\varphi\} \cup \{\vartheta \leftrightarrow h(\vartheta): \vartheta \in \bar{\psi}_1 \cup \bar{\psi}_2\}$$

have both models.

If not, for every h as above either Σ_h^1 or Σ_h^0 is has no model. We then could construct a boolean function B as follows: Put

$$B_h = \bigwedge \{\vartheta: h(\vartheta) = 1\} \wedge \bigwedge \{\neg\vartheta: h(\vartheta) = 0\}.$$

Now we put

$$B = \bigvee \{B_h: \Sigma_h^1 \text{ has a model}\}.$$

Subclaim. $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$ *iff* $B = B(a_1^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2) = 1$, *where* a_k^i *is the truth value of* $\mathfrak{A}_i \models \psi_k^i$.

To see this, assume $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$. Now put $h_0(\psi_k^i) = a_k^i$. Clearly, $B = 1$. Conversely, if $B = 1$, there is h such that Σ_h^1 has a model. So, by our assumption, Σ_h^0 has no model. So $[\mathfrak{A}_1, \mathfrak{A}_2] \models \varphi$.

Using Claim 1, we define H to be the set of functions $h: \bar{\psi}_1 \cup \bar{\psi}_2 \to 2$ such that Σ_h^1 and Σ_h^0 have both models.

We define a filter F_0 on H with filter basis $U_\vartheta = \{h \in H: \vartheta \in \mathrm{dom}(h)\}$ where $\vartheta \in \mathscr{L}[\tau_1] \cup \mathscr{L}[\tau_2]$. Let F be an ultrafilter extending F_0. Now we define a function $g: \mathscr{L}[\tau_1] \cup \mathscr{L}[\tau_2] \to 2$ by $g(\vartheta) = 0$ iff $\{h \in H: h(\vartheta) = 0\} \in F$. Clearly, we have:

Claim 2. *For every pair of sequences $\bar{\psi}_1$, $\bar{\psi}_2$ there is a function $h \in H$ such that $g \upharpoonright \mathrm{dom}(h) = h$.*

Now we define Σ_g^i $(i = 0, 1)$ like the Σ_h^i's. Using (μ, ω)-compactness and Claim 2 we get:

Claim 3. *There are* $\mathfrak{A}_j^i$ $(i = 0, 1, j = 1, 2)$ *such that* $[\mathfrak{A}_1^i, \mathfrak{A}_2^i] \models \Sigma_g^i$.

But the latter contradicts PPP($\mathscr{L}$), since, by the definition of Σ_g^i, $\mathfrak{A}_1^i \equiv_{\mathscr{L}} \mathfrak{A}_1^{1-i}$ $(i = 0, 1)$.

The proof of (iv) is essentially the same. □

Uniform reduction is closely related to the interpolation property. Feferman [1974] derived UR_1 from it and in Makowsky [1978] the converse was observed.

4.2.14 Theorem (Feferman, Makowsky). *Let $\mathscr{L}$ be a logic with finite dependence. Then* $\mathrm{UR}_1(\mathscr{L})$ *iff $\mathscr{L}$ has the interpolation property.*

Proof. (i) Assume $\mathrm{UR}_1(\mathscr{L})$ and let $\mathbf{K}_1, \mathbf{K}_2 \subset \mathrm{Str}(\tau_0)$ be two disjoint classes of τ_0-structures which are PC in $\mathscr{L}$. So there are vocabularies τ_i and sentences $\psi_i \in \mathscr{L}[\tau_i]$ such that $\mathbf{K}_i = \mathrm{Mod}(\psi_i) \upharpoonright \tau_0$. Since $\mathscr{L}$ has finite dependence all the vocabularies can be assumed finite. We now define $R \subset \mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1 \cup \tau_2)$ by $R(\mathfrak{A}, \mathfrak{B})$ iff $\mathfrak{A} \cong \mathfrak{B} \upharpoonright \tau_0$ and $\mathfrak{B} \upharpoonright \tau_1 \in \mathbf{K}_1$ or $\mathfrak{B} \upharpoonright \tau_2 \in \mathbf{K}_2$. Clearly R is $\mathrm{PC}_{\mathscr{L}}$ using an additional predicate for the isomorphism and the fact that τ_0 is finite.

Claim. *Both ψ_1, ψ_2 are invariant in the range of R.*

This follows from the fact that $\mathbf{K}_1 \cap \mathbf{K}_2 = \varnothing$.

Now let ϑ_i be UR sentences for ψ_i, respectively. It is easy to check that $\vartheta_1 \wedge \neg \vartheta_2$ is the desired interpolating sentence.

(ii) Now assume that $\mathscr{L}$ has the interpolation property, R is a $\mathrm{PC}_{\mathscr{L}}$-relation on $\mathrm{Str}(\tau_0) \times \mathrm{Str}(\tau_1)$ and $\varphi \in \mathscr{L}[\tau_1]$ is invariant on the range of R. Assume R is defined by $\psi \in \mathscr{L}[\tau]$. Now put

$$\mathbf{K}_1 = \mathrm{Mod}(\psi \wedge \varphi) \upharpoonright \tau_0 \text{ and } \mathbf{K}_2 = \mathrm{Mod}(\psi \wedge \neg \varphi) \upharpoonright \tau_0.$$

Claim. $\mathbf{K}_1 \cap \mathbf{K}_2 = \varnothing$.

This follows from the fact that φ is invariant on the range of R. So let $\vartheta \in \mathscr{L}[\tau_0]$ be an interpolating sentence. Therefore, whenever $R(\mathfrak{A}, \mathfrak{B})$ we have that $\mathfrak{A} \models \vartheta$ iff $\mathfrak{B} \models \varphi$, in other words, ϑ is an UR sentence for φ. □

Note that in Feferman [1974b] uniform reduction is defined for PC_δ, and Theorem 4.2.14(ii) is stated assuming some compactness properties.

4.2.15 Theorem. (i) *For a logic $\mathscr{L}$ the following are equivalent:*

(a) $\mathrm{UR}_2(\mathscr{L})$.
(b) $\mathrm{UR}_1(\mathscr{L})$ *(or equivalently the interpolation property) together with* URP($\mathscr{L}$).
(c) $\mathrm{UR}_n(\mathscr{L})$ *for* $n \geq 2$.

(ii) *For a compact logic* $\mathscr{L}$ *the following are equivalent:*

(a) $\mathrm{UR}_2(\mathscr{L})$.
(b) $\mathrm{UR}_1(\mathscr{L})$ *(or equivalently the interpolation property) together with* $\mathrm{PPP}(\mathscr{L})$.
(c) $\mathrm{UR}_n(\mathscr{L})$ *for* $n \geq 2$.
(d) $\mathrm{PPPO}(\mathscr{L})$.

(iii) $\mathrm{URP}(\mathscr{L})$ *does not imply* $\mathrm{UR}_1(\mathscr{L})$, *not even for compact logics.*
(iv) $\mathrm{UR}_1(\mathscr{L})$ *does not imply* $\mathrm{URP}(\mathscr{L})$. *(For compact logics this is open.)*

Proof. (i) (a) implies (b) by Theorem 4.2.14 and Remarks 4.2.12. (b) implies (c), since URP allows us to reduce n-ary relations to binary relations, and (c) implies (a) is trivial. To prove (ii) we combine (i) with Theorem 4.2.13.

To prove (iii) we observe that by Example 4.2.10(ii) $\mathscr{L}_{\omega\omega}(Q_\kappa)$ satisfies URP, but, as shown in Counterexamples II.7.1.3, it does not have the interpolation property. So the result follows from Theorem 4.2.14. For a compact counterexample see Remark 4.2.17 below.

To prove (iv) we note that $\mathscr{L}_{\omega_1\omega}$ satisfies the interpolation property, and therefore, by Theorem 4.2.14. $\mathrm{UR}_1(\mathscr{L}_{\omega_1\omega})$ holds. As noted in Example 4.2.10(ii) $\mathrm{URP}(\mathscr{L}_{\omega_1\omega})$ does not hold. $\square$

The last proposition in this section gives us a connection between the tree preservation property and uniform reduction, but it is only interesting for logics which are not recursively generated, because the latter hypothesis together with UR_2 implies recursive compactness, by Theorem 4.2.6(ii).

4.2.16 Proposition. *Assume* $\mathscr{L}$ *is a logic in which* $\langle \omega, \in \rangle$ *is not characterizable by a single sentence with additional predicates and sorts (in particular* $\mathscr{L}$ *is not recursively compact). Then* $\mathrm{UR}_2(\mathscr{L})$ *implies* $\mathrm{TPP}(\mathscr{L})$.

Proof, Clearly, we can use the PC-definition of $\langle \omega, \in \rangle$ to get a PC-definition of the tree construction involved in the tree preservation property. See also Remark 4.1.6(iii). $\square$

4.2.17 Remark. In Section 4.6 we shall present an example of a logic $\mathscr{L}$ which satisfies the Beth property, the pair preservation property, is compact, but does not satisfy the interpolation property.

4.3. The Finite Robinson Property

In Section 3.3 we have seen that the amalgamation property implies compactness therefore (Corollary 3.3.5) that the Robinson property implies compactness. These results depend on some assumptions on the dependence number of the logic. In Chapter XIX the Robinson property is further investigated and instead of the dependence number we have different smallness assumptions on the logic. Here we want to study two weakened version of the Robinson property. They were studied first in Makowsky–Shelah [1979b], and the assumptions on the logics also did not involve the dependence number.

4.3.1 Definition. Let $\mathscr{L}$ be a logic.

(i) $\mathscr{L}$ satisfies the *finite Robinson property* (FROB), if given a complete set Σ of $\mathscr{L}[\tau]$-sentences and two sentences φ_1 $(\varphi_2) \in \mathscr{L}[\tau_1]$ $(\mathscr{L}[\tau_2])$ with $\tau_1 \cap \tau_2 = \tau$ such that $\Sigma \cup \{\varphi_i\}$ has a τ_i-model then $\Sigma \cup \{\varphi_1, \varphi_2\}$ has a $\tau_1 \cup \tau_2$-model.

(ii) $\mathscr{L}$ satisfies the *weak finite Robinson property* (WFROB), if given a complete set Σ of $\mathscr{L}[\tau]$-sentences and two sentences φ_1 $(\varphi_2) \in \mathscr{L}[\tau_1]$ $(\mathscr{L}[\tau_2])$ with $\tau_1 \cap \tau_2 = \tau$ such that $\Sigma \cup \{\varphi_i\}$ has a τ_i-model then $\{\varphi_1, \varphi_2\}$ has a $\tau_1 \cup \tau_2$-model.

4.3.2 Proposition. (i) *Both* FROB *and* WFROB *are consequences of the Robinson property.*

(ii) *Clearly* FROB *implies* WFROB.

(iii) *The interpolation property implies* WFROB.

(iv) *If $\mathscr{L}$ is compact then the Robinson property is equivalent to both* FROB, WFROB *and the interpolation property.*

(v) WFROB *does not imply* FROB.

The proof of (i)–(iii) is left to the reader. For (iv) cf. Chapter II, Theorem 7.1.5. For (v) we note that $\mathscr{L}_{\omega_1\omega}$ has the interpolation property and therefore, by (iii) above the WFROB. That $\mathscr{L}_{\omega_1\omega}$ does not satisfy FROB is shown in Keisler [1971a, p. 22].

Our next aim is to study when the pair preservation property suffices to make FROB equivalent to the Robinson property. The answer is given in Theorem 4.3.8.

4.3.3 Definition. (i) We call a logic $\mathscr{L}$ *tiny*, if for every vocabulary τ with card(τ) smaller than the first uncountable measurable cardinal μ_0

$$\text{card}(\mathscr{L}[\tau]) < \mu_0.$$

(ii) We call a logic $\mathscr{L}$ *small*, if for every vocabulary τ, which is a set, $\mathscr{L}[\tau]$ is a set. (Smallness was already introduced in Chapter II, Theorem 6.1.4). Clearly, if a logic $\mathscr{L}$ is tiny, it is also small, provided measurable cardinals exist. If no uncountable measurable cardinals exist, then tiny and small coincide. There are logics with dependence number $o(\mathscr{L}) = \omega$ which are not small, and it is not dfficult to construct logics which are small but have no dependence number. We leave this as an exercise to the reader. The logic defined in Example 2.2.5(ii) is tiny, but has an dependence number which is bigger than the first uncountable measurable cardinal.

(iii) If a logic $\mathscr{L}$ is small then there is function s on the cardinals such that for every vocabulary τ of cardinality λ, $\lambda \leq \text{card}(\mathscr{L}[\tau]) < s(\lambda)$. We call this function the *size function of* L. If $\mathscr{L}$ is tiny then $\lambda < \mu_0$ implies $s(\lambda) < \mu_0$.

(iv) Recall that a logic $\mathscr{L}$ is said to be *ultimately compact*, if $\mathscr{L}$ is (∞, λ)-compact for some cardinal λ.

4.3.4 Theorem. *If $\mathscr{L}$ has the Robinson property and is tiny then:*

(i) *$\mathscr{L}$ is $[\omega]$-compact; and*
(ii) *$\mathscr{L}$ has the finite dependence property.*

This differs from Corollary 3.3.5 inasmuch as here we do not require that $\mathscr{L}$ has an dependence number, whereas in Corollary 3.3.5 we require that $o(\mathscr{L})$ exists and is smaller than the first uncountable measurable cardinal.

4.3.5 Theorem. *If $\mathscr{L}$ has the pair preservation property, the finite Robinson property and is tiny then:*

(i) *$\mathscr{L}$ is $[\omega]$-compact; and*
(ii) *$\mathscr{L}$ has the finite dependence property.*

Proof. Clearly in both theorems (ii) follows from (i) by Theorem 2.2.1. To prove (i) we proceed in parallel and point out the difference in the appropriate places.

Let B_1, B_2 be two infinite sets of different cardinality β_1, β_2 smaller than the first uncountable measurable cardinal μ_0. Now we fix $\kappa > \max\{\beta_1, \beta_2\}$ but $\kappa < \mu_0$ and put $\mathfrak{A}_\kappa = \langle \mathfrak{H}(\kappa^+), P_1, P_2 \rangle$ where $\mathfrak{H}(\kappa^+)$ is the complete expansion of $\langle \kappa^+, \in \rangle$ and P_1, P_2 are unary predicates of cardinality β_1, β_2, respectively. Let τ_κ be the vocabulary of $\mathfrak{A}_\kappa$ and Σ the complete $\mathscr{L}[\tau_\kappa]$-theory of $\mathfrak{A}_\kappa$. Assuming that $\mathscr{L}$ is not $[\omega]$-compact, we conclude, using the Rabin–Keisler theorem (1.2.3), that Σ is categorical. Let $\mathfrak{B}_i = [\mathfrak{A}_k, B_i]$ for $i = 1, 2$ be τ_i-structures with $\tau_1 \cap \tau_2 = \tau_\kappa$.

Assumption: *$\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\mathscr{L}$-equivalent (after appropriate name changing, so that both are τ_1-structures).*

We first finish the proof from the assumption. Let φ_i be the first-order formula which says that "f_i is a bijection from P_i onto the universe of the second sort." Clearly $\mathfrak{B}_i \models \Sigma \cup \{\varphi_i\}$, but $\Sigma \cup \{\varphi_1, \varphi_2\}$ has no model.

To satisfy the assumption the two proofs differ. In the case of Theorem 4.3.5 we use tinyness and an argument as in the proof of the existence of Hanf numbers (Section II.6.1) to find β_1, β_2 such that for $\tau = \{=\}$ B_1 *and* B_2 are $\mathscr{L}$-equivalent. Since τ is finite we may assume that $\beta_1, \beta_2 < \mu_0$. Now we can use the pair preservation property to conclude that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\mathscr{L}$-equivalent (after appropriate name changing).

In the case of Theorem 4.3.4 we fix a countable universal vocabulary τ_∞ which has countably many relation symbols for every arity. Using enough constants τ_C, we can think of Σ as being written over the vocabulary $\tau_\infty \cup \tau_C$. Let Σ_∞ be $\Sigma \upharpoonright \tau_\infty$. Using tinyness we find, as in the case of Theorem 4.3.4, κ, $\beta_1 \beta_2$ such that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $\mathscr{L}[\tau_\infty]$-equivalent.

Let τ_1 and τ_2 be two disjoint copies of τ_C and put $\Sigma_i = \Sigma \cup \{\varphi_i\}$ written over $\tau_\infty \cup \tau_i$. Clearly $\Sigma_\infty \cup \Sigma_i$ has each a model, but $\Sigma_\infty \cup \Sigma_1 \cup \Sigma_2$ has not. $\square$

The following is an improvement of Theorem 4.3.4.

4.3.6 Theorem*. (i) *The Robinson property implies the joint embedding property.*
(ii) *If a logic $\mathscr{L}$ is small and has the joint embedding property then $\mathscr{L}$ is ultimately compact.*
(iii) *If a logic $\mathscr{L}$ is tiny and has the joint embedding property then $\mathscr{L}$ is $[\omega]$-compact.*

Proof. (i) is proved in a similar way to Theorem 3.1.14. (ii) is Theorem 1.1 from Chapter XIX and (iii) follows from (ii) and the fact that $\mathscr{L}$ was assumed to be tiny. ☐

4.3.7 Examples. (i) If a logic $\mathscr{L}$ has a Löwenheim number $l_1(\mathscr{L})$ then $\mathscr{L}$ is small.
(ii) In Chapter XIX, Theorem 1.1.1 it is shown that if $\mathscr{L}$ is small and satisfies the joint embedding property then $\mathscr{L}$ is ultimately compact.
(iii) If $\mathscr{L}$ has an dependence number and satisfies the amalgamation property then $\mathscr{L}$ is ultimately compact. This holds in particular, if $\mathscr{L}$ satisfies the Robinson property (Theorem 3.3.1).

Our next theorem shows that already the finite Robinson property implies ultimate compactness.

4.3.8 Theorem (Shelah). *Let $\mathscr{L}$ be a tiny logic which satisfies both the preservation property for pairs and the finite Robinson property. Then*
(i) *$\mathscr{L}$ is ultimately compact. In fact, if s is the size function of $\mathscr{L}$ and $2^{s(\omega)} < 2^{\omega_{\alpha+n}}$ then $\mathscr{L}$ is $[\infty, \omega_\alpha]$-compact.*
(ii) *If additionally $\mathscr{L}$ is countably generated or $s(\omega) < \omega_n$ for some $n \in \omega$, then $\mathscr{L}$ is compact and satisfies the uniform reduction properties $\mathrm{UR}_n(\mathscr{L})$.*

For the proof we need a lemma. Parts (ii) and (iii) the author has learned from S. Shelah, though others probably have observed them, too.

4.3.9 Lemma. (i) (Ulam). *Let κ be an infinite cardinal. If $S \subset \kappa^+$ is stationary, S may be decomposed into κ^+ disjoint stationary subsets.*
(ii) *There is a family $\mathbf{S}$ of 2^{κ^+} many stationary subsets of κ^+ such that for any $S_1, S_2 \in \mathbf{S}$ the symmetric difference $S_1 \Delta S_2$ is stationary as well.*
(iii) *There are 2^{κ^+} many stationary subsets of κ^+ such that any finite boolean combination of them is stationary as well.*

Proof. (i) is standard, e.g., Theorem 3.2 in Chapter B.3 of the *Handbook of Mathematical Logic* [Barwise, 1977].

To prove (ii) let $\{S_\alpha: \alpha < \kappa^+\}$ be the disjoint family of stationary sets from (i). Let $X \subset \kappa^+$, $X \neq \varnothing$. Define $T_X = \bigcup_{\alpha \in X} S_{2\alpha} \cup \bigcup_{\alpha \notin X} S_{2\alpha+1}$. Clearly each T_X is stationary and $X \neq Y$ implies that $T_X \Delta T_Y$ is stationary.

The proof of (iii) is similar, but uses a combinatorial result from Engelking–Karlowicz [1965]. ☐

Proof of Theorem 4.3.8. Let κ be as required. We can assume it is regular, by Theorem 1.5.16. Assume $\mathscr{L}$ is not $[\kappa]$-compact, so by Theorem 1.5.16 again, $\mathscr{L}$ is not $[\kappa^+]$-compact, and, by induction, we can assume that κ is such that $2^s(\omega) < 2^\kappa$. Let $C_\kappa = \{\beta: \beta \in \kappa^+ \textit{ and } \mathrm{cf}(\beta) = \kappa\}$. For every $S \subset C_\kappa$ we define a structure $\mathfrak{M}_S = \langle \kappa^+, \in, S\rangle$. By Lemma 4.3.9(ii) there are 2^{κ^+} many stationary sets in C_κ with their symmetric difference stationary, too. So, by our assumption on the size function of $\mathscr{L}$, and by Proposition 2.1.3, there are $S_1, S_2 \in C_\kappa$, with $\mathfrak{M}_{S_1} \equiv_{\mathscr{L}} \mathfrak{M}_{S_2}$. We put now $\mathfrak{A} = \langle \kappa^+, \varepsilon, S_1, S_2, S_3, \mathrm{cf}\rangle$ with $S_3 = S_1 \Delta S_2$ and $\in$, cf membership and cofinality on κ^+. Let $\mathfrak{B}$ be the complete expansion of $\mathfrak{A}$. We note that in $\mathfrak{B}$ every ordinal of cofinality κ or κ^+ is cofinally characterized by the complete $\mathscr{L}$-theory of $\mathfrak{B}$. Using that $\mathscr{L}$ has the pair preservation property, we conclude that $[\mathfrak{B}, \mathfrak{M}_{S_1}] \equiv_{\mathscr{L}} [\mathfrak{B}, \mathfrak{M}_{S_2}]$. Let Σ be the complete theory of $[\mathfrak{B}, \mathfrak{M}_{S_1}]$. We want to build a counterexample to FROB. For this purpose let F_i $(i = 1, 2)$ be new unary function symbols and φ_i be the sentence which says that "F_i is an isomorphism between $\langle \kappa^+, \in, S_i^B\rangle$ and $\mathfrak{M}_{S_i}$. Clearly $\Sigma \cup \{\varphi_i\}$ is each satisfiable but it is not difficult to show that $\Sigma \cup \{\varphi_1, \varphi_2\}$ has no model. □

A complete proof may be found in Makowsky–Shelah [1979b].

A combination of the proofs of Theorem 4.3.8 and Proposition 4.3.2 gives us the following theorem:

4.3.10 Theorem. *Let $\mathscr{L}$ be a logic which is small and satisfies either the Robinson property or the finite Robinson property together with the pair preservation property. Then $\mathscr{L}$ is ultimately compact.*

Combining Theorem 4.3.10 with the hypothesis $A(\infty)$ from Section 1.5 we get:

4.3.11 Corollary (Makowsky–Shelah, Mundici). *For a logic as in Theorem* 4.3.10 *we have*:

(i) *If $A(\infty)$ holds then $\mathscr{L}$ is compact.*
(ii) *If $\mathscr{L}$ is tiny and there are no uncountable measurable cardinals, then $\mathscr{L}$ is compact.*

Proof. Assume $A(\infty)$, so there are no uncountable measurable cardinals, by Theorem 1.5.4(iii). Therefore, if a logic $\mathscr{L}$ is small, then it is tiny and by Theorems 4.3.4 or 4.3.5 $[\omega]$-compact. So Theorem 4.3.8 together with Theorem 1.5.7 give us that $\mathscr{L}$ is compact. This proves both (i) and (ii). □

Let us end this section with an open problem.

4.3.12 Problem. Is there a countable logic, different from first-order logic, which satisfies both the Robinson property and the uniform reduction property (as in Theorem 4.3.8)?

4.4. Constructing Counter Examples to the Beth Property

This last section is devoted to an abstract theorem (4.4.5) whose main use it is to direct us in the construction of possible counterexamples to the Beth property. For compact logics, it gives a sufficient condition, the tree preservation property, for the Beth and the interpolation property to be equivalent. As the example in Theorem 4.6.12 shows, the pair preservation property does not suffice. Experience shows that in many cases where we do not have the interpolation property, we actually can find a counterexample to the weak finite Robinson property. The following theorem gives some indication on how to transform such a counterexample into a counterexample of the Beth property.

4.4.1 Theorem. (i) *Let $\mathscr{L}$ be a logic which satisfies the Beth property and the tree preservation property. Then $\mathscr{L}$ also satisfies the weak finite Robinson property.*

(ii) *If additionally to the tree preservation property $\mathscr{L}$ is compact, then $\mathscr{L}$ has the Beth property iff it has the interpolation property.*

Stated in this form the theorem does not have many applications. But its proof still gives directions on how to construct counterexamples to the Beth property, provided the interpolation property fails. In Makowsky-Shelah [198?b] this approach lead to a proof that $\Delta(\mathscr{L}_{\infty\omega})$ does not have the Beth property. Another way of making Theorem 4.4.1 more useful, is to define all the properties involved for pairs of logics.

4.4.2 Definitions. (i) Let $\mathscr{L}_1$, $\mathscr{L}_2$ be two logics such that $\mathscr{L}_1 \leq \mathscr{L}_2$. We define the various Robinson properties ROB, FROB, WFROB for the pair $\mathscr{L}_1$, $\mathscr{L}_2$ and write ROB($\mathscr{L}_1, \mathscr{L}_2$), FROB($\mathscr{L}_1, \mathscr{L}_2$), WFROB($\mathscr{L}_1, \mathscr{L}_2$), respectively. For ROB this looks explicitly as follows: If Σ is a complete set of formulas in $\mathscr{L}_2(\tau_0)$, Σ_1, Σ_2 are in $\mathscr{L}_1(\tau_1)$, $\mathscr{L}_1(\tau_2)$, respectively, $\tau_1 \cap \tau_2 = \tau$ and $\Sigma \cup \Sigma_i$ $(i = 1, 2)$ have models each, then $\Sigma \cup \Sigma_1 \cup \Sigma_2$ has a model. We leave it to the reader to state the corresponding properties FROB, WFROB.

(ii) Similarly we define the various Beth and interpolation properties BETH, WBETH, INT, Δ-INT for the pair $\mathscr{L}_1$, $\mathscr{L}_2$ and write BETH($\mathscr{L}_1, \mathscr{L}_2$), WBETH($\mathscr{L}_1, \mathscr{L}_2$), INT($\mathscr{L}_1, \mathscr{L}_2$), Δ-INT($\mathscr{L}_1, \mathscr{L}_2$), respectively, if the implicit definition or the formulas to be interpolated are in $\mathscr{L}_1$ and the explicit definition or the interplant is in $\mathscr{L}_2$.

(iii) Similarly we define the various preservation properties PPP, TPP for the pair $\mathscr{L}_1$, $\mathscr{L}_2$ and write PPP($\mathscr{L}_1, \mathscr{L}_2$), TPP($\mathscr{L}_1, \mathscr{L}_2$), if the given structures are $\mathscr{L}_2$-equivalent and the resulting structures are $\mathscr{L}_1$-equivalent.

4.4.3 Examples. (i) While $\mathscr{L}_{\omega\omega}(Q_0)$ does not have the interpolation property by Counterexamples II.7.1.3, INT($\mathscr{L}_{\omega\omega}(Q_0), \mathscr{L}_{\omega_1\omega}$) does hold.

(ii) The logics $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ and $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ both do not satisfy the interpolation property (Makowsky–Shelah [1981, Proposition 6.6] and Counterexamples II.7.1.3) but, as we shall see in Propsotion 4.6.7,

$$\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)}), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$$

does hold.

(iii) $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ does not hold, by Counterexamples II.7.1.3.

4.4.4 Proposition. *Let* PROPERTY *be any of the above defined definability properties, and let* $\mathscr{L}_{10} < \mathscr{L}_{11} < \mathscr{L}_{20} < \mathscr{L}_{21}$ *be logics. Then* $\mathrm{PROPERTY}(\mathscr{L}_{11}, \mathscr{L}_{20})$ *implies* $\mathrm{PROPERTY}(\mathscr{L}_{10}, \mathscr{L}_{21})$.

Proof. Obvious.

With these definitions we can state a slightly stronger theorem.

4.4.5 Theorem. (i) *Let* $\mathscr{L}_1 < \mathscr{L}_2 < \mathscr{L}_3$ *be three logics such that* $\mathrm{BETH}(\mathscr{L}_1, \mathscr{L}_2)$ *and* $\mathrm{TPP}(\mathscr{L}_2, \mathscr{L}_3)$ *hold. Then* $\mathrm{WFROB}(\mathscr{L}_1, \mathscr{L}_3)$ *holds.*

(ii) *If in addition* $\mathscr{L}_3$ *is compact, then* $\mathrm{INT}(\mathscr{L}_1, \mathscr{L}_3)$ *holds.*

Proof. Let φ_1, φ_2 be two formulas of $\mathscr{L}_1(\tau_i)$, respectively, with $\tau_i = \tau_0 \cup_{\text{disjoint}} \{\mathbf{P}_i\}$, which form a counterexample to $\mathrm{WFROB}(\mathscr{L}_1, \mathscr{L}_3)$. Let $\mathfrak{A}_i$ be τ_i-structures such that $\mathfrak{A}_1 \upharpoonright \tau_0 \equiv_{\mathscr{L}_3} \mathfrak{A}_2 \upharpoonright \tau_0$. Without loss of generality we assume that both $\mathbf{P}_i$'s are of the same arity. In case they are unary, we apply Lemma 4.1.7 directly, otherwise we combine it with Remark 4.1.6. So we obtain a formula $\psi \in \mathscr{L}_1(\tau_0 \cup \tau_{\text{tree}} \cup \{\mathbf{P}\})$ which defines $\mathbf{P}$ implicitly. So let $\vartheta \in \mathscr{L}_2(\tau_0 \cup \tau_{\text{tree}})$ be an explicit definition of $\mathbf{P}$. So we get $\mathrm{Tree}^1_P(\mathfrak{A}_1, \mathfrak{A}_2) \models \vartheta(\mathbf{c})$ but $\mathrm{Tree}^0_P(\mathfrak{A}_1, \mathfrak{A}_2) \models \neg\vartheta(\mathbf{c})$ which contradicts

$$\mathrm{Tree}^1_P(\mathfrak{A}_1, \mathfrak{A}_2) \upharpoonright \tau_0 \cup \tau_{\text{tree}} \equiv_{\mathscr{L}_2} \mathrm{Tree}^0_P(\mathfrak{A}_1, \mathfrak{A}_2) \upharpoonright \tau_0 \cup \tau_{\text{tree}}.$$

as were required by $\mathrm{TPP}(\mathscr{L}_2, \mathscr{L}_3)$. □

Stating definability and preservation properties for pairs of logics allows us to sharpen results which were previously proven for absolute logics (and therefore for Karp logics). The reader should also consult Chapter XVII.

4.4.6 Proposition (Barwise). *If* $\mathscr{L}$ *is a logic which satisfies* $\mathrm{WFROB}(\mathscr{L}, \mathscr{L}_{\infty\omega})$, *then it has Löwenheim number* ω.

Proof. Since $\mathscr{L} \subset \mathscr{L}_{\infty\omega}$, $\mathscr{L}$ is a Karp logic. Therefore, if $\mathscr{L}$ properly extends first-order logic, there is a sentence $\varphi \in \mathscr{L}[\tau_1]$ such that the relativized reducts of its models are all countably infinite, by Lemma 2.1.2 of Chapter 3. Assume, for contradiction that there is a sentence $\psi \in \mathscr{L}[\tau_2]$ with $\tau_1 \cap \tau_2 = \{=\}$, which has only uncountable models. Let Σ be the $\mathscr{L}_{\omega\omega}$ theory of infinite sets. So Σ, φ, ψ form a counterexample to $\mathrm{WFROB}(\mathscr{L}, \mathscr{L}_{\infty\omega})$. □

4.4.7 Corollary. *Let $\mathscr{L}$ be a logic which satisfies* BETH($\mathscr{L}, \mathscr{L}_{\infty\omega}$). *Then it satisfies*

(i) WFROB($\mathscr{L}, \mathscr{L}_{\infty,\omega}$).
(ii) *The Löwenheim number $l_1(\mathscr{L})$ of $\mathscr{L}$ is ω.*

Proof. (i) follows from Theorem 4.4.5 and Proposition 4.2.16, and (ii) follows from (i) together with Proposition 4.4.6. □

We end this section with some more concrete examples:

4.4.8 Examples. (i) The logic $\mathscr{L}_{\omega\omega}(Q_1)$ from Chapters II or VII satisfies the tree preservation property, as one proves easily with a back-and-forth argument. By Counterexample II.7.1.3 it does not satisfy the interpolation property and therefore, since it is countably compact, not the weak finite Robinson property. So Theorem 4.4.5 gives us that it does not satisfy the Beth property.

(ii) The logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ is compact and does not satisfy the interpolation property by Counterexample II.7.1.3. It is not too difficult to check that TPP holds for this logic. So again by Theorem 4.4.5, the Beth property fails.

(iii) The logic $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ from Chapter IV does not satisfy the Beth property by Makowsky–Shelah [1981]. This is shown using the ideas in the proof of Theorem 4.4.5, though by Example 4.1.2(iv) $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ does not satisfy even the pair preservation property. To carry through the proof one has only to verify that it holds for specific structures.

(iv) We cannot replace TPP by PPP in Theorem 4.4.5, as the example in Section 4.6 shows.

4.5. *Definability and Existence of Models with Automorphisms*

The aim of this section is to explore further the consequences of the assumption that a logic $\mathscr{L}$ satisfies both PPP($\mathscr{L}$) and ROB($\mathscr{L}$). As stated in Problem 4.3.12, it is an open problem whether such logics exist which properly extend first-order logic. The results below may give us directions in solving that problem. Our main theorem is

4.5.1 Theorem (Shelah). *Let $\mathscr{L}$ be a small logic which has the pair preservation property and the Robinson property. Then every infinite τ-structure $\mathfrak{A}$ has $\mathscr{L}$-extensions with arbitrarily large τ-automorphism groups.*

For first-order logic this is a corollary to the celebrated theorem by Ehrenfeucht and Mostowski concerning indiscernibles. The reader may consult Chang–Keisler [1973, Chapter 3.3] for a detailed exposition. In the proof of Theorem 4.5.1 we discern various possibilities of defining abstract model theoretic properties centering around the existence of various automorphisms. Let us explore these first:

4.5.2 Definition. Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be logics.

(i) We say that the pair of logics $\mathscr{L}_1, \mathscr{L}_2$ has the *homogeneity property* (*homogeneity property for finite vocabularies*), if for every τ-structure (τ finite) $\mathfrak{M}$ and $c_1, c_2 \in M$ such that $\langle \mathfrak{M}, c_1 \rangle \equiv_{\mathscr{L}_2} \langle \mathfrak{M}, c_2 \rangle$ there is model $\langle \mathfrak{N}, c_1^N, c_2^N \rangle$ of $\mathrm{Th}_{\mathscr{L}_1}(\langle \mathfrak{M}, c_1, c_2 \rangle)$ and a τ-automorphism g of $\mathfrak{N}$ such that $g(c_1^N) = c_2^N$. If $\mathscr{L}_1 = \mathscr{L}_2$ we just say that $\mathscr{L}_1$ has the *homogeneity property* (*homogeneity property for finite vocabularies*).

(ii) We say that the pair of logics $\mathscr{L}_1, \mathscr{L}_2$ has the *local homogeneity property*, if for every τ-structure $\mathfrak{M}$ and $c_1, c_2 \in M$ such that $\langle \mathfrak{M}, c_1 \rangle \equiv_{\mathscr{L}_2} \langle \mathfrak{M}, c_2 \rangle$ and every $\varphi \in \mathrm{Th}_{\mathscr{L}_1}(\langle \mathfrak{M}, c_1, c_2 \rangle)$ there is model $\langle \mathfrak{N}, c_1^N, c_2^N \rangle \models \varphi$ and a τ-automorphsim g of $\mathfrak{N}$ such that $g(c_1^N) = c_2^N$. If $\mathscr{L}_1 = \mathscr{L}_2$ we just say that $\mathscr{L}_1$ has the *local homogeneity property*.

(iii) We say that $\mathscr{L}$ has the (*local*) *automorphism property*, if for every τ-structure $\mathfrak{M}$ and infinite subset $P \subset M$, the theory (every sentence φ of the theory) $\mathrm{Th}_{\mathscr{L}}(\langle , P \rangle)$ has a model $\langle \mathfrak{N}, P \rangle$ which has an automorphism g of $\mathfrak{N}$ such that $g \restriction P \neq \mathrm{Id}$.

4.5.3 Remarks. (i) If $\mathscr{L}$ is compact, then the local homogeneity property and the homogeneity property coincide. The same holds for the automorphism property. We shall be mainly interested in the compact case. The local case may be of independent interest for further developments.

(ii) If a logic does not satisfy the Beth property, one may construct its Beth closure in the natural way. Unlike the Δ-closure, studied in Chapter II and Chapter XVII, the Beth closure cannot easily be proven to preserve compactness. In Shelah [1983, Manuscript] the properties of the Beth closure were studied extensively. It turns out that stronger forms of the homogeneity property yield a sufficient condition for the Beth closure to preserve compactness. In Theorem 4.6.12 an example of a compact logic satisfying PPP and the Beth property is presented, whose proof relies on this idea.

4.5.4 Proposition* (Makowsky). (i) *Let $\mathscr{L}$ be a logic which has the automorphism property. Then $\mathscr{L}$ satisfies* REXT($\mathscr{L}$) *and therefore is* $[\omega]$-*compact.*

(ii) *Let $\mathscr{L}$ be a logic which has the local automorphism property. Then $\mathscr{L}$ has well-ordering number $w_1(\mathscr{L}) = \omega$. In particular, if $\mathscr{L}$ is recursively generated then $\mathscr{L}$ is also recursively compact.*

Proof. (i) We show that REXT($\mathscr{L}$), which is equivalent to $[\omega]$-compactness by Theorem 3.2.1. Let $\langle \mathfrak{M}, P^M \rangle$ be a τ-structure with $\mathbf{P} \in \tau$ and P^M infinite. Let τ_1 be a vocabulary, extending τ, giving every element in P^M a different name and let $\mathfrak{M}_1$ be the corresponding expansion. Clearly $\langle \mathfrak{M}_1, P^M \rangle$ still satisfies the hypothesis of the automorphism property. So let $\langle \mathfrak{N}, P^N \rangle$ be a $\mathscr{L}[\tau_1]$-extension of $\langle \mathfrak{M}_1, P^M \rangle$ with the required automorphism. Clearly, $P^M \subsetneq P^N$.

(ii) Here we just use that the standard model of arithmetic is rigid. For the latter remark we apply Remarks 4.2.5. ∎

In general the homogeneity property does not imply compactness.

4.5.5 Example. Let κ be a compact cardinal. The pair of logics $\mathscr{L}_{\kappa\omega}$, $\mathscr{L}_{\kappa\kappa}$ has the homogeneity property. To see this one uses an ultralimit construction as in Hodges–Shelah [1981]. Clearly, for $\lambda < \kappa$, these logics are not $[\lambda]$-compact.

However, for compact logics we have:

4.5.6 Proposition. *If $\mathscr{L}$ is a small and compact logic, which has the homogeneity property, then $\mathscr{L}$ has the automorphism property.*

Proof. Let $\langle \mathfrak{M}, P\rangle$ be a structure with P infinite. Using compactness there are $\mathscr{L}$-extensions $\langle \mathfrak{N}, P\rangle$ with P of arbitrary large cardinality. Using smallness we can find such an extension with $c_1, c_2 \in P$, $c_1 \neq c_2$ satisfying the same $\mathscr{L}$-type. Now we apply the homogeneity property. ☐

Now we are in a position to prove the existence of models with many automorphisms.

4.5.7 Proposition. *Let $\mathscr{L}$ be a compact logic with the automorphism property. Then every $\mathscr{L}$-theory with infinite models has models with arbitrarily large automorphism groups.*

Proof. Let Σ be an $\mathscr{L}$ theory and $\mathfrak{A}$ be an infinite model of Σ. We want to define by induction vocabularies τ_α and theories Σ_α which are sets such that, if $\mathfrak{A} \models \Sigma_\alpha$, then $\mathfrak{A} \upharpoonright \tau \models \Sigma$ and that $\mathfrak{A} \upharpoonright \tau$ has at least card(α) many different automorphisms.

For $\alpha = 0$ we proceed as follows. Since $\mathscr{L}$ is small the complete $\mathscr{L}$-theory Σ_0 of $\mathfrak{A}$ is a set. Again using smallness together with compactness we can find a model $\mathfrak{B}$ and $b, b' \in B$ satisfying the same type. So there is a model $\mathfrak{M}_0$ with a nontrivial automorphism F_0. Now we put Σ_1 to be the complete $\mathscr{L}$-theory of $\langle \mathfrak{M}_0, F_0\rangle$. Clearly, this also works for α successor. For α limit we put $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$. To show that Σ_α has a model we use compactness in the form of Proposition 1.1.1. ☐

4.5.8 Example (Shelah). We define a quantifier binding four variables and acting on two formulas (i.e., of type $\langle 2, 2\rangle$) in the following way: Let $\mathfrak{A}$ be a τ-structure.

$$\mathfrak{A} \models Q^{\mathrm{ibool}}uvwx(\varphi(u, v, \bar{z}), \psi(w, x, \bar{z}))[\bar{a}]$$

if $\langle A_\varphi^{\bar{a}}, R_\varphi^{\bar{a}}\rangle$ and $\langle A_\psi^{\bar{a}}, R_\psi^{\bar{a}}\rangle$ are partially ordered structures, where the order satisfies the axioms of a boolean algebra and

$$\langle A_\varphi^{\bar{a}}, R_\varphi^{\bar{a}}\rangle \cong \langle A_\psi^{\bar{a}}, R_\psi^{\bar{a}}\rangle.$$

By $A_\varphi^{\bar{a}}$ we denote the set $\{b \in A : \mathfrak{A} \models \varphi[b, b, \bar{a}]\}$ and by $R_\varphi^{\bar{a}}$ the relation

$$\{(b, c) \in A^2 : \mathfrak{A} \models \varphi[b, c, \bar{a}]\},$$

and similarly for ψ.

4.5.9 Theorem (Shelah [1983d]). *Assume* GCH. *Then the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$ is compact.*

4.5.10 Proposition. *There is a sentence $\Psi_{\mathrm{rigid}} \in \mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$ such that:*

(i) *Every model of Ψ_{rigid} is rigid, i.e., has no non-trivial automorphisms.*

(ii) *Ψ_{rigid} has models of every infinite cardinality.*

Proof. Let P be a ternary predicate symbol. Define Ψ_{rigid} to be the conjunction of the following formulas:

$$\psi_1 = \forall z Q^{\mathrm{ibool}} xyx'y'(P(x, y, z), P(x', y', z))$$

and

$$\psi_2 = \forall zz'(z \neq z' \rightarrow \neg Q^{\mathrm{ibool}} xyx'y'(P(x, y, z), P(x', y', z'))$$

To prove (i) let $\mathfrak{A}$ be a model of Ψ_{rigid}, $a \in A$ and let h be an automorphism of $\mathfrak{A}$. Clearly, $\langle A_R^a, R_P^a \rangle$ is a boolean algebra by ψ_1. Since h is an automorphism, so is $\langle A_R^{h(a)}, R_P^{h(a)} \rangle$ and they are isomorphic. So, by ψ_2, $h(a) = a$.

To prove (ii), let λ an infinite cardinal and $\{\mathfrak{B}_i = \langle B_i, \leq_i \rangle : i < \lambda\}$ a family of λ many pairwise non-isomorphic boolean algebras of cardinality λ each. Without loss of generality $B_i = \lambda$. We define a model $\mathfrak{A} = \langle A, P^A \rangle$ of Ψ_{rigid} as follows: We put $A = \lambda$ and $P^A = \{(i, a, b) \in \lambda^3 : a \leq_i b\}$. Clearly, $\mathfrak{A} \models \Psi_{\mathrm{rigid}}$. Note that (ii) does not follow from the compactness of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$. On the other hand (ii) does not require GCH, as the proof of compactness. $\square$

4.5.11 Corollary (GCH). *The logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{ibool}})$ is compact but does not satisfy the homogeneity property.*

Proof. By Theorem 4.5.9 the logic is compact. Assume, for contradiction, the homogeneity property. So by Propositions 4.5.6 and 4.5.7 we get models with arbitrarily large automorphism groups, contradicting Proposition 4.5.10. $\square$

4.5.12 Proposition (GCH). *There is a compact logic $\mathscr{L}$ which does not have the automorphism property.*

Proof. This follows from Proposition 4.5.10 and Corollary 4.5.11. $\square$

4.5.13 Theorem (Shelah). *Let $\mathscr{L}$ be a logic.*

(i) *If $\mathscr{L}$ satisfies* PPP($\mathscr{L}$) *and* ROB($\mathscr{L}$), *then $\mathscr{L}$ has the homogeneity property.*

(ii) *If $\mathscr{L}$ satisfies* PPP($\mathscr{L}$) *and* FROB($\mathscr{L}$), *then $\mathscr{L}$ has the homogeneity property for finite vocabularies.*

(iii) *If $\mathscr{L}$ satisfies* PPP($\mathscr{L}$) *and* INT($\mathscr{L}$), *then $\mathscr{L}$ has the local homogeneity property.*

Proof. We prove only (i), the others being similar. Let $\mathfrak{M}$ and $c_1, c_2 \in M$ be as in the hypothesis of the homogeneity property.

Let $\mathfrak{M}'$, c_1', c_2' be disjoint copies. Put $\mathfrak{N} = [\mathfrak{M}, \mathfrak{M}']$. Put

$$T = \mathrm{Th}_{\mathscr{L}}(\langle \mathfrak{N}, c_1, c_2, c_1' \rangle) = \mathrm{Th}_{\mathscr{L}}(\langle \mathfrak{N}, c_1, c_2, c_2' \rangle).$$

The equality holds because of PPP($\mathscr{L}$). Let $\mathbf{c}_1, \mathbf{c}_2$ be constant symbols with interpretations c_1, c_2 and $\mathbf{c}$ be a constant symbol with interpretation c_1' or c_2', respectively. Let $\mathbf{F}$ be a new function symbol. Let $\psi_i (i = 1, 2)$ be the sentence which says that $\mathbf{F}$ is a τ-isomorphism (modulo name changing) mapping the first sort into the second sort which maps $\mathbf{c}_i$ into $\mathbf{c}$. If τ is infinite, we need a set of sentences Ψ_i defined similarly.

Clearly, $T \cup \{\psi_i\}$ has a model. So by ROB($\mathscr{L}$) or, if τ is finite, by WROB($\mathscr{L}$), $T \cup \{\psi_1, \psi_2\}$ has a model $[\mathfrak{M}_1, \mathfrak{M}_1']$ which gives as the required automorphism in $\mathfrak{M}_1$ $\square$

4.5.14 Remarks. (i) In Proposition 4.5.13 above the three cases coincide for compact logics.

(ii) If we assume that the logics are tiny, the hypotheses in the cases 4.5.13(i) and (ii) imply that the logics are $[\omega]$-compact and ultimately compact. Assuming that $\mathscr{L}$ has an dependence number $o(\mathscr{L})$ which is smaller than the first uncountable measurable cardinal, the hypothesis in Theorem 4.5.13(i) actually implies compactness. In Theorem 4.5.13(ii) we need for this, that the logic $\mathscr{L}$ has size function $s(\omega) < \omega_n$ for some $n \in \omega$ (cf. Theorem 4.3.8 and Corollary 3.3.5).

4.5.15 Corollary. *Let $\mathscr{L}$ be a logic with dependence number $o(\mathscr{L})$ smaller than the first uncountable measurable cardinal (or, alternatively, with size function $s(\omega) < \omega_n$ for some $n \in \omega$). If $\mathscr{L}$ satisfies* PPP($\mathscr{L}$) *and* ROB($\mathscr{L}$), *then $\mathscr{L}$ has the automorphism property.*

Proof. We use Remark 4.5.14(ii) above and Proposition 4.5.13. $\square$

This corollary, together with Theorem 4.5.7 gives us a proof of Theorem 4.5.1.

4.6. Some More Examples: Stationary Logic and Its Friends

In this last section we want to discuss, mostly without proofs, some more examples and consistency results, which all come from Shelah [198?e] and Mekler–Shelah [1983, 198?]. They are all concerned with preservation and definability properties of compact or (ω, ω)-compact logics. Our first example concerns extensions of $\mathscr{L}_{\omega\omega}(Q_1)$. Let us recall some facts:

4.6.1 Proposition. *The logic $\mathscr{L}_{\omega\omega}(Q_1)$ has the following properties:*

(i) *$\mathscr{L}_{\omega\omega}(Q_1)$ is (ω, ω)-compact, but not (ω_1, ω)-compact.*
(ii) *$\mathscr{L}_{\omega\omega}(Q_1)$ does satisfy the pair preservation property.*
(iii) *$\mathscr{L}_{\omega\omega}(Q_1)$ does not satisfy the Δ-interpolation property, and therefore neither the interpolation property.*

It remains open whether $\mathscr{L}_{\omega\omega}(Q_1)$ satisfies the weak Beth property. However, there is the following consistency result proved in Mekler–Shelah [198?].

4.6.2 Theorem (Shelah). *Every model* $\mathfrak{M}$ *of* ZFC *has a generic extension* $\mathfrak{M}[G]$ *in which* $\mathscr{L}_{\omega\omega}(Q_1)$ *satisfies the weak Beth property.*

For the stronger definability properties there is a consistency result in the other direction. We want to state, that it is consistent with ZFC, that no "reasonable" extension of $\mathscr{L}_{\omega\omega}(Q_1)$ satisfies both PPP and the interpolation property (or equivalently the uniform reduction property UR_2). For this we need a definition:

4.6.3 Definition (Definable Logics). (i) A logic $\mathscr{L}$ is *definable*, if the relations "$\varphi \in \mathscr{L}[\tau]$" ("$\varphi$ is a $\mathscr{L}[\tau]$-formula") and "$\mathfrak{M} \models \varphi$" ("$\mathfrak{M}$ is a model of φ") are definable by a formula of set theory *without parameters.*

(ii) A logic $\mathscr{L}$ is *λ-definable*, for λ a cardinal, if the relations "$\varphi \in \mathscr{L}[\tau]$" ("$\varphi$ is a $\mathscr{L}[\tau]$-formula") and "$\mathfrak{M} \models \varphi$" ("$\mathfrak{M}$ is a model of φ") are definable by a formula of set theory *with a parameter* $A \subset \lambda$.

4.6.4 Remark. In Chapter XVII *absolute logics* were introduced. This notion is not quite comparable with the above definition. For a logic to be absolute definability *with parameters* is allowed, but definability is restricted to Δ_1-definability.

4.6.5 Examples. (i) Logics of the form $\mathscr{L}_{\omega\omega}(Q^i)_{i\in n}$ are definable, provided the quantifiers are set presentable in the sense of Definition 1.5.8.

(ii) The logics $\mathscr{L}_{\kappa\lambda}$ are definable.

(iii) Not all logics are definable without parameters. Especially some of the fragments $\mathscr{L}_A \subset \mathscr{L}_{\omega_1\omega}$ are not definable, but they are ω_1-definable with parameter $A \subset \omega_1$. If A is a countable admissible fragment which has a code in ω then $\mathscr{L}_A$ is even ω-definable.

(iv) The logic $\mathscr{L}_{F\omega}$ from Section 1.6 is definable, provided the ultrafilter F is definable. The definability of this filter may very well depend on the set-theoretic assumptions under consideration.

4.6.6 Theorem (Shelah). *For every model* $\mathfrak{M}$ *of* ZFC *that there is generic extension* $\mathfrak{M}[G]$ *such that no definable logic* $\mathscr{L}$ *extending* $\mathscr{L}_{\omega\omega}(Q_1)$ *satisfies both* $\mathrm{PPP}(\mathscr{L})$ *and the interpolation property (or, equivalently, the uniform reduction property* UR_2*) in* $\mathfrak{M}[G]$.

It was widely believed that the Δ-closure of $\mathscr{L}_{\omega\omega}(Q_1)$ is a rather untackable logic. That this need not be the case is shown by the next consistency result from Mekler–Shelah [198?]. Let us first recall some facts about the logic $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ from Section IV.4 and Counterexample II.7.1.3.

4.6.7 Proposition. (i) *The logic* $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ *is* (ω, ω)*-compact, r.e. for validity, but does not satisfy the interpolation property.*
(ii) $\mathscr{L}_{\omega\omega}(Q_1)$ *is a sublogic of* $\mathscr{L}_{\omega\omega}(\mathrm{aa})$.
(iii) $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ *does not hold.*

Inspired by Theorem 4.6.2 we can state the following problem:

4.6.8 Problem. (Shelah). Does every model $\mathfrak{M}$ of ZFC have a generic extension $\mathfrak{M}[G]$ in which Δ-$\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ holds?

In Mekler–Shelah [1983] a positive answer is given for Δ-Interpolation on finitely determinate structures. In contrast to this it is shown in Counterexample II.7.1.3 that $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q_1), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$ does not hold.

The next example involves the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$.

4.6.9 Proposition. (i) *The logic* $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ *is compact, r.e. for validity, but does not satisfy the interpolation property.*
(ii) $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ *is a sublogic of* $\mathscr{L}_{\omega\omega}(\mathrm{aa})$.

Proof. From Section II.2.4, and Makowsky–Shelah [1981] we know (i). To see (ii) we axiomatize the class of orderings of cofinality ω by the $\mathscr{L}_{\omega\omega}(\mathrm{aa})$-sentence which says that the ordering has no last element, but that almost every countable set P is unbounded. □

The next theorem shows that $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ behaves more like second-order logic, than originally suspected, since it provides interpolating formulas for the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$. Note that for Hanf number calculations $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ is as strong as the logic which allows unrestricted quantification over countable sets, as shown in Kaufmann–Shelah [198?].

4.6.10 Theorem (Shelah). $\mathrm{INT}(\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)}), \mathscr{L}_{\omega\omega}(\mathrm{aa}))$.

The proof may be found in Mekler–Shelah [198?].

4.6.11 A Generalization. The pair of logics in Theorem 4.6.10 can be generalized to higher cardinals. For $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}(\omega)})$ this gives us the logics $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq\lambda})$ which requires the ordering to be of infinite cofinality less or equal to λ. As shown in Makowsky–Shelah [1981] this logic is still compact, but does not satisfy the interpolation property. For $\mathscr{L}_{\omega\omega}(\mathrm{aa})$ we have to define a logic $\mathscr{L}_{\omega\omega}(\mathrm{aa}_\lambda)$ for an appropriate filter D_λ. A detailed exposition may be found in Mekler–Shelah [198?]. What is important here, is a theorem of Shelah which states that the pair $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq\lambda})$ and $\mathscr{L}_{\omega\omega}(\mathrm{aa}_\lambda)$ satisfies a strong form of the homogeneity property, as defined in Section 4.5. As mentioned in Section 4.5, such homogeneity properties can be used to prove that the Beth closure preserves PPP and compactness.

Using this line of thought Shelah proved the following theorem:

4.6.12 Theorem (Shelah). *The Beth closure $\mathscr{L}$ of the logic $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ is a compact logic which satisfies:*

(i) PPP($\mathscr{L}$) (*and therefore, by compactness,* URP);
(ii) *has the Beth property; but*
(iii) *does not satisfy the interpolation property (and therefore, by compactness, none of the Robinson properties).*

This shows, that in Theorem 4.4.5 the tree preservation property cannot be weakened to the pair preservation property. For otherwise, since the logic is compact, the Beth property would imply the interpolation property. It also shows that the uniform reduction property for pairs does not imply even the uniform reduction property UR_1, which, by Theorem 4.2.12 is equivalent to the interpolation property.

This example is also the first example so far, which exhibits a compact logic satisfying the Beth property. Note that it is easy to construct compact logics, which satisfy the weak Beth property or the Δ-interpolation property by the construction of the Δ-closure or weak Beth closure, as described in Proposition II.7.2.5 and, in more detail, Makowsky–Shelah–Stavi [1976].

Also the Δ-closure of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ has remarkable properties:

4.6.13 Theorem* (Shelah). *The Δ-closure of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ does not have the Beth property.*

A proof will appear in Makowsky–Shelah [198?b].

The following is open:

4.6.14 Problem. Is there a logic $\mathscr{L}$ which satisfies both the Beth property and Δ-interpolation, is compact but does not satisfy the interpolation property? In particular, is the iterated Beth and Δ-closure of $\mathscr{L}_{\omega\omega}(Q^{\mathrm{cf}}_{\leq 2^\omega})$ compact, and if yes, does it satisfy the interpolation property?

4.7. *Which Definability Property?*

The first definability property proven for $\mathscr{L}_{\omega\omega}$ was the Beth property (Beth [1953]). The interpolation property was introduced in Craig [1957b], and is sometimes also called Craig's interpolation property. Its main application was to give a simplified proof of the Beth property. Another proof of the Beth property for $\mathscr{L}_{\omega\omega}$ was given in Robinson [1956a] where the Robinson property, or rather the finite Robinson property, was introduced. The choice of these properties was not really questioned in this period. The weak Beth property was first discussed in Friedman [1973]. Friedman suggested also first that it was the weak Beth property which really mattered in the context of logics different from first-order logic. The first thorough discussion of definability properties for logics in general is in Feferman [1974a, b, 1975].

Feferman focuses the attention on the Δ-interpolation, pointing out its equivalence to the weak projective Beth property. His paper had great impact and the Δ-closure was studied extensively in Barwise [1974], Makowsky–Shelah–Stavi [1976], Hutchinson [1976], Väänanen [1977a, 1979a, 1983], Paulos [1976] and Makowsky–Shelah [198?]. From this it emerged that the Δ-closure may well be a "better" definability property than all the others studied so far. This is especially so, since the Δ-closure of a logic $\mathscr{L}$ preserves compactness and the recursive enumerability of the validities of its finitely generated sublogics.

It was also in Feferman [1974a, b] and in Feferman [1972] that preservation properties were first discussed in the general setting. $\mathrm{UR}_n(\mathscr{L})$ was introduced to unify known preservation theorems and interpolation theorems. In Makowsky [1978] the equivalence of $\mathrm{UR}_1(\mathscr{L})$ and the interpolation property was established. From this one was led to think that the next "reasonable" strengthening of the Δ-interpolation property would be uniform reduction $\mathrm{UR}_2(\mathscr{L})$. Note that the equivalence of non-uniform and uniform reduction for pairs PPP and URP for compact logics, due to Shelah, appears here for the first time.

The finite Robinson property was first discussed in the general setting in Makowsky–Shelah [1976] and the Robinson property in Mundici [1979a]. Mundici suggested that the Robinson property is a "natural" property of logics, since it is equivalent, for finitely generated logics, to compactness and the interpolation property. But, as it emerges in this chapter, it seems to us that it is the Robinson property together with PPP which has more merits: In the case of compact logics they are together again equivalent to $\mathrm{UR}_2(\mathscr{L})$ or to the preservation property for projective operations PPPO.

It should be pointed out here that this comparison of definability properties has still a severe drawback: The lack of an abundance of examples. There are, by now, many compact, and therefore many compact and Δ-closed logics, mostly constructed by Shelah. But there are no interesting examples satisfying any strengthening of the interpolation property, such as uniform reduction or the Robinson property.

TABLES

Table 1. Transfer of Compactness Properties

From	To	Condition	Reference
cf(κ)	κ		1.1.6
κ^+	κ	κ singular κ regular	1.3.11(i) 1.5.4 1.3.11(ii) 1.5.4
κ	ω	$\kappa < \mu_0$ μ_0 first uncountable measurable cardinal	1.5.2

Table 2. The Compactness Spectrum

Form	Condition	Reference
Comp($\mathscr{L}$) is initial segment	$A(\infty)$	1.5.7(i)
Comp($\mathscr{L}$) contains final segment	Vopenka's principle	1.5.16(iv)
First element measurable		1.5.2
Gaps in spectrum		1.6

Table 3. Transfer of Dependence Properties

From	To	Condition	Reference
κ	ω	compactness	5.1.3 in Chapter II
κ	ω	[ω]-compactness $\kappa < \mu_0$ μ_0 first uncountable measurable cardinal	2.2.1
κ	Finite dependence structure	[ω]-compactness	2.4.3

Table 4. Compactness and Extensions

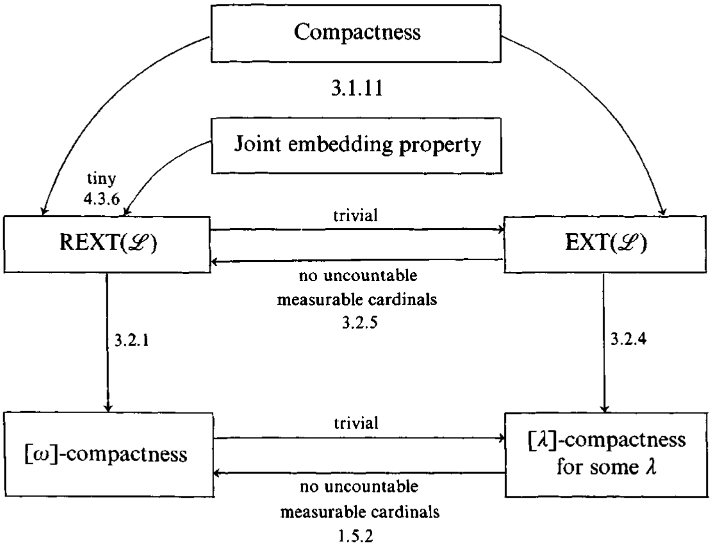


Table 5. Amalgamation, Joint Embeddings, and Compactness

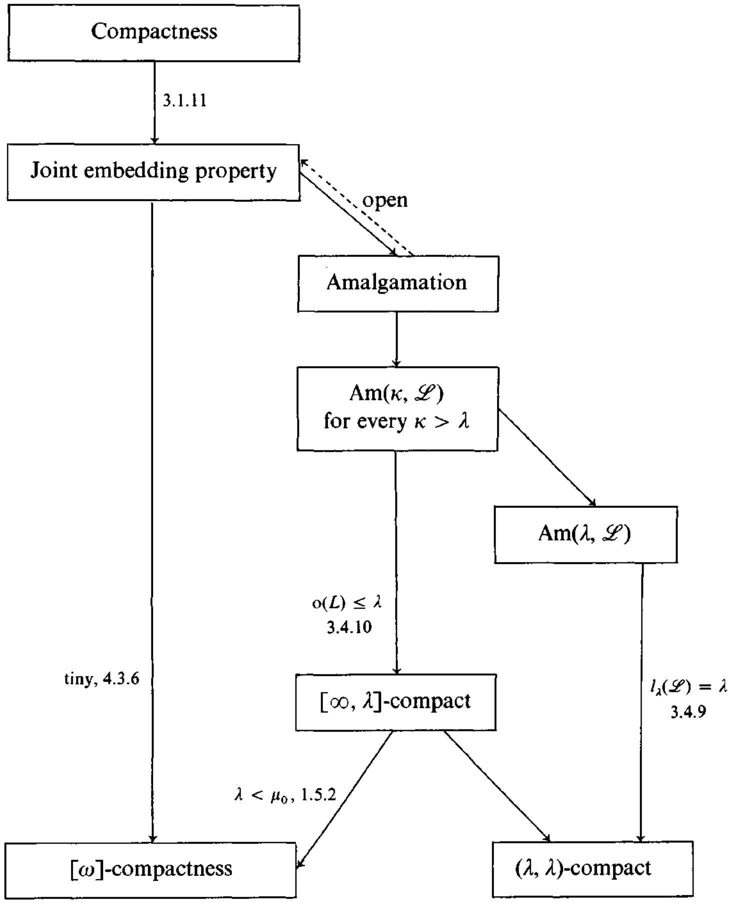


Table 6. Compactness, Definability, and Automorphisms (for logics with finite dependence)

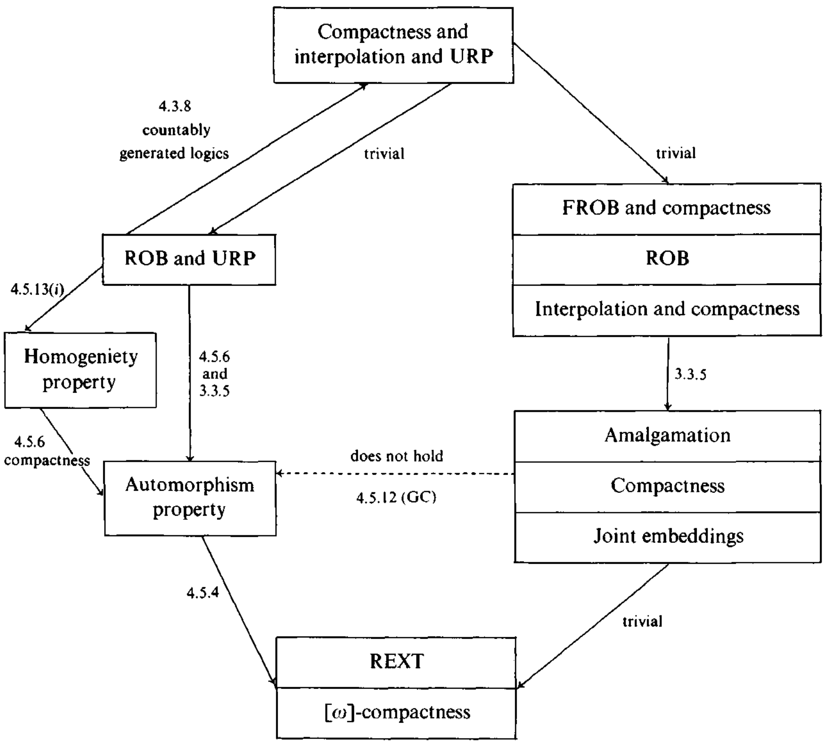


Table 7. Definability Properties

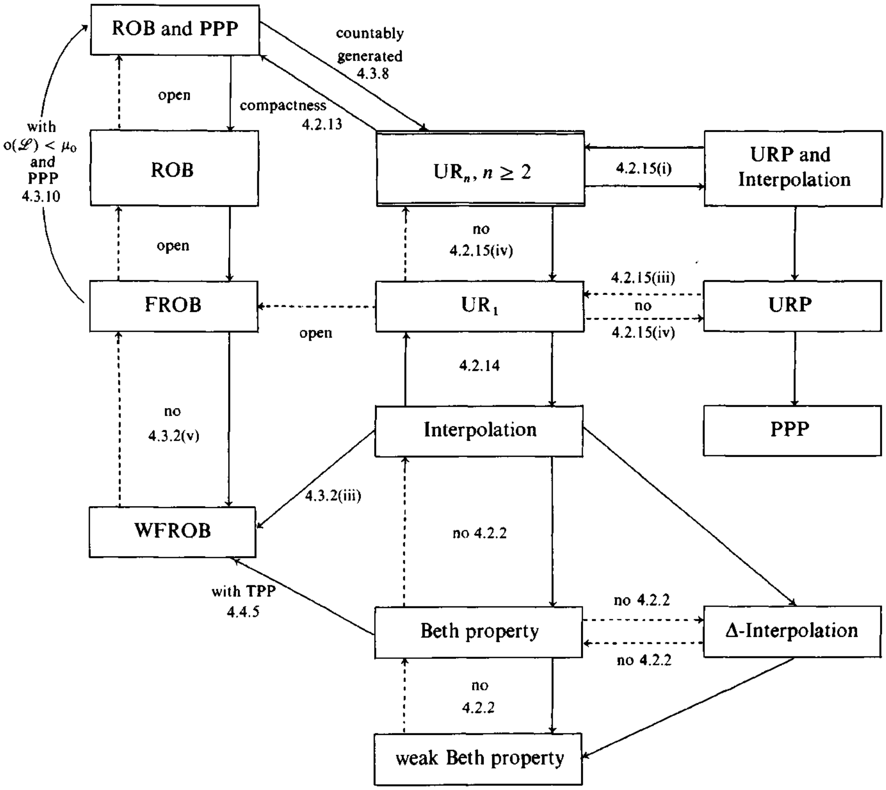
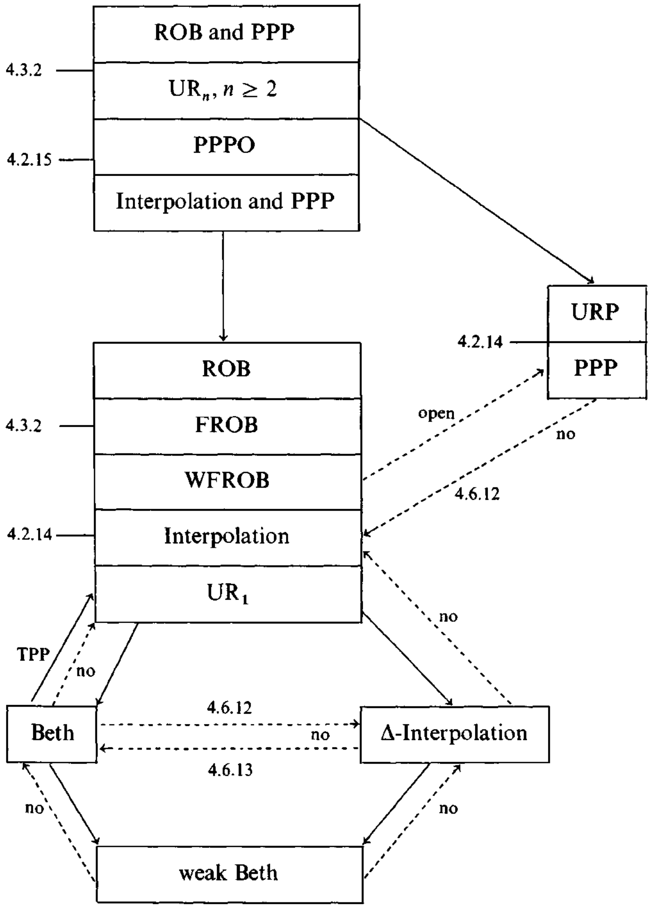


Table 8. Definability for Compact Logics



Chapter XIX
Abstract Equivalence Relations

by J. A. MAKOWSKY and D. MUNDICI

For a regular logic $\mathscr{L}$, let $\sim\ =\ \equiv_{\mathscr{L}}$ be the equivalence relation obtained by saying that two structures are $\sim$-equivalent iff they satisfy the same sentences of $\mathscr{L}$. The isomorphism relation $\cong$ is automatically a refinement of $\sim$—that is, isomorphic structures are $\sim$-equivalent— $\sim$ itself is a refinement of elementary equivalence $\equiv$, and $\sim$ is preserved under both renaming and reduct. This last property simply means that upon renaming, or taking reducts of $\sim$-equivalent structures, we obtain $\sim$-equivalent structures. Furthermore, if $\mathscr{L}[\tau]$ is a set for every vocabulary τ, then the collection of equivalence classes given by $\sim$ on $\mathrm{Str}(\tau)$ has a cardinality. (Briefly, we say that $\sim$ is *bounded*). This paper is mainly concerned with abstract equivalence relations $\sim$ on $\bigcup_\tau \mathrm{Str}(\tau)$, having the above-mentioned properties as well as the *Robinson property* so that for every $\mathfrak{M}$, $\mathfrak{N}$, and τ with $\tau = \tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}}$,

$$\begin{aligned}&\text{if}\quad \mathfrak{M}\restriction\tau \sim \mathfrak{N}\restriction\tau \quad \text{then for some } \mathfrak{A},\\ &\mathfrak{M} \sim \mathfrak{A}\restriction\tau_{\mathfrak{M}} \quad\text{and}\quad \mathfrak{N} \sim \mathfrak{A}\restriction\tau_{\mathfrak{N}}.\end{aligned}$$

If $\sim\ =\ \equiv_{\mathscr{L}}$, then $\sim$ has the Robinson property iff $\mathscr{L}$ satisfies the Robinson consistency theorem. If, in addition, $\mathscr{L}[\tau]$ is a set for all τ, and if all sentences in $\mathscr{L}$ have a finite vocabulary, then the Robinson consistency theorem holds in $\mathscr{L}$ iff $\mathscr{L}$ is compact and has the interpolation property (see Corollary 1.4). Every bounded equivalence relation $\sim$ with the Robinson property satisfies the equation $\sim\ =\ \equiv_{\mathscr{L}}$ for *at most one* logic $\mathscr{L}$ (see Corollary 3.4). This result can be extended to equivalence relations corresponding to compact logics (see Theorem 3.11). Moreover, we have that $\sim\ =\ \equiv_{\mathscr{L}}$ for *exactly one* logic $\mathscr{L}$ iff $\sim$ is *separable* by quantifiers, in the sense that whenever $\mathfrak{M}$ and $\mathfrak{N}$ are not $\sim$-equivalent, there is a quantifier Q such that $\sim$ is a refinement of $\equiv_{\mathscr{L}(Q)}$ and $\mathfrak{M} \not\equiv_{\mathscr{L}(Q)} \mathfrak{N}$ (see (ii) of Theorem 3.10). Even if $\sim$ is not separable by quantifiers, there is still a strongest logic $\mathscr{L}$ such that $\sim$ refines $\equiv_{\mathscr{L}}$. This $\mathscr{L}$ is compact and can be written as $\mathscr{L} = \mathscr{L}\{Q \mid \sim \text{ is a refinement of } \equiv_{\mathscr{L}(Q)}\}$ (see Corollary 3.3 and (i) of Theorem 3.10).

The Robinson property of $\mathscr{L}$ can also be coupled with such properties as $[\omega]$-incompactness. Then $\equiv_{\mathscr{L}}$ will coincide with $\cong$ below the first uncountable measurable cardinal μ_0 (see Theorem 1.7), and the infinitary logic $\mathscr{L}_{\mu_0\omega}$ can be interpreted in $\mathscr{L}$ in some natural sense (refer to Theorem 1.12).

Some of the results in Section 1 can be extended to logics for enriched structures, such as topological, uniform, and proximity structures, as discussed in Section 2.

With any logic $\mathscr{L}$ we can associate an embedding relation $\to_{\mathscr{L}}$, where $\mathfrak{A} \to_{\mathscr{L}} \mathfrak{N}$ means that $\tau_{\mathfrak{N}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \equiv_{\mathscr{L}} \mathfrak{N}^+$ for some expansion $\mathfrak{N}^+$ of $\mathfrak{N} \upharpoonright \tau_{\mathfrak{A}}$, with $\mathfrak{A}_A$ denoting, as usual, the diagram expansion of $\mathfrak{A}$. In Definition 4.1 we define embedding relations by abstracting these properties of the $\to_{\mathscr{L}}$ relation. Any such relation $\to$ generates an equivalence relation $\sim\; = \;\to^*$ by writing $\mathfrak{A} \sim \mathfrak{N}$ iff $\mathfrak{A}$ and $\mathfrak{N}$ are connected by a finite path of arrows. Conversely, every equivalence relation $\sim$ generates an embedding relation $\to\; = \;\sim^*$ by writing $\mathfrak{A} \to \mathfrak{N}$ iff $\tau_{\mathfrak{N}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \sim \mathfrak{N}^+$, for some expansion $\mathfrak{N}^+$ of $\mathfrak{N} \upharpoonright \tau_{\mathfrak{A}}$. The mapping $*$ sends equivalence relations with the Robinson property into embedding relations with the *expanded amalgamation property* (AP^+) in a one–one way, the latter being a natural strengthening of the usual amalgamation property (AP). The mapping $*$ becomes a bijection with $** =$ identity, provided we restrict ourselves to embedding relations with AP^+ and such that $\to^{**} = \to$ (see Theorem 4.8). In particular, first-order logic $\mathscr{L}_{\omega\omega}$ is uniquely determined by the familiar elementary embeddability relation $\precsim$ (see Theorem 4.9).

Every countably generated logic $\mathscr{L} = \mathscr{L}(Q^i)_{i<\omega}$ determines, for each finite vocabulary τ, a sequence $\{\simeq_\tau^n\}_{n<\omega}$ of finite partitions over $\mathrm{Str}(\tau)$, by writing $\mathfrak{M} \simeq_\tau^n \mathfrak{N}$ iff $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same $\mathscr{L}[\tau]$-sentences of quantifier rank $\leq n$. We study an abstract notion of back-and forth systems (see Definition 5.1); the latter generalize the celebrated Fraïssé–Ehrenfeucht games for $\equiv$ (see Examples 5.2 and Theorem 5.3) and are in one–one correspondence with their associated logics, under the Robinson assumption (see Theorem 5.4). By use of Theorems 3.11 and 5.7 and the argument in Theorem 5.4 this correspondence can be extended to the realm of compact logics.

Any back-and-forth game G for $\mathscr{L}$-elementary equivalence determines not only a back-and-forth system in the above sense, but also a game $G(\mathfrak{A}, \mathfrak{B})$ for pairs of structures, or—equivalently—a decreasing sequence of sets of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. We regard the former as a global version of G (since each partition acts on the whole of $\mathrm{Str}(\tau)$), and the latter as a local version of G. Global and local versions have the same extreme generality (see Theorems 5.7 and 5.10) and are closely related, as is discussed in Section 5.

As this chapter will show, the Robinson property is very strong. Indeed, one of the main open problems of abstract model theory—a problem originally posed by Feferman—asks whether compactness and interpolation together are strong enough to characterize first-order logic. A negative answer would exhibit a proper extension of $\mathscr{L}_{\omega\omega}$ still having many important features in common with $\mathscr{L}_{\omega\omega}$ (by the very results of this chapter) while a positive answer would characterize $\mathscr{L}_{\omega\omega}$ in terms of properties which are generally reputed to be desirable for a logic $\mathscr{L}$. As a matter of fact, compactness is related to the finiteness of sentences and proofs in $\mathscr{L}$ and makes available a number of methods for constructing models; interpolation (together with its most notable consequence, Δ-closure—or equivalently—truth-maximality) is related to the equilibrium between syntax and semantics in $\mathscr{L}$.

Whatever the ultimate answer to this problem, the techniques and results of this chapter can be applied to logics for enriched structures (see Section 2). Furthermore, even for ordinary structures, several theorems originally stated under the Robinson assumption, can now be proved under the (weaker) compactness, or

JEP assumption (see Theorems 1.1, 2.4, 3.11, Lemma 3.11.1, Corollary 3.12, Remark 4.10, and Theorem 5.7) by simply refining the methods developed for the study of the Robinson property. Sometimes there are even applications to first-order logic itself (see Corollary 3.5, Theorem 4.9, and Corollary 5.5).

Throughout this chapter logics are assumed to satisfy the occurrence axiom, stating that for each sentence φ in $\mathscr{L}$ there is a smallest $\tau = \tau_\varphi$ such that $\varphi \in \mathscr{L}[\tau]$. We will write $\mathscr{L}(Q^i)_{i \in I}$ instead of $\mathscr{L}_{\omega\omega}(Q^i)_{i \in I}$ and will always assume that Q^i is a quantifier with built in relativization, as in Proposition II.4.1.5. Given equivalence relations $\sim$ and $\sim'$, instead of saying that $\sim$ is a refinement of $\sim'$ we will usually say that $\sim$ is *finer* than $\sim'$ (or, that $\sim'$ is *coarser* than $\sim$). We will constantly work with many-sorted structures and logics, so that our expansions may very well involve new sorts. Vocabularies and universes of structures are always assumed to be sets, while $\mathscr{L}[\tau]$ may be a proper class. However, when we want to exclude this possibility for $\mathscr{L}$ we will simply say that $\mathscr{L}[\tau]$ is a set for each τ.

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1. Logics with the Robinson Property

In Chapter XVIII we saw that logics satisfying the amalgamation property are compact, provided they have the finite dependence property, or even if the dependence number is smaller than the first uncountable measurable cardinal. The amalgamation property is both a consequence of the Robinson property, and of the joint embedding property. In this section we will review the relationship between the latter two properties and compactness, since under these stronger hypotheses many of the proofs are simpler and generalize to the case of logics whose underlying structures need not be first-order structures. Recall that a logic $\mathscr{L}$ has the *joint embedding property* (abbreviated JEP) iff whenever $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ and $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are jointly embeddable in some structure $\mathfrak{M}$. That is, $\mathfrak{M} \models_{\mathscr{L}} \mathrm{Th}_{\mathscr{L}}(\mathfrak{A}_A) \cup \mathrm{Th}_{\mathscr{L}}(\mathfrak{B}_B)$, where $\mathfrak{A}_A$ (resp., $\mathfrak{B}_B$) is the diagram expansion of $\mathfrak{A}$ (resp., of $\mathfrak{B}$) in vocabulary $\tau_A = \tau \cup \{c_a\}_{a \in A}$ (resp., $\tau_B = \tau \cup \{c_b\}_{b \in B}$), and $\tau_A \cap \tau_B = \tau$.

1.1 Theorem. *Let $\mathscr{L}$ be a regular logic such that $\mathscr{L}[\tau]$ is a set for every τ. Assume that for every countable τ_0, $|\mathscr{L}[\tau_0]| \leq \lambda$ for some fixed λ. If $\mathscr{L}$ satisfies the JEP, then there are at most 2^λ many regular cardinals κ such that $\mathscr{L}$ is not $[\kappa]$-compact.*

Proof. Let S be a set of regular cardinals such that $\mathscr{L}$ is not $[\kappa]$-compact for each $\kappa \in S$. By Definition XVIII.1.2.1, κ is cofinally characterizable in $\mathscr{L}$; hence, there is a structure $\mathfrak{A}_\kappa = \langle \kappa, <, \ldots \rangle$ whose diagram expansion

$$\mathfrak{A}_A = \langle \kappa, <, \{c_\alpha\}_{\alpha<\kappa}, \ldots \rangle$$

has the following property:

(*) whenever $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}_A$ the set $\{c_\alpha^{\mathfrak{B}}\}_{\alpha<\kappa}$ is unbounded in the order $<^{\mathfrak{B}}$.

Adding, if necessary, more sorts and elements, and using the regularity properties of $\mathscr{L}$, we can safely assume that the vocabulary τ_κ of $\mathfrak{A}_\kappa$ is countable (we can code n-ary relations into a single "universal" $(n+1)$-ary relation). Indeed, we can safely assume that for some fixed countable vocabulary $\tau \supseteq \{<\}$, the τ_κ are all equal to τ, for any $\kappa \in S$. Let T_κ be the complete $\mathscr{L}$-theory of $\mathfrak{A}_\kappa$ in vocabulary τ. By hypothesis, there are at most 2^λ such theories. Therefore, if $|S| > 2^\lambda$, then for two regular cardinals $\mu < \nu$ in S, we must have that $\mathfrak{A}_\mu \equiv_{\mathscr{L}} \mathfrak{A}_\nu$. Let $\mathfrak{M} = \mathfrak{A}_\mu$, $\mathfrak{N} = \mathfrak{A}_\nu$. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are joint embeddable in a structure $\mathfrak{D}$ (*absurdum* hypothesis), say $\mathfrak{D} \models \mathrm{Th}(\mathfrak{M}_M) \cup \mathrm{Th}(\mathfrak{N}_N)$, and let $\tau_M = \tau \cup \{c_m\}_{m\in M}$ and $\tau_N = \tau \cup \{g_n\}_{n\in N}$ be the vocabularies of $\mathfrak{M}_M$ and $\mathfrak{N}_N$, respectively, with $\tau_M \cap \tau_N = \tau$. Since

$$\mathfrak{M}_M = \langle \mu, <, \{c_\alpha\}_{\alpha<\mu}, \ldots \rangle$$

the set $\{c_m\}_{m\in M}$ contains a subset $\{c_\alpha\}_{\alpha<\mu}$ whose interpretation in $\mathfrak{M}_M$ are the ordinals $\alpha < \mu$. Similarly, $\{g_n\}_{n\in N}$ contains a subset $\{g_\beta\}_{\beta<\nu}$ whose interpretation in $\mathfrak{N}_N$ are the ordinals $\beta < \nu$. Now consider the linear order $<^{\mathfrak{D}}$. Since $\mathfrak{D} \models \mathrm{Th}\ \mathfrak{M}_M$ then $\langle \mu, < \rangle \cong \langle \{c_\alpha^{\mathfrak{D}}\}_{\alpha<\mu}, <^{\mathfrak{D}} \rangle$, and the set $\{c_\alpha^{\mathfrak{D}}\}_{\alpha<\mu}$ is unbounded in $<^{\mathfrak{D}}$, by (*). Similarly, from $\mathfrak{D} \models \mathrm{Th}\ \mathfrak{N}_N$, we obtain that $\langle \nu, < \rangle \cong \langle \{g_\beta^{\mathfrak{D}}\}_{\beta<\nu}, <^{\mathfrak{D}} \rangle$ and the set $\{g_\beta^{\mathfrak{D}}\}_{\beta<\nu}$ is unbounded in $<^{\mathfrak{D}}$. Therefore, we get that μ is cofinal in $\nu > \mu$, thus contradicting the assumed regularity of ν. Therefore, the JEP fails in $\mathscr{L}$ if $|S| > 2^\lambda$. □

To be able to prove that no incompactness exists, we will need some set-theoretic hypotheses, as discussed in Section XVIII.1.3. However, if we assume the Robinson property, we can get even more. Recall that a logic $\mathscr{L}$ satisfies the *Robinson consistency theorem* (for short, $\mathscr{L}$ has the *Robinson property*) iff for arbitrary vocabularies τ, τ', τ'' and classes of sentences T, T', T'', if T is complete in τ and T' and T'' are consistent extensions of T in τ' and τ'', respectively, with $\tau = \tau' \cap \tau''$, then $T' \cup T''$ is consistent; (that is, $T' \cup T''$ has some model). Equivalently, we might assume also that T' and T'' in the above definition are complete. In fact, the Robinson property only depends on the complete theories of $\mathscr{L}$ and may thus be regarded as a property of the equivalence relation $\equiv_{\mathscr{L}}$. This notion will be pursued further in later sections, for in this section we will only be concerned with the effect of the Robinson property on logics.

1.2 Theorem. *Let $\mathscr{L}$ be a regular logic with dependence number $o(\mathscr{L}) \leq$ the first uncountable measurable cardinal μ_0—if it exists—or $o(\mathscr{L}) < \infty$ otherwise. If $\mathscr{L}$ has the Robinson property, then $\mathscr{L}$ has the finite dependence property.*

Proof. The proof follows immediately from Chapter XVIII, Corollary 3.3.5, Theorem 2.2.1, and Proposition 2.1.2. ☐

For logics of the form $\mathscr{L}(Q^i)_{i \in I}$ there is an easy, self-contained proof that the Robinson property implies compactness. This is given in the following result.

1.3 Theorem. *Let $\mathscr{L}$ be a regular logic. Assume that each sentence of $\mathscr{L}$ is of finite vocabulary—or even assume that $o(\mathscr{L})$ satisfies the hypotheses of Theorem* 1.2. *If $\mathscr{L}$ has the Robinson property, then $\mathscr{L}$ is compact.*

Proof. In the light of Theorem 1.2, it suffices to prove the theorem under the assumption that sentences in $\mathscr{L}$ are of finite vocabulary. Now assume that $\mathscr{L}$ has the Robinson property and is not compact (*absurdum* hypothesis). Let κ be the smallest cardinal such that $\mathscr{L}$ is not (κ, ω)-compact. There is a vocabulary τ and a set of sentences $T = \{\varphi_\alpha | \alpha < \kappa\} \subseteq \mathscr{L}[\tau]$ such that T has no model, while for each $\beta < \kappa$, the subtheory $T_\beta = \{\varphi_\alpha | \alpha < \beta\}$ does have a model $\mathfrak{A}_\beta$. In $\mathscr{L}$ we can replace function by relation symbols (by regularity); constant symbols are eliminable by using instead unary relations which represent singletons. This can be done in the usual manner for $\mathscr{L}_{\omega\omega}$ without using the substitution property. Thus, replace, for example, $\psi(c, d)$ by $\exists! cRc \wedge \exists! dSd \wedge \forall c, d(Rc \wedge Sd \rightarrow \psi(c, d))$. For arbitrary ψ', we proceed similarly, recalling that $|\tau_{\psi'}| < \omega$, where $\tau_{\psi'}$ is the smallest vocabulary of ψ', as given by the occurrence axiom. For the sake of notational simplicity we will also assume that τ is single-sorted (the proof for the many-sorted case only requires some additional notation). Without loss of generality the $\mathfrak{A}_\beta$'s have pairwise disjoint universes. Recalling that τ may be assumed to contain only relation symbols, define the disjoint union $\mathfrak{A} = \langle A, \ldots \rangle$ of the $\mathfrak{A}_\beta$'s by

$$A = \bigcup_{\beta < \kappa} A_\beta, \qquad R^{\mathfrak{A}} = \bigcup_{\beta < \kappa} R^{\mathfrak{A}_\beta} \quad \text{for each} \quad R \in \tau.$$

Define the function $f: A \rightarrow \kappa$ by $f(a) = \beta$ iff $a \in A_\beta$, for each $a \in A$, $\beta < \kappa$, and let $\mathfrak{M}$ be the two-sorted structure given by

$$\mathfrak{M} = [\mathfrak{A}, \langle \kappa, <, c_\beta \rangle_{\beta < \kappa}, f],$$

where, as usual, symbols are identified with their natural interpretation; and, in particular, $c_\beta^{\mathfrak{M}} = \beta$ for every $\beta < \kappa$.

Claim. *Whenever $\mathfrak{N} \equiv_{\mathscr{L}} \mathfrak{M}$, the set $\{c_\beta^{\mathfrak{N}}\}_{\beta < \kappa}$ is unbounded in $<^{\mathfrak{N}}$.*

Proof of Claim. Otherwise (*absurdum* hypothesis) let $\mathfrak{N}$ be a counterexample so that, for some fixed element n in the second sort of $\mathfrak{N}$, we have

$$\langle \mathfrak{N}, n \rangle \models_{\mathscr{L}} c_\beta < n \quad \text{for each} \quad \beta < \kappa.$$

For every $\beta < \kappa$, let ψ_β be the sentence of $\mathscr{L}$ given by

$$\psi_\beta \underset{\text{def}}{=} \forall z(c_\beta < z \rightarrow \varphi_\beta^{\{x | f(x) = z\}}).$$

This is the only place in this proof where we use the assumption that $\mathscr{L}$ is closed under relativization. Indeed, we only need that $\mathscr{L}$ be closed under relativization to atomic sentences. Observe that for each $\beta < \kappa$, we have $\mathfrak{M} \models_{\mathscr{L}} \psi_\beta$. Hence, $\mathfrak{N} \models_{\mathscr{L}} \psi_\beta$. Then for each $\beta < \kappa$,

$$\langle \mathfrak{N}, n \rangle \models_{\mathscr{L}} \varphi_\beta^{\{x \mid f(x) = n\}},$$

which implies that $\mathfrak{N} \mid \{x \in N \mid \langle \mathfrak{N}, x \rangle \models_{\mathscr{L}} f(x) = n\} \models_{\mathscr{L}} T$. This contradicts the assumed inconsistency of T and our claim is thus established. Now expand $\mathfrak{M}$ to $\mathfrak{M}' \in \mathrm{Str}(\tau')$, $\tau' = \tau_{\mathfrak{M}} \cup \tau_0$ with $\tau_0 = \{P_\beta\}_{\beta < \kappa}$ a set of new unary relations, to be interpreted in $\mathfrak{M}'$ as initial segments,

$$P_\beta^{\mathfrak{M}'} = \{\alpha \mid \alpha < \beta\} \quad \text{for each} \quad \beta < \kappa.$$

Let $T' = \mathrm{Th}_{\mathscr{L}}(\mathfrak{M}) \cup \{\forall x(P_\beta x \leftrightarrow x < c_\beta) \mid \beta < \kappa\}$, and observe that $\mathfrak{M}' \models_{\mathscr{L}} T'$. On the other hand, let $\tau'' = \tau_0 \cup \{c\}$, with c a new constant, and let

$$T'' = \{\neg P_\beta c \mid \beta < \kappa\}.$$

Consider the structure $\mathfrak{M}''$ of vocabulary τ'' given by

$$\mathfrak{M}'' = \langle \kappa \cup \{c\}, P_\beta^{\mathfrak{M}'} \rangle_{\beta < \kappa};$$

that is, $\mathfrak{M}''$ is obtained by adding one element at the end of κ and by interpreting each P_β exactly as in $\mathfrak{M}'$. Then we have that $\mathfrak{M}'' \models_{\mathscr{L}} T''$. For every finite $\tau^* \subseteq \tau_0$, we have that $\mathfrak{M}' \restriction \tau^* \cong \mathfrak{M}'' \restriction \tau^*$ (it is easy to get an isomorphism). Hence,

$$\mathfrak{M}' \restriction \tau^* \equiv_{\mathscr{L}} \mathfrak{M}'' \restriction \tau^*$$

by the isomorphism property of logics. Whence $\mathfrak{M}' \restriction \tau_0 \equiv_{\mathscr{L}} \mathfrak{M}'' \restriction \tau_0$, recalling that each sentence of $\mathscr{L}$ is of finite vocabulary. Now $\tau_{\mathfrak{M}'} \cap \tau_{\mathfrak{M}''} = \tau' \cap \tau'' = \tau_0$. Hence, by the assumed Robinson property of $\mathscr{L}$, there is $\mathfrak{D}$ of vocabulary $\tau' \cup \tau''$ such that $\mathfrak{D} \restriction \tau' \equiv_{\mathscr{L}} \mathfrak{M}'$ and $\mathfrak{D} \restriction \tau'' \equiv_{\mathscr{L}} \mathfrak{M}''$. In particular, $\mathfrak{D} \models_{\mathscr{L}} T' \cup T''$, and $c^{\mathfrak{D}}$ is a strict upper bound for the $\{c_\beta^{\mathfrak{D}}\}_{\beta < \kappa}$. But $\mathfrak{D} \restriction \tau_{\mathfrak{M}} \equiv_{\mathscr{L}} \mathfrak{M}$ then stands as a counterexample to our claim. Therefore, $\mathscr{L}$ is compact. ☐

1.4 Corollary. *Let $\mathscr{L}$ be a logic satisfying the hypotheses of Theorem* 1.3. *Assume further that $\mathscr{L}[\tau]$ is a set for every τ. Then $\mathscr{L}$ has the Robinson property iff $\mathscr{L}$ is compact and satisfies Craig's interpolation theorem.*

Proof. This proof requires use of Theorem 1.3 and Proposition II.7.1.5. The assumption that $\mathscr{L}[\tau]$ is a set for every τ is needed in the proof that compactness plus Robinson property imply interpolation. In order to apply compactness, we must guarantee that complete theories are *sets* of sentences. ☐

1.5 Remark. Although it is stated only for regular $\mathscr{L}$, Theorem 1.3 still holds if the relativization axiom is replaced by the weaker requirement that $\mathscr{L}$ allow relativizations to atomic sentences. Also, the substitution axiom can be relaxed for the requirement that in $\mathscr{L}$ we are allowed to replace a function f by a relation R representing the graph of f. This will be important in the sequel (see Section 3.11.2, and Theorem 5.4).

1.6 Corollary. *Assume that $\mathscr{L}$ is a logic with the Robinson property, and that $\mathscr{L}[\tau]$ is a set for every τ, and $|\tau_\varphi| < \omega$ for every sentence φ. Assume further that $\mathscr{L}$ is closed under the atom, Boole, and particularization property of Definition* II.1.2.1. *Then the following are equivalent:*

(i) *$\mathscr{L}$ is closed under relativization to atomic sentences and allows elimination of function symbols;*
(ii) *$\mathscr{L}$ is regular.*

Proof. That (ii) implies (i) is evident in the light of Definition II.1.2.3. To prove that (i) implies (ii), we first note that, by Remark 1.5, Theorem 1.3 can be applied to $\mathscr{L}$. Since $\mathscr{L}[\tau]$ is always a set, then $\mathscr{L}$ satisfies Craig's interpolation theorem, by Corollary 1.4; and, in particular, $\mathscr{L}$ is Δ-closed (Definition II.7.2.1), whence regularity follows immediately. $\square$

We now look at logics $\mathscr{L}$ which satisfy the Robinson property but are not $[\omega]$-compact. In contrast to the above results, no restriction is here imposed on the size of $\mathscr{L}[\tau]$ or on that of $o(\mathscr{L})$. On the other hand, we require that relativization in $\mathscr{L}$ incorporate τ-closure; that is, $\mathfrak{B} \models_{\mathscr{L}} \varphi^{\{x|\alpha(x)\}}$ implies that the set $B' = \{b \in B \mid \langle \mathfrak{B}, b\rangle \models_{\mathscr{L}} \alpha(x)\}$ contains all the constants of τ_φ; and, for each $f \in \tau_\varphi$, if $b_1, \ldots, b_n \in B'$, then $f(b_1, \ldots, b_n) \in B'$ (see Barwise [1974a, p. 235], and Flum [1975b, p. 294]). All the infinitary logics mentioned in the literature have this property; for logics in which all sentences have a finite vocabulary, the present form of relativization is exactly the same as the usual relativization as defined in Definition II.1.2.2, since τ-closure is expressible by a first-order sentence whenever τ is finite.

1.7 Theorem. *If $\mathscr{L}$ is a regular logic with the Robinson property and $\mathscr{L}$ is not $[\omega]$-compact, then for every $\mathfrak{A}$, $\mathfrak{B}$ with $|\mathfrak{A}|$ of cardinality $< \mu_0$, we have that $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ implies $\mathfrak{A} \cong \mathfrak{B}$. If no uncountable measurable cardinal exists, then $\equiv_{\mathscr{L}} = \cong$.*

Proof. To prove this theorem, we establish three formal claims.

Claim 1. *$\langle \omega, <, c_n\rangle_{n<\omega}$ is characterized up to isomorphism by its own complete theory in $\mathscr{L}$.*

Proof of Claim 1. The proof is reminiscent of the proof of the first part of Theorem 1.3. Let the pair $T = \{\varphi_i \mid i < \omega\}$, Δ be a counterexample to $[\omega]$-compactness in $\mathscr{L}$. For every $m < \omega$, let $T_m = \{\varphi_i \mid i < m\} \cup \Delta$, and let $\mathfrak{A}_m \models T_m$. For the moment, assume the vocabulary τ of $T \cup \Delta$ is single-sorted and has only relations. Assume further that the universes of the $\mathfrak{A}_m$'s are pairwise disjoint. Define the disjoint union $\mathfrak{A} = \langle A, \ldots\rangle$ of the $\mathfrak{A}_m$'s by

$$A = \bigcup_{m<\omega} A_m, \qquad R^{\mathfrak{A}} = \bigcup_{m<\omega} R^{\mathfrak{A}_m} \quad \text{for each} \quad R \in \tau.$$

Let $f: A \to \omega$ be defined by $f(a) = m$ iff $a \in A_m$. Let finally the two-sorted structure $\mathfrak{M}$ be given by $\mathfrak{M} = [\mathfrak{A}, \langle \omega, <, c_m \rangle_{m<\omega}, f]$. By arguing as in the proof of Claim 1 in Theorem 1.3, we see that whenever $\mathfrak{N} \equiv_{\mathscr{L}} \mathfrak{M}$, the $\{c_m^{\mathfrak{N}}\}_{m<\omega}$ are unbounded in $<^{\mathfrak{N}}$. We can now prove that whenever $\mathfrak{D} \equiv_{\mathscr{L}} \langle \omega, <, c_n \rangle_{n<\omega}$, we also have $\mathfrak{D} \cong \langle \omega, <, c_n \rangle_{n<\omega}$. As a matter of fact, if this were not the case and $\mathfrak{D}$ were a counterexample, then we expand $\mathfrak{D}$ to $\mathfrak{D}^+ = \langle \mathfrak{D}, c, g \rangle$, where g maps the set W of predecessors of c one-one onto $W \cup \{c\}$. We expand $\langle \omega, <, c_n \rangle_{n<\omega}$ to the structure $\mathfrak{M}$ defined above. Using the Robinson property of $\mathscr{L}$, we exhibit $\mathfrak{B}$ such that $\mathfrak{B} \restriction \tau_{\mathfrak{M}} \equiv_{\mathscr{L}} \mathfrak{M}$ and $\mathfrak{B} \restriction \tau_{\mathfrak{D}^+} \equiv_{\mathscr{L}} \mathfrak{D}^+$. Then the $\{c_n^{\mathfrak{B}}\}_{n<\omega}$ are unbounded in $<^{\mathfrak{B}}$, by the discussion above. But they are also bounded by $c^{\mathfrak{B}}$, because $\mathfrak{B} \models_{\mathscr{L}} c > c_n$ for all $n < \omega$—a contradiction which establishes our claim in case τ is single-sorted and only contains relation symbols. The many-sorted case (for τ only containing relations) can be established, with the help of additional notation. We now consider in detail the case in which some sentence ψ of T is such that the set τ_ψ of symbols occurring in ψ, as given by the occurrence axiom, also contains constants (but no functions). If there are only finitely many such constants, then we can get rid of them by using unary relations and renamings, without using the substitution property of $\mathscr{L}$ (see the proof of Theorem 1.3). Otherwise, if τ_ψ has infinitely many constants, display them as $\{b_\alpha\}_{\alpha<\kappa}$, for some $\kappa \geq \omega$. Recalling that we assumed that relativization incorporates τ_ψ-closure, whenever U is a new relation, we have

(1) $\qquad (\psi \vee \neg\psi)^{\{x|Ux\}}$ is equivalent to $Ub_0 \wedge Ub_1 \wedge \cdots$.

Similarly, letting θ be the sentence in $\mathscr{L}$ given by

(2) $\qquad \theta \underset{\text{def}}{=} \forall y \neg((\psi \vee \neg\psi)^{\{x|x\neq y\}})$,

we must have

(3) $\qquad \theta$ is equivalent to $\forall y(y = b_0 \vee y = b_1 \vee \cdots)$.

Add a new relation V and let theory Γ be given by

(4) $\qquad \Gamma \underset{\text{def}}{=} \{Vb_\beta \mid \beta < \omega\} \cup \{\neg Vb_\gamma \mid \gamma \geq \omega\}$.

By (2) and (3), for every structure $\mathfrak{S}$, we have

(5) $\qquad \mathfrak{S} \models \Gamma \cup \{\theta\}$ implies $V^{\mathfrak{S}} = \{b_\beta^{\mathfrak{S}}\}_{\beta<\omega}$.

Let T' be defined by $T' \underset{\text{def}}{=} \mathrm{Th}_{\mathscr{L}}\langle \omega, <, c_n \rangle_{n<\omega} \cup \Gamma \cup \{\theta, \eta\}$, where η says that f is a one-one mapping from the new sort of the $\{c_n\}_{n<\omega}$ onto $V = \{b_\beta\}_{\beta<\omega}$. Then, by (5), each model of T' will be an expansion of $\langle \omega, <, c_n \rangle_{n<\omega}$, the latter being defined on a new sort. We now complete the proof of Claim 1. Assume $\mathrm{Th}_{\mathscr{L}}\langle \omega, <, c_n \rangle_{n<\omega}$ has a non-standard model $\mathfrak{D}$ (*absurdum* hypothesis). Expand $\mathfrak{D}$ to $\mathfrak{D}' = \langle \mathfrak{D}, c, g \rangle$, where c is a strict upper bound for the $\{c_n\}_{n<\omega}$, and g maps the set K of predecessors of c one-one onto $K \cup \{c\}$. Then $\mathrm{Th}_{\mathscr{L}}\, \mathfrak{D}' \cup T'$ has no models, thus contradicting the Robinson property of $\mathscr{L}$. This completes the proof of Claim 1 (the case in which τ_ψ has function symbols can be reduced to the cases considered above).

Claim 2. *Let $\kappa \geq \omega$ be an arbitrary cardinal. Assume $\langle \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$ is characterized (up to isomorphism by its own complete theory in $\mathscr{L}$), and also each $\mathfrak{A}'$ with $|A'| < \kappa$ is characterized. Then every $\mathfrak{A}$ with $|A| = \kappa$ is characterized.*

Proof of Claim 2. To establish this claim, we consider two cases, the first being the

Special Case. Here $\mathfrak{A} = \langle \kappa, <, c_\alpha, R^{\mathfrak{A}}, \ldots \rangle_{\alpha<\kappa}$ is a single-sorted expansion of $\langle \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$. Then let $\tau = \tau_{\mathfrak{A}}$ and assume $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}$, but $\mathfrak{B} \not\cong \mathfrak{A}$ (*absurdum* hypothesis). By assumption (and by the reduct and isomorphism axioms given in Definition II.1.1.1) we can safely write $\mathfrak{B} = \langle \kappa, <, c_\alpha, R^{\mathfrak{B}}, \ldots \rangle_{\alpha<\kappa}$. Since $\mathfrak{B} \not\cong \mathfrak{A}$, then without loss of generality we must have $R^{\mathfrak{A}} \neq R^{\mathfrak{B}}$. For the sake of definiteness assume that R is a unary relation (the other cases being treated similarly). We then have that for some $\beta < \kappa$, $R^{\mathfrak{A}}\beta$ holds and $R^{\mathfrak{B}}\beta$ does not (or vice versa). Now by the assumed characterizability of $\langle \beta, <, c_\alpha \rangle_{\alpha<\beta}$ we have that $c_\beta^{\mathfrak{A}} = c_\beta^{\mathfrak{B}} = \beta$, so that $\mathfrak{A} \models_{\mathscr{L}} Rc_\beta$ and $\mathfrak{B} \models_{\mathscr{L}} \neg Rc_\beta$, thus contradicting $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$. Consider now the

General Case. Here we assume that $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$, $|A| = \kappa$. For the moment, let $\mathfrak{A}$ be single-sorted; let $|B| = \lambda$. Then we must have that $\lambda \leq \kappa$; for, otherwise, by expanding $\mathfrak{A}$ to $\mathfrak{A}^+ = \langle \mathfrak{A}, \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$ and $\mathfrak{B}$ to $\mathfrak{B}' = \langle \mathfrak{B}, b_\beta \rangle_{\beta<\lambda}$, using the Robinson property, we exhibit $\mathfrak{M}$ with $\mathfrak{M} \restriction \tau_{\mathfrak{A}^+} \equiv_{\mathscr{L}} \mathfrak{A}^+$ and $\mathfrak{M} \restriction \tau_{\mathfrak{B}'} \equiv_{\mathscr{L}} \mathfrak{B}'$. Hence, by hypothesis $\mathfrak{M} \restriction \{<\} \cong \langle \kappa, < \rangle$ and $|M| \geq \lambda > \kappa$, since $b_\beta^{\mathfrak{M}} \neq b_\alpha^{\mathfrak{M}}$ for $\beta \neq \alpha$. This is a contradiction. Having seen that $|B| = \lambda \leq \kappa$, we now expand $\mathfrak{B}$ to $\mathfrak{B}^+ = \langle \mathfrak{B}, \lambda, <', d_\beta \rangle_{\beta<\lambda}$, where $<'$ is a new binary relation symbol having the natural interpretation in $\mathfrak{B}^+$. By the Robinson property, we let $\mathfrak{N}$ be such that $\mathfrak{N} \restriction \tau_{\mathfrak{A}^+} \equiv_{\mathscr{L}} \mathfrak{A}^+$ and $\mathfrak{N} \restriction \tau_{\mathfrak{B}^+} \equiv_{\mathscr{L}} \mathfrak{B}^+$. Now $\mathfrak{A}^+$ is taken care of by the special case just considered, and so is $\mathfrak{B}^+$—unless $\lambda < \kappa$, in which case $\mathfrak{B}^+$ is characterized up to isomorphism by hypothesis. In definitive, we have that $\mathfrak{N} \restriction \tau_{\mathfrak{A}^+} \cong \mathfrak{A}^+$ and $\mathfrak{N} \restriction \tau_{\mathfrak{B}^+} \cong \mathfrak{B}^+$. By taking reducts, we finally obtain $\mathfrak{A} \cong \mathfrak{N} \restriction \tau_{\mathfrak{A}} = \mathfrak{N} \restriction \tau_{\mathfrak{B}} \cong \mathfrak{B}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are many-sorted, one proceeds similarly, by first excluding the possibility of $\mathfrak{B}$ having sorts of cardinality $> \kappa$, and by adding one copy of $\langle |S|, <', \ldots \rangle$ over each sort S in $\mathfrak{B}$. This completes the proof of Claim 2.

Claim 3. *All structures of cardinality $< \mu_0$ are characterized.*

Proof of Claim 3. Let κ be the least cardinal such that there are two $\mathscr{L}$-equivalent non-isomorphic structures $\mathfrak{A}'$ and $\mathfrak{A}''$, with $\kappa = |A'| \leq |A''|$. Clearly $\kappa \geq \omega$, and by Claim 2 it follows that $\mathfrak{A} = \langle \kappa, <, c_\alpha \rangle_{\alpha<\kappa}$ is not characterized. By Claim 1, we see that κ is uncountable. We will now prove that κ is measurable. Let $\mathfrak{B} = \langle B, <, c_\alpha \rangle_{\alpha<\kappa}$ with $\mathfrak{B} \equiv_{\mathscr{L}} \mathfrak{A}$ and $\mathfrak{B} \not\cong \mathfrak{A}$. Using standard arguments of model theory, along with the characterizability of each ordinal $\beta < \kappa$, we conclude that there must be some $b \in B$ such that $\mathfrak{B}^+ \models_{\mathscr{L}} b > c_\alpha$ for all $\alpha < \kappa$, where $\mathfrak{B}^+ = \langle \mathfrak{B}, b \rangle$. Expand $\mathfrak{A}$ to $\mathfrak{A}^+$, adding symbols for all unary functions and relations on κ, as follows:

$$\mathfrak{A}^+ = \langle \kappa, <, c_\alpha, U_s, f_j \rangle_{\alpha<\kappa, s\in P(\kappa), j\in {}^\kappa\kappa}$$

with

$$U_s^{\mathfrak{A}^+} = s \quad \text{and} \quad f_j^{\mathfrak{A}^+} = j,$$

where $P(\kappa)$ is the power set of κ. Using the Robinson property, let $\mathfrak{M}$ be such that $\mathfrak{M} \upharpoonright \tau_{\mathfrak{B}^+} \equiv_{\mathscr{L}} \mathfrak{B}^+$ and $\mathfrak{M} \upharpoonright \tau_{\mathfrak{A}^+} \equiv_{\mathscr{L}} \mathfrak{A}^+$. Define $D \subseteq P(\kappa)$ by

$$s \in D \quad \text{iff} \quad s \subseteq \kappa \quad \text{and} \quad \mathfrak{M} \models_{\mathscr{L}} U_s(b);$$

that is, $s \in D$ iff the unary relation U_s whose interpretation is s in $\mathfrak{A}^+$ has $b^{\mathfrak{M}}$ among its elements when interpreted in $\mathfrak{M}$. Clearly D is a nonprincipal ultrafilter on κ. We now show that D is κ-complete. If not (*absurdum* hypothesis), D is μ-descendingly incomplete for some $\mu < \kappa$; that is, there is a descending chain $D' = \{s'_\alpha\}_{\alpha<\mu}$ with $s'_\alpha \in D$ and $\bigcap_{\alpha<\mu} s'_\alpha \notin D$. Hence, without loss of generality, $\bigcap_{\alpha<\mu} s'_\alpha = \emptyset$. Without loss of generality, we may also assume that for every limit ordinal $\varepsilon < \mu$, $\bigcap_{\alpha<\varepsilon} s'_\alpha = s'_\varepsilon$. Define $h: \kappa \to \mu$ by $h(\beta) = \alpha$ iff $\beta \in s'_\alpha \setminus s'_{\alpha+1}$, for $\beta < \kappa$, $\alpha < \mu$, so that intuitively h tells us how long an element $\beta \in \kappa$ stays in the descending, and eventually vanishing, chain D'. h is well defined, by our assumption that for every β the first η such that $\beta \notin s'_\eta$ is a successor ordinal. Let $U_\alpha = U_{s'_\alpha}$ $(\alpha < \mu)$. Then, for every $\alpha < \mu$, we have

$$\begin{aligned}
&\mathfrak{A}^+, \mathfrak{M} \models_{\mathscr{L}} \forall x(h(x) \leq c_\alpha \to \neg U_{\alpha+1}(x)),\\
&\qquad \mathfrak{M} \models_{\mathscr{L}} h(b) > c_\alpha \quad \text{since} \quad \mathfrak{M} \models_{\mathscr{L}} U_{\alpha+1}(b),\\
&\qquad \mathfrak{M} \models_{\mathscr{L}} \forall x(x < c_\mu \to h(b) > x) \quad \text{since } \mu < \kappa \text{ is characterizable},\\
&\mathfrak{M}, \mathfrak{A}^+ \models_{\mathscr{L}} \exists y(\forall x(x < c_\mu \to h(y) > x)),
\end{aligned}$$

so that $\bigcap_{\alpha<\mu} s'_\alpha \neq \emptyset$—a contradiction. Therefore, D is κ-complete, and κ is measurable, indeed uncountable and measurable. This completes the proof of Claim 3 and of the theorem as well. □

1.8 Corollary. *If $\mathscr{L}$ is a regular logic with the Robinson property, and there are $<\mu_0$ many sentences in the pure identity language of $\mathscr{L}$, then $\mathscr{L}$ is $[\omega]$-compact.* □

1.9 Examples. (a) The logic $\mathscr{L}_{\infty\infty} = \mathscr{L}$ has the Robinson property and is not $[\omega]$-compact. Here $\mathscr{L}[\tau]$ is a proper class and $\equiv_{\mathscr{L}} = \cong$.

(b) If κ is an extendible cardinal, then $\mathscr{L}^{\text{II}}_{\kappa\kappa}$, infinitary logic with conjunctions and quantifications of elements and relations of length $<\kappa$, has the Robinson property and is not $[\omega]$-compact. Here $\mathscr{L}^{\text{II}}_{\kappa\kappa}$-equivalence coincides with isomorphism below κ, and $\kappa \geq \mu_0$. Indeed $\kappa \geq$ the first supercompact cardinal (see Magidor [1971], and Examples XVIII.3.3.7).

The above examples, together with Theorem 1.7, simply tell us that if $\mathscr{L}$ fares well with the interpolation or definability properties, but does not do so with compactness, then its expressive power is extremely strong below some measurable cardinal. The prototype of this sort of result is Scott's theorem which yields for each countable structure $\mathfrak{A}$ a sentence $\varphi_{\mathfrak{A}}$ of $\mathscr{L}_{\omega_1\omega}$ whose countable models are exactly those which are isomorphic to $\mathfrak{A}$ (see Theorem VIII.4.1.1). A partial converse is given by the following result.

1.10 Theorem. *Let $\mathscr{L}$ be a logic such that for every countable structure $\mathfrak{A}$ there is a sentence $\varphi_{\mathfrak{A}}$ of vocabulary $\tau_{\mathfrak{A}}$ having the property that for any countable $\mathfrak{B} \in \mathrm{Str}(\tau_{\mathfrak{A}})$, $\mathfrak{B} \models_{\mathscr{L}} \varphi_{\mathfrak{A}}$ implies $\mathfrak{B} \cong \mathfrak{A}$. Then $\Delta\mathscr{L}$ is an extension of $\mathscr{L}_{\omega_1\omega}$.*

Proof. See Section XVII.3.2. □

Since $\mathscr{L}_{\omega_1\omega}$ is Δ-closed, Theorem 1.10 implies that $\mathscr{L}_{\omega_1\omega}$ can be characterized as the smallest Δ-closed logic satisfying Scott's theorem. Using Theorem 1.7, we can now prove an analogue of Theorem 1.10.

1.11 Definition. Let $\mathscr{L}$, $\mathscr{L}'$ be logics. We say that $\mathscr{L}'$ is *weakly interpretable in* $\mathscr{L}$ iff for every sentence $\varphi \in \mathscr{L}'[\tau]$ there is a vocabulary $\sigma \supseteq \tau$ and a set of sentences $\Sigma \subseteq \mathscr{L}[\sigma]$ such that $\mathrm{Mod}(\varphi) = (\mathrm{Mod}(\Sigma)) \restriction \tau$.

1.12 Theorem. *If $\mathscr{L}$ is a regular logic with the Robinson property which is not* $[\omega]$-*compact, then:*

(i) $\mathscr{L}_{\mu_0\omega}$ *is weakly interpretable in* $\mathscr{L}$; *and,*
(ii) $\equiv_{\mathscr{L}}$ *is finer than* $\equiv_{\mathscr{L}_{\mu_0\omega}}$; *that is,* $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$ *implies* $\mathfrak{A} \equiv_{\mathscr{L}_{\mu_0\omega}} \mathfrak{B}$.

Proof. For (i). If $\varphi \in \mathscr{L}_{\mu_0\omega}[\tau]$, then we can assume that $\varphi \in H(\kappa)$, for some $\kappa < \mu_0$. We now follow Feferman [1974a, b, 1975] and find an expansion $\mathfrak{M}_\varphi$ of $\langle H(\kappa), \in\rangle$ and a set $\Sigma_\varphi \subseteq \mathscr{L}[\sigma]$, for some $\sigma \supseteq \tau$, such that for all $\mathfrak{A} \in \mathrm{Str}(\tau)$ we have that

$$\mathfrak{A} \models \varphi \quad \text{iff the pair} \quad \langle \mathfrak{A}, \mathfrak{M}_\varphi \rangle \models \Sigma_\varphi.$$

The existence of $\mathfrak{M}_\varphi$ and Σ_φ with the required properties is guaranteed by Theorem 1.7.

Deny (ii). Then there is $\mathfrak{A} \in \mathrm{Str}(\tau)$ and $\varphi \in \mathscr{L}_{\mu_0\omega}[\tau]$ such that if we let $T = \mathrm{Th}_{\mathscr{L}}(\mathfrak{A})$, then both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ have a model. Let $\sigma, \sigma' \supseteq \tau, \Sigma_\varphi \subseteq \mathscr{L}[\sigma]$ and $\Sigma_{\neg\varphi} \subseteq \mathscr{L}[\sigma']$ be as in the proof of (i), and $\sigma \cap \sigma' = \tau$. By (i) each of $T \cup \Sigma_\varphi$ and $T \cup \Sigma_{\neg\varphi}$ has a model, and by the Robinson property of $\mathscr{L}$ we can write $\mathfrak{B} \models_{\mathscr{L}} T \cup \Sigma_\varphi \cup \Sigma_{\neg\varphi}$, for some $\mathfrak{B}$. Hence, $\mathfrak{B} \models T \cup \{\varphi, \neg\varphi\}$—a contradiction. □

To some extent, Theorem 1.12 clarifies how a non-$[\omega]$-compact logic with the Robinson property resembles an infinitary logic built on a measurable cardinal. Indeed, the only known examples of such logics involve an extendible cardinal (see Example 1.9(b)). Shelah has constructed a logic $\mathscr{L}$ with the amalgamation property, (a property which is weaker than the Robinson property) and which still does not contain $\mathscr{L}_{\omega_1\omega_1}$. This result was given in a private communication, and it seems an interesting problem to explore it with the view of making possible improvements of Theorem 1.12.

1.13 Notes and Remarks. A more detailed proof of Theorem 1.1 can be extracted from Mundici [1982b, pp. 64–66], where it is shown that compactness = JEP (for logics where $\mathscr{L}[\tau]$ is a set) under such set-theoretical hypotheses as $V = L$ or $\neg 0^\#$. Theorem 1.2 was originally proved by Makowsky–Shelah [1983]. Theorem 1.3 and Corollary 1.4 are independently due to Mundici [1982b], and Makowsky–Shelah [1983]. The proof presented here is given by Lindström in a private communication. Theorem 1.7 is due to Mundici [1982f] (see also [1982a] for results on the many-sorted case). The proof given here uses a number of ingredients from

Rabin [1959], Keisler [1963b], Lindström [1968] and Makowsky-Shelah [1979b]. In this latter reference, a variant of Corollary 1.8 was proven using a weaker notion of Robinson property together with the Feferman-Vaught property and different assumptions about $|\mathscr{L}[\tau]|$. For Example 1.9(b) see Magidor [1971] and Makowsky-Shelah [1983]. Theorem 1.10 is due to Makowsky [1973] and Barwise [1974a]. Actually, the theorem still holds under the weaker hypothesis that we can characterize by a sentence of $\mathscr{L}$ every structure of the form $\langle \omega, <, P \rangle$ with P an arbitrary subset of ω (see Makowsky-Shelah-Stavi [1976]). Theorem 1.12 is an unpublished result of Makowsky.

2. *Abstract Model Theory for Enriched Structures*

This short section is devoted to extending the results of Section 1 to logics for enriched structures, such as topological, uniform, proximity structures (see Chapter XV). The reader who is only interested in the usual (first-order) structures may safely proceed to Section 3 at first reading.

For an arbitrary nonempty set B, the *superstructure* V^B_ω of B is given by $V^B_0 = B$, $V^B_{n+1} = V^B_n \cup PV^B_n$, $V^B_\omega = \bigcup_n V^B_n$, where P is the power-set operation. An *enriched structure* of vocabulary τ is a pair $\mathfrak{M}' = \langle \mathfrak{M}, \mu \rangle$ where $\mathfrak{M} \in \mathrm{Str}(\tau)$ is an *ordinary* structure (as defined in Chapter II), and $\mu \in V^M_\omega$. The many-sorted case is an immediate generalization of this notion. Examples of enriched structures are topological, weak, uniform, monotone, proximity, ordinary structures, as well as the structures studied in Chang [1973] in the framework of modal model theory. The *forgetful functor* $\|\cdot\|$ transforms $\mathfrak{M}'$ into $\mathfrak{M}$; the operations of reduct, renaming, diagram expansion, disjoint union (for structures of disjoint vocabularies) are the same as in the ordinary case. A *strict expansion* of $\mathfrak{M}' = \langle \mathfrak{M}, \mu \rangle$ is any structure $\mathfrak{M}'' = \langle \mathfrak{M}^+, \mu \rangle$, where $\mathfrak{M}^+$ is an expansion of $\mathfrak{M}$. The *ordinary semantic domain* is the function $\mathscr{O}$ assigning to every vocabulary τ the category $\mathscr{O}(\tau) = \langle \mathrm{Str}(\tau), \mathrm{Emb}(\tau) \rangle$ whose arrows are the isomorphic embeddings equipped with composition. More generally we consider

2.1 Definition. A *semantic domain* is a function $\mathscr{C}$ assigning to every vocabulary τ a category $\mathscr{C}(\tau) = \langle \mathrm{Ob}(\tau), \mathrm{Ar}(\tau) \rangle$ whose objects are enriched structures of vocabulary τ and whose arrows, called the *isomorphic embeddings* of $\mathscr{C}$, are functions equipped with composition, satisfying the following seven conditions:

(a) $\|\cdot\|$ preserves identities and commutative diagrams;

(b) $\cup_\tau \mathrm{Ob}(\tau)$ is closed under reduct, renaming, strict expansion, formation of disjoint pairs, and *disjoint union*; that is, for every set $\{\mathfrak{B}_\alpha\}_{\alpha<\kappa} \subseteq \mathrm{Ob}(\tau)$, τ without constants, there are $\mathfrak{B} \in \mathrm{Ob}(\tau)$ and arrows $g_\alpha : \mathfrak{B}_\alpha \to \mathfrak{B}$ $(\alpha < \kappa)$ having pairwise disjoint ranges whose union is B (here we essentially require that all the operations on structures used in the proof of Theorem 1.3 are also available for our enriched structures);

(c) for each ordinary structure $\mathfrak{A}$ there is a structure $\mathfrak{B}$ in $\mathscr{C}$ with the same vocabulary and such that $\|\mathfrak{B}\| = \mathfrak{A}$ (this amounts to requiring that $\mathscr{C}$ extends the ordinary semantic domain);
(d) $g: \mathfrak{M} \to \mathfrak{N}$ iff $g: \mathfrak{M}^\rho \to \mathfrak{N}^\rho$ for any renaming ρ;
(e) $g: \mathfrak{M} \to \mathfrak{N}$ iff $g: \mathfrak{M} \upharpoonright \tau \to \mathfrak{N} \upharpoonright \tau$ for all finite $\tau \subseteq \tau_{\mathfrak{M}} = \tau_{\mathfrak{N}}$;
(f) $f: \mathfrak{M} \to \mathfrak{N}$, $g: \mathfrak{A} \to \mathfrak{B}$ and $\tau_{\mathfrak{M}} \cap \tau_{\mathfrak{A}} = \varnothing$ imply $f \cup g: [\mathfrak{M}, \mathfrak{A}] \to [\mathfrak{N}, \mathfrak{B}]$;
(g) $g: \mathfrak{M} \to \mathfrak{N}$ implies $g: \mathfrak{M}_M \to \mathfrak{N}_{g(M)}$, $\mathfrak{M}_M =$ diagram expansion of $\mathfrak{M}$.

We also say that $\mathscr{C}$ *has substructures* iff $\mathscr{C}$ satisfies the following two additional conditions:

(h) whenever $B' \subseteq B$ is the range of an isomorphic embedding into $\|\mathfrak{B}\|$ (with respect to $\mathscr{O}$), then B' is also the range of some isomorphic embedding into the whole of $\mathfrak{B}$ (with respect to $\mathscr{C}$);
(i) whenever $\mathfrak{M} \xrightarrow[f]{} \mathfrak{B} \xleftarrow[g]{} \mathfrak{N}$ and range$(f) \subseteq$ range(g), then there exists $h: \mathfrak{M} \to \mathfrak{N}$ such that $f = g \circ h$.

2.2 Examples. The following are semantic domains with substructures: the category of topological structures with homeomorphic embeddings (see Chapter XV); the monotone structures with monotone embeddings (see Makowsky–Tulipani [1977]); the uniform structures with uniformly continuous embeddings (see Flum–Ziegler [1980]), the proximity structures with proximity-preserving embeddings.

The notion of a logic $\mathscr{L}$ over a semantic domain $\mathscr{C}$ is exactly the same as for the ordinary case (see Chapter II), except for the definition of relativization, which requires a little more care:

2.3 Definition. A logic $\mathscr{L}$ over $\mathscr{C}$ has *relativization* iff $\mathscr{C}$ has substructures and for every boolean combination α of atomic sentences with $\tau_\alpha \supseteq \{x\}$, and every sentence $\varphi \in \mathscr{L}[\tau_\varphi]$ there is $\psi \in \mathscr{L}[\tau']$ (with $\tau' = \tau_\varphi \cup (\tau_\alpha \setminus \{x\})$), denoted $\psi = \varphi^{\{x|\alpha\}}$, such that for all $\mathfrak{B} \in \mathrm{Str}(\tau')$, $\mathfrak{B} \models_{\mathscr{L}} \psi$ iff $\{b \in B \mid \|\langle\mathfrak{B}, b\rangle\| \models \alpha\}$ is the range of an isomorphic embedding $g: \mathfrak{N} \to \mathfrak{B} \upharpoonright \tau_\varphi$, for some $\mathfrak{N} \in \mathrm{Str}(\tau_\varphi)$ with $\mathfrak{N} \models_{\mathscr{L}} \varphi$.

The assumption that $\mathscr{C}$ has substructures ensures that $\mathfrak{N}$ in the above definition is unique up to the isomorphism relation in $\mathscr{C}$. Furthermore we have incorporated τ-closure in relativization. In other words, $\mathfrak{B} \models_{\mathscr{L}} \varphi^{\{x|\alpha\}}$ implies that the substructure $\mathfrak{B} | \alpha^{\mathfrak{B}}$ contains all the constants of τ_φ and is closed under all the functions of τ_φ. We can recover the ordinary definition given in Definition II.1.2.2 simply by noting that for ordinary structures the following holds, for any isomorphic embedding g:

$$g: \mathfrak{N} \to \mathfrak{B} \upharpoonright \tau_\varphi \quad \text{iff} \quad \mathfrak{N} \cong (\mathfrak{B} \upharpoonright \tau_\varphi) | \mathrm{range}(g).$$

In Chapter XV the reader encountered a logic $\mathscr{L}^t$ which stands to topological structures as $\mathscr{L}_{\omega\omega}$ stands to ordinary structures. In Chapter III it is shown that large portions of ordinary abstract model theory can be extended to the realm of enriched structures. As for extensions of the results of Section 1, we have:

2.4 Theorem. *For $\mathscr{C}$ an arbitrary semantic domain with substructures, and $\mathscr{L}$ a logic over $\mathscr{C}$ obeying the hypotheses of Theorem* 1.1, *the conclusion of the theorem still holds.*

Proof. See Mundici [1982c, II and 198?b]. □

It remains an open problem whether or not the main results of Chapter XVIII—notably, the implication AP $\Rightarrow$ compactness—or even Theorem 1.3 above can be extended to logics over arbitrary $\mathscr{C}$. With the help of such axioms as $V = L$ or $\neg O^{\#}$ we can strengthen Theorem 1.1 to the effect that if $\mathscr{L}$ is not compact, then there is a proper class of regular cardinals κ such that $\mathscr{L}$ is not $[\kappa]$-compact. By using Theorem 2.4, the proof of this fact for $\mathscr{O}$ can be extended to arbitrary $\mathscr{C}$. Hence, we have

2.5 Theorem ($V = L$, *or even* $\neg O^{\#}$). *For $\mathscr{C}$ an arbitrary semantic domain with substructures and $\mathscr{L}$ a regular logic over $\mathscr{C}$, assume that $\mathscr{L}[\tau]$ is a set for every τ and that $|\tau_\varphi| < \omega$ for every sentence φ. Then if $\mathscr{L}$ has the Robinson property, $\mathscr{L}$ is compact.*

Proof. The reader is referred to Mundici [1982c, II and 198?b]. Actually, the theorem is proven there under the weaker assumption (denoted $\natural$) that for every infinite regular cardinal κ and for every uniform ultrafilter D over κ, D is λ-descendingly incomplete for all infinite $\lambda \leq \kappa$. For a proof that $\natural$ is weaker than $\neg O^{\#}$, the reader should consult D. Donder, R. B. Jensen, and B. J. Koppelberg; *Lecture Notes in Mathematics*, **872** (1981), p. 91. □

3. Duality Between Logics and Equivalence Relations

We now return to ordinary (first-order) structures. As we remarked in Section 1, the Robinson property of a logic $\mathscr{L}$ only depends on $\equiv_{\mathscr{L}}$. In general, for $\sim$ an arbitrary equivalence relation on the class of all structures, we say that $\sim$ has the *Robinson property* iff for every $\mathfrak{A}' \in \mathrm{Str}(\tau')$, $\mathfrak{A}'' \in \mathrm{Str}(\tau'')$, if $\mathfrak{A}' \upharpoonright \tau \sim \mathfrak{A}'' \upharpoonright \tau$ and $\tau = \tau' \cap \tau''$ then there is $\mathfrak{M} \in \mathrm{Str}(\tau' \cup \tau'')$ such that $\mathfrak{M} \upharpoonright \tau' \sim \mathfrak{A}'$ and $\mathfrak{M} \upharpoonright \tau'' \sim \mathfrak{A}''$. It is immediately seen that whenever $\sim \; = \; \equiv_{\mathscr{L}}$ for some logic $\mathscr{L}$, the relation $\sim$ has the Robinson property iff $\mathscr{L}$ satisfies the Robinson consistency theorem. Among the equivalence relations with the Robinson property, we mention elementary equivalence $\equiv$, isomorphism $\cong$, equality $=$ and $\equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}^{\mathrm{II}}_{\kappa\kappa}$ (see Example 1.9(b)). All the equivalence relations considered in this paper will satisfy a few natural prerequisites which the attentive reader may find reminiscent of the simplest axiomatic properties of logics.

3.1 Definition. Let $\sim$ be an equivalence relation on the class of all structures. Then $\sim$ is said to be *regular* iff $\sim$ satisfies the following conditions (for every two structures $\mathfrak{M}$ and $\mathfrak{N}$):

vocabulary: $\mathfrak{M} \sim \mathfrak{N}$ implies $\tau_{\mathfrak{M}} = \tau_{\mathfrak{N}}$;
renaming: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M}^\rho \sim \mathfrak{N}^\rho$ for any $\rho: \tau_{\mathfrak{M}} \to \tau'$;
reduct: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M} \upharpoonright \tau \sim \mathfrak{N} \upharpoonright \tau$ for any $\tau \subseteq \tau_{\mathfrak{M}}$;
isomorphism: $\mathfrak{M} \cong \mathfrak{N}$ implies $\mathfrak{M} \sim \mathfrak{N}$;
expressiveness: $\mathfrak{M} \sim \mathfrak{N}$ implies $\mathfrak{M} \equiv \mathfrak{N}$.

Moreover, we say that $\sim$ is *bounded* iff for every vocabulary τ there is a *set* $S_\tau \subseteq \mathrm{Str}(\tau)$ such that for every $\mathfrak{A} \in \mathrm{Str}(\tau)$ there is $\mathfrak{B} \in S_\tau$ with $\mathfrak{B} \sim \mathfrak{A}$. Thus, all equivalence classes have a representative in S_τ. Observe that this has nothing to do with "bounded" logics. When $\sim$ is a regular equivalence relation on the class of all structures and has the Robinson property, then we simply say that $\sim$ is a *Robinson equivalence relation.* Of the four equivalence relations given above, $\equiv$ and $\equiv_{\mathscr{L}^{\mathrm{II}}_{\kappa\kappa}}$ are bounded Robinson equivalence relations. If $\mathscr{L}[\tau]$ is a set for every τ, then $\equiv_{\mathscr{L}}$ is (regular and) bounded. Conversely, if $\sim\, = \,\equiv_{\mathscr{L}}$ then $\mathscr{L}$ is (equivalent to) a logic where $\mathscr{L}[\tau]$ is a set for all τ, provided $\sim$ is bounded. Finally, we say that $\sim$ has the *finite vocabulary property* iff for every τ and $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$, we have that $\mathfrak{A} \sim \mathfrak{B}$ iff $\mathfrak{A} \upharpoonright \tau_0 \sim \mathfrak{B} \upharpoonright \tau_0$ for each finite vocabulary $\tau_0 \subseteq \tau$.

As remarked in the introduction, an open problem of abstract model theory is whether $\equiv$ is the only bounded Robinson equivalence relation $\sim$ having the finite vocabulary property and satisfying $\sim\, = \,\equiv_{\mathscr{L}}$ for some logic $\mathscr{L}$. In the following pages we will see that if any such relation $\sim\, \neq\, \equiv$ exists, then that relation $\sim$ has many properties in common with $\equiv$.

3.2 Theorem (Relative Compactness Theorem). *Let $\sim$ be a Robinson equivalence relation having the finite vocabulary property. Let $\mathscr{L}' = \mathscr{L}(Q^i)_{i\in I}$ and $\mathscr{L}'' = \mathscr{L}(Q^j)_{j\in J}$ be logics, with $\equiv_{\mathscr{L}'}$ and $\equiv_{\mathscr{L}''}$ both coarser than $\sim$. Let $\psi \in \mathscr{L}''[\tau]$ and $\Gamma \subseteq \mathscr{L}'[\tau]$ be an arbitrary set. If $\Gamma \models \psi$, then $\Gamma_0 \models \psi$, for some finite $\Gamma_0 \subseteq \Gamma$, where $\Gamma \models \psi$ means* $\mathrm{Mod}_{\mathscr{L}'}\,\Gamma \subseteq \mathrm{Mod}_{\mathscr{L}''}\,\psi$.

Proof. Assume that $\Gamma \models \psi$ holds but for no finite $\Gamma_0 \subseteq \Gamma$ do we have $\Gamma_0 \models \psi$ (*absurdum* hypothesis). Since Γ is a set, we can write $\Gamma = \{\varphi_\alpha | \alpha < \kappa\}$. We can safely assume κ is minimal, so that each $T_\beta = \{\neg\psi\} \cup \{\varphi_\alpha | \alpha < \beta\}$ has a model $\mathfrak{A}_\beta$, for each $\beta < \kappa$. Now, construct the disjoint union $\mathfrak{A}$ of the $\mathfrak{A}_\beta$ and let $\mathfrak{M} = [\mathfrak{A}, \langle\kappa, <, c_\beta\rangle_{\beta<\kappa}, f]$ exactly as in the proof of Theorem 1.3 (here we use the hypothesis that all sentences in $\mathscr{L}'$ and $\mathscr{L}''$ have a finite vocabulary). The claim in the proof of Theorem 1.3 now reads as follows:

Whenever $\mathfrak{N} \sim \mathfrak{M}$, the $\{c_\beta^{\mathfrak{N}}\}_{\beta<\kappa}$ are unbounded in the order $<^{\mathfrak{N}}$.

To prove the present claim, for each $\beta < \kappa$, let ψ'_β and ψ''_β be defined by

$$\psi'_\beta \underset{\mathrm{def}}{=} \forall z(c_\beta < z \to \varphi_\beta^{\{x|f(x)=z\}}), \qquad \psi''_\beta \underset{\mathrm{def}}{=} \forall z(c_\beta < z \to (\neg\psi)^{\{x|f(x)=z\}}).$$

Observe that since $\mathfrak{M} \models_{\mathscr{L}'} \psi'_\beta$ and $\mathfrak{M} \models_{\mathscr{L}''} \psi''_\beta$, then so does $\mathfrak{N} \sim \mathfrak{M}$, since $\equiv_{\mathscr{L}'}$ and $\equiv_{\mathscr{L}''}$ are assumed to be coarser than $\sim$. Therefore, if $\mathfrak{N}$ were a counterexample to the claim, i.e., for some $n \in N$, $\langle \mathfrak{N}, n\rangle \models c_\beta < n$ (for all $\beta < \kappa$), then

$$\mathfrak{N} \mid \{x \in N \mid \langle \mathfrak{N}, x\rangle \models f(x) = n\}$$

would provide a model of $\Gamma \cup \{\neg \psi\}$. But this is impossible and our claim is thus established.

At this point, we consider $\mathfrak{M}'$ and $\mathfrak{M}''$, of vocabulary τ' and τ'', respectively, exactly as in the final part of the proof of Theorem 1.3, where $\tau' = \tau_{\mathfrak{M}} \cup \tau_0$ and $\tau_0 = \{P_\beta\}_{\beta<\kappa}$, and $\tau'' = \tau_0 \cup \{c\}$. Using the assumed finite vocabulary property of $\sim$, we must have that $\mathfrak{M}' \restriction \tau_0 \sim \mathfrak{M}'' \restriction \tau_0$. By the Robinson property of $\sim$, there is $\mathfrak{D}$ with $\mathfrak{D} \restriction \tau' \sim \mathfrak{M}'$ and $\mathfrak{D} \restriction \tau'' \sim \mathfrak{M}''$. In particular, $\mathfrak{M}''$, $\mathfrak{D} \models \neg P_\beta c$, for every $\beta < \kappa$, so that $c^{\mathfrak{D}}$ is a strict upper bound for the set $\{c_\beta^{\mathfrak{D}}\}_{\beta<\kappa}$. In definitive, $\mathfrak{D} \restriction \tau_{\mathfrak{M}} \sim \mathfrak{M}$ is counterexample to our claim. Having thus obtained a contradiction, we conclude the proof of the theorem. □

3.3 Corollary. *Let $\sim$ be a Robinson equivalence relation with the finite vocabulary property. For a set I, let $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ be a logic with $\equiv_{\mathscr{L}}$ coarser than $\sim$. Then $\mathscr{L}$ is compact.*

Proof. The proof follows immediately from Theorem 3.2. □

The following corollary is a "unique representability" result:

3.4 Corollary. *Let $\sim$ be a bounded Robinson equivalence relation. Then there is at most one (up to equivalence) logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ such that $\equiv_{\mathscr{L}} = \sim$. Furthermore, if any such $\mathscr{L}$ exists, then $\mathscr{L}$ is compact and has the interpolation property.*

Proof. Let $\mathscr{L}' = \mathscr{L}(Q^j)_{j\in J}$ be such that $\equiv_{\mathscr{L}'} = \equiv_{\mathscr{L}} = \sim$. Observe that $\sim$ necessarily has the finite vocabulary property. Also, I and J may be assumed to be sets, for $\sim$ is bounded. For arbitrary φ in $\mathscr{L}[\tau_\varphi]$, φ has the same models as

$$\bigcup_{\mathfrak{A} \models_{\mathscr{L}} \varphi,\, \mathfrak{A} \in \mathrm{Str}(\tau_\varphi)} \ \bigcap_{\psi \in \mathrm{Th}_{\mathscr{L}'}(\mathfrak{A})} \mathrm{Mod}_{\mathscr{L}'}\, \psi.$$

Using Theorem 3.2 and noting that $\mathrm{Th}_{\mathscr{L}'}(\mathfrak{A})$ is a set, we see that for every $\mathfrak{A} \models_{\mathscr{L}} \varphi$ there is $\psi_{\mathfrak{A}} \in \mathrm{Th}_{\mathscr{L}'}(\mathfrak{A})$ such that $\mathrm{Mod}_{\mathscr{L}'}(\psi_{\mathfrak{A}}) \subseteq \mathrm{Mod}_{\mathscr{L}}\, \varphi$. Applying Theorem 3.2 to $\neg\varphi$, there exist $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that φ has the same models as $\psi_{\mathfrak{A}_1} \vee \cdots \vee \psi_{\mathfrak{A}_n}$. Therefore, φ is equivalent to some sentence in $\mathscr{L}'$. Finally, the fact that $\mathscr{L}$ is compact and has the interpolation property now follows from Theorem 1.3 and Corollary 1.4 (recall that I and J are sets). □

3.5 Corollary. *Up to equivalence, first-order logic is the only logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ such that $\equiv_{\mathscr{L}} = \equiv$.* □

For an alternative proof of Corollary 3.5, see Theorem III.2.1.4. Observe also that the (generalized downward) Löwenheim–Skolem theorem coupled with Lindström's theorem (see Theorem III.1.1.4) implies that $\mathscr{L}_{\omega\omega}$ is the only *countably* *compact* logic $\mathscr{L}$ with $\equiv_{\mathscr{L}} = \equiv$. The point of Corollary 3.5 is that $\equiv$ uniquely characterizes $\mathscr{L}_{\omega\omega}$ among *all* logics $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$.

Corollary 3.4 shows that *at most one* logic $\mathscr{L}$ exists with $\equiv_{\mathscr{L}} = \sim$, whenever $\sim$ is a bounded Robinson equivalence relation. The problem of whether *at least one* such $\mathscr{L}$ exists will be settled in the remainder of this section.

3.6 Notational Convention. If φ is a sentence in $\mathscr{L}$ of vocabulary $\tau \cup \{c_1, \ldots, c_n\}$ with $c_1, \ldots, c_n \notin \tau$, then for $\mathfrak{A} \in \mathrm{Str}(\tau)$ we define the set $\varphi^{\mathfrak{A}}$ by

$$\varphi^{\mathfrak{A}} \underset{\text{def}}{=} \{(a_1, \ldots, a_n) \in A^n \mid \langle \mathfrak{A}, a_1, \ldots, a_n \rangle \models_{\mathscr{L}} \varphi\}.$$

3.7 Lemma. *Let $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$ be a logic such that $\equiv_{\mathscr{L}}$ is coarser than a Robinson equivalence relation $\sim$. Given $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ with $\mathfrak{A} \sim \mathfrak{B}$, and*

$$\varphi \in \mathscr{L}[\tau \cup \{c_1, \ldots, c_n\}],$$

let R be a new n-ary relation symbol and let structures $\mathfrak{A}^+, \mathfrak{B}^+ \in \mathrm{Str}(\tau \cup \{R\})$ be defined by $\mathfrak{A}^+ = \langle \mathfrak{A}, R^{\mathfrak{A}^+} \rangle$ and $\mathfrak{B}^+ = \langle \mathfrak{B}, R^{\mathfrak{B}^+} \rangle$, where $R^{\mathfrak{A}^+} = \varphi^{\mathfrak{A}}$ and $R^{\mathfrak{B}^+} = \varphi^{\mathfrak{B}}$. Then $\mathfrak{A}^+ \sim \mathfrak{B}^+$.

Proof. Let R_1, R_2 be new n-ary relation symbols, and let ρ_1 be the renaming on $\tau \cup \{R\}$ which maps R into R_1 and is equal to the identity on τ. Let ρ_2 similarly, map R into R_2. Let $\rho_1(\mathfrak{A}^+)$ and $\rho_2(\mathfrak{B}^+)$ be the correspondingly renamed structures (see Definition II.1.1.1). By the assumed Robinson property of $\sim$, there exists $\mathfrak{N} \in \mathrm{Str}(\tau \cup \{R_1, R_2\})$ such that

(1) $$\mathfrak{N} \upharpoonright \tau \cup \{R_1\} \sim \rho_1(\mathfrak{A}^+) \quad \text{and} \quad \mathfrak{N} \upharpoonright \tau \cup \{R_2\} \sim \rho_2(\mathfrak{B}^+).$$

Therefore, we have

(2) $$\begin{aligned} &\mathfrak{N}, \rho_1(\mathfrak{A}^+) \models_{\mathscr{L}} \forall c_1, \ldots, c_n(\varphi \leftrightarrow R_1 c_1, \ldots, c_n), \quad \text{and} \\ &\mathfrak{N}, \rho_2(\mathfrak{B}^+) \models_{\mathscr{L}} \forall c_1, \ldots, c_n(\varphi \leftrightarrow R_2 c_1, \ldots, c_n), \end{aligned}$$

whence $R_1^{\mathfrak{N}} = R_2^{\mathfrak{N}}$. Now using (1) and the renaming property of $\sim$ we get

(3) $$\mathfrak{A}^+ \sim \rho_1^{-1}(\mathfrak{N} \upharpoonright \tau \cup \{R_1\}) = \rho_2^{-1}(\mathfrak{N} \upharpoonright \tau \cup \{R_2\}) \sim \mathfrak{B}^+. \quad \square$$

3.8 Corollary. *Let $\mathscr{L}' = \mathscr{L}(Q^i)_{i \in I}$ and $\mathscr{L}'' = \mathscr{L}(Q^j)_{j \in J}$, where $I \cap J = \varnothing$, be logics such that $\equiv_{\mathscr{L}'}$ and $\equiv_{\mathscr{L}''}$ are both coarser than a Robinson equivalence relation $\sim$. Let $\mathscr{L} = \mathscr{L}(Q^k)_{k \in I \cup J}$. Then $\equiv_{\mathscr{L}}$ is coarser than $\sim$.*

Proof. Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ with $\mathfrak{A} \sim \mathfrak{B}$. We must prove that $\mathfrak{A} \models_{\mathscr{L}} \varphi$ iff $\mathfrak{B} \models_{\mathscr{L}} \varphi$, for every φ in $\mathscr{L}$. To this purpose, it suffices to show that for every ψ in $\mathscr{L}$ of vocabulary $\tau \cup \{c_1, \ldots, c_n\}$ we have $\langle \mathfrak{A}, \psi^{\mathfrak{A}} \rangle \sim \langle \mathfrak{B}, \psi^{\mathfrak{B}} \rangle$, as structures of vocabulary $\tau \cup \{R\}$ (recall the notational convention (3.6) and the notation of Lemma 3.7). We proceed by induction on the complexity (quantifier rank) of ψ. The only nontrivial step is when, say,

$$\psi \underset{\text{def}}{=} Q^t x_0 \vec{x}_1, \ldots, \vec{x}_m\ \varphi_0(x_0), \varphi_1(\vec{x}_1), \ldots, \varphi_m(\vec{x}_m) \quad \text{for some} \quad t \in I.$$

By the induction hypothesis, we have

$$\langle \mathfrak{A}, \varphi_0^{\mathfrak{A}}, \varphi_1^{\mathfrak{A}}, \ldots, \varphi_m^{\mathfrak{A}} \rangle \sim \langle \mathfrak{B}, \varphi_0^{\mathfrak{B}}, \varphi_1^{\mathfrak{B}}, \ldots, \varphi_m^{\mathfrak{B}} \rangle,$$

as structures of vocabulary $\{R_0, R_1, \ldots, R_m\} \cup \tau$. By Lemma 3.7, after noting that $Q^i x_0 \vec{x}_1, \ldots, \vec{x}_m R_0 x_0 R_1 \vec{x}_1, \ldots, R_m \vec{x}_m$ is a sentence in $\mathscr{L}'$, we have

$$\langle \mathfrak{A}, \varphi_0^{\mathfrak{A}}, \varphi_1^{\mathfrak{A}}, \ldots, \varphi_m^{\mathfrak{A}}, \psi^{\mathfrak{A}} \rangle \sim \langle \mathfrak{B}, \varphi_0^{\mathfrak{B}}, \varphi_1^{\mathfrak{B}}, \ldots, \varphi_m^{\mathfrak{B}}, \psi^{\mathfrak{B}} \rangle$$

as structures of vocabulary $\tau \cup \{R_0, R_1, \ldots, R_m, R\}$. Finally, by the reduct property of $\sim$, we have the desired conclusion. □

3.9 Definition. We say that a regular equivalence relation $\sim$ is *separable by quantifiers* iff whenever $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and not-$\mathfrak{A} \sim \mathfrak{B}$, then there is a quantifier Q such that $\equiv_{\mathscr{L}(Q)}$ is coarser than $\sim$, and $\mathfrak{A} \not\equiv_{\mathscr{L}(Q)} \mathfrak{B}$.

Notice that if $\sim$ is representable as $\sim \; = \; \equiv_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$, then $\sim$ is separable by quantifiers. The next theorem shows that separability is not only necessary, but also sufficient for the representability of $\sim$, provided $\sim$ has the Robinson property and is bounded.

3.10 Theorem. *Let $\sim$ be an arbitrary bounded Robinson equivalence relation. Let $\mathscr{L}^* = \mathscr{L}\{Q \mid \equiv_{\mathscr{L}(Q)}$ is coarser than $\sim\}$. Then we have:*

(i) *$\mathscr{L}^*$ is the strongest logic $\mathscr{L}$ of the form $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$ such that $\equiv_{\mathscr{L}}$ is coarser than $\sim$.*
(ii) *The identity $\sim \; = \; \equiv_{\mathscr{L}^*}$ holds iff $\sim$ is separable by quantifiers. If this is the case, then $\mathscr{L}^*$ is uniquely determined by $\sim$ (up to equivalence) and is a compact logic with the interpolation property.*

Proof. The assertion in (i) is immediate from Corollary 3.8. As for (ii) clearly, if $\sim$ is separable by quantifiers, then $\sim$ is coarser than $\equiv_{\mathscr{L}^*}$. Hence, $\sim \; = \; \equiv_{\mathscr{L}^*}$. Conversely, if $\sim \; = \; \equiv_{\mathscr{L}^*}$, not-$\mathfrak{A} \sim \mathfrak{B}$, and $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$, then $\mathfrak{A} \not\equiv_{\mathscr{L}^*} \mathfrak{B}$ so that $\mathfrak{A} \models_{\mathscr{L}^*} \psi$ and $\mathfrak{B} \models_{\mathscr{L}^*} \neg \psi$ for some ψ in $\mathscr{L}^*$. Let Q_ψ be the quantifier given by the class $\mathrm{Mod}_{\mathscr{L}^*}(\psi)$. Then $\mathscr{L}(Q_\psi) \leq \mathscr{L}^*$ by the regularity properties of logics generated by quantifiers (see Section II.4.1), hence $\equiv_{\mathscr{L}(Q_\psi)}$ is coarser than $\sim$ and $\mathfrak{A} \not\equiv_{\mathscr{L}(Q_\psi)} \mathfrak{B}$. Therefore, $\sim$ is separable by quantifiers. To conclude the proof, the uniqueness of $\mathscr{L}^*$ follows from Corollary 3.4, while the compactness and interpolation properties of $\mathscr{L}^*$ follow from Theorem 1.3 and Corollary 1.4 upon noting that since $\sim$ is bounded, then $\mathscr{L}^*[\tau]$ is a set for all τ. □

Can the duality given by (Corollary 3.4 and) Theorem 3.10 be extended beyond the realm of logics and equivalence relations with the Robinson property? The answer is partially affirmative. As a matter of fact, using the equivalence between JEP and compactness (see Chapter XVIII), we have that the bijection given by Theorem 3.10 can be extended to an injection from compact logics into equivalence relations via the following generalization of Corollary 3.4:

3.11 Theorem. *Let $\sim$ be an arbitrary regular equivalence relation such that $\sim \; = \; \equiv_{\mathscr{L}^*}$ for some logic $\mathscr{L}^* = \mathscr{L}(Q^i)_{i \in I}$, where I is a set. If $\mathscr{L}^*$ is compact (or,* *equivalently, if $\mathscr{L}^*$ has the* JEP) *then $\mathscr{L}^*$ is uniquely determined by $\sim$ up to equivalence.*

Proof. First observe that the JEP is indeed equivalent to compactness (see Examples 4.2 below and Theorem XVIII.3.3.3). We now prove the following lemma, which is of independent interest:

3.11.1 Lemma. *Let $\mathscr{L}' = \mathscr{L}(Q^j)_{j\in J}$, $\mathscr{L}'' = \mathscr{L}(Q^k)_{k\in K}$, J, K disjoint sets; let $\mathscr{L}$ be the weakest logic closed under existential quantification and boolean operations with $\mathscr{L} \geq \mathscr{L}'$ and $\mathscr{L} \geq \mathscr{L}''$. If $\equiv_{\mathscr{L}'}$ is finer than $\equiv_{\mathscr{L}''}$, then $\equiv_{\mathscr{L}} = \equiv_{\mathscr{L}'}$.*

Proof of Lemma. Assume $\equiv_{\mathscr{L}} \neq \equiv_{\mathscr{L}'}$, so that for some $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M} \equiv_{\mathscr{L}'} \mathfrak{N}$, we have $\mathfrak{M} \models_{\mathscr{L}} \psi$ and $\mathfrak{N} \models_{\mathscr{L}} \neg\psi$ for some ψ in $\mathscr{L}$. It is easy to see that ψ can be written in the form

$$\psi = Q_1 y_1, \ldots, Q_r y_r\, B(\varphi'_1, \ldots, \varphi'_p, \varphi''_1, \ldots, \varphi''_q),$$

where $Q_n \in \{\exists, \forall\}$ for each $n = 1, \ldots, r$, B is a boolean function, that is, a finite composition of $\wedge$, $\vee$, $\neg$, each φ'_i is a sentence in $\mathscr{L}'$ and each φ''_j is in $\mathscr{L}''$. Let $R_1, \ldots, R_p$ be new r-ary relation symbols, and let $\mathfrak{M}^+ = \langle \mathfrak{M}, R_1, \ldots, R_p \rangle$, $\mathfrak{N}^+ = \langle \mathfrak{N}, R_1, \ldots, R_p \rangle$ be given by

$$\text{(1)} \qquad \mathfrak{N}^+, \mathfrak{M}^+ \models_{\mathscr{L}'} \bigwedge_{i=1}^{p} \forall \vec{y}(\varphi'_i \leftrightarrow R_i \vec{y}), \quad \text{with} \quad \vec{y} = (y_1, \ldots, y_r).$$

Let sentence δ in $\mathscr{L}''$ be defined by $\delta \underset{\text{def}}{=} Q_1 y_1, \ldots, Q_r y_r\, B(R_1, \ldots, R_p, \varphi''_1, \ldots, \varphi''_q)$, and observe that these substitutions are legitimate and $\mathfrak{M}^+ \models_{\mathscr{L}''} \delta$, $\mathfrak{N}^+ \models_{\mathscr{L}''} \neg\delta$. Since $\equiv_{\mathscr{L}'}$ is finer than $\equiv_{\mathscr{L}''}$ and $\mathfrak{M}^+ \not\equiv_{\mathscr{L}''} \mathfrak{N}^+$, then, for some sentence χ in $\mathscr{L}'$, we have $\mathfrak{M}^+ \models_{\mathscr{L}'} \chi$ and $\mathfrak{N}^+ \models_{\mathscr{L}'} \neg\chi$. Define sentence θ in $\mathscr{L}'$ by

$$\theta \underset{\text{def}}{=} \chi(\varphi'_1/R_1, \ldots, \varphi'_p/R_p);$$

that is, θ is obtained from χ by replacing each occurrence of R_i in χ by φ'_i. Again, these substitutions are allowed in $\mathscr{L}'$. In conclusion, recalling (1), we have $\mathfrak{M}^+ \models_{\mathscr{L}'} \theta$ and $\mathfrak{N}^+ \models_{\mathscr{L}'} \neg\theta$. Whence $\mathfrak{M} \models_{\mathscr{L}'} \theta$ and $\mathfrak{N} \models_{\mathscr{L}'} \neg\theta$, which contradicts $\mathfrak{M} \equiv_{\mathscr{L}'} \mathfrak{N}$. □

3.11.2 *End of Proof of Theorem* 3.11. Assume that both $\mathscr{L}^*$ and $\mathscr{L}'' = \mathscr{L}(Q^k)_{k\in K}$ have $\equiv_{\mathscr{L}^*} = \equiv_{\mathscr{L}''} = \sim$. Let $\mathscr{L}$ be as in Lemma 3.11.1 (with regard to $\mathscr{L}^*$ and $\mathscr{L}''$). Using this lemma twice, we get $\equiv_{\mathscr{L}} = \sim$. Now, $\mathscr{L}[\tau]$ is a set for every τ, as can be seen by examining the form of any sentence ψ in $\mathscr{L}$, according to the proof of Lemma 3.11.1. Moreover, $\mathscr{L}$ is closed under relativizations to boolean combinations of atomic sentences, and functions can be replaced by relations in $\mathscr{L}$. Now the fact that $\mathscr{L}$ has the joint embedding property is enough to prove that $\mathscr{L}$ is compact (our assertions in Remarks 1.5 can be extended to the present case, to the effect that the results in Theorem XVIII.3.3.3 can be applied to $\mathscr{L}$). By a familiar finite cover argument such as the one given in Theorem III.1.1.5 we finally conclude that $\mathscr{L}$, $\mathscr{L}''$, and $\mathscr{L}^*$ are equivalent. □

3.12 Corollary. *Let $\mathscr{L}$ be an arbitrary logic with $\mathscr{L} \leq \Delta\mathscr{L}(Q^{\mathrm{cf}\,\omega})$. Then $\equiv_{\mathscr{L}'} = \equiv_{\mathscr{L}}$ iff $\mathscr{L}'$ is equivalent to $\mathscr{L}$.*

Proof. The Δ-closure of any compact logic is still compact (see Proposition II.7.2.5), and sublogics of compact logics are compact; $\mathscr{L}(Q^{\mathrm{cf}\,\omega})$ is compact (see Theorem II.3.2.3). □

3.13 Notes and Remarks. Regular equivalence relations in abstract model theory were introduced in Nadel [1980a]. In Theorem 7 of his paper, we proves that whenever $\sim\ =\ \equiv_{\mathscr{L}_0}$, for some logic $\mathscr{L}_0$ (i) if $\sim$ is bounded, then there is a strongest logic $\mathscr{L}$ with $\sim\ =\ \equiv_{\mathscr{L}}$ and which is closed under negation, conjunction and disjunction. By constrast, he also shows (ii) that no such strongest $\mathscr{L}$ exists if the transitive closures $\langle \bar{x}, \in\rangle$ and $\langle \bar{y}, \in\rangle$ of any two sets $x \neq y$ are never $\sim$-equivalent. Nadel's logics are systems of sentences obeying only the basic axioms given in Definition II.1.1.1. He also has a number of results about logics closed under Scott sentences, that is, logics $\mathscr{L}$ in which each $\equiv_{\mathscr{L}}$-equivalence class of structures is $\mathrm{EC}_{\mathscr{L}}$.

Corollaries 3.4 and 3.5, and the duality theorem (Theorem 3.10) of this section were orginally proved in Mundici [1982a]. The assumption used there that there are no uncountable measurable cardinals is unnecessary and was subsequently dropped (see Mundici [1982e, Section 1.1]). In Mundici [1982c, II and 198?b], Theorem 3.10 is extended to logics and equivalence relations for enriched structures (see Section 2). For instance, it is proved that topological, monotone, uniform logics are uniquely determined by their own elementary equivalence relations. The proof of Theorem 3.10 given here depends on Theorem 3.2, Corollary 3.3, Lemma 3.7 and Corollary 3.8, which were given by Flum in a private communication. Theorem 3.11 is due to Lipparini [1982].

4. *Duality Between Embedding and Equivalence Relations*

The notion of $\mathscr{L}$-(elementary) equivalence is generalized in Definition 3.1; the notion of $\mathscr{L}$-(elementary) embedding is generalized in the following:

4.1 Definition. An arbitrary binary relation $\to$ on the class of all structures is called an (abstract) *embedding relation* iff $\to$ satisfies the following axioms (for every two structures $\mathfrak{M}, \mathfrak{N}$):

vocabulary:	$\mathfrak{M} \to \mathfrak{N}$ implies $\tau_{\mathfrak{M}} \subseteq \tau_{\mathfrak{N}}$;
	$\mathfrak{M} \to \mathfrak{N} \upharpoonright \tau_{\mathfrak{M}}$ iff $\mathfrak{M} \to \mathfrak{N}$;
renaming:	$\mathfrak{M} \to \mathfrak{N}$ implies $\mathfrak{M}^{\rho} \to \mathfrak{N}^{\rho}$ for any renaming ρ of $\tau_{\mathfrak{N}}$;
reduct:	$\mathfrak{M} \to \mathfrak{N}$ implies $\mathfrak{M} \upharpoonright \tau \to \mathfrak{N} \upharpoonright \tau$ for all $\tau \subseteq \tau_{\mathfrak{M}}$;
isomorphism:	$\mathfrak{M} \cong \mathfrak{N}$ implies $\mathfrak{M} \to \mathfrak{N}$;
expressiveness:	$\mathfrak{M} \to \mathfrak{N}$ implies $\mathfrak{M}_M \equiv \mathfrak{N}^+$ for some expansion $\mathfrak{N}^+$ of $\mathfrak{N} \upharpoonright \tau_{\mathfrak{M}}$;
transitivity:	$\mathfrak{M} \to \mathfrak{N}$ and $\mathfrak{N} \to \mathfrak{B}$ implies $\mathfrak{M} \to \mathfrak{B}$.

Recall that $\mathfrak{M}_M$ denotes the diagram expansion of $\mathfrak{M}$. An embedding relation $\to$ has the *expanded amalgamation property*, denoted by AP^+ (resp., the *amalgamation property*, denoted by AP) iff whenever $\mathfrak{A} \leftarrow \mathfrak{N} \to \mathfrak{B}$ and $\tau_{\mathfrak{A}} \cap \tau_{\mathfrak{B}} = \tau_{\mathfrak{N}}$ (resp., $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}} = \tau_{\mathfrak{N}}$), then $\mathfrak{A} \to \mathfrak{M} \leftarrow \mathfrak{B}$ for some structure $\mathfrak{M}$. Given an embedding $\to$ and an equivalence relation $\sim$, we say that the pair $(\sim, \to)$ has the *joint embedding property*, denoted as before by JEP, iff whenever $\mathfrak{A} \sim \mathfrak{B}$ then $\mathfrak{A} \to \mathfrak{M} \leftarrow \mathfrak{B}$ for some $\mathfrak{M}$. When $\sim \; = \; \equiv_{\mathscr{L}}$ this agrees with Section 1. If $\sim$ is a regular equivalence relation (on the class of all structures), then $\sim$ *generates* an embedding relation $\to$ by stipulating that $\mathfrak{A} \to \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \sim \mathfrak{B}^+$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. We denote by $\sim^*$ the embedding relation generated by $\sim$. Conversely, any embedding relation $\to$ *generates* a regular equivalence relation $\sim$ by stipulating that $\mathfrak{A} \sim \mathfrak{B}$ iff $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and there is a finite path:

$$\mathfrak{A} = \mathfrak{N}_0 \underset{1}{-} \mathfrak{N}_1 \underset{2}{-} \cdots \underset{k}{-} \mathfrak{N}_k = \mathfrak{B},$$

with $\tau_{\mathfrak{N}_0} = \cdots = \tau_{\mathfrak{N}_k}$ and $\underset{i}{-}$ being either $\to$ or $\leftarrow$, depending on i ($i = 1, \ldots, k$). We denote by $\to^*$ the regular equivalence relation generated by $\to$.

4.2 Examples. (a) If $\mathscr{L}$ is a logic, define $\to_{\mathscr{L}}$ by stipulating that $\mathfrak{A} \to_{\mathscr{L}} \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}_A \equiv_{\mathscr{L}} \mathfrak{B}^+$ for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. Then $\to_{\mathscr{L}}$ is an embedding relation, called *$\mathscr{L}$-embedding*. Observe that $\to_{\mathscr{L}} = (\equiv_{\mathscr{L}})^*$. For the particular case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, we have that $\mathfrak{A} \to_{\mathscr{L}} \mathfrak{B}$ iff $\mathfrak{A} \precsim \mathfrak{B} \restriction \tau_{\mathfrak{A}}$; that is, $\mathfrak{A}$ is elementarily embedded into $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$. Returning now to the general case, assume that $\to \; = \; \to_{\mathscr{L}}$, for $\mathscr{L} = \mathscr{L}(Q^i)_{i \in I}$, where I is a set. Let $\sim \; = \; \equiv_{\mathscr{L}}$ (so that $\to \; = \; \sim^*$). Then $\mathscr{L}$ is compact iff $\to$ has the AP, iff the pair $(\sim, \to)$ has the JEP. For a proof of this fact see Theorem XVIII.3.3.3. The above equivalences—originally proved in Mundici [1982b] (compactness = JEP) and, independently, in Makowsky–Shelah [1983] (compactness = AP = JEP)—enable us to regard the notion of compactness as an algebraic property of embedding or equivalence relations in much the same way as compactness + interpolation is algebraized via the Robinson property. The latter, in turn, has an equivalent counterpart for embeddings in terms of the AP^+, as will be shown in Theorem 4.8.

(b) If $\mathscr{L}$ is a logic, define $\to_{\mathscr{L}}^{\#}$ by stipulating that $\mathfrak{A} \to_{\mathscr{L}}^{\#} \mathfrak{B}$ iff $\tau_{\mathfrak{B}} \supseteq \tau_{\mathfrak{A}}$ and $\mathfrak{A}^{\#} \equiv_{\mathscr{L}} \mathfrak{B}''$ for some expansion $\mathfrak{B}''$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$, where $\mathfrak{A}^{\#}$ denotes the complete expansion of $\mathfrak{A}$ (see Section XVIII.1.2). Then $\to_{\mathscr{L}}^{\#}$ is an embedding relation, called the *$\mathscr{L}$-complete embedding relation.* In case $\mathscr{L} = \mathscr{L}_{\omega\omega}$ it is well known that $\to_{\mathscr{L}}^{\#}$ has AP^+. Indeed, $(\equiv, \to_{\mathscr{L}}^{\#})$ has the JEP; also, $(\to_{\mathscr{L}}^{\#})^* = \; \equiv$.

We now begin consideration of the (preservation) properties of the map $*$.

4.3 Proposition. *Let $\sim$ be a regular equivalence relation. Let $\to \; = \; \sim^*$, and $\approx \; = \; \to^*$. Then $\approx$ is finer than $\sim$.*

Proof. First observe that if $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and $\mathfrak{A} \sim^* \mathfrak{B}$ then $\mathfrak{A} \sim \mathfrak{B}$. As a matter of fact, $\mathfrak{A} \sim^* \mathfrak{B}$ means that $\mathfrak{A}_A \sim \mathfrak{B}^+$, for some expansion $\mathfrak{B}^+$ of $\mathfrak{B} \restriction \tau_{\mathfrak{A}}$ ($=\mathfrak{B}$, in the present case). Therefore, by the reduct axiom, $\mathfrak{A} = \mathfrak{A}_A \restriction \tau_{\mathfrak{A}} \sim \mathfrak{B}^+ \restriction \tau_{\mathfrak{A}} = \mathfrak{B}$. Now, to conclude the proof of our proposition, if $\mathfrak{M} \approx \mathfrak{N}$, then by definition there is a path

$$\mathfrak{M} = \mathfrak{A}_0 \underset{1}{-} \mathfrak{A}_1 \underset{2}{-} \cdots \underset{k}{-} \mathfrak{A}_k = \mathfrak{N},$$

with $\tau_{\mathfrak{A}_0} = \cdots = \tau_{\mathfrak{A}_k}$ and $\underset{i}{-} = \rightarrow$ or $\underset{i}{-} = \leftarrow$; by the above initial remark we have that $\mathfrak{A}_0 \sim \cdots \sim \mathfrak{A}_k$, as required. $\square$

4.4 Proposition. *Let $\rightarrow$ be an embedding relation with* AP$^+$. *Let $\sim\ = \rightarrow^*$; then we have:*

(i) *the pair $(\rightarrow^*, \rightarrow)$ has the* JEP;
(ii) *$\sim$ is a regular Robinson equivalence relation.*

Proof. For (i), we assume $\mathfrak{M} \sim \mathfrak{N}$, and let $\tau = \tau_{\mathfrak{M}} = \tau_{\mathfrak{N}}$. By definition there is a path:

$$(+) \qquad \mathfrak{M} = \mathfrak{A}_0 \underset{1}{-} \mathfrak{A}_1 \underset{2}{-} \cdots \underset{n}{-} \mathfrak{A}_n = \mathfrak{N},$$

with $\tau_{\mathfrak{A}_i} = \tau$, for each $i = 0, \ldots, n$, and $\underset{i}{-}$ being either $\rightarrow$ or $\leftarrow$. If $n = 1$, then let $\mathfrak{D} = \mathfrak{N}$ or $\mathfrak{D} = \mathfrak{M}$, according to whether $\underset{1}{-} = \rightarrow$ or $\underset{1}{-} = \leftarrow$ is the case; then $\mathfrak{M} \rightarrow \mathfrak{D} \leftarrow \mathfrak{N}$, and we are done. Proceeding now by induction on n, we obtain from $(+)$:

$$(++) \qquad \mathfrak{M} \rightarrow \mathfrak{B} \leftarrow \mathfrak{A}_{n-1} \underset{n}{-} \mathfrak{N}.$$

Now, if $\underset{n}{-} = \leftarrow$, then by transitivity we see that $\mathfrak{M} \rightarrow \mathfrak{B} \leftarrow \mathfrak{N}$. If $\underset{n}{-} = \rightarrow$, then by the AP$^+$ (actually only the AP is needed here) we have

$$(+++) \qquad \begin{tikzcd} \mathfrak{M} \arrow[r] & \mathfrak{B} \arrow[dr] & \mathfrak{A}_{n-1} \arrow[l] \arrow[r] & \mathfrak{N}; \arrow[dl] \\ & & \mathfrak{D} & \end{tikzcd}$$

hence $\mathfrak{M} \rightarrow \mathfrak{D} \leftarrow \mathfrak{N}$, as required.

As for (ii), we see that the regularity of $\rightarrow^*$ is an immediate consequence of Definition 4.1. Let $\mathfrak{M} \restriction \tau \sim \mathfrak{N} \restriction \tau$, where $\tau = \tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}}$. From (i) above and the regularity properties of $\sim$, we must have, for some $\mathfrak{D} \in \mathrm{Str}(\tau)$, that

$$\mathfrak{M} \leftarrow \mathfrak{M} \restriction \tau \rightarrow \mathfrak{D} \leftarrow \mathfrak{N} \restriction \tau \rightarrow \mathfrak{N}.$$

By repeated application of the AP$^+$, we obtain, for some $\mathfrak{A} \in \mathrm{Str}(\tau_{\mathfrak{M}})$, $\mathfrak{B} \in \mathrm{Str}(\tau_{\mathfrak{N}})$ and $\mathfrak{S} \in \mathrm{Str}(\tau_{\mathfrak{M}} \cup \tau_{\mathfrak{N}})$:

$$\begin{tikzcd} \mathfrak{M} \arrow[r] & \mathfrak{A} \arrow[dr] & \mathfrak{D} \arrow[l] \arrow[r] & \mathfrak{B} \arrow[dl] & \mathfrak{N} \arrow[l] \\ & & \mathfrak{S} & & \end{tikzcd}$$

We thus conclude that $\mathfrak{M} \to \mathfrak{S} \restriction \tau_{\mathfrak{M}}$ and $\mathfrak{N} \to \mathfrak{S} \restriction \tau_{\mathfrak{N}}$. By definition of $\sim$, we finally obtain that $\mathfrak{M} \sim \mathfrak{S} \restriction \tau_{\mathfrak{M}}$ and $\mathfrak{N} \sim \mathfrak{S} \restriction \tau_{\mathfrak{N}}$, thus showing that $\sim$ has the Robinson property. ∎

4.5 Proposition. *Let $\sim$ be a regular Robinson equivalence relation. Let $\to\ = \ \sim^*$. Then we have:*

(i) *the pair $(\sim, \sim^*)$ has the* JEP;
(ii) *$\to$ is an embedding relation with the* AP$^+$.

Proof. If $\mathfrak{M} \sim \mathfrak{N}$, let $\mathfrak{M}_M$ and $\mathfrak{N}_N$ be obtained by using different constants so that $\tau = \tau_{\mathfrak{M}} = \tau_{\mathfrak{N}} = \tau_{\mathfrak{M}_M} \cap \tau_{\mathfrak{N}_N}$ and $\mathfrak{M}_M \restriction \tau \sim \mathfrak{N}_N \restriction \tau$. Using the Robinson property of $\sim$, we let $\mathfrak{A}$ be such that $\mathfrak{A} \restriction \tau_{\mathfrak{M}_M} \sim \mathfrak{M}_M$ and $\mathfrak{A} \restriction \tau_{\mathfrak{N}_N} \sim \mathfrak{N}_N$. By definition of $\to$, $\mathfrak{M} \to \mathfrak{A} \leftarrow \mathfrak{N}$. Turning now to (ii) Assume $\mathfrak{M} \leftarrow \mathfrak{B} \to \mathfrak{N}$ with $\tau_{\mathfrak{M}} \cap \tau_{\mathfrak{N}} = \tau_{\mathfrak{B}}$. By the initial remark in the proof of Proposition 4.3 we automatically have that $\mathfrak{M} \restriction \tau_{\mathfrak{B}} \sim \mathfrak{B} \sim \mathfrak{N} \restriction \tau_{\mathfrak{B}}$. If different constants are used in the diagram expansions of $\mathfrak{M}$ and $\mathfrak{N}$, we also have that $\mathfrak{M}_M \restriction \tau_{\mathfrak{B}} \sim \mathfrak{B} \sim \mathfrak{N}_N \restriction \tau_{\mathfrak{B}}$, and, by the Robinson property of $\sim$, there is some $\mathfrak{D}$ such that $\mathfrak{D} \restriction \tau_{\mathfrak{N}_N} \sim \mathfrak{N}_N$ and $\mathfrak{D} \restriction \tau_{\mathfrak{M}_M} \sim \mathfrak{M}_M$. From the definition of $\to$, we obtain $\mathfrak{M} \to \mathfrak{D} \leftarrow \mathfrak{N}$, which establishes the desired AP$^+$ property for $\to$. ∎

4.6 Proposition. (i) *If $\sim$ is a regular Robinson equivalence relation then $(\sim^*)^* = \ \sim$; and*
(ii) *If $\sim_1$ and $\sim_2$ are different regular Robinson equivalence relations, then $\sim_1^*$ is different from $\sim_2^*$.*

Proof. For (i), we observe that in view of Proposition 4.3, it suffices to show that $\sim$ is finer than $\sim^{**}$. Now, if $\mathfrak{M} \sim \mathfrak{N}$, then for some $\mathfrak{D}$ we have $\mathfrak{M} \to \mathfrak{D} \leftarrow \mathfrak{N}$, by Proposition 4.5(i), where $\to\ = \ \sim^*$. From the definition of $\to^*$, we thus have $\mathfrak{M} \sim^{**} \mathfrak{N}$, as required.

Turning now to (ii), we assume $\mathfrak{M} \sim_1 \mathfrak{N}$ and not-$\mathfrak{M} \sim_2 \mathfrak{N}$. Let $\to_1\ = \ \sim_1^*$ and $\to_2\ = \ \sim_2^*$. We also that assume $\to_1\ = \ \to_2$ (*absurdum* hypothesis). By Proposition 4.5(i), for some $\mathfrak{D}$, we have $\mathfrak{M} \to_1 \mathfrak{D} \ {}_1\!\!\leftarrow \mathfrak{N}$. Hence, $\mathfrak{M} \to_2 \mathfrak{D} \ {}_2\!\!\leftarrow \mathfrak{N}$, whence it follows that $\mathfrak{N} \sim_2 \mathfrak{M}$ (by the first remark in Proposition 4.3). This contradicts our assumption. ∎

4.7 Remark. The counterpart of Proposition 4.6(i) and (ii) does not hold for embeddings with the AP$^+$ in place of Robinson equivalence relations. For example, the complete embedding relation $\to_{\mathscr{L}}^{\#}$ arising from $\mathscr{L} = \mathscr{L}_{\omega\omega}$ (see Example 4.2(b)) generates $\equiv$, and $\equiv$ in turn generates $\to_{\mathscr{L}}$, which is different from $\to_{\mathscr{L}}^{\#}$. To obtain the analogue of Proposition 4.6, we must restrict attention to *involutive* embedding relations $\to$ with AP$^+$ (where $\to$ is involutive iff $\to\ = \ \to^{**}$). An example of involutive embedding relation with AP$^+$ is $\to_{\mathscr{L}_{\omega\omega}}$. Indeed, we have the following quite general fact:

4.8 Theorem. *Let $\mathscr{R}$ be the family of all regular Robinson equivalence relations; let $\mathscr{A}$ be the family of all involutive embedding relations with* AP$^+$. *Then $*$ maps $\mathscr{A}$ one-one onto $\mathscr{R}$, and vice versa. Furthermore, $**$ is the identity function on $\mathscr{A} \cup \mathscr{R}$.*

Proof. Map $*$ sends elements of $\mathscr{R}$ into elements of $\mathscr{A}$ by Proposition 4.5(ii), and by noting that $(\sim^*)^{**} = (\sim^{**})^* = \sim^*$, see Proposition 4.6(i). Also, $*$ is injective from $\mathscr{R}$ into $\mathscr{A}$ by Proposition 4.6(ii). Map $*$ sends elements of $\mathscr{A}$ into elements of $\mathscr{R}$ by Proposition 4.4(ii) and is injective from $\mathscr{A}$ into $\mathscr{R}$. As a matter of fact, if $\to_1$ and $\to_2$ are in $\mathscr{A}$ and $\to_1^* = \to_2^*$, then also $\to_1^{**} = \to_2^{**}$. Whence it follows that $\to_1 = \to_2$, by definition of $\mathscr{A}$. Map $**$ is the identity on $\mathscr{A}$ by definition, and is the identity on $\mathscr{R}$ by Proposition 4.6(i). Finally, $*$ maps $\mathscr{A}$ onto $\mathscr{R}$, and $\mathscr{R}$ onto $\mathscr{A}$, because every element in $\mathscr{A} \cup \mathscr{R}$ is the $*$-image of its own $*$-image. ☐

From Example 4.2(a) we now recall the definition of $\mathscr{L}_{\omega\omega}$-embedding, $\to_{\mathscr{L}_{\omega\omega}}$ in terms of $\precsim$:

4.9 Theorem. *First-order logic is the only (up to equivalence) logic $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$ such that $\to_{\mathscr{L}} = \to_{\mathscr{L}_{\omega\omega}}$.*

Proof. Assume $\mathscr{L}$ is a logic with $\to_{\mathscr{L}} = \to_{\mathscr{L}_{\omega\omega}}$. By definition of $\to_{\mathscr{L}}$, we have that $\equiv_{\mathscr{L}}^* = \equiv^*$. Hence $\equiv_{\mathscr{L}}^{**} = \equiv^{**} = \equiv$ (the fact that $\equiv^{**} = \equiv$ is a consequence of Proposition 4.6(i), since $\equiv$ has the Robinson property). By Proposition 4.3, $\equiv_{\mathscr{L}}$ is coarser than $\equiv_{\mathscr{L}}^{**} = \equiv$. Conversely, $\equiv_{\mathscr{L}}$ is finer than $\equiv$, as $\mathscr{L} \geq \mathscr{L}_{\omega\omega}$. Therefore, we have $\equiv_{\mathscr{L}} = \equiv$. We now apply Corollary 3.5 to conclude that $\mathscr{L}$ is equivalent to first-order logic. ☐

4.10 Remarks. Abstract embedding relations were introduced in Mundici [1982d, 1983a and 198?a]. The results of the present section are extracted from the last paper. Notice that if we delete the expressiveness axiom from both definitions of $\sim$ and $\to$, the duality between (the resulting, weaker) embedding and equivalence relations can still be shown to hold exactly as in Theorem 4.8. In Mundici [198?a], Theorem 4.8 is partially extended, replacing the Robinson (or the AP^+) assumption by the weaker requirement that $(\sim, \sim^*)$ has the JEP.

5. Sequences of Finite Partitions, Global and Local Back-and-Forth Games

The separability assumption in Theorem 3.10(ii) can be neglected in the important case of equivalence relations associated with countably generated compact logics with interpolation. In general, countably generated logics are given by sequences of finite partitions on structures; and these are, in turn, related to the back-and-forth games for $\mathscr{L}$-elementary equivalence. Throughout this section, the vocabularies will only contain relation and constant symbols, for the sake of simplicity.

5.1 Definition. A *back-and-forth system* is a function $\simeq$ assigning to every finite vocabulary τ a sequence $\{\simeq_\tau^n\}_{n<\omega}$, with $\simeq_\tau^n$ a finite partition on $\operatorname{Str}(\tau)$, that is, an equivalence relation with finitely many classes, coarser than isomorphism and satisfying the following conditions, for every $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$:

renaming: $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ implies $\mathfrak{A}^\rho \simeq_{\tau'}^n \mathfrak{B}^\rho$ for any $\rho: \tau \to \tau'$;

reduct: $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ implies $\mathfrak{A} \upharpoonright \tau' \simeq_{\tau'}^n \mathfrak{B} \upharpoonright \tau'$ for any $\tau' \subseteq \tau$;

atomic: $\mathfrak{A} \simeq_\tau^0 \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same atomic sentences of vocabulary τ;

diagram: $\mathfrak{A} \simeq_\tau^{n+1} \mathfrak{B}$ implies $\forall a \in A\, \exists b \in B$ with $\langle \mathfrak{A}, a\rangle \simeq_{\tau'}^n \langle \mathfrak{B}, b\rangle$, where τ' is obtained from τ by adding one constant symbol;

substructure: $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ implies

$$\mathfrak{A} | \{a \in A \mid \langle \mathfrak{A}, a\rangle \models \alpha(a)\} \simeq_\tau^n \mathfrak{B} | \{b \in B \mid \langle \mathfrak{B}, b\rangle \models \alpha(b)\},$$

whenever $\alpha(x)$ is a boolean combination of atomic sentences of vocabulary $\tau_\alpha \subseteq \tau \cup \{x\}$, $x \notin \tau$. $\mathfrak{A}|A'$ is the substructure of $\mathfrak{A}$ generated by $A' \subseteq A$.

Note that the diagram condition together with the reduct axiom imply that $\simeq_\tau^{n+1}$ is finer than $\simeq_\tau^n$.

5.2 Examples. In Theorem 5.3 we will see that every countably generated logic $\mathscr{L} = \mathscr{L}(Q^i)_{i<\omega}$ determines a back-and-forth system $\simeq$, if we let $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ mean that $\tau_\mathfrak{A} = \tau_\mathfrak{B} = \tau$ finite, and $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathscr{L}$ of vocabulary τ and quantifier rank $\leq n$. In the particular case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, we get the *Fraïssé–Ehrenfeucht* back-and-forth system, which can be equivalently obtained by writing "iff" instead of "implies" in the diagram axiom above; and, if this is done, the substructure axiom becomes superfluous. Back-and-forth systems are a natural generalization of the familiar games for $\mathscr{L}$-equivalence. For the case $\mathscr{L} = \mathscr{L}_{\omega\omega}$, see Section II.4.2 and Section IX.4. For many other $\mathscr{L}$'s the reader should consult Weese [1980], Caicedo [1979], Makowsky–Shelah [1981], Flum–Ziegler [1980]. In a final subsection we shall relate the back-and-forth games existing in the literature to our present back-and-forth systems. Given a logic $\mathscr{L}$, the question of the existence and uniqueness of a back and forth system characterizing $\equiv_\mathscr{L}$ arises. In Theorem 5.4 we will use the Robinsion assumption to establish a one–one correspondence between back-and-forth systems and countably generated logics. Before defining the proper uniqueness notion for back-and-forth systems, however, let us remark that any such system $\simeq$ *generates* a bounded regular equivalence relation $\sim$ on the class of all structures by letting $\mathfrak{A} \sim \mathfrak{B}$ mean that $\tau_\mathfrak{A} = \tau_\mathfrak{B}$ and $\mathfrak{A} \upharpoonright \tau \simeq_\tau^n \mathfrak{B} \upharpoonright \tau$ for every finite $\tau \subseteq \tau_\mathfrak{A}$ and all $n < \omega$. Now, given two back-and-forth systems $\simeq'$ and $\simeq''$, we say that $\simeq''$ is *finer* than $\simeq'$ iff for every finite τ and $n < \omega$, there is $m < \omega$ such that $\simeq_\tau''^m$ is finer than $\simeq_\tau'^n$. In case $\simeq'$ is finer than $\simeq''$ and vice versa, we say that $\simeq'$ and $\simeq''$ are *equivalent*.

The great generality of the notion of a back-and-forth system is shown by the following result.

5.3 Theorem. *Let $\mathscr{L}$ be a countably generated logic. Then $\equiv_\mathscr{L}$ is generated by some back-and-forth system.*

Proof. Write $\mathscr{L}$ as $\mathscr{L}(Q^i)_{i<\omega}$ and assign a rank $r_i = 2 + i$ to each Q^i, the rank 1 being assigned to $\exists$ and to $\forall$. Then the sentences of $\mathscr{L}$ inherit a quantifier rank as in Definition II.4.2.5. Notice that for any finite τ and $n < \omega$, there are in $\mathscr{L}[\tau]$ only a finite number of pairwise inequivalent sentences with quantifier rank $\leq n$. Define $\simeq_\tau^n$ by

$\mathfrak{A} \simeq_\tau^n \mathfrak{B}$ iff $\tau_\mathfrak{A} = \tau_\mathfrak{B} = \tau$, τ finite, and $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathscr{L}[\tau]$ with quantifier rank $\leq n$.

Then the equivalence relation $\simeq_\tau^n$ on $\mathrm{Str}(\tau)$ has finitely many equivalence classes and is coarser than isomorphism. Moreover, the reduct, renaming and atomic properties follow immediately from the basic closure properties of $\mathscr{L}$. As to the diagram axiom, let $\mathfrak{A} \simeq_\tau^{n+1} \mathfrak{B}$ and $a \in A$; let $T = \{\psi_1, \ldots, \psi_t\}$ display, without repetitions of equivalent sentences, the finitely many sentences of $\mathscr{L}[\tau]$ having quantifier rank $\leq n$, and which are satisfied by $\langle \mathfrak{A}, a \rangle$. Since $\langle \mathfrak{A}, a \rangle \models_\mathscr{L} \psi_1 \wedge \cdots \wedge \psi_t$, then $\mathfrak{A} \models_\mathscr{L} \exists a(\psi_1 \wedge \cdots \wedge \psi_t)$; since the quantifier rank of this latter sentence is $\leq n + 1$, then by assumption, $\mathfrak{B}$ is among its models. Hence $\langle \mathfrak{B}, b \rangle \models_\mathscr{L} \psi_1 \wedge \cdots \wedge \psi_t$, for some $b \in B$, whence $\langle \mathfrak{B}, b \rangle \simeq_{\tau'}^n \langle \mathfrak{A}, a \rangle$, thus establishing the diagram property of $\simeq$ (τ' is given by τ plus one constant). Concerning the substructure axiom, let $\mathfrak{A} \simeq_\tau^n \mathfrak{B}$. Assume further that $\varphi \in \mathscr{L}[\tau]$ is an arbitrary sentence with quantifier rank $\leq n$, such that $\mathfrak{A}_0 = \mathfrak{A} | \{a \in A | \langle \mathfrak{A}, a \rangle \models \alpha(a)\} \models_\mathscr{L} \varphi$. It then follows that $\mathfrak{A} \models_\mathscr{L} \varphi^{\{x|\alpha(x)\}}$. But the latter sentence has the same quantifier rank as φ: to see this, we first note that τ-closure amounts to saying that all the constants of τ satisfy α, and this can be expressed by an atomic sentence in light of the finiteness of τ and of our assumption that τ has no function symbols. Moreover, as $\mathscr{L}$ is generated by quantifiers, we see that writing down $\varphi^{\{x|\alpha(x)\}}$ involves the conjunction of α with the sentences in the scope of Q^i (if φ is of the form $Q^i\chi$). Thus, the quantifier rank is not increased; in case φ is of the form $\neg\chi$, or $\chi \vee \psi$, or $\chi \wedge \psi$, the quantifier rank of the relativization to α is still not increased. In definitive, $\varphi^{\{x|\alpha(x)\}}$ has quantifier rank $\leq n$. Hence, by assumption, $\mathfrak{B} \models_\mathscr{L} \varphi^{\{x|\alpha(x)\}}$ so that $\mathfrak{B}_0 = \mathfrak{B} | \{b \in B | \langle \mathfrak{B}, b \rangle \models \alpha(b)\} \models_\mathscr{L} \varphi$. Since φ is arbitrary, we have proven that $\mathfrak{A}_0 \simeq_\tau^n \mathfrak{B}_0$, which yields the substructure property of $\simeq$. Finally, it is clear that $\simeq$ generates $\equiv_\mathscr{L}$, for two structures $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathscr{L}$-equivalent iff they satisfy the same sentences of quantifier rank $\leq n$ and vocabulary τ for all $n < \omega$ and all finite $\tau \subseteq \tau_\mathfrak{M} = \tau_\mathfrak{N}$. $\square$

When $\sim$ has the Robinson property we have a strong converse of the above theorem, as follows.

5.4 Theorem. *For $\sim$ an arbitrary Robinson equivalence relation the following are equivalent:*

(i) $\sim\ =\ \equiv_\mathscr{L}$ *for some countably generated logic $\mathscr{L}$;*
(ii) $\sim\ =\ \equiv_\mathscr{L}$ *for a unique (up to equivalence) countably generated logic $\mathscr{L}$; further, $\mathscr{L}$ is compact and has the interpolation property;*
(iii) $\sim$ *is generated by some back-and-forth system;*
(iv) $\sim$ *is generated by precisely one (up to equivalence) back-and-forth system.*

Proof. The implications (ii) $\Rightarrow$ (i), and (iv) $\Rightarrow$ (iii) are trivial. The implication (i) $\Rightarrow$ (iii) has been shown in Theorem 5.3. In order to prove that (iii) implies (ii) we proceed as follows: In the light of Theorem 1.3 and Corollary 3.4, it suffices to prove that (iii) implies (i). To this purpose, let $\simeq$ be a back-and-forth system generating $\sim$.

Define $[\mathscr{L}, \models_{\mathscr{L}}]$ by

$(*)$ $\varphi \in \mathscr{L}[\tau]$ iff φ is a union of equivalence classes of $\simeq^n_{\tau_\varphi}$ for some $n < \omega$ and some (necessarily unique and finite) vocabulary $\tau_\varphi \subseteq \tau$; and,

$(\overset{*}{*})$ $\mathfrak{A} \models_{\mathscr{L}} \varphi$ iff $\varphi \in \mathscr{L}[\tau_{\mathfrak{A}}]$ and $\mathfrak{A} \upharpoonright \tau_\varphi \in \varphi$.

Then clearly $\mathscr{L}$ satisfies the isomorphism, (finite) occurrence, renaming, reduct axioms for logics (the reader is referred to Definition II.1.1.1), and $\mathscr{L}$ contains the classes of models of atomic sentences and is closed under the boolean operations. To prove that $\mathscr{L}$ is closed under $\exists$, we assume that φ is a union of equivalence classes of $\simeq^n_{\tau_\varphi}$. It now suffices to prove that $\exists c\varphi$ is also a union of equivalence classes of $\simeq^{n+1}_\tau$, where $\tau = \tau_\varphi \backslash \{c\}$. Here we pose a denial (*absurdum* hypothesis) so that for some $\mathfrak{A}$ and $\mathfrak{B}$, with $\mathfrak{A} \simeq^{n+1}_\tau \mathfrak{B}$ we have that $\mathfrak{A} \in \exists c\varphi$ and $\mathfrak{B} \notin \exists c\varphi$. Now $\langle \mathfrak{A}, a\rangle \in \varphi$ for some $a \in A$. The assumed diagram property of $\simeq$ assures us that $\langle \mathfrak{B}, b\rangle \simeq^n_{\tau_\varphi} \langle \mathfrak{A}, a\rangle$ for some $b \in B$. Hence $\langle \mathfrak{B}, b\rangle \in \varphi$, whence we have that $\mathfrak{B} \in \exists c\varphi$—a contradiction. To prove that $\mathscr{L}$ is closed under relativization, we first show that $\mathscr{L}$ is closed under relativization to any boolean combination α of atomic sentences. Assume then that φ is a union of equivalence classes of $\simeq^n_{\tau_\varphi}$, then it suffices to show that $\varphi^{\{x|\alpha(x)\}}$ is also a union of equivalence classes of $\simeq^n_\tau$, with $\tau = \tau_\varphi \cup (\tau_\alpha \backslash \{x\})$. Again, we pose a denial (*absurdum* hypothesis) so that for some $\mathfrak{A}$ and $\mathfrak{B}$, with $\mathfrak{A} \simeq^n_\tau \mathfrak{B}$, we have $\mathfrak{A} \in \varphi^{\{x|\alpha(x)\}}$ and $\mathfrak{B} \notin \varphi^{\{x|\alpha(x)\}}$. By definition of relativization, $\mathfrak{A}_0 \in \varphi$ and $\mathfrak{B}_0 \notin \varphi$, where $\mathfrak{A}_0 = \mathfrak{A}|\{a \in A | \langle \mathfrak{A}, a\rangle \models \alpha(a)\} \upharpoonright \tau_\varphi$, and $\mathfrak{B}_0 = \mathfrak{B}|\{b \in B | \langle \mathfrak{B}, b\rangle \models \alpha(b)\} \upharpoonright \tau_\varphi$. In contrast, however, the substructure together with the reduct axiom for $\simeq$ are to the effect that $\mathfrak{A}_0 \simeq^n_{\tau_\varphi} \mathfrak{B}_0$; hence, $\mathfrak{A}_0 \in \varphi$ iff $\mathfrak{B}_0 \in \varphi$. But this is a contradiction, which proves that $\mathscr{L}$ is closed under relativization to α, as required. By conditions $(*)$ and $(\overset{*}{*})$, $\equiv_{\mathscr{L}}$ is coarser than $\sim$. On the other hand, if $\tau_{\mathfrak{A}} = \tau_{\mathfrak{B}}$ and not-$\mathfrak{A} \sim \mathfrak{B}$, then, since $\sim$ is generated by $\simeq$, there is a finite $\tau \subseteq \tau_{\mathfrak{A}}$ and $n < \omega$ such that not-$\mathfrak{A} \upharpoonright \tau \simeq^n_\tau \mathfrak{B} \upharpoonright \tau$. Thus, there is a $\varphi \in \mathscr{L}[\tau]$ such that $\mathfrak{A} \models_{\mathscr{L}} \varphi$ and $\mathfrak{B} \models_{\mathscr{L}} \neg\varphi$, whence it follows that $\mathfrak{A} \not\equiv_{\mathscr{L}} \mathfrak{B}$ and $\equiv_{\mathscr{L}} = \sim$. We now prove that $\mathscr{L}$ is countably generated. Hence, let ψ be an $\mathscr{L}[\tau]$-sentence, with $\tau = \{R_1, \ldots, R_n\}$ (without constants for the sake of notational simplicity). Also, let Q_ψ be the quantifier given by $\mathrm{Mod}_{\mathscr{L}}(\psi)$, and let θ be the sentence of $\mathscr{L}^+ = \mathscr{L}(Q_\psi) \cup \mathscr{L}$ given by

$$\theta \underset{\text{def}}{=} Q_\psi x_0 \vec{x}_1, \ldots, \vec{x}_n\ \varphi_0(x_0), \varphi_1(\vec{x}_1), \ldots, \varphi_n(\vec{x}_n),$$

where the φ_i are arbitrary sentences in $\mathscr{L}$. By definition of Q_ψ, we have $\mathfrak{M} \models_{\mathscr{L}^+} \theta$ iff $\mathfrak{M}$ has an expansion $\mathfrak{N} = [\mathfrak{M}, \langle s, R_1, \ldots, R_n\rangle, f]$ with the following properties: s is a new sort, $\langle s, R_1, \ldots, R_n\rangle \models_{\mathscr{L}} \psi$, f maps sort s one–one onto $\varphi_0^{\mathfrak{M}}$ (recalling

Notational Convention 3.6), and $\mathfrak{N} \models_{\mathscr{L}} \eta \wedge \delta$, where

$$\delta \underset{\text{def}}{=} \bigwedge_{i=1}^{n} \forall \vec{y}_i R_i(\vec{y}_i) \leftrightarrow \varphi_i(f(\vec{y}_i)), \quad \text{and}$$

$$\eta \underset{\text{def}}{=} \bigwedge_{i=1}^{n} \forall \vec{x}_i(\varphi_i(\vec{x}_i) \to \text{the coordinates of } \vec{x}_i \text{ satisfy } \varphi_0).$$

To conform to our stipulation that function symbols are absent in this section, we regard f as a binary relation symbol. The above shows that $\mathrm{Mod}_{\mathscr{L}^+}(\theta)$ is $\mathrm{RPC}_{\mathscr{L}}$ (see Definition II.3.1.1). Similarly, we prove that $\mathrm{Mod}_{\mathscr{L}^+}(\neg\theta)$ is also $\mathrm{RPC}_{\mathscr{L}}$. Now Corollary 1.6 and Theorem 1.3 can be applied to $\mathscr{L}$, since $\mathscr{L}$ is closed under relativization to atomic sentences (and there are no function symbols whatsoever). Therefore, $\mathscr{L}$ is compact; whence the Robinson property also implies that $\mathscr{L}$ obeys Craig's interpolation theorem (see Corollary 1.4), and so, *a fortiori*, $\mathscr{L}$ is Δ-closed (see Section II.3.1). In particular, θ must be a sentence of $\mathscr{L}$, which shows that application of Q_ψ in $\mathscr{L}$ does not lead beyond $\mathscr{L}$; in short, $\mathscr{L}(Q_\psi) \leq \mathscr{L}$. Observe also that as a Δ-closed logic $\mathscr{L}$ is closed under full relativization and substitution. Now let ψ range over all sentences of $\mathscr{L}[\tau]$. Because of the finiteness of each partition $\simeq_\tau^n$, there exists a countable set Z_τ of quantifiers such that every sentence of $\mathscr{L}[\tau]$ can be written down using only the quantifiers in Z_τ. By the renaming and reduct properties of $\simeq$, we are now able to exhibit a countable set Z of quantifiers such that every sentence of $\mathscr{L}$ (no matter the τ involved) can be expressed using only the quantifiers in Z. In other words, $\mathscr{L}$ has been shown to be countably generated, as was required to complete the proof that (iii) implies (ii).

Finally, we must prove that (iii) implies (iv). Assume that both $\simeq'$ and $\simeq''$ are back-and-forth systems generating $\sim$. Observe first of all that $\sim$ is a bounded Robinson equivalence relation. Now, as in the above proof of (iii) $\Rightarrow$ (ii), let $\mathscr{L}'$ and $\mathscr{L}''$ arise from $\simeq'$ and $\simeq''$, respectively, via definitions (∗) and ($\underset{*}{*}$). By Corollary 3.4 $\mathscr{L}'$ and $\mathscr{L}''$ are equivalent, since $\equiv_{\mathscr{L}'} = \equiv_{\mathscr{L}''} = \sim$. Now let ε be an equivalence class of $\sim_\tau'^n$. By clause (∗), ε is also a sentence of $\mathscr{L}'[\tau]$ and is (equivalent to) a sentence of $\mathscr{L}''[\tau]$. Whence it follows that ε is a union of equivalence classes of $\simeq_\tau''^{m_\varepsilon}$, for some $m_\varepsilon < \omega$. Letting ε range over all the equivalence classes of $\simeq_\tau'^n$, there will be a fixed $m < \omega$ providing an upper bound for the totality of the m_ε's. Indeed, $\simeq_\tau'^n$ has only finitely many equivalence classes. Therefore, $\simeq_\tau''^m$ is finer than $\simeq_\tau'^n$. Reversing the roles of $\simeq'$ and $\simeq''$, we finally establish that $\simeq'$ and $\simeq''$ are equivalent. This completes the proof of our theorem. ☐

5.5 Corollary. *Elementary equivalence is generated by a unique (up to equivalence) back-and-forth system, namely the Fraïssé–Ehrenfeucht system of Example* 5.2. ☐

5.6 Remarks. Abstract back-and-forth systems were introduced in Mundici [1982e], where Theorem 5.4 is also proven. We might wonder whether the above duality between countably generated logics and systems of sequences of finite partitions can be extended in the absence of the Robinson property. Lipparini [1982] considers special back-and-forth systems satisfying the following additional condition:

Expansion Axiom: For any finite τ, $\vec{c} = (c_1, \ldots, c_r)$, $R \notin \tau$ an r-ary relation symbol, φ a union of components of the partition $\simeq^n_{\tau \cup \{\vec{c}\}}$, if $\mathfrak{A} \simeq^{m+n}_\tau \mathfrak{B}$, then $\langle \mathfrak{A}, R^{\mathfrak{A}} \rangle \simeq^m_{\tau \cup \{R\}} \langle \mathfrak{B}, R^{\mathfrak{B}} \rangle$, where, e.g., $R^{\mathfrak{A}} = \{\vec{a} \in A^r \mid \langle \mathfrak{A}, \vec{a} \rangle \in \varphi\}$.

We then have the following converse of Theorem 5.3, namely

5.7 Theorem. *For every equivalence relation* $\sim$ *we have that* $\sim \, = \, \equiv_{\mathscr{L}}$ *for some countably generated logic* $\mathscr{L}$ *iff* $\sim$ *is generated by some back-and-forth system with the expansion property.*

Proof. See Lipparini [1982]. $\square$

Using Theorems 3.11 and 5.7, Theorem 5.4 can be extended to yield a bijection between countably generated compact logics and back-and-forth systems with the expansion property such that $(\sim, \sim^*)$ has the JEP (where $\sim$ is the equivalence relation generated by $\simeq$, and $\sim^*$ is the embedding relation generated by $\sim$, see Definition 4.1). Thus the expansion property seems to be the right counterpart of the substitution axiom for logics in all general contexts where the latter property is not taken care of by the Robinson property.

5.8 Global Versus Local Versions of Back-and-Forth Games. The celebrated Fraïssé–Ehrenfeucht game G for elementary equivalence determines a sequence of finite partitions on $\mathrm{Str}(\tau)$, for each finite τ, as was remarked in Example 5.2. For more details the reader should consult Lemma II.4.2.6, where each partition is related to (the models of) the so-called Scott–Vaught–Hintikka sentences of the corresponding quantifier rank. We may regard this system of partitions as a *global* version of G, since each partition is defined over the whole of $\mathrm{Str}(\tau)$. On the other hand, given structures $\mathfrak{A}$ and $\mathfrak{B}$, G also determines a game $G(\mathfrak{A}, \mathfrak{B})$ or, equivalently, a decreasing sequence of sets of partial isomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$ (see also Section IX.4 for further information on this matter), and this may be regarded as a *local* version of G. Passing now to an arbitrary logic $\mathscr{L}(Q^i)_{i<\omega}$, we may fruitfully use the notion of back-and-forth system (see Definition 5.1) to study the global aspects of back-and-forth games in the general case. For example, Theorem 5.4 or Theorem 5.7 might be the starting point for investigating the abstract model-theoretical counterparts of the notion of *subformula*.

Is there a corresponding local version of back-and-forth game having the same degree of generality? To give an affirmative answer to this question we must first make the latter precise. We will restrict attention to $\mathscr{L}(Q)$ with Q an s-ary quantifier. Q determines a function which assigns to each structure $\mathfrak{A}$ a set $Q\mathfrak{A} \subseteq P(A^s)$ of s-ary relations on A; and (recalling Notational Convention 3.6) we have the familiar clause:

$$\mathfrak{A} \models Q\vec{x}\varphi(\vec{x}) \quad \text{iff} \quad \varphi^{\mathfrak{A}} \in Q\mathfrak{A}.$$

Now let $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}(\tau)$ with τ finite. We let A^* be the set of finite words over A, namely $A^* = \{\varnothing\} \cup A \cup A^2 \cup \cdots$. Arbitrary words over A will be denoted by a, x, t, and $|a|$ is the length of an arbitrary word a. Similarly, b, y, u will be arbitrary words over B, and w, w' arbitrary elements of $A^* \cup B^*$. Following Caicedo [1979], we give the following

5.9 Definition. With the above notation, a *back-and-forth game from* $\langle \mathfrak{A}, Q\mathfrak{A}\rangle$ *to* $\langle \mathfrak{B}, Q\mathfrak{B}\rangle$ is a sequence $\{\sim^p\}_{p<\omega}$, where each $\sim^p$ is a partition (i.e. an equivalence relation) on $A^* \cup B^*$ and, for all $p < \omega$, we have

(i) $w \sim^p w'$ implies $|w| = |w'|$;
(ii) $\varnothing \sim^p \varnothing$;
(iii) $a \sim^p b$ implies that the assignment $a_i \mapsto b_i$ is a partial isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ (as structures of vocabulary τ);
(iv) whenever $a \sim^{p+s} b$, there is a map $f: A^s \to B^s$ obeying conditions (iv′) and (iv″) below:
  (iv′) $ax \sim^p bf(x)$ for all $x \in A^s$, where ax denotes the juxtaposition of a and x;
  (iv″) for any $X \subseteq A^s$, if $\{t \in A^s \mid at \sim^p ax$ for some $x \in X\} \in Q\mathfrak{A}$, then $\{u \in B^s \mid bu \sim^p by$, for some $y \in f(X)\} \in Q\mathfrak{B}$;
(v) same as (iv) with the roles of A and B interchanged.

5.10 Theorem. *For arbitrary* $\mathscr{L} = \mathscr{L}(Q)$, $\mathfrak{A}$, $\mathfrak{B} \in \mathrm{Str}(\tau)$, τ *finite, the following are equivalent*:

(i) $\mathfrak{A} \equiv_{\mathscr{L}} \mathfrak{B}$;
(ii) *there is a back-and-forth game from* $\langle \mathfrak{A}, Q\mathfrak{A}\rangle$ *to* $\langle \mathfrak{B}, Q\mathfrak{B}\rangle$.

Proof. See Caicedo [1979, Section 3.5]. □

Actually, Caicedo [1979] proves Theorem 5.10 for the general case $\mathscr{L} = \mathscr{L}(Q^i)_{i\in I}$. Indeed, he also gives a back-and-forth characterization of $\mathscr{L}_{\infty\omega}(Q^i)_{i\in I}$, using the notion of a back-and-forth game from $\langle \mathfrak{A}, Q_i\mathfrak{A}\rangle_{i\in I}$ to $\langle \mathfrak{B}, Q_i\mathfrak{B}\rangle_{i\in I}$. The latter is still a sequence $\{\sim^p\}_{p<\omega}$ of equivalence relations on $A^* \cup B^*$ satisfying, roughly, the cartesian product of Definition 5.9 and I (see Caicedo [1979, Section 2.1]). Caicedo's (local) equivalence relations on $A^* \cup B^*$ generalize back-and-forth technology for specific quantifiers as developed by Fraïssé, Ehrenfeucht, Lipner, Brown, Vinner, Slomson, Krawczyk, Krynicki, Badger, Makowsky, Shelah, Tulipani, Kaufmann, and others. Weese [1980] proves an analogue of Theorem 5.10 for sets of monotone quantifiers (see Section II.4.2). Summing up the results of this section: Theorem 5.10 yields a map from logics onto (local) back-and-forth games for sets of quantifiers; with the help of Theorem 5.7 we now have a map from global onto local versions of back-and-forth games for countably generated logics.

Chapter XX
Abstract Embedding Relations

by J. A. Makowsky

Abstract model theory deals with the generalization of the concept of a logic. A logic consists of a family of objects called formulas, a family of objects called structures and a binary relation between them, called satisfaction. Various properties of logics, however, can be phrased without direct reference to the formulas, but rather, by considering as the basic concept the class of structures which are the models of some (complete) theory. The previous two chapters have given plenty of evidence for this. In Section XVIII.3 we studied amalgamation properties and in Chapter XIX, the Robinson property, both of which fit this approach. In Chapter XIX we even went a step further: we looked into the possibility of axiomatizing abstract equivalence relations between structures, such as they arise naturally from logics in the form of $\mathscr{L}$-equivalence. There we studied the question under which circumstances such an equivalence relation does indeed come from a logic $\mathscr{L}$.

Algebra, on the other hand, deals with classification of algebraic structures and their extensions. The paradigm of algebraic classification theory, and, for that matter, the paradigm of model-theoretic classification theory, is Steinitz' theory of fields and their algebraic and transcendental extensions. But many of the examples studied in algebra, such as locally finite groups or Banach spaces, are not fit for first-order axiomatizations. Though classes of algebras can be axiomatized, if necessary, with the help of generalized quantifiers, this approach does not necessarily help us to axiomatize the corresponding notion of extensions.

In this chapter we axiomatize the notion of $\mathscr{L}$-extensions, but, contrary to the approach in Chapter XIX, we are not that much interested in the case where it is derived from a logic $\mathscr{L}$. We are rather interested in the question: Under which conditions can certain constructions and proofs from model theory be carried out in a framework which resembles more that of universal algebra or algebra in general?

Very often, axiomatizations grow out of a better understanding of proofs. First, they serve only to structure and clarify the flow of reasoning, but sometimes they gain their own significance and reach maturity. If this happens, new branches of mathematical activity emerge.

Examples from history are the emergence of Hilbert and Banach spaces; universal algebra and model theory of first-order logic, abstract model theory, and here especially, the framework of abstract classes. The abstract classes have their origin in the attempt to better understand certain constructions of models, as they occur in the classification theory of models of first-order theory, and in trying to generalize those constructions so as to fit classes which are not first-order definable. The constructions we have in mind divide sharply into two cases: In the case in which amalgamation fails in an abstract class K, they allow us to construct maximally many non-isomorphic structures of a given cardinality and to show that no universal structures of a given cardinality exist, or, as a combination of both, that there are maximally many structures such that no two of them are mutually embeddable. On the other hand, if some form of amalgamation holds, they allow us to obtain a structure in a higher cardinality. It turns out that the presence or absence of various forms of the amalgamation property acts like a watershed. This is similar to the effect of stability or superstability in first-order classification theory. The transfer of all the technical knowledge of the classification theory of models of first-order theories to models of abstract classes, however, poses challenging difficulties. This chapter presents some of the initial steps towards this aim. The completion of such a program remains the task of future research.

But the axiomatic framework has yet another advantage: It allows us to discern more clearly the set-theoretic and combinatorial structure of the proofs and to separate their combinatorial from their structural contents. Such proofs are usually based on a property P of our abstract class K which is inherently connected to the very definition of K, and a set-theoretic part, whose application does not require more than an axiomatic description of some of the basic aspect of $\mathfrak{K}$ together with the property P. We have encountered such situations in the case of locally finite groups, such as in Giorgetta–Shelah [1983] or in the model theory of ω_1-categorical sentences of extensions of $\mathscr{L}_{\omega\omega}(Q_1)$. It would be interesting to see, if the same applies to recent results in Banach space theory, cf. Bourgain–Rosenthal–Schechtman [1981], for instance, where $\mathfrak{K}$ is the class of all separable Banach spaces with the Radon–Nikodym property.

However, the present chapter is not concerned with such deep results of a very specialized character. Our subject here is the axiomatization of the framework which allows the use of the set-theoretic machinery. What we present are the first steps of a theory still to be developed. The chapter is an exposition of and introduction to three papers by S. Shelah (Shelah [1983b, c, 198?c]), and improvements or elaborations in its exposition due to S. Fuchino, R. Grossberg, and the author. An early version of this chapter consisted of lectures the author and D. Giorgetta have given on the subject in Oberwolfach in January 1980. It contains, for completeness and historical accuracy, also early results of Mal'cev and Jonsson, and some additional material which we include to stress some analogies or give more examples.

In detail the chapter is organized as follows. In Section 1 we present the axiomatic framework and variations thereof. In Section 1.1 we define our program in detail and in Section 1.2 we state and motivate the axioms. The main results of this section, presented in Section 1.3, are various forms of axiomatizability theorems which assert the existence of certain standard logics, in which such classes can be described. One of them, Shelah's presentability theorem, provides us with some cardinal parameters, on which the development of the theory depends. It also gives rather surprising results on the Hanf numbers of abstract classes, depending on those parameters, presented in Section 1.4.

In Section 2 we study the effect of the presence or absence of amalgamation properties in an abstract class $\mathfrak{K}$. In the case when an abstract class has the amalgamation property and the joint embedding property they are called Jonsson classes and were introduced already in 1962 by M. Morley and R. Vaught. It should be mentioned here, that R. Fraïssé was seemingly the first to study amalgamation properties of classes of structures, cf. Fraïssé [1954]. We give a brief survey on what we know about Jonsson classes in Section 2.1 for the sake of completeness and proper perspective. The main advantage of Jonsson classes consists in the existence of universal, homogeneous models, though not necessarily in every cardinality. A substitute of saturated models in many of our constructions, is the limit model, which is introduced in Section 2.2, and some basic properties of limit and superlimit models are proved. Our main interest here, however, is in the absence of amalgamation properties. The thesis, put forward in Shelah's work and in this chapter, states that amalgamation properties should not be part of the axioms, and that, basically, Jonsson's axioms, without amalgamation and joint embedding, provide us with the correct framework for a structure/non-structure theory. The main result, presented in Section 2.3, is Shelah's non-structure theorem for abstract classes and some conjectures for further developments. The non-structure theorem presupposes some weak instance of the GCH, connected to the combinatorial principle weak diamond. In Section 2.4 we present an example which shows that this is necessary. In Section 2.5 we collect the set-theoretic background about the weak diamond, necessary to prove the non-structure theorem. The easier parts of its proof are presented in Section 2.6 and the more complex parts in Section 2.7. The reader interested in the missing proofs will have to get involved with the technical details and conceptual intricacies of Shelah [1984a, b].

In Section 3 we study ω-presentable classes, which, by the presentability theorem, are closely connected to the model theory of $\mathscr{L}_{\omega_1\omega}$. In Section 3.1 we present the present state of art in classification theory for ω-presentable classes and classes defined by a $\mathscr{L}_{\omega_1\omega}$-sentence, and we state some conjectures on how the latter should be true also for ω-presentable classes in general. The main results proved in the sequel are Shelah's reduction theorem and Shelah's abstract ω_1-categoricity theorem. For the proofs of the other theorems the reader will have to consult Shelah [198?c]. In Section 3.2 we present the "soft" aspects of the proof of the abstract ω_1-categoricity theorem, and in Section 3.3 the parts which are more related to the model theory of $\mathscr{L}_{\omega_1\omega}$. In Section 3.4 we prove the reduction theorem. In Section 3.5, finally, we give a narrative account of some aspects of the proof of the existence of superlimits in ω_1.

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1. The Axiomatic Framework

1.1. Prolegomena

In Chapter XVIII we have seen that various $\mathscr{L}$-extension properties play a fruitful role in abstract model theory. In Chapter XIX we have seen how one can replace, under certain circumstances, a logic $\mathscr{L}$ by an abstract equivalence relation or an abstract embedding relation. However, in both cases we still retained the idea of dealing with a logic with various regularity properties concerning the passage from one vocabulary to another. The type of results obtained there also requires such assumptions. If we deal with properties of a fixed class of τ-structures, we are more in the framework of universal algebra. In fact, some of the classical theorems of universal algebra can be viewed as precursors of abstract model theory. Let us elaborate on this a bit.

The first theorem along these lines is Birkhoff's theorem characterizing varieties.

1.1.1 Definitions. (i) A class V of τ-structures closed under isomorphic images, cartesian products, substructures, and homomorphic images is called a *τ-variety*.

(ii) A class of τ-structures K is *automatically definable* if $K = \mathrm{Mod}(\Sigma)$ for some set of atomic τ-formulas Σ.

1.1.2 Theorem (Birkhoff). *The τ-varieties are exactly the atomically definable classes of τ-structures.*

1.1.3 Definitions. (i) A class V of τ-structures closed under isomorphic images, cartesian products, and substructures is called a *quasi-variety*.

(ii) An *infinitary Horn formula* is a formula of the form $\bigwedge_{i \in I} \varphi_i \to \psi$, where I is any set and φ_i, ψ are atomic formulas.

(iii) A class of τ-structures K is *Horn definable*, if $K = \mathrm{Mod}(\Sigma)$ for a (possibly proper) class of infinitary Horn formulas.
(iv) An *infinitary clause* is a formula of the form $\bigvee_{i \in I} \varphi_i$, where I is any set and the φ_i's are atomic or negated atomic (i.e., *basic*) formulas.
(v) A class of τ-structures K is *clause definable* if $K = \mathrm{Mod}(\Sigma)$ for a (possibly proper) class of infinitary clauses over τ.
(vi) A class of τ-structures is *basic compact*, if for every set Σ of basic formulas over some vocabulary τ_1, $\tau \subset \tau_1$, such that every finite subset $\Sigma_0 \subset \Sigma$, Σ_0 has a model $\mathfrak{A}$ with $\mathfrak{A} \restriction \tau \in K$, the Σ has too.

1.1.4 Theorem. (i) (Cudnovskii). *A class K of τ-structures closed under isomorphisms and substructures iff K is clause definable.*
(ii) (Cudnovskii [1968]). *The quasi-varieties are exactly the Horn definable classes.*
(iii) (McKinsey [1943]). *If additionally K is basic compact then the class defining K is a set of finitary clauses or finitary Horn formulas.*
(iv) (E. Fisher [1977]). *The assumption that in* (i) *or* (ii) *K is always definable by a set* (*Horn*) *clauses is equivalent to Vopenka's principle.*

For a definition of Vopenka's principle see Section XVIII.1.3. Similar theorems hold for classes closed under unions of chains and other closure properties.

Quasi-varieties are particularly interesting because they allow the construction of *free objects* (*initial objects*) and Mal'cev [1954] has given the following characterization of quasi-varieties.

1.1.5 Definition (Free Structures). (i) Let K be a class of structures for a vocabulary τ, $\mathfrak{A} \in K$ and $X \subset A$ such that $\mathfrak{A}$ is generated (as a substructure) by X. We say that $\mathfrak{A}$ is *free* in K, if for every $\mathfrak{B} \in K$ and any relation preserving mapping $f\colon X \to B$ there is a homomorphism $g\colon \mathfrak{A} \to \mathfrak{B}$ extending f.
(ii) Let a class K of τ-structures be called *free*, if for every variety V of τ'-structures such that $K \cap V \neq \varnothing$, $K \cap V$ has a $\tau \cup \tau'$-structure which is free in $K \cap V$.

1.1.6 Theorem (Mal'cev [1954]). *A class K is free iff it is a quasi-variety.*

For a discussion of Mal'cev's theorem cf. also Mahr–Makowsky [1983].

1.1.7 Stating the Problem. The aim of this chapter is to give an introduction in to a sequence of papers by S. Shelah entitled "Classification theory for non-elementary classes Ia, Ib, and II." (Shelah [1983b, c, 198?c]). The idea here is very simple. Instead of having a logic $\mathscr{L}$ we are given a class K of τ-structures satisfying certain properties. We would like to ask questions concerning the existence of various models in such a class K. In the following we list the paradigms of our questions together with a typical instance of a theorem answering such a question in some special case.

1.1.8 Categoricity. Under what conditions is K categorical in some cardinal?

The paradigm of such questions concerns categoricity in ω for first-order model theory. There the characterization theorem due independently to Engeler, Ryll–Nardzewski, and Svenonious, connects categoricity of a theory with its Lindenbaum algebras being atomic. In the case of $\mathscr{L}_{\omega_1\omega}$ Scott's theorem states that every complete sentence is categorical in ω. For other cardinalities characterization of categoricity is more connected to transfer properties, such as Morley's theorem, stating that a countable first-order theory is categorical in one uncountable cardinal iff it is categorical in every uncountable cardinal. Attempts to generalize this to $\mathscr{L}_{\omega_1\omega}$ have only partially succeeded, cf. Keisler [1971]. Much of Section 3 is devoted to related questions.

1.1.9 The Spectrum. More generally, denote by $I(K, \kappa)$ the number of isomorphism types of models in K of cardinality κ. If $K = \text{Mod}(T)$ for some first-order theory, we write $I(T, \kappa)$ instead of $I(K, \kappa)$. What can we say about $I(K, \kappa)$?

In the case of countable first-order theory, twenty years of research have led to the following theorem of Shelah, proving therewith a conjecture due to Morley.

1.1.10 Theorem (Shelah). *Let T be a countable first-order theory. Then $I(T, \kappa)$ is not-decreasing on uncountable cardinals and, in fact, either*:

(i) $I(T, \kappa) = 2^\kappa$; or
(ii) $I(T, \omega_\alpha) < \beth_{\omega_1}(\text{card}(\alpha))$.

The proof of this theorem was complete with Shelah [1982f], based on Shelah [1978a].

Much of Section 2 is devoted to prove similar theorems for abstract classes.

1.1.11 Rigid Models. A model is *rigid*, if it has no non-trivial automorphisms. Let $R(K, \kappa)$ be the number of isomorphism types of rigid structures in K of cardinality κ. Interest in rigid models arose, after it was shown by Ehrenfeucht and Mostowski, that every first-order theory has models with many automorphisms. Generalizations of this to abstract model theory are discussed in Section XVIII.4.5. The following theorem shows, unfortunately, that very little can be said about the function $R(T, \kappa)$ in the case of first-order logic.

1.1.12 Theorem (Shelah [1976b]). *Assume $\lambda^\omega \leq \lambda^+$ for every λ. For every Σ^1_2-class $\mathbf{C}$ of cardinals there is a sentence $\varphi \in \mathscr{L}_{\omega\omega}$ such that $\mathbf{C} = \{\kappa \in \text{Card}\colon R(\varphi, \kappa) \neq 0\}$.*

This refutes a conjecture of Ehrenfeucht, which tried to describe $R(T, \kappa)$. It seems that one should ask for rigid models which are also $\text{card}(T)^+$-saturated. In Shelah [1983d] there are partial results indicating that at least the existence of rigid models in some class K can be settled in an abstract framework. In this chapter we shall not deal with rigid models, but we would like to draw attention to this promising direction of research. A sample theorem is the following result due to Shelah, refuting a conjecture (unpublished) of H. Salzmann, suggesting that every rigid real closed field is archimedian:

1.1.13 Theorem (Shelah). *Assume GCH. There are arbitrarily large rigid ω_1-saturated real closed fields.*

1.1.14 Problem. Characterize the abstract classes (defined below) which have arbitrarily large rigid models.

1.1.15 Homogeneous and Saturated Models. Similar problems can be stated for homogeneous models. In Section 2 we shall study this question. Theorem 2.1.11 gives some information about the spectrum of homogeneous models $H(K, \kappa)$. In first-order model theory saturated models are suitable described as universal and homogeneous. Already in the early days of classification theory, axiomatic frameworks have been studied. Jonsson [1956, 1960] and Fraïssé [1954] proposed axioms for the existence of universal and homogeneous structures in a class K and Morley–Vaught [1962] used this framework to construct saturated structures. We shall return to a detailed discussion of these axioms in Section 1.2 and for Jonsson's work in Section 2.1. What we want to note here, is that the construction of the saturated model heavily depends on the *amalgamation* property of K. We shall see that there are good reasons for this. The question arises if there is a suitable substitute for saturated models? One of the key notions introduced in this chapter is the *limit model.* The similarity consists less in the definition, than in its use in various proofs. Section 2.2 gives the definitions and its presence is felt through the rest of the chapter.

1.2. The Axioms

Here K is a class of τ-structures and $<_K$ is a two-place relation between members $\mathfrak{A}$, $\mathfrak{B}$ of K. If the context is clear we omit the K in $<_K$ and assume that all structures $\mathfrak{A}, \mathfrak{B} \in K$.

The axioms presented below are modeled after various examples of model theory. It is good to have some of these at disposal when reading the axioms, so we present them before stating the axioms.

1.2.1 Examples. (i) Let T be a complete first-order theory over some vocabulary τ and put $K_T = \mathrm{Mod}(T)$ and $<$ be first-order elementary extension.

(ii) Let K_{wo} be the class of well-orderings and $\mathfrak{A} <_{\mathrm{wo}} \mathfrak{B}$ hold if $\mathfrak{B}$ is an end extension of $\mathfrak{A}$, i.e., every $b \in B - A$ is bigger than every $a \in A$.

(iii) Let $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_{\omega_1})$ be the logic with the quantifier "there exist uncountably many." Let a weak τ-model $(\mathfrak{A}, q)$ consist of a τ-structure together with a family q of subsets of A. Let the formulas of $\mathscr{L}_{\mathrm{weak}}$ be as for $\mathscr{L}$ but define $\mathfrak{A} \models_{\mathrm{weak}} Qx\varphi(x)$ if $\{a \in A : \mathfrak{A} \models_{\mathrm{weak}} \varphi(a)\} \in q$. Let K be the class of all weak τ-models for some fixed vocabulary τ. We define $<^{**}$ as in Keisler [1970, 1971a] by $\mathfrak{A} <^{**} \mathfrak{B}$ iff $\mathfrak{A} <_{\mathscr{L}\mathrm{weak}} \mathfrak{B}$ and for every $\bar{a} \in A^m$ and for every formula $\varphi = \varphi(x, \bar{y}) \in \mathscr{L}(\tau)$ we have that if $\mathfrak{A} \models \neg Qx\varphi(x, \bar{a})$ then $\{b \in A : \mathfrak{A} \models \varphi(b, \bar{a})\} = \{b \in B : \mathfrak{B} \models \varphi(b, \bar{a})\}$.

We shall return to this example in more detail in Section 4.

(iv) The category of *universal locally finite groups, with* K_{ULF} *the class* of those groups and $<_{\mathrm{ULF}}$ the ordinary subgroup relation, cf. Kegel–Wehrfritz [1973]. The model theory of uncountable universal locally finite groups was studied in Macintyre–Shelah [1976] and Grossberg–Shelah [1983].

(v) (Elementary Classes with Omitting Types). Let τ be a fixed vocabulary, T be a first-order theory over τ, i.e., $T \subset \mathscr{L}_{\omega\omega}(\tau)$, and Γ be a set of types over τ. Let $K = \{\mathfrak{A} \in \mathrm{Str}(\tau)\colon \mathfrak{A} \models T$ *and* $\mathfrak{A}$ *omits every* $p \in \Gamma\}$ and $\mathfrak{A} <_K \mathfrak{B}$ if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$. It is easy to see (cf. Keisler [1970]) that example (iii) is a special case of this.

We shall return to this example in Section 1.3.

Having these examples in mind, we now state the axioms. They come in several groups of various degree of strength. First some (almost) trivial axioms concerning transitivity of our embedding relation:

Axiom 1 (Substructure Axiom). If $\mathfrak{A} < \mathfrak{B}$ then $\mathfrak{A} \subset \mathfrak{B}$, i.e., $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

1.2.2 Definitions. (i) If $\mathfrak{A} \subset \mathfrak{B}$ are τ-structures, and f is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, say that f *is an K-embedding*, if $f(\mathfrak{A}) <_K \mathfrak{B}$.

(ii) If $\mathfrak{A}$, $\mathfrak{B}$ are τ-structures and f_{AB} is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, we denote by $[\mathfrak{A}; \mathfrak{B}, f_{AB}]$ the two-sorted structure consisting of the two structures $\mathfrak{A}$, $\mathfrak{B}$ expanded by a function symbol F interpreted by the embedding f_{AB} and a new unary predicate symbol U, both not in τ, such that $\mathfrak{B} \restriction U \cong f_{AB}(\mathfrak{A})$. If $\mathfrak{A} \subset \mathfrak{B}$ and f_{AB} is the identity on $\mathfrak{A}$ we just write $[\mathfrak{A}; \mathfrak{B}]$. Note the difference between our notation $[\mathfrak{A}; \mathfrak{B}]$ and $[\mathfrak{A}, \mathfrak{B}]$ for the disjoint pair construction in Chapter XVIII.

Axiom 2 (Isomorphism Axiom). (i) If $\mathfrak{A} \in K$ and $\mathfrak{A}_1 \cong \mathfrak{A}$ then $\mathfrak{A}_1 \in K$.

(ii) If $\mathfrak{A} < \mathfrak{B}$ and $[\mathfrak{A}; \mathfrak{B}] \cong [\mathfrak{A}_1; \mathfrak{B}_1]$ then $\mathfrak{A}_1 < \mathfrak{B}_1$.

Axiom 3 (Transitivity Axiom). (i) If $\mathfrak{A}_1 < \mathfrak{A}_2 < \mathfrak{A}_3$ then $\mathfrak{A}_1 < \mathfrak{A}_3$.

(ii) If $\mathfrak{A}_1 \subset \mathfrak{A}_2 < \mathfrak{A}_3$ and $\mathfrak{A}_1 < \mathfrak{A}_3$ then $\mathfrak{A}_1 < \mathfrak{A}_2$.

Clearly examples (i)–(v) satisfy these axioms.

1.2.3 Definition. Let $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ be a family of structures in $\mathfrak{K}$.

(i) $\mathfrak{A}_\alpha$ is *K-increasing* if $\alpha > \beta < \gamma$ implies that $\mathfrak{A}_\alpha < \mathfrak{A}_\beta$.

(ii) $\mathfrak{A}_\alpha$ is *continuous* if for every limit ordinal $\delta < \gamma$ we have $\mathfrak{A}_\delta = \bigcup_{\alpha <} \mathfrak{A}_\alpha$.

(iii) $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ is a *K-chain* if it is both K-increasing and continuous.

Axiom 4 (Chain Axiom). (i) If $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ is a K-chain then $\mathfrak{A}_0 < \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$.

(ii) If $\mathfrak{A}_\alpha$ $(\alpha < \gamma)$ is a K-chain, $\mathfrak{N} \in K$ and for each $\alpha < \gamma$ $\mathfrak{A}_\alpha < \mathfrak{N}$ then $\bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha < \mathfrak{N}$.

Again all our examples from above satisfy this axiom.

We denote by K_λ ($K_{<\lambda}$, $K_{\leq\lambda}$) the class of structures of K of cardinality *exactly* (*less than, less than or equal to*) λ.

Our next axiom is an analogue of the Löwenheim–Skolem–Tarski theorem for first-order logic and introduces a cardinal parameter, which we shall call the *Löwenheim number* $l(K)$ of $\langle K, < \rangle$.

Axiom 5 (Existence of Löwenheim Number). There is first a cardinal $l(K) \geq \operatorname{card}(\tau(K))$ such that:

(i) $K_{l(K)} \neq \emptyset$; and
(ii) whenever $\mathfrak{A} \in K$ and X is a subset of the universe A of $\mathfrak{A}$ then there is a $\mathfrak{B} \in K$ such that $X \subset B$, $\operatorname{card}(\mathfrak{B}) \leq l(K) + \operatorname{card}(X)$ and $\mathfrak{B} < \mathfrak{A}$.

1.2.4 Examples. (i) In the example of well-orderings with end-extensions (Example 1.2.1(ii)) has no Löwenheim number. To see this, take any well ordering $\mathfrak{A}$ of cofinality ω of cardinality κ. If X is a countable cofinal set then for every $\mathfrak{B} <_{\mathrm{wo}} \mathfrak{A}$ with $X \subset B$ we have $\mathfrak{B} = \mathfrak{A}$.

(ii) The Löwenheim numbers of Examples 1.2.1(i), (iii), (iv), and (v) are ω.

(iii) If we modify Example 1.2.1(iii) such that the interpretation of the quantifier $Qx\varphi(x)$ ensures that the set defined by φ is uncountable, then the Lowenheim number is ω_1.

(iv) In Gurevic [1982] Löwenheim properties of general categories are studied. The situation described there consists of a logic $\mathscr{L}$ and an abstract class K together with a family $\mathbf{H}$ of homomorphisms. Supposing that K has Löwenheim number λ and $\mathscr{L}$ has Löwenheim number μ, we are interested in the existence of a cardinal $g(\lambda, \mu)$ such that for every $\mathfrak{A} \in K$ there is $\mathfrak{B} < \mathfrak{A}$ with $\operatorname{card}(B) < g(\lambda, \mu)$ such that for every $H \in \mathbf{H}$ we have $H(\mathfrak{B}) < H(\mathfrak{A})$ is also an $\mathscr{L}$-embedding.

1.2.5 Remark. We could state Axiom 5 only for $X \subset A$ with $\operatorname{card}(X) \leq l(K)$ and use Axiom 4 to prove Axiom 5 from this weaker assumption.

1.2.6 Definitions. (i) A class K together with a relation $<_K$ satisfying the Axioms 1–4 is called a *abstract class*.

(ii) A class K together with a relation $<_K$ satisfying the Axioms 1–5 is called a *abstract class of Löwenheim number* $l(K)$.

(iii) Let K_i be abstract classes over vocabularies τ_i, $i \in I$. We define the intersection $K = \bigcap_{i \in I} K_i$ to be the class of $\bigcup_{i \in I} \tau_i$-structures such that for $\mathfrak{A}, \mathfrak{B} \in K$, $\mathfrak{A} < \mathfrak{B}$ iff $\mathfrak{A} \upharpoonright \tau_i < \mathfrak{B} \upharpoonright \tau_i$ holds in K_i, $i = 1, 2$.

1.2.7 Proposition. (i) *The intersection of any family of abstract classes is again an abstract class.*

(ii) *If K_i, $i \in I$ is a family of abstract classes of Löwenheim number κ_i then the intersection $\bigcap_{i \in I} K_i$ is an abstract class of Löwenheim number $\sum_{i \in I} \kappa_i$.*

1.2.8 Remark. Unions of abstract classes need not be abstract classes. It is easy to construct examples violating Axiom 3(i) and also Axiom 3(ii). But then Axiom 4(i) and 4(ii) become meaningless. For disjoint unions only Axiom 3(ii) may be violated, but unions of disjoint abstract classes are admittedly uninteresting.

1.3. Presentability of Abstract Classes

Our next theorem establishes a connection between abstract classes of a given Löwenheim number and some infinitary logics and will give as a more precise cardinal parameter than the Löwenheim number.

1.3.1 Definitions. Let τ be a fixed vocabulary, T be a first-order theory over τ, i.e., $T \subset \mathscr{L}_{\omega\omega}(\tau)$, and Γ be a set of types over τ.

(i) A class $K = \mathrm{MOT}(T, \Gamma)$ if $K = \{\mathfrak{A} \in \mathrm{Str}(\tau)\colon \mathfrak{A} \models T$ and $\mathfrak{A}$ *omits every* $p \in \Gamma\}$.
MOT(T, Γ) stands for *Models of T Omitting the Types* from Γ.

We say that K is an *elementary class omitting some types* and write $K \in \mathrm{ECOT}$ if there are T, Γ such that $K = \mathrm{MOT}(T, \Gamma)$.

(ii) If $\tau_0 \subset \tau$ and K is a class of τ_0-structures we write $K = \mathrm{MOT}_{\tau_0}(T, \Gamma)$ if $K = \{\mathfrak{A} \in \mathrm{Str}(\tau_0)\colon \mathfrak{A}$ *has an expansion* $\mathfrak{A}' \in \mathrm{MOT}(T, \Gamma)\}$.

We say that K is a *projective class omitting some types* and write $K \in \mathrm{PCOT}$ if there are T, Γ, τ_0 such that $K = \mathrm{MOT}_{\tau_0}(T, \Gamma)$.

(iii) We say that $K \in \mathrm{ECOT}(\lambda, \mu)$ or $K \in \mathrm{PCOT}(\lambda, \mu)$ if for T, Γ as above we have that $\mathrm{card}(T) \le \lambda$, $\mathrm{card}(\Gamma) \le \mu$.

(iv) If $\langle K, <_K\rangle$ is an abstract class, we say that K is (λ, μ)-*presentable* if $K \in \mathrm{PCOT}(\lambda, \mu)$ and $K_< = \{[\mathfrak{A}, \mathfrak{B}]\colon \mathfrak{A} <_K \mathfrak{B}\} \in \mathrm{PCOT}(\lambda, \mu)$.

If $\lambda = \mu$ we omit μ and just speak of λ-*presentable classes.*

1.3.2 Examples. From the examples in 1.2.1 in the previous section, (i) and (iii) are ω-presentable and (ii) is not presentable for any cardinals λ, μ. This follows from the non-characterizability of the class of well-orderings in $\mathscr{L}_{\infty\omega}$ (cf. Theorem 3.3.1) and the theorem below. However, they are axiomatizable in $\mathscr{L}_{\omega_1\omega_1}$.

Clearly (λ, μ)-presentable classes are projective classes in the logic $\mathscr{L}_{\nu\omega}$ with $\nu = (\sup(\lambda, \mu)^+)$, but from the infinitary operations we only use once universal quantification over infinitary formulas. Example (v) is just an instance of an PCOT-class.

Clearly, a λ-presentable class has Löwenheim number λ.

1.3.3 Theorem (Shelah's Presentability Theorem). *Let* $\langle K, <\rangle$ *be an abstract class over a vocabulary* τ, $\mathrm{card}(\tau) = \lambda$, *and with Löwenheim number* $\mu \ge \lambda$. *Then* $\langle K, >\rangle$ is $(\mu, 2^\mu)$-*presentable.*

Proof. The proof uses two lemmas.

1.3.4 Lemma (Direct Limit Lemma). *Let I be a directed set (i.e., partially ordered by $\leq$, such that any two elements have a common upper bound). Let $\langle K, <\rangle$ be an abstract class and $\mathfrak{M}_i$ ($i \in I$) be a family of structures in K with $i, j \in I$, $i \leq j$ implies that $\mathfrak{M}_i < \mathfrak{M}_j$. Then*

(i) *for every $i \in I$ the structure $\mathfrak{M}_i < \bigcup_{j \in I} \mathfrak{M}_j$ and*
(ii) *if $\mathfrak{N} \in K$ and for every $j \in I$, $\mathfrak{M}_j < \mathfrak{N}$ then $\bigcup_{j \in I} \mathfrak{M}_j < \mathfrak{N}$.*

Proof. We prove (i) and (ii) simultaneously by induction on card(I). If I is finite there is nothing to prove, since I has a maximal element.

Suppose card(I) $= \mu$ and we have proved the lemma for card(I) $< \mu$. We can find a family I_α ($\alpha < \mu$) such that:

(a) $\operatorname{card}(I_\alpha) < \operatorname{card}(I)$;
(b) $\alpha < \beta < \mu$ implies that $I_\alpha \subset I_\beta \subset I$;
(c) $\bigcup_{\alpha < \mu} I_\alpha = I$;
(d) for every limit ordinal $\delta > \mu$ $\bigcup_{\alpha < \delta} I_\alpha = I_\delta$; and
(e) for each $\alpha < \mu$ I_α is directed and non-empty.

Let $\mathfrak{M}^\alpha = \bigcup_{j \in I_\alpha} \mathfrak{M}_j$. So by indication hypothesis from (i), $j \in I_a$ implies $\mathfrak{M}_j < \mathfrak{M}^\alpha$ and by induction hypothesis from (ii) $\mathfrak{M}^\alpha < \mathfrak{N}$. If $\alpha > \beta$ then $j \in I_\alpha$ implies $\mathfrak{M}_j < \mathfrak{M}^\beta$. Hence, by the induction hypothesis from (ii) $\mathfrak{M}^\alpha = \bigcup_{j \in I_\alpha} \mathfrak{M}_j < \mathfrak{M}^\beta$. So by the chain axiom $\mathfrak{M}^\alpha < \bigcup_{\beta < \mu} \mathfrak{M}^\beta = \bigcup_{j \in I} \mathfrak{M}_j$, and as $j \in I_\alpha$ implies $\mathfrak{M}_j < \mathfrak{M}^\alpha$, we can conclude by the transitivity axiom that $\mathfrak{M}_j < \bigcup_{i \in I} \mathfrak{M}_D$. To conclude that $\bigcup_{i \in I} \mathfrak{M}_i = \bigcup_{\alpha < \mu} \mathfrak{M}^\alpha < \mathfrak{N}$ we use the second part of the chain axiom. □

1.3.5 Lemma (Skolemization Lemma). *Let $\langle K, >\rangle$ be an abstract class over a vocabulary τ with Löwenheim number $l(K)$ and let $\tau_1 = \tau \cup \{F_i^n : i < l(K), n \in \omega\}$ a new vocabulary where all the F_i^n are n-place function symbols not in τ. If $\mathfrak{M}$ is a τ-structure and $\mathfrak{M}^*$ is an expansions of $\mathfrak{M}$ to an τ_1-structure and $\bar{a} \in M^n$ we denote by $\mathfrak{M}^*_{\bar{a}}$ the minimal substructure of $\mathfrak{M}^*$ containing $\bar{a}$ and put $\mathfrak{M}_{\bar{a}} = \mathfrak{M}^*_{\bar{a}} \restriction \tau$. Then every $\mathfrak{M} \in K$ has an expansion $\mathfrak{M}^*$ such that for every $n \in \omega$ and $\bar{a} \in M^n$:*

(i) $\mathfrak{M}_{\bar{a}} < \mathfrak{M}$;
(ii) $\operatorname{card}(\mathfrak{M}_{\bar{a}}) \leq l(\mathfrak{K})$;
(iii) *if $\bar{b}$ is a subsequence of $\bar{a}$ then $\mathfrak{M}_{\bar{b}} < \mathfrak{M}_{\bar{a}}$; and*
(iv) *for every τ_1-substructure $\mathfrak{N}^*$ of $\mathfrak{M}^*$ we have that $\mathfrak{N}^* \restriction \tau < \mathfrak{M}$.*

Proof. We define by induction on $n \in \omega$ for every $\bar{a} \in M^n$ the values of $f_i(\bar{a})$, the interpretation of $F_i^n(\bar{a})$, where $i < l(K)$. By our assumption on the Löwenheim number of K there is for every subsequence $\bar{b}$ of $\bar{a}$ an $\mathfrak{M}_{\bar{b}}$ of cardinality less or equal than $l(K)$ such that $\mathfrak{M}_{\bar{b}} < \mathfrak{M}$. So we can find $\mathfrak{M}_{\bar{a}}$ of cardinality less or equal than $l(K)$ such that:

(a) $\mathfrak{M}_{\bar{a}} < \mathfrak{M}$;
(b) for every subsequence $\bar{b}$ of $\bar{a}$, $\mathfrak{M}_{\bar{b}} < \mathfrak{M}_{\bar{a}} < \mathfrak{M}$; and
(c) the choice of $\mathfrak{M}_{\bar{a}}$ does not depend on the order of $\bar{a}$.

To secure (b) we need Axiom 3(ii).

Now let $\{c_i: i < j \leq l(K)\}$ be an enumeration of the universe of $\mathfrak{M}_{\bar{a}}$ and put $f_i^n(\bar{a}) = c_i$ for $i < j$ and $f_i^n(\bar{a}) = c_0$ for $j < i < l(K)$.

Clearly, (i)–(iii) hold for $\mathfrak{M}^*$. To verify (iv) we use Lemma 1.3.4. □

1.3.6 Proof of Theorem 1.3.3. Let $\mathfrak{M}^*$ be as in Lemma 1.3.5 and let Γ_n be the set of complete n-types $p = p(x_0, \ldots, x_{n-1})$ in $\mathscr{L}_{\omega\omega}(\tau_1)$ such that:

(a) if $\bar{a} \in M^n$ realizes p in $\mathfrak{M}^*$ and $\bar{b}$ is a subsequence of $\bar{a}$ then $\mathfrak{M}_{\bar{b}} <_K \mathfrak{M}_{\bar{a}}$.

Clearly, (a) can be expressed by a first-order type over τ_1.

Now let Γ the set of complete n-types in $\mathscr{L}_{\omega\omega}(\tau_1)$ which are not in $\bigcup_{m \in \omega} \Gamma_m$ and put $K' = \mathrm{MOT}(\varnothing, \Gamma)$.

Claim 1. *If $\mathfrak{A} \in K'$ then $\mathfrak{A} \upharpoonright \tau \in K$.*

If $\mathfrak{A}$ is finitely generated, this is true since the only types realized in $\mathfrak{A}$ take care of this. Otherwise we write $\mathfrak{A}$ as the union of its finitely generated substructures and apply Lemma 1.3.4.

Claim 2. *If $\mathfrak{A} \in K$ then it has an expansion $\mathfrak{A}^* \in K'$.*

This clearly follows from Lemma 1.3.5.

This proves that $K \in \mathrm{PCOT}$. To prove that $\{[\mathfrak{A}; \mathfrak{B}]: \mathfrak{A} <_K \mathfrak{B}\}$ is also in PCOT we repeat the same proof for pairs of structures. □

Shelah's presentability theorem uses additional function symbols, even in the case where $<_K$ is just the substructure relation. On the other hand it guarantees axiomatizability in $\mathscr{L}_{\kappa\omega}$ for some κ depending on the Löwenheim number of K. One should compare this with the following easy generalization of the classical Chang–Łos–Suszko theorem:

1.3.7 Proposition*. *Let κ be a strongly inaccessible cardinal, τ a vocabulary with* $\mathrm{card}(\tau) < \kappa$ *and K an abstract class of τ-structures with Löwenheim number $l(K) < \kappa$ and $<_K$ the ordinary substructure relation. Then there is a prenex $\forall\exists$-sentence $\varphi \in \mathscr{L}_{\kappa\kappa}(\tau)$ such that $K = $* $\mathrm{Mod}(\varphi)$.

1.4. Hanf Numbers

Hanf numbers were defined in Chapter II for arbitrary logics. In Section IX.3.2 Hanf numbers for infinitary logics are studied. We want to apply these results together with the presentability theorem and characterizability theorem to abstract classes. We first define Hanf numbers for abstract classes and recall some material from Chapter IX.

1.4.1 Definitions (Hanf Numbers). (i) Let K be any class of structures closed under isomorphisms. We define the *Hanf number* $h(K)$ to be

$$h(K) = \bigcup \{\mathrm{card}(\mathfrak{A})^+ : \mathfrak{A} \in K\}.$$

If $h(K) > \kappa$ for every cardinal κ we write $h(K) = \infty$.

(ii) If $\mathbf{C}$ is a family of classes of structures closed under isomorphisms, we define the *Hanf number* $h(\mathbf{C})$ to be

$$h(\mathbf{C}) = \bigcup \{h(K): K \in \mathbf{C} \text{ and } h(K) < \infty\}.$$

$h(\mathbf{C})$ is the smallest cardinal κ such that if some some $K \in \mathbf{C}$ has a model of cardinality κ then it has arbitrary large models.

The concept of a Hanf number is only interesting for families of classes K, such as Jonsson classes, abstract classes with Löwenheim number $l(K) = \lambda$, ECOT(λ, μ), PCOT(λ, μ), etc.

1.4.2 Examples. (i) If all models $\mathfrak{A} \in K$ are of cardinality strictly less than κ then $h(K) \leq \kappa$.

(ii) If K is $\mathrm{PC}_{\mathscr{L}_{\omega_1\omega}}$ then $h(K) \leq \beth_{\omega_1}$, by Theorem VIII.6.4.4.

(iii) If K is PCOT(λ, μ) and $\lambda \leq \mu$ then $h(K) < \beth_{(2^\mu)^+}$, by corollary IX.3.2.14.

1.4.3 Theorem*. *Let K be an abstract class over a vocabulary τ with* $\mathrm{card}(\tau) = \lambda$ *and with Löwenheim number $l(K) = \mu$. Put $\kappa_0 = 2^{\lambda+\mu}$ and $\kappa = \beth_{(2^{\kappa_0})^+}$. Then $h(K) < \kappa$.*

Proof. Use the presentability theorem (1.3.3) and Example 1.4.1(iv) above. ▯

2. Amalgamation

2.1. Jonsson Classes and Universal and Homogeneous Models

We did not require in our definition of abstract classes any form of amalgamation. In fact, the point of our approach is, that amalgamation is *not* needed to get a nice structure/non-structure theory. It turns out that the presence or absence of amalgamation is like a watershed: The resulting model theories differ considerably. In this section we look at the case where amalgamation is true for any triple of models, a case which had been studied in the literature already in Jonsson [1956]. In Morley–Vaught [1962] they are called Jonsson classes. Jonsson classes are special cases of our abstract classes in the sense that the axioms of abstract classes are part of the axioms of Jonsson classes which we shall discuss now. Note that our terminology will differ slightly from the terminology scattered in the literature.

Let K be an abstract class. We shall introduce some more axioms:

Axiom 6 (Amalgamation). If $\mathfrak{A}_i \in K$, $i = 0, 1, 2$ and $\mathfrak{A}_0 < \mathfrak{A}_j$, $j = 1, 2$ then there is $\mathfrak{A} \in K$ such that $\mathfrak{A}_j < \mathfrak{A}$, $j = 1, 2$ and such that the diagram of the embeddings commutes.

Axiom 7 (Joint Embedding). If $\mathfrak{A}_j \in K$, $j = 1, 2$ then there is $\mathfrak{A} \in K$ such that $\mathfrak{A}_j < \mathrm{A}$.

Axiom 8 (Unboundedness). K contains structures of arbitrarily unbounded cardinality.

2.1.1 Definitions. (i) K is a *weak Jonsson class* (*with Löwenheim number* κ) if K is an abstract class (with Löwenheim number κ) satisfying additionally Axiom 6.

(ii) K is a *Jonsson class* (*with Löwenheim number* κ) if K is an abstract class (with Löwenheim number κ) satisfying additionally Axioms 6 and 7.

(iii) K is an *unbounded Jonsson class* (*with Löwenheim number* κ) if K is an abstract class (with Löwenheim number κ) satisfying additionally Axioms 6, 7, and 8.

2.1.2 Proposition*. (i) *Every weak Jonsson class K is a disjoint union of (possibly a proper class) of Jonsson classes.*

(ii) *If K is a weak Jonsson class and $l(K) = \lambda$ then K is a disjoint union of at most 2^{λ} many Jonsson classes.*

Proof. To see (i), we define an equivalence relation $\mathfrak{A} \equiv \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in K$ by: $\mathfrak{A} \equiv \mathfrak{B}$ if there is $\mathfrak{C} \in K$ such that $\mathfrak{A} < \mathfrak{C}$ and $\mathfrak{B} < \mathfrak{C}$. By the amalgamation axiom this is indeed an equivalence relation and every such equivalence class is a Jonsson class.

(ii) is obvious. ☐

2.1.3 Examples. (i) If Σ is a complete set of first-order sentences with an infinite model, then $\mathrm{Mod}(\Sigma)$ with the elementary embedding $<$ is a unbounded Jonsson class.

(ii) Jonsson classes are not necessarily unbounded: Let $K(\alpha)$ be the class of well-orderings embeddable into $\langle \alpha, < \rangle$ with end-extensions. As noted already in Example 1.2.1(ii) this gives rise to an abstract class and amalgamation and joint embedding hold trivially.

Unbounded Jonsson classes are the right framework for the construction of universal, homogeneous, and saturated structures. A fair exposition of this approach may be found in Bell–Slomson [1969, Chapter 10].

However we note that Jonsson classes are rather rare. In fact we have:

2.1.4 Proposition. *Let $\mathscr{L}$ be a logic with occurrence number below the first uncountable measurable cardinal such that for every complete set of sentences $\Sigma \subset \mathscr{L}(\tau)$ with an infinite model,* $\mathrm{Mod}(\Sigma)$ *together with $\mathscr{L}$-extensions is a weak Jonsson class. Then $\mathscr{L} \equiv \mathscr{L}_{\omega\omega}$.*

Proof. From the abstract amalgamation theorem (Theorem XVIII.3.4.2) we get that $\mathscr{L}$ is compact. Now we apply Theorem 3.1.9 also from Chapter XVIII. ☐

2.1.5 Definitions. Let K be a fixed abstract class with Löwenheim number $l(K) = \lambda$ and $\kappa \geq \lambda$.

(i) A structure $\mathfrak{M} \in K$ is *(K, κ)-universal*, if whenever $\mathfrak{A} \in K$ is of cardinality strictly less than κ then there is a K-embedding of $\mathfrak{A}$ into $\mathfrak{M}$.
(ii) A structure $\mathfrak{M} \in K$ is *K-universal*, if it is $\text{card}(\mathfrak{M})^+$-universal.
(iii) A structure $\mathfrak{M} \in K$ is *(K, κ)-homogeneous*, if whenever $\mathfrak{A} <_K \mathfrak{B} <_K \mathfrak{M}$, $\text{card}(\mathfrak{B}) < \kappa$ and $f: \mathfrak{A} \to \mathfrak{M}$ is a K-embedding, then there is a K-embedding $f': \mathfrak{B} \to \mathfrak{M}$ such that $f' \upharpoonright \mathfrak{A} = f$.
(iv) A structure $\mathfrak{M} \in K$ is *K-homogeneous*, if its $\text{card}(\mathfrak{M})$-homogeneous.

The following theorem is at the origin of Jonsson classes. It was first proved in Jonsson [1960] for countable vocabularies. The general treatment occurs first in Morley–Vaught [1962]. A fair treatment is in Bell–Slomson [1969] and Comfort–Negrepontis [1974].

2.1.6 Theorem (Jonsson). *Let K be a unbounded Jonsson class with Löwenheim number $l(K) = \lambda$. Let further $\kappa \geq \lambda$ be a regular beth number. Then there is $\mathfrak{M} \in K$ which is K-homogeneous and K-universal and $\mathfrak{M}$ is unique up to isomorphism.*

If the Jonsson class K is not unbounded, we can still get universal and homogeneous structures, even if we relax the amalgamation axiom a bit.

2.1.7 Definitions. Let K be an abstract class.

(i) Let $\mathfrak{A} \in K$. We say that $\mathfrak{A}$ *is an (λ, μ)-amalgamation base for K*, if for every $\mathfrak{B}_1, \mathfrak{B}_2 \in K$ with $\text{card}(\mathfrak{B}_1) = \lambda$, $\text{card}(\mathfrak{B}_2) = \mu$, $\mathfrak{A} <_K \mathfrak{B}_i$ $(i = 1, 2)$ there is $\mathfrak{M} \in K$ and K-embeddings $f_i: \mathfrak{B}_i \to \mathfrak{M}$ such that $f_1 \upharpoonright \mathfrak{A} = f_2 \upharpoonright \mathfrak{A}$. We call $\mathfrak{M}$ also an *amalgamating structure for* $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$.
(ii) We say that K *has the (κ, λ, μ)-amalgamation property*, if every $\mathfrak{A} \in K$ with $\text{card}(\mathfrak{A}) = \kappa$ is a (λ, μ)-amalgamation base.
(iii) If $\kappa = \lambda$ we just speak of the (λ, μ)-amalgamation property. If $\kappa = \lambda = \mu$ we just say that K_λ has the amalgamation property.
(iv) We write $(<\lambda, \mu)$-amalgamation property, if K has the (λ', μ)-amalgamation property for every $\lambda' < \lambda$ and similarily for the other parameters.

The precise theorem on the existence of homogeneous and universal models, using basically the same proof, is the following:

2.1.8 Theorem (Shelah). *Let K be an abstract class with Löwenheim number $l(K)$, $\kappa \leq \lambda$ and $\lambda = \lambda^{<\kappa}$.*

(i) *If K has the $(<\kappa, \lambda)$-amalgamation property, then for every $\mathfrak{A} \in K$ of cardinality λ there is κ-homogeneous model $\mathfrak{M}$ of cardinality λ such that $\mathfrak{A} <_K \mathfrak{M}$.*
(ii) *If in* (i) *$\kappa = \lambda$ and additionally, K has the joint embedding property* (*i.e., satisfies Axiom* 7), *then there is a universal, homogeneous model $\mathfrak{M}$ of cardinality λ.*
(iii) *If in* (i) *additionally $l(K) < \lambda$ and $\kappa = \lambda$ then the universal and homogeneous model of cardinality λ is unique up to isomorphism.*

2.1.9 Remarks. (i) If K is an unbounded Jonsson class then the universal and homogeneous model of cardinality λ has a proper K-extension. In fact, if λ is regular and $\lambda > l(K)$, then it is a (λ, λ)-limit, as defined in the next section.

(ii) If K is not unbounded, then the universal and homogeneous model can be rigid and have no proper K-extensions. Take, for example, the class of well-orderings of order type less or equal to some fixed cardinal κ together with end-extensions. Then $\langle \kappa, \in \rangle$ has all the above properties.

(iii) If we drop Axiom 4(ii) in our definition of abstract classes we still can prove an analogue to Theorem 2.1.8(ii), losing universality only. More precisely, there is a homogeneous, $(<\lambda)$-universal model in K which is smooth, i.e., the union of a continuous K-chain of models of cardinality strictly smaller than λ. Axiom 4(ii) is used to get the universality from $(<\lambda)$-universality and smoothness. An example of a class K, where this situation applies, is given in Section XVIII.3.4.

(iv) In the literature before 1980 Axiom 4(ii) is usually not required for the definition of a Jonsson class. Presentations of the original theory of Jonsson classes may be found in Bell–Slomson [1969] and Comfort–Negrepontis [1974]. The latter also contains detailed historical remarks.

Given an abstract class K we might also be interested in the number of homogeneous models K has in a given cardinality:

2.1.10 Definition. Let K be an abstract class. We denote by $H(K, \lambda)$ the number of isomorphism classes of K-homogeneous models of cardinality λ.

2.1.11 Theorem (Shelah). *Let K be an abstract class (over a vocabulary τ) with Löwenheim number $l(K) = \lambda$ and $\kappa > \lambda$. Then $H(K, \kappa) \leq 2^{\lambda + \mathrm{card}(\tau)}$.*

Outline of Proof. We observe that two K-homogeneous structures $\mathfrak{A}$, $\mathfrak{B}$ of cardinality $\kappa > \lambda$ are isomorphic iff they have the same substructures of cardinality λ. □

It remains an open problem to characterize $H(K, \kappa)$ further.

We conclude this subsection with a theorem on the existence of universal models in big cardinals.

2.1.12 Theorem (Grossberg–Shelah [1983]). *Let κ be a compact cardinal and $\lambda > \kappa$ with λ strong limit and of cofinality ω. Let K be an abstract class with Löwenheim number $l(K) < \kappa$ which satisfies the joint embedding property. Then there is a universal model in K_λ.*

There are non-trivial applications of the above theorem in the case of locally finite groups.

Proof. In Grossberg–Shelah [1983] this is proved for K the class of all models of some $\mathscr{L}_{\kappa\kappa}$-sentence φ which satisfies the joint embedding property for 2^κ many models simultaneously. In the case of an abstract class the latter can be replaced by the simple JEP, using Axiom 4 (unions of chains). It is easy to see how the proof in

Grossberg–Shelah [1983] can be adapted to abstract classes: We use the Lowenheim number and the presentability theorem to get that K is a projective class in $\mathscr{L}_{\kappa\kappa}$. Next we observe that the proof in Grossberg–Shelah [1983] also works for projective classes. $\square$

2.2. Limit Models

One of the more powerful tools in classical model theory is the use of saturated or special models. Their construction can be carried out in the context of Jonsson classes as described in the previous section. However, we have also seen there that Jonsson classes are very rare outside of first-order model theory. So we need a substitute for saturated models whose existence does not depend on the amalgamation axiom.

2.2.1 Definition ((λ, κ)-Limit Models). Let K be an abstract class with $<_K$.

(i) A model $\mathfrak{N} \in K$ is a *weak λ-limit* if the following properties (a), (b), and (c) are satisfied.

(a) $\operatorname{card}(\mathfrak{N}) = \lambda$.
(b) $\mathfrak{N}$ has a proper extension $\mathfrak{M}$ with $\mathfrak{N} <_K \mathfrak{M}$.
(c) For every $\mathfrak{M} \in K$ such that $\operatorname{card}(\mathfrak{M}) = \lambda$ and $\mathfrak{N} <_K \mathfrak{M}$ there is a $\mathfrak{N}' \in K$ such that $\mathfrak{N} \cong \mathfrak{N}'$ and $\mathfrak{M} <_K \mathfrak{N}'$.

(ii) A structure $\mathfrak{N} \in K$ is a *(λ, κ)-limit model in K*, if it is a weak λ-limit and additionally the following property (d) holds.

(d) If $\{\mathfrak{N}_i : i < \kappa \leq \lambda\}$ is a K-chain and for each $i < \kappa$, $\mathfrak{N}_i \cong \mathfrak{N}$, then $\bigcup_{i<\kappa} \mathfrak{N}_i \cong \mathfrak{N}$.

(iii) A model $\mathfrak{N} \in K$ is a *λ-superlimit* if it is a (λ, κ)-limit for every $\kappa \leq \lambda$.

Superlimits are closely related to saturated models:

2.2.2 Proposition. (i) *If $\mathfrak{M}$ is saturated or special and of cardinality λ then $\mathfrak{M}$ is a (λ, $\operatorname{cf}(\lambda)$)-limit in $K = \{\mathfrak{A} \in \operatorname{Str}(\tau) : \mathfrak{M} \equiv_{\mathscr{L}_{\omega\omega}} \mathfrak{A}\}$ with elementary embeddings.*

(ii) *If K is an abstract class and $\mathfrak{N} \in K$ is K-universal and K-homogeneous of cardinality λ, then $\mathfrak{N}$ is weak λ-limit iff $\mathfrak{N}$ is not K-maximal.*

Proof. (i) We have to verify (a), (b), (c), and (d). (a) is true by hypothesis. (b) follows from the compactness of first-order logic and (c) follows from the fact that saturated models are universal. For (d) we have to show that if for every $i < \operatorname{cf}(\lambda)$ $\mathfrak{M}_i$ is saturated then $\bigcup_{i<\operatorname{cf}(\lambda)} \mathfrak{M}_i$ is saturated, too. For λ regular this is easy (Chang–Keisler [1973, Exercise 5.1.1]). For λ singular, see Shelah [1978a]. From this, together with the uniqueness of saturated models, we conclude that $\bigcup_{i<\operatorname{cf}(\lambda)} \mathfrak{M}_i \cong \mathfrak{M}$. The proof for special models is similar and left to the reader.

(ii) is trivial. $\square$

The following two simple propositions will be used in the later sections.

2.2.3 Proposition. *Let K be an abstract class with Löwenheim number $l(K) \leq \lambda^+$ which has a weak λ-limit model $\mathfrak{N} \in K$. Then there is a model $\mathfrak{M} \in K$ with* $\operatorname{card}(\mathfrak{M}) = \lambda^+$.

Proof. By (b) there is $\mathfrak{M}' \in K$ such that $\mathfrak{N} <_K \mathfrak{M}'$. If $\operatorname{card}(\mathfrak{M}') > \lambda^+$ we get $\mathfrak{M}$ from the Löwenheim number. If $\operatorname{card}(\mathfrak{M}') = \lambda$ we apply (c) to get $\mathfrak{N}' \cong \mathfrak{N}$ with $\mathfrak{N} <_K \mathfrak{N}'$ and use this to construct a K-chain of length λ^+. Now we apply the chain axiom. $\square$

2.2.4 Proposition. *Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ which has, up to isomorphism, exactly one model $\mathfrak{N} \in K$ of cardinality λ. Then $\mathfrak{N}$ is λ-superlimit iff K has a model $\mathfrak{M}$ of cardinality strictly bigger than λ.*

Proof. If $\mathfrak{N}$ is weak λ-limit we can apply Proposition 2.2.3. So assume that $\mathfrak{M} \in K$ is of cardinality strictly bigger than λ. Using the Löwenheim number we can get $\mathfrak{M}_0 <_K \mathfrak{M}_1 <_K \mathfrak{M}$ with both $\mathfrak{M}_0, \mathfrak{M}_1$ of cardinality λ and isomorphic to $\mathfrak{N}$. This proves (b) of the definition of the superlimit (Definition 2.2.1). Properties (c) and (d) are trivial under the hypothesis of categoricity in λ $\square$

We conclude this section with a few observations on the uniqueness of superlimits, whose proofs are trivial.

2.2.5 Proposition. *Let K be an abstract class with a λ-superlimit $\mathfrak{M}$.*

(i) *If $\mathfrak{N}$ is also a λ-superlimit then $\mathfrak{N} \cong \mathfrak{M}$ iff either $\mathfrak{N} < \mathfrak{M}$ or $\mathfrak{M} < \mathfrak{N}$ (modulo some K-embedding).*
(ii) *If K has the joint embedding property, then the superlimit is unique, up to isomorphism.*
(iii) *If $\mathfrak{M}$ is universal, then it is unique.*

2.2.6 Example. Here is an example of an abstract class K_P which has exactly $\alpha + 1$ λ-superlimits of cardinality ω_α. Let K consist of structures with one unary predicate R, whose interpretation is infinite. We put $\langle A, R_A \rangle < \langle B, R_B \rangle$ iff $A \subset B$ and $R_A = R_B$. Clearly $\langle A, R_A \rangle$ is λ-superlimit iff $A - R_A$ has cardinality λ.

We shall often deal with a situation where an abstract class K with Löwenheim number $l(K) < \lambda$ has a λ-superlimit $\mathfrak{M}$ which is universal, homogeneous and is an amalgamation basis for K_λ. Clearly then, only by universality and homogeneity, K has the $(<\lambda, \lambda)$-amalgamation property.

2.2.7 Problem. Does K_λ in this case also have the (λ, λ)-amalgamation property?

In Section 2.3 we state a conjecture, whose proof would follow from a positive answer to this problem.

2.3. Counting Models in the Absence of an Amalgamation Bases

In this section we assume K is an abstract class, which is not a Jonsson class and therefore does not have the amalgamation property, but still does have a λ-superlimit $\mathfrak{M} \in K_\lambda$. Our main theorem of this section is:

2.3.1 Theorem (Shelah's Non-structure Theorem for Abstract Classes). *Assume* $2^\lambda < 2^{\lambda^+}$. *Let K be an abstract class such that:*

(i) *there is a λ-superlimit $\mathfrak{M} \in K_\lambda$;*
(ii) *$\mathfrak{M}$ is not an amalgamation basis for K_{λ^+}.*

Then $I(K, \lambda^+) = 2\lambda^+$ and there is no universal model in K_{λ^+}.

At this point it is appropriate to state some conjectures. The first one deals with the existence of universal and homogeneous superlimits.

2.3.2 Conjecture (Shelah). Let K be an abstract class with Löwenheim number $l(K) < \lambda$ such that $I(K, \lambda^+) < 2^{\lambda^+}$.

(i) If K additionally satisfies the joint embedding property (Axiom 7), then there is K-universal and K-homogeneous λ-superlimit $\mathfrak{M} \in K$.
(ii) If K has arbitrarily large models, then there is a K-universal and K-homogeneous λ-superlimit $\mathfrak{M} \in K$.

(It may be enough to assume that there is a model of cardinality bigger than 2^{λ^+}.)

An instance of this conjecture is Theorem 3.1.8, with $\lambda = \omega_1$ and K ω-presentable. A proof of this conjecture would give us, with the help of the previous theorem, also a proof of the following conjecture:

2.3.3 Conjecture. Assume GCH. Let K be an abstract class with Löwenheim number $l(K) = \omega$ which has arbitrary large models and such that for every $\lambda > \omega$, $I(K, \lambda) < 2^\lambda$. Then K has the amalgamation property and therefore is a weak Jonsson class.

2.3.4 Problem. Could we replace $l(K) = \omega$ by arbitrary λ in the above conjecture?

Finally we state a conjecture which presents an improvement on Theorem 2.3.1.

2.3.5 Conjecture. Assume $2^\lambda < 2^{\lambda^+}$. Let K be an abstract class with Löwenheim number $l(K) < \lambda$ such that:

(i) there is a universal and homogeneous λ-superlimit $\mathfrak{M} \in K_\lambda$;
(ii) $I(K, \lambda^+) < 2^{\lambda^+}$.

Then K_λ has the amalgamation property.

Clearly, from Theorem 2.3.1, $\mathfrak{M}$ is an amalgamation basis for K_λ, and, by universality and homogeneity, K has the $(<\lambda, \lambda)$-amalgamation property.

We conclude this section with another conjecture, generalizing Morley's categoricity theorem for first-order logic to λ-presentable classes.

2.3.6 Conjecture (Shelah). Let K be an abstract λ-presentable class and let h_λ be the Hanf number for λ-presentable classes. If $I(K, \kappa) = 1$ for some $\kappa > h_\lambda$ then $I(K, \kappa) = 1$ for every $\kappa > h_\lambda$.

Added in Proof. Recently R. Grossberg and S. Shelah announced the following Theorem:

2.3.7 Theorem. *Let K be an unbounded abstract λ-presentable class. If there is a $\mu > \lambda$ such that for every $n \in \omega$ $I(K, \mu^{+n}) = 1$ then for every $\kappa > \lambda$ $I(K, \kappa) = 1$.*

2.4. Martin's Axiom Disproves the Non-structure Theorem

Before we discuss the proof of Theorem 2.3.1 we want to comment on its set-theoretic hypothesis $2^\lambda = \lambda^+$. For this we have to recall Martin's Axiom MA from set theory.

2.4.1 Definitions(Partial Orders). (i) A *partial order* is a pair $\langle P, \leq \rangle$ such that P is not empty and $\leq$ is a transitive and reflexive relation on P.

(ii) Given $p, q \in P$ we say that *p and q are compatible* if there is $r \in P$ such that $r \leq p$ and $r \leq q$ and *p and q are incompatible* if they are not compatible. A *antichain in P* is a set $A \subset P$ such that for every $p, q \in P$ either $p = q$ or p and q are incompatible.

(iii) A partial order $\langle P, \leq \rangle$ satisfies the *countable chain condition* (c.c.c) if every antichain in P is countable.

(iv) A set $D \subset P$ is *dense*, if for every $p \in P$ there is a $q \in D$ such that $q \leq p$.

(v) A set $G \subset P$ is *filter in P*, if any two elements in G are compatible and whenever $p \in G$ and $q \geq p$ then $q \in G$.

2.4.2 Martin's Axiom. (i) MA(κ) is the statement: If $\langle P, \leq \rangle$ is a partial order satisfying c.c.c and $\{D_i : i < \kappa\}$ is a family of dense subsets of P then there is a filter G in P such that for every $i < \kappa$, $D_i \cap G \neq \varnothing$.

(ii) MA is the statement: For every $\kappa < 2^\omega$ MA(κ).

For more references the reader may consult Kunen [1980] or Shelah [1982c].

2.4.3 Proposition (Shelah). *Assume* ZFC + MA + $\neg$CH *(and therefore $2^\omega = 2^{\omega_1}$). Then there is an ω-presentable abstract class $K_0 \in$* ECOT *such that*:

(i) *$I(K, \kappa) = 1$ for every $\kappa < 2^\omega$*;

(ii) *$I(K, \kappa) = \varnothing$ for every $\kappa > 2^\omega$*; but

(iii) *K_ω does not have the amalgamation property.*

Proof. We first define the class K_0. The vocabulary τ_0 consist of a binary predicate **C** and a unary predicate **P**. A τ_0-structure $\mathfrak{A} = \langle A, E, P \rangle$ is in K_0 if:

(1) P is countable.
(2) If xEy then $x \in P$ but $y \notin P$.

For every $y \notin P$ we define $S_y = \{x \in A : xEy\}$. Clearly $S_y \subset P$.

(3) (Extensionality of E). If $x \neq y$, $x, y \notin P$ then $S_x \neq S_y$.
(4) For every $x \notin P$ and for every finite set $C \subset P$ there is a $y \notin P$ such that the symmetric difference $S_x \Delta\, S_y = C$.

We define an equivalence relation on $A - P$ by $x \equiv y$ iff $S_x \Delta\, S_y$ is finite. Clearly every equivalence class is countable. Let the number of such equivalence classes be the dimension $\dim(\mathfrak{A})$ of $\mathfrak{A}$. Now we require that:

(5) If $x_1, x_2, \ldots, x_n$ are mutually inequivalent and not in P, then every finite boolean combination of the sets $S_{x_1}, S_{x_2}, \ldots, S_{x_n}$ is infinite.

This concludes the definition of K_0.

Next we define the substructure relation $<_0$ for $\mathfrak{A} = \langle A, E_A, P_A \rangle$, $\mathfrak{B} = \langle B, E_B, P_B \rangle$, both in K_0 by $\mathfrak{A} <_0 \mathfrak{B}$ if $\mathfrak{A} \subset \mathfrak{B}$ and $P_A = P_B$.

We have to verify that this defines an abstract class with (i)–(iii). We leave the verification of the axioms to the reader. To verify (i) and (ii) we prove five claims:

Claim 1. *There are no models of cardinality greater than* 2^ω.

By (1) P is countable and by (4) every element is either in P or in some S_x for $x \in P$. So the claim follows from (3). This proves Proposition 2.4.3(ii).

Claim 2. K_0 *is categorical in* ω.

This one can prove using (5) and a Cantor-style back-and-forth argument.

Claim 3. $\mathrm{MA}(\kappa)$ *implies that* K_0 *is categorical in* $\kappa < 2^\omega$.

Clearly $\mathfrak{A}, \mathfrak{B} \in K_0$ have the same cardinality iff they have the same dimension. So let $\mathfrak{A}, \mathfrak{B} \in K_0$ be of the same dimension $\kappa < 2^\kappa$. So let $E_i^A, E_i^B, i < \kappa$ be an enumeration of the equivalence classes in $\mathfrak{A}, \mathfrak{B}$, respectively. Let F be the family of all finite partial isomorphisms $f: \mathfrak{A} \to \mathfrak{B}$ such that additionally to the isomorphism conditions we have:

(a) for every $x \in \mathrm{dom}(f)$, $x \in E_i^A$ iff $f(x) \in E_i^B$; and
(b) if $x, y \in \mathrm{dom}(f)$, $x, y \notin P$, and $x \equiv y$ then the finite set $S_x \Delta\, S_y \subset \mathrm{dom}(f)$.

Clearly F is a partial order by the natural extension relation of partial isomorphisms: $f \leq g$ iff f extends g. To show that F satisfies c.c.c., we show:

Claim 4. *If* $\{f_i : i < \kappa\} \subset F$ *for some* κ *such that* $\omega < \kappa < 2^\omega$ *then there is* $I \subset \kappa$, $\mathrm{card}(I) = \kappa$, *such that* $\{f_i : i \in I\}$ *are all compatible.*

This follows from the fact that all the sets P_A, E_i^A are countable.

Now we define $D_a = \{f \in F: a \in \text{dom}(f)\}$ and $D_b = \{f \in F: b \in \text{rg}(f)\}$. Clearly, all the D_a, D_b are dense in F. So let G be a filter in F which intersects all the D_a, D_b with $a \in P_A$ and $b \in P_B$. Such a filter exists by MA(κ). Next, we define $g = \bigcup_{f \in G} f$.

Claim 5. *$g: \mathfrak{A} \to \mathfrak{B}$ is an isomorphism.*

g is one–one and onto by our choice of the D_a, D_b, and g is an isomorphism, since every finite restriction of g has an extension in F.

So Claims 3–5 prove Proposition 2.4.3(i).

We still have to prove Proposition 2.4.3(iii). For this let $\mathfrak{A} = \langle A, E_A, P_A \rangle \in K_0$ be countable. Let $S^1 \subsetneqq S^2 \subsetneqq P_A$ be two generic subsets different from all the S_x, $x \in A - P_A$. We now form $\mathfrak{A}_i$ $(i = 1, 2)$ by adding the necessary new points to $A - P_A$ to ensure that S^i is of the form S_x for some $x \in A_i - P_A$ and to make (2)–(4) true. No points are added in P_A. Clearly $\mathfrak{A}_i$ can be constructed to be countable and in K_0, and $\mathfrak{A} <_0 \mathfrak{A}_i$. Now assume $\mathfrak{B}$ is an amalgamating structure. Then there are $z_i \in B - P_A$ $(i = 1, 2)$ such that $S_{z_i} = S^i$ and $S_{z_1} \cap S_{z_2} = \emptyset$, contradicting (5). Therefore $\mathfrak{B} \notin K_0$. $\square$

2.5. *Preliminaries for the Weak Diamond*

In this section we collect the set-theoretic preliminaries needed in Section 2.6. They are concerned about the relation between various instances of the GCH and combinatorial principles related to $\diamondsuit$. First we present a variation of Ulam's theorem (cf. Lemma XVIII.4.3.9). Recall that an *ideal J on a set I* is the dual of a filter F on the set I, and that an ideal is *normal*, if the dual filter is normal. A subset $S \subset I$ is called *J-positive*, if $S \notin J$. Since the filter D_κ of closed and unbounded sets on κ is normal, the stationary sets on κ are D_κ-positive.

2.5.1 Ulam's Theorem. *Let J be a normal ideal on κ^+.*

(i) (Ulam). *Let κ be an infinite cardinal. If $S \subset \kappa^+$, $S \notin J$, S may be decomposed into κ^+ disjoint J-positive subsets.*

(ii) *There is a family $\mathbf{S}$ of 2^{κ^+} many J-positive subsets of κ^+ such that for any $S_1, S_2 \in \mathbf{S}$ the symmetric difference $S_1 \Delta S_2$ is J-positive as well.*

Proof. (i) is standard, e.g., Theorem 3.2 in Chapter B.3 of the *Handbook of Mathematical Logic* [Barwise 1977], where it is stated for stationary rather than J-positive sets. But the same proof works for this generalized version.

To prove (ii) let $\{S_\alpha: \alpha < \kappa^+\}$ be the disjoint family of J-positive sets from (i). Let $X \subset \kappa^+$, $X \neq \emptyset$. Define $T_X = \bigcup_{\alpha \in X} S_{2\alpha} \cup \bigcup_{\alpha \notin X} S_{2\alpha+1}$. Clearly each T_X is J-positive and $X \neq Y$ implies that $T_X \Delta T_Y$ is J-positive. $\square$

2.5.2 Jensen's $\diamondsuit$. Jensen's $\diamondsuit$ for $\omega_1(\diamondsuit_{\omega_1})$ can be formulated as: There exists a family of functions $\{g_\alpha: \alpha \to \alpha: \alpha < \omega_1\}$ such that for every $f: \omega_1 \to \omega_1$ we have that $\{\alpha < \omega_1 : f \upharpoonright \alpha = g_\alpha\}$ is stationary.

2.5.3 The Principles Φ and Θ of Devlin and Shelah. Let F be a function which maps (0, 1)-sequences of length $\alpha < \lambda$ into $\{0, 1\} = 2$, and let $S \subset \lambda$.

(i) The principle $\Phi_\lambda^2(S)$ says that for every such function F there is a function $g: \lambda \to 2$ such that for every other function $f: \lambda \to 2$ the set

$$\{\alpha \in S: F(f \restriction \alpha) = g(\alpha)\}$$

is stationary on λ.

(ii) The principle Φ is just $\Phi_{\omega_1}^2(\omega_1)$.

(iii) The principle $\Phi_\lambda^\kappa(S)$ is obtained from Φ_λ^2 by replacing every occurrence of 2 by κ, both in the range and domain of F as well as in the range of g and the domain of f.

(iv) If $S = \lambda$ we omit it.

(v) The principle Θ says that if $\{f_\eta: \eta \in {}^{\omega_1}2\}$ is a family of functions with each $f_\eta: \omega_1 \to 2^\omega$, then there is $\eta \in {}^{\omega_1}2$ such that the set

$$\{\delta \in \omega_1: (\exists\rho \in {}^{\omega_1}2)[f_n \restriction \delta = f_\eta \restriction \delta \text{ and } \rho \restriction \delta = \eta \restriction \delta \text{ and } \rho(\delta) \neq \eta(\delta)]\}$$

is stationary.

2.5.4 Theorem. (i) (Jensen). $\diamond_{\omega_1}$ *implies* $2^\omega = \omega_1$.

(ii) (Devlin–Shelah). $\diamond_{\omega_1}$ *implies the principle* Φ.

(iii) (Devlin–Shelah). $2^\omega < 2^{\omega_1}$ *implies* Θ.

A proof of (i) may be found in textbooks like Kunen [1980]. (ii) and (iii) are proved in Devlin–Shelah [1978]. The important fact about the principle Φ is the following theorem:

2.5.5 Theorem (Devlin–Shelah [1978]). *The principle* Φ *is equivalent to* $2^\omega < 2^{\omega_1}$.

2.5.6 Definition (Small Sets). (i) A subset $S \subset \lambda$ is (λ, κ)-*small*, if $\Phi_\lambda^\kappa(S)$ fails.

(ii) Let us denote by $\mathrm{S}(\lambda, \kappa)$ the set of all (λ, κ)-small subsets of λ.

2.5.7 Remarks. (i) Clearly, (λ, κ)-small sets are stationary in λ.

(ii) The principle Φ is equivalent to $\omega_1 \notin \mathrm{S}(\omega_1, 2)$.

2.5.8 Proposition (Shelah). (i) $\mathrm{S}(\lambda, \kappa)$ *forms a normal ideal on* λ.

(ii) Φ_λ^κ *holds iff* $\mathrm{S}(\lambda, \kappa)$ *forms a non-trivial normal ideal on* λ.

Proof. (i) is a special case of Lemma 14.1.9 in Shelah [1982, Book] and (ii) follows trivially from the definitions and (i). □

2.5.9 Strong Negations of Φ. First we write out the negation of $\Phi_{\omega_1}^2$: there is a function F which maps (0, 1)-sequences of length $\alpha < \lambda$ into $\{0, 1\} = 2$, such that for every function $g: \lambda \to 2$ there is a function $f: \lambda \to 2$ such that the set

$$\{\alpha < \lambda: F(f \restriction \alpha) = g(\alpha)\}$$

is closed and unbounded in λ. We want to generalize and parametrize this further.

Let λ be a regular cardinal and $\bar{\mu} = \langle \bar{\mu}(i) : i < \lambda \rangle, \bar{\chi} = \langle \bar{\chi}(i) : i < \lambda \rangle$ be sequences of cardinals. We want to generalize the above negation of Φ for a function F with domain

$$\mathrm{dom}(F) = D(\bar{\mu}) = \bigcup_{\alpha < \lambda} \prod_{i < \alpha} \bar{\mu}(i).$$

Now we denote by $\mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$ the statement:

There is a function F such that:

(a) for every $\alpha < \lambda$, if $\eta \in \prod_{i<\alpha} \bar{\mu}(i)$, then $F(\eta) < \bar{\chi}(\alpha)$; and
(b) for every $h \in \prod_{\alpha<\lambda} \bar{\chi}(\alpha)$ there exists $\eta \in \prod_{\alpha<\lambda} \bar{\mu}(\alpha)$ such that

$$\{\alpha < \lambda : F(\eta \restriction \alpha) = h(\alpha)\}$$

is closed and unbounded in λ.

Such a function F is said to *exemplify* $\mathrm{Unif}(\lambda, \bar{\mu}, \bar{\chi})$.

If $\bar{\mu}, \bar{\chi}$ are singletons we use the obvious notation. If $\bar{\mu} = \mu(0), \mu(1)$ we use the obvious abuse of notation. In 14.1.5 of Shelah [1982c] it is proved that we can always assume that $\bar{\mu}$ is a sequence of length two. Clearly $\mathrm{Unif}(\lambda, 2, 2)$ is just the negation of Φ^2_λ, and $\mathrm{Unif}(\lambda, \kappa, \kappa)$ is just the negation of Φ^κ_λ.

The version of the weak diamond needed in Section 2.7, and its connection to the continuum hypothesis, is captured in the following proposition:

2.5.10 Proposition. *Assume* $2^\kappa < 2^{\kappa^+}$.

(i) $\Phi^2_{\kappa^+}$ holds.
(ii) $\mathrm{Unif}(\kappa^+, \mu, 2, 2)$ *fails for every* μ *with* $\mu^\omega < 2^{\kappa^+}$.

This proposition follows from the following two results from Shelah [1982c]:

2.5.11 Theorem (Shelah). *Assume that* λ *is regular and*

(i) $2^{<\lambda} < 2^\lambda$;
(ii) $\mu^\omega < 2^\lambda$.

Then $\mathrm{Unif}(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ *fails.*

Proof. Shelah [1982c, Theorem 14.1.10]. □

2.5.12 Proposition (Shelah). $\mathrm{Unif}(\lambda^+, \mu, 2, 2)$ *implies* $\mathrm{Unif}(\lambda^+, \mu, 2^\lambda, 2^\lambda)$.

Proof. Shelah [1982c, Lemma 14.1.7(1)] for the case λ replaced by λ^+. □

Proof of Proposition 2.5.10. We prove (ii) since (i) follows from (ii) by putting $\mu = 1$. We apply Theorem 2.5.11 with $\lambda = \kappa^+$ and therefore $2^{<\lambda} = 2^\kappa$. So we get that $\mathrm{Unif}(\kappa^+, \mu, 2^\kappa, 2^\kappa)$ fails, for every $\mu^\omega < 2^{\kappa^+}$. So by Proposition 2.5.12 $\mathrm{Unif}(\kappa^+, \mu, 2, 2)$ fails, for every $\mu^\omega < 2^{\kappa^+}$. □

2.6. Proof of the Non-structure Theorem: The Countable Case

The purpose of this section is to prove completely some special cases of the non-structure theorem (2.3.1) and to understand some of the intricacies of its complete proof. The complete proof appears in Section 2.7. For expository (and historical) reasons we shall work our way from the moderately simple case to the more difficult.

2.6.1 Theorem. *Assume $2^\omega < 2^{\omega_1}$. Let K be an abstract class with Löwenheim number $l(K) = \omega$ such that:*

(i) $I(K, \omega) = 3$;
(ii) $I(K, \omega_1) \neq \varnothing$; *and*
(iii) K_ω *does not have the amalgamation property.*

Then there is no universal model in K_{ω_1} and therefore $I(K, \omega_1) > 1$.

Outline of Proof. The proof consists of several stages: A construction of a system of countable models, a construction of uncountable models, and a verification that no model of cardinality ω_1 is universal. The same pattern will be followed in subsequent proofs, so we try to give this first proof a modular structure.

2.6.2 Construction of Countable Models. Clearly, by (i) and (ii) and Proposition 2.2.4 the unique countable model $\mathfrak{N} \in K$ is an ω-superlimit. Let $\mathfrak{M}^* \in K$ be of cardinality ω_1, so without loss of generality we can assume that its domain $M^* = \omega_1$. By (iii) there are countable $\mathfrak{M} <_K \mathfrak{M}_i$ $(i = 0, 1)$ which exemplify the failure of the amalgamation property and $\mathfrak{M} \cong \mathfrak{M}_i \cong \mathfrak{N}$. Since the Löwenheim number $l(K) = \omega$ we can assume that $\mathfrak{M} <_K \mathfrak{M}^*$. We shall show that $\mathfrak{M}^*$ is not universal.

For this we define by induction on $\alpha < \omega_1$ countable models $\mathfrak{M}_\eta$ where $\eta \in {}^\alpha 2$, i.e., η ranges over sequences of 0's and 1's of length $l(\eta) \leq \alpha$. If η, ν are two such sequences, we write $\eta \subset \nu$ if η is an initial segment of ν, and if $\beta < \alpha$ we denote by $\eta \restriction \beta$ the restriction of η to β.

Now we require that:

(1) $\mathfrak{M}_\eta$ is countable and the universe $M_\eta = \omega(1 + l(\eta))$.
(2) If $\eta \subset \nu$ then $\mathfrak{M}_\eta <_K \mathfrak{M}_\nu$.
(3) If $\delta \in \omega_1$ is a limit ordinal and η is a sequence of length δ then

$$\mathfrak{M}_\eta = \bigcup_{\alpha < \delta} \mathfrak{M}_{\eta \restriction \alpha}.$$

To construct all the $\mathfrak{M}_\eta$'s we put for $\alpha = 0$, $\mathfrak{M}_\phi \cong \mathfrak{N}$, and for the $\alpha = \delta$ limit we take the limits, as K is an abstract class. For $\alpha = \beta + 1$ we have to work a bit. For each η of length $\leq \beta$ we choose an isomorphism f_η from $\mathfrak{M}$ onto $\mathfrak{M}_\eta$. Here we use (i). Now we define functions f_η^i and models $\mathfrak{M}_{\eta^\frown\langle i \rangle}$ such that f_η^i extends f_η and is an isomorphism from $\mathfrak{M}_i$ onto $\mathfrak{M}_{\eta^\frown\langle i \rangle}$. In other words, at every stage we copy our original counterexample to the amalgamation property.

2.6.3 Construction of Models in ω_1. Now we construct models of cardinality ω_1. For every η of length ω_1 we put $\mathfrak{M}_\eta = \bigcup_{\alpha<\omega_1} \mathfrak{M}_{\eta \restriction \alpha}$.

2.6.4 $\mathfrak{M}^*$ is not Universal. Assume, for contradiction, that $\mathfrak{M}^*$ were universal. Then for each η of length ω_1 there is an embedding g_η from $\mathfrak{M}_\eta$ into $\mathfrak{M}^*$ such that $g_\eta(\mathfrak{M}_\eta) <_K \mathfrak{M}^*$.

Now we use the principle Θ and get two sequences η, ν of length ω_1 and $\alpha < \omega_1$ with $\alpha = \omega\alpha$ such that:

(4) $\eta \restriction \alpha = \nu \restriction \alpha, \eta(\alpha) = 0, \nu(\alpha) = 1$; and
(5) $g_\eta \restriction \mathfrak{M}_{\eta \restriction \alpha} = g_\nu \restriction \mathfrak{M}_{\nu \restriction \alpha}$.

But this shows that $\mathfrak{M}_0$, $\mathfrak{M}_1$ can be amalgamated over $\mathfrak{M}$ with amalgamating structure $\mathfrak{M}^*$ by setting

$$h_0 = (g_\eta \restriction \mathfrak{M}_{\eta \restriction (\alpha+1)}) f^0_{\eta \restriction \alpha} \colon \mathfrak{M}_0 \to \mathfrak{M}^*,$$

and

$$h_1 = (g_\nu \restriction \mathfrak{M}_{\nu \restriction (\alpha+1)}) f^1_{\eta \restriction \alpha} \colon \mathfrak{M}_1 \to \mathfrak{M}^*,$$

a contradiction to our choice of $\mathfrak{M}, \mathfrak{M}_0, \mathfrak{M}_1$. $\square$

This proof describes the basic structure of all the further proofs. We have, till now, avoided two problems: How to get maximally many models in $\lambda^+ = \omega_1$, rather than just no universal models, and how to replace ω by general cardinals λ. Historically, Shelah solved these two problems one after the other, and the proof of the general theorem evolved while various versions of Shelah [1983b, c, 198?c] were written. For instance, the following theorem can be proven with just slightly more combinatorial effort:

2.6.5 Theorem. *Assume $2^\omega < 2^{\omega_1}$ and let K be an abstract class with Löwenheim number $l(K) = \omega$ such that:*

(i) $I(K, \omega) = 1$;
(ii) $I(K, \omega_1) \neq \emptyset$; *and*
(iii) K_ω *does not have the amalgamation property.*

Then $I(K, \omega_1) > 2^\omega$.

The best possible results for $\lambda = \omega$ was first proved using the additional hypothesis that K be ω-presentable. However, in the following section we present the general case with a complete proof, taken from Shelah [1983b, c].

2.7. *Proof of the Non-structure Theorem: The General Case*

In the general case we have to analyze closer, how the λ-superlimit fails to be an amalgamation basis. We distinguish two cases:

2.7.1 Definitions (Failures of Amalgamation). Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ and $\mathfrak{N} \in K$ be a λ-superlimit, which is not an amalgamation basis for K.

(i) We say that $\mathfrak{M}$, $\mathfrak{M}_0$, $\mathfrak{M}_1$, all isomorphic to $\mathfrak{N}$, form a *maximal counterexample*, if $\mathfrak{M} <_K \mathfrak{M}_i$ cannot be amalgamated in K, but for every $\mathfrak{A}_{i,k} \in K_\lambda$ $(i, k = 0, 1)$ such that $\mathfrak{M}_i <_K \mathfrak{A}_{i,k}$ there is an amalgamating structure $\mathfrak{B}_i \in K$ for $\mathfrak{M} <_K \mathfrak{A}_{i,k}$ $(k = 0, 1)$.

(ii) We say that $\mathfrak{N}$, $\mathfrak{M}$, $\mathfrak{M}_0$, $\mathfrak{M}_1$, form an *extendible counterexample*, if they are all isomorphic to $\mathfrak{N}$, $\mathfrak{N} <_K \mathfrak{M}$ and for every $\mathfrak{B} \in K_\lambda$ with $\mathfrak{N} <_K \mathfrak{B}$ there are K-embeddings $f_i^B\colon \mathfrak{B} \to \mathfrak{M}_i$ such that $\mathfrak{N} <_{K, f_i^B} \mathfrak{M}_i$ has no amalgamating structure.

2.7.2 Lemma. *Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ and $\mathfrak{N} \in K$ be a λ-superlimit, which is not an amalgamation basis for K. Then either:*
Case 1. *There is a maximal counterexample; or*
Case 2. *There is an extendible counterexample.*

Proof. Let $\overline{\mathfrak{M}}, \overline{\mathfrak{M}}_i$ $(i = 0, 1)$ be a counterexample which is not maximal. So without loss of generality for every $\mathfrak{K}$-extension $\mathfrak{B} \in \mathfrak{K}_\lambda$ of $\overline{\mathfrak{M}}_0$ there are $\mathfrak{K}$-extensions $\mathfrak{B}_i \in \mathfrak{K}_\lambda$ $(i = 0, 1)$ of $\mathfrak{B}$ such that $\overline{\mathfrak{M}}$, $\mathfrak{B}_i$ have no amalgamating structure. Put $\mathfrak{N} = \overline{\mathfrak{M}}$ and $\mathfrak{M} = \overline{\mathfrak{M}}_0$. Clearly, using property (c) of the definition of superlimits (Definition 2.2.1), for every $\mathfrak{K}$-extension $\mathfrak{B} \in \mathfrak{K}_\lambda$ of $\overline{\mathfrak{M}}_0$ there are $\mathfrak{M}_i$ $(i = 0, 1)$ isomorphic to $\mathfrak{N}$ and embeddings f_i^B such that $\mathfrak{N} <_{K, f_i^B} \mathfrak{M}_I$ has no amalgamating structure. So $\mathfrak{N}$, $\mathfrak{M}$, $\mathfrak{M}_i$ is our extendible counterexample. □

This lemma is the key to the proof of the non-structure theorem for abstract classes. For the sake of readability we state it once more, in a sharpened form:

2.7.3 Theorem (Shelah's Non-structure Theorem for Abstract Classes). *Assume $2^\lambda < 2^{\lambda^+}$. Let K be an abstract class with Löwenheim number $l(K) \leq \lambda$ and $\mathfrak{N} \in K$ be a λ-superlimit, which is not an amalgamation basis for K. Then $I(K, \lambda^+) = 2^{\lambda^+}$. (In fact there are 2^{λ^+} many structures in K_{λ^+} such that for no two of them is there a K-embedding from one into the other.)*

Proof. The proof uses Lemma 2.7.2 and therefore treats the two cases separately. In each case the proof proceeds along the pattern of the proof of Theorem 2.6.1: Construction of a system of models is of cardinality λ, each of them isomorphic to the superlimit; construction of models is of cardinality λ^+ and the verification that there are many non-isomorphic models.

2.7.4 Case 1: Construction of Models of Cardinality λ. We define by induction on $\alpha < \lambda^+$ models $\mathfrak{M}_\eta$ indexed by (0, 1)-sequences $\eta \in {}^\alpha 2$ such that:

(1) $\mathfrak{M}_\eta$ is isomorphic to the λ-superlimit $\mathfrak{M}$ the universe of $\mathfrak{M}_\eta$ is the set $M_\eta = \lambda(1 + l(\eta))$.
(2) If $\eta < \nu$ then $\mathfrak{M}_\eta <_K \mathfrak{M}_\nu$.
(3) If $\delta \in \lambda^+$ is a limit ordinal and η is a sequence of length δ then

$$\mathfrak{M}_\eta = \bigcup_{\alpha<\delta} \mathfrak{M}_{\eta \restriction \alpha}.$$

The construction is the same as in subsection 2.6.2, using the maximal counterexample and the properties of the λ-superlimit.

2.7.5 Case I: Construction of Models of Cardinality λ^+. For every (0, 1)-sequence $\eta \in {}^{\lambda^+} 2$ we put $\mathfrak{M}_\eta = \bigcup_{\alpha<\lambda^+} \mathfrak{M}_{\eta \restriction \alpha}$.

2.7.6 Case I: Counting the Models of Cardinality λ^+. Let $\delta < \lambda^+$ be such that $\lambda\delta = \delta$, $\eta, \nu \in {}^\delta 2$ $\mathfrak{M}_\eta, \mathfrak{M}_\nu$ models of cardinality λ as constructed above, and $h: \mathfrak{M}_\eta \to \mathfrak{M}_\nu$, a $\mathfrak{K}$ embedding. We now define a function $F(\eta, \nu, h)$ such that $F(\eta, \nu, h) = 1$ iff $\mathfrak{M}_\eta <_K \mathfrak{M}_{\eta\langle 0\rangle}$ and $\mathfrak{M}_\eta <_{K,h} \mathfrak{M}_{\eta\langle 0\rangle}$ have an amalgamating structure, and $F(\eta, \nu, h) = 0$, otherwise. Note that, by our assumptions on δ the universe of $\mathfrak{M}_\eta$ is the set $M_\eta = \delta$. Use now $2^\lambda < 2^{\lambda^+}$ and Proposition 2.5.10 to conclude that λ^+ is not $(\lambda^+, 2)$-small. Then apply Ulam's theorem (2.5.1) and Proposition 2.5.8(ii) to partition λ^+ into a family $\{S_\alpha : \alpha < \lambda^+\}$ of disjoint non-$(\lambda^+, 2)$-small subsets. Now apply $\Phi^2_{\lambda^+}$ to find a family $\{\rho_\alpha \in {}^{\lambda^+} 2 : \alpha < \lambda^+\}$ such that for each $\alpha < \lambda^+$, $\eta, \nu \in \lambda^+$ and $h: \lambda^+ \to \lambda^+$ the set $\{\delta < \lambda^+ : F(\eta \restriction \delta, \nu \restriction \delta, h \restriction \delta) = \rho_\alpha(\delta)\}$ is stationary in λ^+.

For each $I \subset \lambda^+$ we define a (0, 1)-sequence $\eta_I \in {}^{\lambda^+} 2$ such that $\eta_I(i) = \rho_\alpha(i)$ if $i \in \bigcup_{\alpha \in I} S_\alpha$ and $\eta_I(i) = 0$, otherwise. This is well defined since the S_α's form a partition of λ^+.

Our next goal is:

2.7.7 Lemma. *Given $I, J \subset \lambda^+$, $I - J \neq \emptyset$, then there is no K-embedding*

$$h: \mathfrak{M}_{\eta_I} \to \mathfrak{M}_{\eta_J}.$$

Proof of Lemma. Assume, for contradiction, that $h: \mathfrak{M}_\eta \to \mathfrak{M}_\eta$ is a K-embedding, but there is $\gamma \in I - J$. Clearly the set

$$C = \{\delta < \lambda^+ : h \restriction \delta \text{ is a function into } \delta \text{ and } \lambda\delta = \delta\}$$

is closed and unbounded. Look at S_γ. We use C and F, ρ defined above, to define

$$S'_\gamma = \{\delta \in S_\gamma : F(\eta_I \restriction \delta, \eta_J \restriction \delta, h \restriction \delta) = \rho_\gamma(\delta)\} \cap C.$$

By the choice of ρ above we conclude that $S'_\gamma \neq \emptyset$. Now we choose $\delta \in S'_\gamma$ and put $\eta = \eta_I \restriction \delta$ and $\nu = \eta_J \restriction \delta$. From the definition of η and the fact that $\{S_\alpha : \alpha < \lambda^+\}$ forms a partition, it is clear that $\eta_J(\delta) = 0$.

We now proceed to show that both possibilities, $\eta_I(\delta) = 0$ and $\eta_I(\delta) = 1$, lead to a contradiction.

Case 1: $\eta_I(\delta) = 0$.

Then $\rho_\gamma(\delta) = 0$ and, since $\delta \in S'_\gamma$, we have $F(\eta, \nu, h \upharpoonright \delta) = 0$. But by the choice of h and δ we know that $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 0\rangle}$ and $\mathfrak{M}_\eta <_{h\upharpoonright\delta} \mathfrak{M}_{\nu^\frown\langle 0\rangle}$ have an amalgamating structure, contradicting the definition of the function F.

Case 2: $\eta_I(\delta) = 1$.

Then both $\rho_\gamma(\delta) = 1$ and $F(\eta, v, h \upharpoonright \delta) = 1$, and, by the definition of F, $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 0\rangle}$ and $\mathfrak{M}_\eta <_{h\upharpoonright\delta} \mathfrak{M}_{\nu^\frown\langle 0\rangle}$ have an amalgamating structure. On the other hand we have

Fact 1: $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 1\rangle}$ and $\mathfrak{M}_\eta <_{h\upharpoonright\delta} \mathfrak{M}_{\nu^\frown\langle 0\rangle}$ have an amalgamating structure inside $\mathfrak{M}_\eta$.

But $h \upharpoonright \delta: \mathfrak{M}_\eta \to \mathfrak{M}_\nu$ is a K-embedding, by the choice of δ.

We now construct two models $\mathfrak{N}_1$, $\mathfrak{N}_2$ of cardinality λ such that:

(i) $\mathfrak{M}_{\nu^\frown\langle 0\rangle} < \mathfrak{N}_1$ and $\mathfrak{M}_{\eta^\frown\langle 0\rangle}$ is embeddable into $\mathfrak{N}_1$ by some h_0 extending $h \upharpoonright \delta$.
(ii) $\mathfrak{N}_2 \overset{\text{def}}{=} \mathfrak{M}_{\eta_J\upharpoonright\gamma}$ for some γ with $\delta + 1 < \gamma < \lambda^+$ and $\mathfrak{M}_{\eta^\frown\langle 1\rangle}$ is embeddable into $\mathfrak{N}_2$ by some mapping h_1 extending $h \upharpoonright \delta$.

To get (i) is trivial. To get (ii) we use Fact 1.

Fact 2. $\mathfrak{M}_{\nu^\frown\langle 0\rangle} < \mathfrak{N}_1$, and $\mathfrak{M}_\nu < \mathfrak{N}_1$ and $\mathfrak{M}_\nu < \mathfrak{N}_2$ have an amalgamating structure.

This follows from (i) and (ii) and the fact that our construction is based on a maximal counterexample.

But we have

Fact 3. $\mathfrak{M}_{\eta^\frown\langle 0\rangle} <_{h_0} \mathfrak{N}_1$, $\mathfrak{M}_{\eta^\frown\langle 1\rangle} <_{h_1} \mathfrak{N}_1$ and $h_0 \upharpoonright \delta = h_1 \upharpoonright \delta = h \upharpoonright \delta$.

Furthermore, since our construction is based on counterexamples to amalgamation, we have

Fact 4. $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 0\rangle}$ and $\mathfrak{M}_\eta < \mathfrak{M}_{\eta^\frown\langle 1\rangle}$ have no amalgamating structure.

But Facts 2 and 3 contradict Fact 4, which concludes the proof of the lemma.

2.7.8 Case 2: Construction of Models of Cardinality λ. We define by induction on $\alpha < \lambda^+$ models $\mathfrak{M}_\eta$ indexed by (0, 1)-sequences $\eta \in {}^\alpha 2$ such that:

(1) $\mathfrak{M}_\eta$ is isomorphic to the λ-superlimit $\mathfrak{M}$ and for the empty sequence $\langle\ \rangle$ we put $\mathfrak{M}_{\langle\ \rangle} = \mathfrak{M}$.
(2) If $\eta \subset \nu$ then $\mathfrak{M}_\eta <_K \mathfrak{M}_\nu$.
(3) If $\delta \in \omega_1$ is a limit ordinal and η is a sequence of length δ then

$$\mathfrak{M}_\eta = \bigcup_{\alpha<\delta} \mathfrak{M}_{\eta\upharpoonright\alpha}.$$

(4) For each η the structures $\mathfrak{N}$, $\mathfrak{M}_{\eta^\frown\langle 0\rangle}$, $\mathfrak{M}_{\eta^\frown\langle 1\rangle}$ have no amalgamating structure.

The definition of Case 2 is ready tailored for the construction of the $\mathfrak{M}_\eta$'s. The construction of models of cardinality λ^+ is the same as in Case 1.

2.7.9 Case 2: Counting the Models of Cardinality λ^+. If $\eta, \nu \in {}^{\lambda^+}2$, $\eta \neq \nu$, there is no K-embedding $f\colon \mathfrak{M}_\eta \to \mathfrak{M}$ such that $f \upharpoonright \mathfrak{N} = \mathrm{id}$. For, otherwise, let α be minimal such that $\eta(\alpha) \neq \nu(\alpha)$ and put $\delta = \eta \upharpoonright \alpha$. Then f would allow us to find an amalgamating structure for $\mathfrak{N}$, $\mathfrak{M}_{\delta\langle 0\rangle}$, $\mathfrak{M}_{\delta\langle 1\rangle}$.

So there are 2^{λ^+} many models of cardinality λ^+ which are not isomorphic over $\mathfrak{N}$. Since $\mathfrak{N}$ has cardinality λ there are at most $(\lambda^+)^\lambda = 2^\lambda$ many ways of interpreting $\mathfrak{N}$ in $\mathfrak{M}_\eta$. Since we assumed that $2^\lambda < 2^{\lambda^+}$, we conclude that $I(K, \lambda^+) = 2^{\lambda^+}$. This completes the proof of Theorem 2.7.3, and therefore the nonstructure theorem. The statement in the parentheses now follows with a subtle counting argument which we leave to the reader. $\square$

2.7.10 Remark. If we just want to show that there are at least 2^λ many non-isomorphic models in K_{λ^+} we can use Proposition 2.5.10(ii) instead of 2.5.10(i) and simplify the proof a bit. We change the definition of the function F to be a function of four variables where the new variable ranges over the indexes of a list of $\mu < \mu^+ = 2^\lambda$ many non-isomorphic models of K. Instead of using first Ulam's theorem to partition λ^+ we can now apply Unif$(\lambda^+, \mu, 2, 2)$ directly.

3. ω-Presentable Classes

3.1. Classification Theory for ω-Presentable Classes

In this section we shall study some examples which illustrate that some of the classification theory of first-order model theory can be carried over to abstract classes, provided they are ω-presentable. For $\mathscr{L}_{\omega_1\omega}$ this was initiated by G. Cudnovskii, J. Keisler, and S. Shelah, cf. Keisler [1971] and was carried out to considerable extent in Shelah [1984a, b, c]. It seems that, with enough effort and ingenuity, many results should be provable, in some form or another, also for λ-presentable classes. This is still in the making, but we think that this direction of future research is among the most challenging tasks of "higher model theory."

The first two theorems along these lines are direct descendants of two theorems in Shelah [1975c]. The proofs, which we are going to sketch, appear, in this streamlined form, here for the first time in print.

3.1.1 Theorem (Shelah's Reduction Theorem). *Let K with $<_K$ be an abstract ω-presentable class over a vocabulary τ such that:*

$$I(K, \omega_1) < 2^{\omega_1}.$$

Then there is a ω-presentable abstract class K with $<_{K'}$ over a vocabulary τ', $\tau \subset \tau'$ such that

(i) *if* $\mathfrak{A} \in K'$ *then* $\mathfrak{A} \upharpoonright \tau \in K$;
(ii) *if* $\mathfrak{A}, \mathfrak{B} \in K'$ *and* $\mathfrak{A} <_{K'} \mathfrak{B}$ *then* $\mathfrak{A} \upharpoonright \tau <_K \mathfrak{B} \upharpoonright \tau$;
(iii) *if* $\mathfrak{A}, \mathfrak{B} \in K'$ *and* $\mathfrak{A} <_{K'} \mathfrak{B}$ *then* $\mathfrak{A} <_{\infty\omega} \mathfrak{B}$; *and still*
(iv) $I(K, \omega_1) \neq \varnothing$ *iff* $(K', \omega_1) \neq \varnothing$.

In particular, $I(K', \omega) = 1$ *by* (iii).

Recall that $\mathfrak{A} <_{\infty\omega} \mathfrak{B}$ here means that for every finite set of constant symbols A_0 the expansion $\langle \mathfrak{A}, A_0 \rangle \equiv \langle \mathfrak{B}, A_0 \rangle$ in the logic $\mathscr{L}_{\infty\omega}$.

3.1.2 Remarks. (i) The reduction theorem allows us to construct Scott sentences of uncountable structures. We shall return to this in Section 3.4.
(ii) In Shelah [1975c] the reduction theorem is proved by constructing what is called there "nice" sentences.
(iii) In the reduction theorem above, we can replace the assumption

$$I(K, \omega_1) < 2^{\omega_1}$$

by the assumption that K has arbitrary large models and Löwenheim number ω, and get the same result.

3.1.3 Theorem (Shelah's Abstract ω_1-Categoricity Theorem, 1977). *Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $I(K, \omega_1) = 1$.

Then $I(K, \omega_2) \neq \varnothing$.

3.1.4 Corollary (Shelah). *Let φ be a sentence of the logic $\mathscr{L}_{\omega_1\omega}(Q_1)$ which has exactly one model of cardinality ω_1. Then φ has a model of cardinality ω_2.*

3.1.5 Historical Remark. Corollary 3.1.4 shows that there are no theories in $\mathscr{L} = \mathscr{L}_{\omega\omega}(Q_1)$ which have exactly one uncountable model. This had been asked by J. T. Baldwin (Friedman [1975c]) and actually was the origin of Theorem 3.1.3. In Shelah [1975c, Corollary 3.1.4] was proved with the additional set-theoretic hypothesis $\diamond$, and in Shelah [1983b, c] under the hypothesis $2^\omega < 2^{\omega_1}$. Without any set-theoretic hypothesis Corollary 3.1.4 was proved by S. Shelah in 1976 (my personal notes).

3.1.6 Theorem (Shelah 1977). *Assume that $2^\omega < 2^{\omega_1} < 2^{\omega_2}$. Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $1 \leq I(K, \omega_1) < 2^{\omega_1}$.

Then $I(K, \omega_2) \neq \varnothing$.

Assumption (i) in the two theorems above is not essential. Though it does not follow from (ii), we can always replace K satisfying (ii) by K which also satisfies (i) using the reduction theorem.

The main tool in the proof of Theorem 3.1.6 is the use of a ω_1-superlimit. The concept of superlimit models was introduced with generalizations in mind. The following theorem guarantees its existence. If our only purpose was to prove Theorem 3.1.6 we could also avoid the construction of superlimits. S. Fuchino [1983] has presented such a direct proof.

3.1.7 Theorem (Existence of Superlimits). *Assume that $2^{\omega} < 2^{\omega_1} < 2^{\omega_2}$. Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $I(K, \omega_1) < 2^{\omega_1}$;
(iii) $I(K, \omega_2) < 2^{\omega_2}$.

Then there is a ω_1-superlimit model $\mathfrak{M}$ in K_{ω_1} which is homogeneous and universal.

Clearly, Theorem 3.1.6 follows from Theorem 3.1.7 together with Proposition 2.2.3. Actually we shall only need that there is a weak limit in ω_1. We shall give a narrative account of the proof of Theorem 3.1.7 in Section 3.5. The existence of a weak limit in ω_1 will be proved as Claim 3.5.2.

3.1.8 Corollary. *Assume K is as in the theorem above. Then the ω_1-superlimit model $\mathfrak{M}$ is an amalgamation basis for K_{ω_1}.*

Proof. Use the non-structure theorem (2.3.1) together with Theorem 3.1.7. □

This corollary is somehow not satisfactory. What we really would like to obtain is the following conjecture:

3.1.9 Conjecture (ω_1-Amalgamation Conjecture). Assume that $2^{\omega} < 2^{\omega_1} < 2^{\omega_2}$. Let K with $<_K$ be an abstract ω-presentable class such that:

(i) $I(K, \omega) = 1$; and
(ii) $I(K, \omega_1) < 2^{\omega_1}$;
(iii) $I(K, \omega_2) < 2^{\omega_2}$.

Then K_{ω_1} has the amalgamation property.

Note that this conjecture follows from Conjecture 2.3.2.

In the remainder of this section we shall prove Theorems 3.1.1 and 3.1.3 completely, and sketch the proof of Theorem 3.1.7, from which Theorem 3.1.6 follows.

We conclude this section with the statement of the main theorem of the classification theory for $\mathscr{L}_{\omega_1\omega}$ (Shelah [1984a, b]) and a conjecture on how this should generalize for ω-presentable classes.

3.1.10 Theorem (Shelah's Classification Theorem for $\mathscr{L}_{\omega_1\omega}$). *Assume $2^{\omega_n} = \omega_{n+1}$ for every $n < \omega$. Let $K = \mathrm{Mod}(\psi)$ for some sentence $\psi \in \mathscr{L}_{\omega_1\omega}$. If K has an uncountable model then at least one of the following is true. Either:*

(i) *for some $n > 0$ $I(K, \omega_1) = 2^{\omega_n}$; or*
(ii) *K has models in every infinity cardinality, and if it is categorical in some $\lambda > \omega_1$ then it is categorical in every $\mu \geq \omega_1$.*

3.1.11 Remark. Theorem 3.1.10 is not true, when we replace K by some $\mathrm{PC}_{\mathscr{L}_{\omega_1\omega}}$-class. To see this consider the class K of structures (with equality only) of cardinality at most ω_1. Clearly K is categorical in every infinite power and has no models bigger than ω_1. Using the fact that the natural numbers are characterizable in $\mathscr{L}_{\omega_1\omega}$, one easily sees that the class of ω_1-like orderings in $\mathrm{PC}_{\mathscr{L}_{\omega_1\omega}}$. Therefore also $K \in \mathrm{PC}_{\mathscr{L}_{\omega_1\omega}}$. For a discussion of categoricity in $\mathscr{L}_{\omega_1\omega}$ see Keisler [1971, p. 91ff.].

For generalizations of Theorem 3.1.10 we shall finally discuss several conjectures:

3.1.12 Conjecture (Shelah). If an abstract ω-presentable class K has one uncountable model then it has at least 2^{ω_1} many non-isomorphic uncountable models.

3.1.13 Comments. Possibly one has to use some set-theoretic hypothesis such as in the classification theorem for $\mathscr{L}_{\omega_1\omega}$, or $2^{\omega_n} < 2^{\omega_{n+1}}$ for every $n \in \omega$ to prove this conjecture. Theorem 3.1.7 was proved in Shelah [198?c] as a basis for a proof of Conjecture 3.1.12. A special case of this conjecture consists in showing, for example, that if K has exactly one model in ω_2 then it has a model in ω_3. As we shall see in the next section, however, there is one application of the non-characterizability of well-orderings (Theorem 3.2.1), which cannot be adapted in an obvious way: We cannot prove that the superlimit $\mathfrak{M}$, whose existence is stated in Theorem 3.1.7, can be embedded into itself such that it forms a dense pair as defined in the next section (Definition 3.2.4). Only a deeper analysis of the types realized in models in K reveals that such dense pairs do not exist. What one really does is more in the spirit of stability theory, than in the original spirit of abstract model theory. But it seems that this is where the future lies: To use the concepts and methods of stability theory in the framework of abstract classes. The following remarks show, however, that this is more complicated than one might be ready to believe at first glance.

Next we look at the logic $\mathscr{L}_{\omega\omega}(\mathrm{pos})$, which was introduced in Chapter II, and its infinitary extensions $\mathscr{L}_{\omega_1\omega}(\mathrm{pos})$. These logics were studied in Makowsky–Shelah [1981] and Makowsky [1978a]. $\mathscr{L}_{\omega\omega}(\mathrm{pos})$ is a countably compact extension of $\mathscr{L}_{\omega\omega}(Q_1)$ which is properly contained in $\mathscr{L}_{\omega\omega}(\mathrm{aa})$. The reader may also want to consult Chapter IV.

3.1.14 Conjecture (Classification Theorem for $\mathscr{L}_{\omega_1\omega}(\mathrm{pos})$, Makowsky–Shelah)). Assume $2^{\omega_n} = \omega_{n+1}$ for every $n < \omega$. Let $K = \mathrm{Mod}(\psi)$ for some sentence

$$\psi \in \mathscr{L}_{\omega_1\omega}(\mathrm{pos}).$$

If K has an uncountable model then at least one of the following is true. Either:

(i) for some $n > 0$, $I(K, \omega_n) = 2^{\omega_n}$; or
(ii) K has models in every infinite cardinality, and if it is categorical in some $\lambda > \omega_1$ then it is categorical in every $\mu \geq \omega_1$.

3.1.15 Remarks. (i) The straightforward notions of stability theory (Shelah [1978a]) do not adapt readily to our situation. In fact, it is consistent with ZFC + $2^\omega = \omega_2$ that there is an ω-presentable abstract class which is categorical in ω_1 but is unstable. Also all its models are of cardinality at most 2^ω. Take the $\mathscr{L}_{\omega\omega}(\text{pos})$ sentence which says that $<$ is a dense linear order with no first or last element, that each interval is uncountable, but that there is a dense countable subset. Categoricity in ω_1 follows from Baumgartner [1973], the bound on the cardinality of the models and instability are obvious.

(ii) Conjecture 3.1.14 becomes false for $\mathscr{L}_{\omega\omega}(\text{aa})$: There is a sentence

$$\psi \in \mathscr{L}_{\omega\omega}(\text{aa})$$

such that ψ has, up to isomorphism, exactly one model and this model is of cardinality ω_1. To see this, let ψ be the sentence which says that $<$ is a dense linear order with no first or last element, each initial segment is countable, but the model is not, and aa$s\, \exists x\, \forall y(s(y) \leftrightarrow y < x)$. The only model of ψ is, up to isomorphism, the structure $\langle \eta \times \omega_1, < \rangle$. (See also Remark IV.4.1.2(v).)

(iii) The analogue of Theorem 3.1.6 for $\mathscr{L}_{\omega_1\omega}(\text{pos})$ has been proved in Makowsky–Shelah [198?a]. At the time of completion of this chapter, this paper was still in the process of being checked.

3.2. Extensions With and Without First Elements

Let K with $<_K$ be an abstract ω-presentable class such that: (i) $I(K, \omega) = 1$, and (ii) $I(K, \omega_1) = 1$. We want to show that $I(K, \omega_2) \neq \emptyset$. For this purpose we show first:

3.2.1 Lemma. *Under the above hypotheses the following are equivalent:*

(i) $I(K, \omega_2) \neq \emptyset$.
(ii) *There are* $\mathfrak{A}, \mathfrak{B} \in K_{\omega_1}$ *such that* $\mathfrak{A} < \mathfrak{B}$ *and* $\mathfrak{A} \neq \mathfrak{B}$, *i.e.*, $\mathfrak{A} \in K_{\omega_1}$ *is not maximal.*

Proof. (i) $\to$ (ii) We just apply Axiom 5.

(ii) $\to$ (i) Since here $\mathfrak{A} \cong \mathfrak{B}$ we can construct a K-chain of length ω_2 which gives us the required model. $\square$

3.2.2 Definition. A structure $\mathfrak{A}$ in an abstract class K (K_λ) is *K-maximal* (*K_λ-maximal*), if there is no $\mathfrak{B} \in K$ $(\mathfrak{B} \in K_\lambda)$ such that $\mathfrak{A} < \mathfrak{B}$ and $\mathfrak{A} \neq \mathfrak{B}$.

In this section we write $\mathfrak{A} < \mathfrak{B}$ only for proper extensions, and we shall use $\mathfrak{A} \leq \mathfrak{B}$ if we allow also the identity.

What we really prove to get Theorem 3.1.3 is the following:

3.2.3 Theorem. *Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$;
(ii) $I(K, \omega_1) \neq 0$; *and*
(iii) *every* $\mathfrak{A} \in K_{\omega_1}$ *is* K_{ω_1}*-maximal.*

Then $I(K, \omega_1) = 2^{\omega_1}$.

Clearly, in the above situation, the structures in K_ω are not maximal, since there is an uncountable model.

3.2.4 Definitions. Let K with $<$ be an abstract class, λ a cardinal, and $\mathfrak{A} < \mathfrak{B}$ with $b \in B - A$ and $\mathfrak{A}, \mathfrak{B} \in K_\lambda$.

(i) We say that b is *a first element for* $\mathfrak{A} < \mathfrak{B}$ if for every $\mathfrak{A}_1$, $\mathfrak{B}_1$ such that $\mathfrak{A} < \mathfrak{A}_1 < \mathfrak{B}_1, \mathfrak{B} < \mathfrak{B}_1$ we have that $b \in A_1$. (We assume here for simplicity that the embeddings are the identity. The reader can easily formulate the definition for the more general case.)

Note that, if there is no first element for $\mathfrak{A} < \mathfrak{B}$, we can think of this as an amalgamation property: For every $b \in B - A$ there is a structure $\mathfrak{A}_1 \in K$ and an amalgamating structure $\mathfrak{B}_1$ such that $b \notin A_1$. If there is no first element for $\mathfrak{A} < \mathfrak{B}$, this can happen in a strong form:

(ii) We say that $\mathfrak{A} < \mathfrak{B}$ is *a dense pair* if for every $b \in B - A$ there is a structure $\mathfrak{A}_1$ in K_λ such that $\mathfrak{A} < \mathfrak{A}_1 < \mathfrak{B}$ with $b \in B - A_1$.

The above definitions are our key tools in the proof of Theorem 3.2.3 and therefore of the abstract categoricity theorem.

3.2.5 Example. To illustrate the proof idea let us recall a simple theorem about the number of non-isomorphic dense linear orderings of cardinality ω_1. We take here K_{end} to be the class of all dense linear orderings without extremal elements, and define for $\mathfrak{A}, \mathfrak{B} \in K_{\text{end}}$ the substructure relation $\mathfrak{A} <_{\text{end}} \mathfrak{B}$ as the end-extensions. Clearly $K_{\text{end},\omega}$ has, up to isomorphism, only one element.

3.2.6 Proposition. *There are 2^{ω_1} many non-isomorphic linear dense ω_1 like orderings.*

Proof. Let $I \subset \omega_1$. We define $\mathfrak{A}_I = \bigcup_{\alpha \in \omega_1} \mathfrak{A}_\alpha$ where each $\mathfrak{A}_\alpha$ is isomorphic to a copy of the rationals $\mathfrak{Q} = \langle Q, < \rangle$. Let $\mathfrak{Q}_{\text{first}} = \langle [b, 1), < \rangle$ be a copy of the rationals with a first element b and put $\mathfrak{Q}_1 = \mathfrak{Q} + \mathfrak{Q}_{\text{first}}$ and $\mathfrak{Q}_2 = \mathfrak{Q} + \mathfrak{Q}$. Clearly $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_2$ is a dense pair and b is a first element for $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_1$. Now we put $\mathfrak{A}_0 = \mathfrak{Q}$ and $\mathfrak{A}_\delta = \bigcup_{\alpha < \delta} \mathfrak{A}_\alpha$ for δ a limit ordinal. To get $\mathfrak{A}_{\alpha+1}$ we make $\mathfrak{A}_\alpha <_{\text{end}} \mathfrak{A}_{\alpha+1}$ isomorphic to $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_1$ if $\alpha \in I$ and isomorphic to $\mathfrak{Q} <_{\text{end}} \mathfrak{Q}_2$ if $\alpha \notin I$.

Let $I, J \subset \omega_1$ and F be the c.u.b. filter on ω_1. We claim that $\mathfrak{A}_I \simeq \mathfrak{A}_J$ implies that $I = J \pmod F$. By Ulam's theorem (cf. Theorem 2.5.1 or Lemma XVIII.4.3.9) there are 2^{ω_1} many non-equivalent stationary subsets of ω_1, hence the result. ☐

The next lemmas will allow us to copy this proof for our abstract classes.

3.2.7 Lemma. *Let K with $<$ be an abstract class with Löwenheim number λ and*

(i) $I(K, \lambda) = 1$;
(ii) $I(K, \lambda^+) \neq \varnothing$;
(iii) *every* $\mathfrak{A} \in K_{\lambda^+}$ *is K-maximal.*

Then there are $\mathfrak{A}, \mathfrak{B} \in K_\lambda$ and $b \in B - A$ such that b is a first element for $\mathfrak{A} < \mathfrak{B}$.

In other words, if no pair $\mathfrak{A} < \mathfrak{B}$ of structures from K_λ has a first element, then there is a non-maximal $\mathfrak{A}_1 \in K_{\lambda^+}$.

Proof. Assume for contradiction that $\mathfrak{A}_0 < \mathfrak{B}_0$ are given in K_λ with no $b \in B_0 - A_0$ a first element. Fix $b_0 \in B_0 - A_0$. So there are $\mathfrak{A}_1 < \mathfrak{B}_1$ with $\mathfrak{A}_0 < \mathfrak{A}_1$ and $\mathfrak{B}_0 < \mathfrak{B}_1$ and $b_0 \in B_1 - A_1$.

From this situation we construct K-chains $\mathfrak{A}_\alpha, \mathfrak{B}_\alpha$ $(\alpha < \lambda^+)$ with $b_0 \in B_\alpha - A_\alpha$, using that $I(K, \lambda) = 1$. Now we put $\mathfrak{A} = \bigcup_\alpha \mathfrak{A}_\alpha$ and $\mathfrak{B} = \bigcup_\alpha \mathfrak{B}_\alpha$ and find that $\mathfrak{A}, \mathfrak{B} \in K_{\lambda^+}$, $\mathfrak{A} < \mathfrak{B}$ and $b_0 \in B - A$. □

3.2.8 Lemma. *Let K with $<_K$ be an abstract ω-presentable class such that:*

(i) $I(K, \omega) = 1$; *and*
(ii) $I(K, \omega_1) \neq \varnothing$.

Then there is a dense pair $\mathfrak{A} < \mathfrak{B}$ in K_ω.

The proof of Lemma 3.2.8 consists in an application of the Morley–Lopez-Escobar theorem on the non-expressibility of well-orderings in $\mathscr{L}_{\infty\omega}$ which was first used in Shelah [1975]. We shall return to this in Section 3.3.

Proof of Theorem 3.2.3. We are now in a position to copy the proof of Proposition 3.2.6. We put now $\mathfrak{Q}$ to be the only countable model of K, $\mathfrak{Q}_1$ a countable extension of $\mathfrak{Q}$ with $b \in Q_1$ a first element (Lemma 3.2.6), and $\mathfrak{Q}_2$ a countable extension of $\mathfrak{Q}$ such that $\mathfrak{Q} <_K \mathfrak{Q}_2$ is a dense pair (Lemma 3.2.8). The rest of the argument remains unchanged. □

3.3. Some Model Theory for $L_{\omega_1\omega}$

In Section 1.3 Shelah's presentability theorem tells us that every ω presentable class K is actually a PC-class in $\mathscr{L}_{\omega_1\omega}$. Some of the model theory of $\mathscr{L}_{\omega_1\omega}$ has been developed in Chapter VIII, but for the reader's sake we make this section as self-contained as possible. Our aim here is to prove Lemma 3.2.8 and Shelah's reduction theorem (3.1.1). Both theorems use heavily the non-characterizability of the class of well-orderings as a PC-class in $\mathscr{L}_{\omega_1\omega}$, which we state here precisely (cf. Section VIII.1.3, Section II.5.2 and Proposition IX.3.2.16)

3.3.1 Theorem (Non-characterizability of Well-Orderings). *Let $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$ and let $U, < \in \tau$ be a unary and a binary relation symbol of τ. Suppose that for each $\alpha \in \omega_1$, φ has a model $\mathfrak{A} = \langle A, U, <, \ldots\rangle$ such that $<$ linearly orders U and $\langle \alpha, <\rangle \subset \langle U, <\rangle$. Then:*

(i) *φ has a countable model $\mathfrak{B} = \langle B, V, <, \ldots \rangle$ such that $<$ linearly orders V and $\langle V, < \rangle$ contains a copy of the rationals $\langle Q, < \rangle$;*

(ii) φ has an uncountable model $\mathfrak{B} = \langle B, V, <, \ldots \rangle$ *such that $<$ linearly orders V and $\langle V, < \rangle$ contains a copy of the rationals $\langle Q, < \rangle$.*

(i) is due to Morley [1965] and Lopez-Escobar [1966]. A proof may be found in Keisler [1971a]. (ii) can be proved by combining (i) with the construction and characterization of the existence of suitable end-extensions, as described in Keisler [1971a]. But it was Shelah who first observed that this theorem can be used in many situations as a substitute for compactness. This is the main theme of this section. We shall use Theorem 3.3.1(ii) to construct, in certain situations, Scott sentences of uncountable models, and also, if such Scott sentences exist, to construct dense pairs of countable models. Let us recall some definitions:

3.3.2 Definition (Scott Sentences). (i) Let $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$. We say that φ is a *Scott sentence*, if all models of φ are $\mathscr{L}_{\infty\omega}$-equivalent.

(ii) Let $\varphi \in \mathscr{L}_{\omega_1\omega}(\tau')$ and $\tau \subset \tau'$. We say that φ is a *weak Scott sentence (for τ)*, if all τ-reducts of models of φ are $\mathscr{L}_{\infty\omega}$-equivalent.

(iii) If $\mathfrak{A}$ is a τ-structure then we say that $\mathfrak{A}$ *has a Scott sentence*, if there is a Scott sentence $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$ with $\mathfrak{A} \models \varphi$. Similarly for weak Scott sentences.

(iv) If $\mathfrak{A}$ is a τ-structure which has a (weak) Scott sentence φ, we denote by $\sigma(\mathfrak{A})$ a formula logically equivalent to φ.

In Theorem VIII.4.1.6 these definitions are justified.

3.3.3 Lemma. *If a τ-structure $\mathfrak{A}$ has a weak Scott sentence σ_w over a vocabulary τ' then it has also a Scott sentence σ.*

Proof. Let $\mathfrak{B}'$ be a countable model of σ_w and $\mathfrak{B} = \mathfrak{B}' \restriction \tau$. Put $\sigma = \sigma(\mathfrak{B})$. By the completeness theorem for $\mathscr{L}_{\omega_1\omega}$ $\sigma_\omega \models \sigma$, so $\mathfrak{A} \models \sigma$. ☐

3.3.4 Definition. (i) (Fragments of $\mathscr{L}_{\omega_1\omega}$). A *countable fragment $\mathscr{L}$ of $\mathscr{L}_{\omega_1\omega}$* is a countable subset of $\mathscr{L}_{\omega_1\omega}$ closed under taking subformulas, name changing, applying the finitary connectives and quantification.

(ii) ($\mathscr{L}$-embeddings). Let $\mathscr{L}$ be a fragment of $\mathscr{L}_{\omega_1\omega}$ and $\mathfrak{A}$, $\mathfrak{B}$ two τ-structures. We say that $\mathfrak{A}$ *is an $\mathscr{L}[\tau]$-substructure of* $\mathfrak{B}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for every finite subset $A_0 \subset A$ the expansions by constants for elements of A_0, $\langle \mathfrak{A}, A_0 \rangle$ and $\langle \mathfrak{B}, A_0 \rangle$, are $\mathscr{L}$-equivalent.

(iii) (Karp Substructures). Let $\mathfrak{A}$, $\mathfrak{B}$ two τ-structures. We say that $\mathfrak{A}$ *is a Karp-substructure of* $\mathfrak{B}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and for every finite subset $A_0 \subset A$ the expansions by constants for elements of A_0, $\langle \mathfrak{A}, A_0 \rangle$ and $\langle \mathfrak{B}, A_0 \rangle$, are $\mathscr{L}_{\infty\omega}$-equivalent.

(iv) (ω-Presentable Substructure Relation). Let $\propto$ be a binary relation between τ-structures such that $\mathfrak{A} \propto \mathfrak{B}$ implies that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. We say that $\propto$ *is an ω-presentable substructure relation, if*:

(a) for every $\mathfrak{A}$ we have $\mathfrak{A} \propto \mathfrak{A}$;
(b) $\propto$ satisfies the transitivity axiom;
(c) $\propto$ satisfies the chain axiom; and
(d) the class of τ_{sr}-structures $[\mathfrak{A}; \mathfrak{B}]$ such that $\mathfrak{A} \propto \mathfrak{B}$ is PCOT(ω, ω).

Obviously we define τ_{sr} such that the universe of $\mathfrak{A}$ is the interpretation of a distinguished unary predicate of τ_{sr}. Note that (d) ensures that we have Lowenheim number ω.

3.3.5 Lemma. (i) *Let $\mathscr{L}$ be a countable fragment of $\mathscr{L}_{\omega_1\omega}$. Then the notion of a $\mathscr{L}$-substructure gives rise to an ω-presentable substructure relation.*
(ii) *The notion of a* Karp-*substructure is also an ω-presentable substructure relation.*

Proof. Both statements are easy coding exercises. For (i) we use the truth adequacy of $\mathscr{L}_{\omega_1\omega}$ for countable fragments. Details are discussed in Section XVII.1. For (ii) we use the characterization of $\mathscr{L}_{\infty\omega}$-equivalence in terms of partial isomorphisms, as described in Section II.4 and Chapter VIII. The questions which interest us now, are whether an uncountable structure $\mathfrak{A}$ has a (weak) Scott sentence, and under what conditions a Scott sentence has uncountable models? The following is a variation on a special case of Theorem XVIII.7.3.1, which is due to Gregory [1973].

3.3.6 Theorem. *Let φ be a weak Scott sentence. Then the following are equivalent:*

(i) *φ has an uncountable model;*
(ii) *for every countable fragment $\mathscr{L}$ containing φ there are countable models $\mathfrak{B}$, $\mathfrak{C}$ of φ such that $\mathfrak{B} \precneqq_{\mathscr{L}} \mathfrak{C}$ and $\mathfrak{B} \cong \mathfrak{C}$;*
(iii) *for every ω-presentable substructure relation $\propto$ there are countable models $\mathfrak{B}$, $\mathfrak{C}$ of φ such that $\mathfrak{B} \propto \mathfrak{C}$, $\mathfrak{B}$ is a proper substructure of $\mathfrak{C}$ and $\mathfrak{B} \cong \mathfrak{C}$.*

Proof. (ii) implies (i) trivially (in contrast to the proof of Theorem XVIII.7.3.1), since $\mathfrak{A} \cong \mathfrak{B}$ allows us to construct a chain of length ω_1 whose limit is the desired model.

(iii) implies (ii) by the lemma above.

So assume (i). To prove (iii) we just use the reflexivity of $\propto$ and the Lowenheim–Skolem theorem for $\mathscr{L}_{\omega_1\omega}$ together with the properties of the weak Scott sentence. $\square$

For weak Scott sentences with uncountable models we can already construct dense pairs for any countable fragment $\mathscr{L}$ of $\mathscr{L}_{\omega_1\omega}$.

3.3.7 Theorem. *Let φ be a weak Scott sentence with an uncountable model $\mathfrak{D}$ and $\propto$ an ω-presentable substructure relation. Then there are two countable τ-structures $\mathfrak{B}$, $\mathfrak{C}$ such that $\mathfrak{B} \propto \mathfrak{C}$ is a dense pair for $\propto$.*

Proof. We can write $\mathfrak{D}$ as the union of a $\propto$-chain of length ω_1 $\{\mathfrak{D}_\alpha: \alpha \in \omega_1\}$ with $\mathfrak{D}_\alpha \propto \mathfrak{D}_\beta$ for every $\alpha < \beta < \omega_1$. We can code this situation in a model $\mathfrak{M}$ and describe it by a formula $\vartheta \in \mathscr{L}_{\omega_1\omega}[\tau']$ over some vocabulary τ' extending τ, which satisfies the hypothesis of Theorem 3.3.1. Here we use the ω-presentability of $\propto$. The universes of the models $\mathfrak{D}_\alpha$ are coded by a binary predicate symbol and constants $R(-, c_\alpha)$. The second argument of R ranges over some linearly ordered set $\langle U, <\rangle$, the index set.

Now we apply Theorem 3.3.1(ii) and get a model $\mathfrak{N}$ such that a copy of the rationals $\langle Q, <\rangle$ can be embedded into the index set. Let $\{d_n: n \in \omega\}$ be a decreasing sequence in $\mathfrak{N}$ and d be a lower bound for it. Put now $\mathfrak{C}$ to be the model defined in $\mathfrak{N}$ by $R(-, d_0)$ and $\bigcup_{a<d_n} R(-, a) = \bigcap_{n\in\omega} R(-, d_n) = \mathfrak{B}$. This is not empty, since the structure defined by $R(-, d)$ is contained in it. Clearly, ϑ can be chosen such that $\mathfrak{B} \propto \mathfrak{C}$ is a dense pair. ▯

3.3.8 Proof of lemma 3.2.8. Our first step in the proof is the construction of a Scott sentence. So let K be an ω-presentable class with $K_{\omega_1} \neq \emptyset$ and $I(K, \omega_1) < 2^{\omega_1}$. Then there is a $\mathfrak{B} \in K_{\omega_1}$ which has a weak Scott sentence σ. To see this, we apply Shelah's reduction theorem (3.1.1) to K. So let K' be as in Theorem 3.1.1 and let $\mathfrak{B} \in K'_{\omega_1}$. Since K' is ω-presentable, there is a countable $\mathfrak{A} \in K'$ with $\mathfrak{A} <_{K'} \mathfrak{B}$ and therefore $\mathfrak{A} <_{\mathscr{L}_{\infty\omega}} \mathfrak{B}$. Let $\sigma = \sigma(\mathfrak{A})$. Clearly, $\mathfrak{B} \models \sigma$. Now the lemma follows from Theorem 3.3.7 ▯

3.4. *Constructing Scott Sentences for Uncountable Models*

Our second application of Theorem 3.3.1(ii) is the proof of the reduction theorem (3.1.1). First we need a lemma on the minimal number of types realized in models in K_{ω_1}. Let us recall the definition of types.

3.4.1 Definition. (i) ($\mathscr{L}[\tau]$-types). Let $\mathfrak{M}$ be a τ-structure, $A \subset M$ a subset of the universe of $\mathfrak{M}$, $\bar{a} \in M^m$ and let $\varphi(\bar{x})$ range over $\mathscr{L}[\tau]$-formulas with all the free variables among $\bar{x} = (x_0, x_1, \ldots)$. For $b \in A$ let $\mathbf{b}$ be a constant symbol whose interpretation in $\mathfrak{M}$ is b. We define

$$\operatorname{tp}(\bar{a}, A, \mathscr{L}, \mathfrak{M}) = \{\varphi(\bar{x}, \mathbf{b}): \varphi \in \mathscr{L}(\tau), \mathfrak{M} \models \varphi(\bar{x}, \bar{y})[\bar{a}, \bar{b}], \bigwedge \bar{b} \in A^n\}$$

be the *m-type of $\bar{a}$ in $\mathfrak{M}$ over A*.

(ii) If $t = \operatorname{tp}(\bar{a}, A, \mathscr{L}, \mathfrak{M})$ is a countable type we define by ϑ_t the conjunction of all the formulas of t. Note that ϑ_t is not necessarily a formula of $\mathscr{L}$.

3.4.2 Lemma. *Let $\tau \subset \tau'$, $\psi \in K_{\omega_1\omega}[\tau']$ and $\mathscr{L}$ a countable fragment of $\mathscr{L}_{\omega_1\omega}$. Put $K = \operatorname{Mod}(\psi) \restriction \tau$. Then:*

(i) (Keisler [1970, Theorem 5.10]). *If in some uncountable model $\mathfrak{M}$ of ψ uncountably many $\mathscr{L}[\tau]$-types are realized, then $I(K, \omega_1) = 2^{\omega_1}$.*

(ii) (Shelah). *Here we assume* $2^{\omega_1} > 2^{\omega}$. *If in some uncountable model* $\mathfrak{M}$ *of* ψ *there is a countable subset* $A \subset M$ *such that in* $\mathfrak{M}$ *uncountably many* $\mathscr{L}[\tau]$-*types are realized over* A, *then* $I(K, \omega_1) = 2^{\omega_1}$.

Proof. To prove (ii) from (i) we observe that there are at most $(\omega_1)^{\omega} = 2^{\omega}$ many ways of interpreting countably many constants in a model of cardinality ω_1. More details may be found in Shelah [1978a, Chapter 8, Lemma 1.3]. ☐

The next theorem extends this to $\mathscr{L}_{\omega_1\omega}$ proper.

3.4.3 Theorem (Shelah). *Let* $\tau \subset \tau'$ *be two vocabularies,* $\psi \in \mathscr{L}_{\omega_1\omega}[\tau']$ *a formula, and* $\mathfrak{M}$ *be a* τ'-*structure of cardinality* ω_1 *such that* $\mathfrak{M} \models \psi$.

(i) *If for every countable fragment* $\mathscr{L}$ *only countably many* $\mathscr{L}[\tau]$-*types are realized in* $\mathfrak{M}$, *then* ψ *has a model* $\mathfrak{N}$ *of cardinality* ω_1 *in which only countably many* $\mathscr{L}_{\omega_1\omega}[\tau]$-*types are realized.*
(ii) *If for every countable fragment* $\mathscr{L}$ *and for every countable subset* $A \subset M$ *only countably many* $\mathscr{L}[\tau]$-*types are realized in* $\mathfrak{M}$ *over* A, *then* ψ *has a model* $\mathfrak{N}$ *of cardinality* ω_1 *in which over every countable* $A \subset N$ *only countably many* $\mathscr{L}_{\omega_1\omega}[\tau]$-*types are realized over* A.
(iii) *If* ψ *has a model* $\mathfrak{N}$ *of cardinality* ω_1 *in which over every countable* $A \subset N$ *only countably many* $\mathscr{L}_{\omega_1\omega}[\tau]$-*types are realized over* A, *then* $\mathfrak{N} \upharpoonright \tau$ *has a Scott sentence* $\sigma = \sigma(\mathfrak{N} \upharpoonright \tau)$.

Proof. (i) For every $\alpha < \omega_1$ we define a countable fragment $\mathscr{L}_\alpha$ of $\mathscr{L}_{\omega_1\omega}$. $\mathscr{L}_0 = \mathscr{L}_{\omega\omega}$ and $\mathscr{L}_\delta = \bigcup_{\beta<\delta} \mathscr{L}_\beta$ for δ a limit ordinal. $\mathscr{L}_{\alpha+1}$ is the minimal fragment of $\mathscr{L}_{\omega_1\omega}$ containing $\mathscr{L}_\alpha$ and for every $\bar{a} \in M^m$ the formula $\vartheta_{t(\bar{a})}$ where $t(\bar{a}) = \mathrm{tp}(\bar{a}, \emptyset, \mathscr{L}_\alpha, \mathfrak{M})$. Clearly, for every $\alpha < \omega_1$ the fragment $\mathscr{L}_\alpha$ is indeed countable. Let τ'' be $\tau' \cup \{\mathbf{C}_n, \mathbf{F}_n : n \in \omega\}$. We now expand $\mathfrak{M}$ to a τ''-structure $\mathfrak{M}''$ in the following way:

First we assume without loss of generality that $M = \omega_1$. Now

$$\mathfrak{M}'' = \langle \mathfrak{M}, <, E_0, \ldots, E_n, \ldots, F_0, \ldots, F_n, \ldots \rangle_{n\in\omega},$$

where

(a) $<$ is the natural ordering on ω_1.
(b) E_n is a $(2n + 1)$-ary relation and $(\alpha, \bar{a}, \bar{b}) \in E_n$ iff $\bar{a}, \bar{b} \in M_n$ and

$$\mathrm{tp}(\bar{a}, \emptyset, \mathscr{L}_\alpha, \mathfrak{M}) = \mathrm{tp}(\bar{b}, \emptyset, \mathscr{L}_\alpha, \mathfrak{M}).$$

(c) F_n is an $(n + 1)$-ary function with the finite ordinals as its range and $F_n(\alpha, \bar{a}) = F_n(\alpha, \bar{b})$ iff $(\alpha, \bar{a}, \bar{b}) \in E_n$. Such an F_n can be chosen because the number of $\mathscr{L}_\alpha$-types realized in $\mathfrak{M}$ is countable by our hypothesis.

We note the following facts:

Fact 1. Every E_n defines a family of equivalence relations $E_{\alpha,n}$ on n-tuples of $\mathfrak{M}$ indexed by the first argument.

Fact 2. If $\alpha < \beta$ then $E_{\beta,n}$ refines $E_{\alpha,n}$.

Fact 3. Each $E_{\alpha,n}$ has at most countably many equivalence classes.

Fact 4. $<$ is an ordering with a first element, which we call 0, and $(0, \bar{a}, \bar{b}) \in E_n$ iff the $\mathscr{L}_0$-types of $\bar{a}$ and $\bar{b}$ are equal.

Fact 5. If $(\alpha + 1, \bar{a}, \bar{b}) \in E_n$ then for every $c \in M$ there is a $d \in M$ such that

$$(\alpha, \bar{a}, c, \bar{b}, d) \in E_{n+1}.$$

Clearly, Facts 1–5 can be expressed by a sentence $\chi \in \mathscr{L}_{\omega_1\omega}[\tau'']$. To express Fact 3 we need the functions F_n.

Now we apply Theorem 3.3.1(ii) to the sentence $\psi \wedge \chi$. We get a model $\mathfrak{N}'' \models \psi \wedge \chi$ of cardinality ω_1 where $<$ contains a copy of the rationals. Put $\mathfrak{N} = \mathfrak{N}'' \upharpoonright \tau$. Let $\{d_n : n \in \omega\}$ be an infinite decreasing sequence of elements in $\mathfrak{N}''$. We use it to define equivalence relations E_n^+ on n-tuples of $\mathfrak{N}''$ by putting

$$(\bar{a}, \bar{b}) \in E_n^+ \text{ iff } \mathfrak{N}'' \models \mathbf{C}_n(d_m, \bar{a}, \bar{b})$$

for some $m \in \omega$. It is easy to check, that for this equivalence relation we have

Fact 6. If $(\bar{a}, \bar{b}) \in E_n^+$ then for every $c \in N$ there is a $d \in N$ such that $(\bar{a}, c, \bar{b}, d) \in E_{n+1}^+$; and

Fact 7. Each $E_{\alpha,n}$ has at most countably many equivalence classes.

We just use the fact that $\mathfrak{N}'' \models \psi \wedge \chi$ and the definition of E_n^+.

Using Fact 6 we can show by induction on φ:

Fact 8. For every $\varphi \in \mathscr{L}_{\omega_1\omega}[\tau]$, if $(\bar{a}, \bar{b}) \in E_n^+$ then $\mathfrak{N} \models \varphi(\bar{a})$ iff $\mathfrak{N}' \models \varphi(\bar{b})$.

This together with Fact 7 shows that in $\mathfrak{N}$ only countably many $\mathscr{L}_{\omega_1\omega}[\tau]$-types are satisfied. This ends the proof of (i).

To prove (ii) we repeat the same proof but change the definition of the fragments such as to include the constants required.

To prove (iii) we remark that $\psi \wedge \chi$ is a weak Scott sentence. To obtain a Scott sentence we apply Lemma 3.3.3. $\square$

3.4.4 Corollary (Shelah). *Let K be a* PC-*class in $\mathscr{L}_{\omega_1\omega}$ with at least one, but less than 2^{ω_1}, many models of cardinality ω_1. Then there is an uncountable model $\mathfrak{A} \in K$ which has a Scott sentence.*

This corollary was proved by different methods (admissible sets) in Makkai [1977] under the stronger hypothesis that there are less than 2^{ω} many models of cardinality ω_1.

We are now in a position to prove Theorem 3.1.1.

3.4.5 Proof of the Reduction Theorem. Assume that $2^{\omega} < 2^{\omega_1}$. Let K with $<_K$ be an abstract ω-presentable class over a vocabulary τ such that $I(K, \omega_1) < 2^{\omega_1}$. Let $\psi \in \mathscr{L}_{\omega_1\omega}[\tau']$ be the sentence defining K. By our assumption on K we can apply

Lemma 3.4.1 and find an uncountable model $\mathfrak{M} \models \psi$ such that the hypothesis of Theorem 3.4.2(ii) is satisfied. So we can use Theorem 3.4.2(iii) to find a model $\mathfrak{N} \models \psi$ of cardinality ω_1 such that $\mathfrak{N} \upharpoonright \tau$ has a Scott sentence σ.

We have to show that there is a ω-presentable abstract class K' with $<_{K'}$ over a vocabulary τ', $\tau \subset \tau'$ such that:

(i) if $\mathfrak{A} \in K'$ then $\mathfrak{A} \upharpoonright \tau \in K$;
(ii) if $\mathfrak{A}, \mathfrak{B} \in K'$ and $\mathfrak{A} <_{K'} \mathfrak{B}$ then $\mathfrak{A} \upharpoonright \tau <_K \mathfrak{B} \upharpoonright \tau$;
(iii) if $\mathfrak{A}, \mathfrak{B} \in K'$ and $\mathfrak{A} <_{K'} \mathfrak{B}$ then $\mathfrak{A} <_{\infty\omega} \mathfrak{B}$; and still
(iv) $I(K, \omega_1) \neq \varnothing$ iff $I(K', \omega_1) \neq \varnothing$.

So we put K' to be $\mathrm{Mod}(\psi \wedge \sigma)$. Clearly, (i) is satisfied. To define $<_{K'}$ we define it as an ω-presentable substructure relation such that $\mathfrak{A} <_{K'} \mathfrak{B}$ iff $\mathfrak{A} <_K \mathfrak{B}$ and $\mathfrak{A}$ is a *Karp*-substructure of $\mathfrak{B}$, applying Theorem 3.3.5(iii). Clearly, this ensures that (ii), (iii), and (iv) are now satisfied. ☐

3.5. How to Construct Super Limits

The purpose of this section is to give a brief survey on the difficulties in the proof of Theorem 3.1.7. Let us state it once more:

3.5.1 Theorem (Existence of Superlimits). *Assume that* $2^{\omega} < 2^{\omega_1} < 2^{\omega_2}$. *Let* K *with* $<_K$ *be an abstract* ω*-presentable class such that*:

(i) $I(K, \omega) = 1$; *and*
(ii) $1 \leq I(K, \omega_1) < 2^{\omega_1}$;
(iii) $I(K, \omega_2) < 2^{\omega_2}$.

Then there is a ω_1*-superlimit model* $\mathfrak{M}$ *in* K_{ω_1} *which is homogeneous and universal.*

3.5.2 Amalgamation and Joint Embedding Property in ω. First we observe that the unique countable model $\mathfrak{M}_\omega$ of K is a ω-superlimit, by Proposition 2.2.4, since K has uncountable models and is ω-categorical. Therefore, using Theorem 2.3.1, $\mathfrak{M}_\omega$ is an amalgamation basis for K_ω. Again by ω-categoricity, K_ω has the joint embedding property.

Now we are in a position to apply Theorem 2.1.8. We need the above hypothesis to ensure that $\omega_1 = \lambda = \lambda^{<\lambda}$. So there is a universal and homogeneous model $\mathfrak{M}$ in K_{ω_1}.

We would like next to prove the following:

3.5.3 Claim. $\mathfrak{M}$ *is a weak-limit.*

Note that Claim 3.5.3 is enough to prove Theorem 3.1.6 as pointed out immediately after Theorem 3.1.7.

Proof. We have to verify the conditions (a–d) of Definition 2.2.1. Clearly, the cardinality of $\mathfrak{M}$ is ω_1, so (a) is satisfied.

To verify (b), i.e., to show that $\mathfrak{M}$ is not maximal, we use a modification of Lemma 3.2.7, respectively, Theorem 3.2.3, stating that if $\mathfrak{M} \in K_{\omega_1}$ is universal and maximal, then $I(K, \omega_1) = 2^{\omega_1}$.

We recall property (c): Given $\mathfrak{N} \in K_{\omega_1}$ with $\mathfrak{M} <_K \mathfrak{N}$ there is $\mathfrak{M} \cong \mathfrak{M}'$ such that $\mathfrak{M} <_K \mathfrak{M}'$. To construct $\mathfrak{M}'$ we write $\mathfrak{N}$ as a union of an increasing K-chain of isomorphic copies of the ω-superlimit and reconstruct a universal and homogeneous model in K_{ω_1} along this chain. Then we use the uniqueness of the universal and homogeneous model (Theorem 2.1.8(iii)). □

Next we want to establish the following claim:

3.5.4 Claim. *$\mathfrak{M}$ is a (ω_1, ω_1)-limit.*

We only have to show that unions of K-chains of ω_1 many isomorphic copies of $\mathfrak{M}$ are again isomorphic to $\mathfrak{M}$. To see this, we show that such an union is again homogeneous. For this, we use the homogeneity of $\mathfrak{M}$ and the following lemma:

3.5.5 Lemma. *If $\mathfrak{M}_0, \mathfrak{M}_1 \in K_{\omega_1}$ are both homogeneous and $\mathfrak{A} \in K_\omega$ with $\mathfrak{A} <_K \mathfrak{M}_0$, then every K-embedding of $\mathfrak{A}$ into $\mathfrak{M}_1$ can be extended to an isomorphism from $\mathfrak{M}_0$ onto $\mathfrak{M}_1$.*

Proof. Besides homogeneity, we use that K is ω-categorical and that K also satisfies closure under directed systems. □

To end the proof of Claim 3.5.4 we apply the lemma cofinally often along the chain and use that every countable substructure already appears in an element of this chain. □

3.5.6 Types and Forcing. The main difficulty in the proof of Theorem 3.5.1 is to prove that it gives a (ω_1, ω)-limit. For this we need a better description of the homogeneous model $\mathfrak{M}$ in K_{ω_1}. We would like to build $\mathfrak{M}$ as a union of countable models $\mathfrak{A}_\alpha$, $\alpha < \omega_1$, such that in every $\mathfrak{A}_{\alpha+1}$ all the types over $\mathfrak{A}_\alpha$, satisfied in $\mathfrak{M}$, are already satisfied. This leads us to a natural, but rather complicated, definition of forcing, a corresponding definition of "types" and a machinery to apply techniques connected with non-forking, symmetry, and finite bases of types, stationarization, etc., as in Shelah [1978a].

3.5.7 The Big Two-Dimensional Picture. All this machinery is needed to cope with the following situation. Let $\mathfrak{M}_i$, $i \in \omega$ be a countable K-chain of isomorphic copies of $\mathfrak{M}$ and let each $\mathfrak{M}_i = \bigcup_\alpha \mathfrak{N}_{\alpha,i}$ be the union of countable models. To show that $\bigcup_i \mathfrak{M}_i \cong \mathfrak{M}$ we have to verify that various finite configurations of countable models in this system allow amalgamation within this system. This is needed to replace this two-dimensional presentation of $\bigcup_i \mathfrak{M}_i$ by an ω_1 long K-chain of countable models from the two-dimensional presentation, which will enable us to show that $\mathfrak{M} \cong \bigcup_i \mathfrak{M}_i$.

3.5.8 The Underlying Philosophy. The underlying philosophy in all of this is, that instead of types, as in first-order model theory, we have to deal with certain generalizations of amalgamation properties of countable structures. Proving the existence of uncountable structures with certain properties is then reduced to proving more and more complicated countable amalgamation properties.

A proof of Theorem 3.1.6 which does not use Theorem 3.5.1 can be found in Fuchino [1983]. There also the condition $2^{\omega_1} < 2^{\omega_2}$ is not needed.

3.5.9 A Gourmet End (Joint Work with Irit M. Manskleid). In the tradition of some of the books of this *Perspectives of Mathematical Logic*, I would like to conclude this last chapter with a gourmet treat. The following recipe is connected with my work with Saharon Shelah in two ways: In 1974, when we started to work together on abstract model theory, I also visited Florence, Italy. There I dined at Sabatini's, a restaurant renowned for its combination of Italian and French cooking. Italian cooking puts the emphasis on the main ingredients of a dish by letting them have their optimal gustatory and olfactory effects; French cooking is famous for refining the ingredients of a dish by the addition of ornamental, but dominant, components, especially sauces. The most exciting dish I tasted at Sabatini's was "vermicelli colla salsa di tartufi" (homemade, very thin spaghetti with a truffle cream sauce). Truffles are ugly, potato-shaped mushrooms, but inside they hide, like many of Shelah's proofs, a delicate core. In a multitude of attempts I tried to find appropriate truffles and to reconstruct the dish. Here is the result.

3.5.10 The Truffles (Fungus; Tuber, Hebrew: Kmehin). Truffles are famous, rare, and expensive, especially the French and Italian kind. They grow on calcarious ground in symbiosis with oaks, beeches, or some desert bushes. Less fancy truffles grow in North Africa, the Carmel mountain, and the Negev desert. They also grow, though rarely, in California and Oregon. These truffles are much cheaper but they are good enough, if pickled for one month in dry white wine with bay leaves. We need about 200 gr of them, after washing and peeling. If they come from sandy areas, such as the Carmel or the Negev, this is equivalent to more than half a kilo bought on the market.

3.5.11 The Vermicelli. Prepare a dough (standard pasta dough, possibly with half the flour whole grain). Let it rest. Using a pasta machine, roll as thin as possible and cut into the thinnest possible slices. Separate them by hand and let them dry for an hour. This is like counting to ω_1, naming every ordinal. You will get acquainted with every slice personally. Boil in water with salt and olive oil added. You need two tablespoons of olive oil per liter of water and half a liter of water per 100 gr of pasta.

3.5.12 The Sauce (The quantities are for half a kilo pasta). A quarter liter of sweet (fat) cream is heated with 100 gr of butter till the butter is melted. Add the truffles, chopped very thin. Simmer for about ten minutes. Add 150 gr of ground dry cheese (parmesan). Stir well till the cheese is melted like in cheese fondue. Add salt and fresh ground pepper.

3.5.13 Serving. When the pasta is ready (*al dente*, not too soft), pour it into a sieve, but do not rinse in cold water. Return the pasta into a heatable dish and add the hot sauce. Stir well and reheat if necessary. Eat and enjoy. Serves four to six.

3.5.14 Postscript. This recipe may look complicated. But here is another analogy to many of the proofs in this chapter: Once you are through, you understand that it was worth it, and moreover, that it was the appropriate way to do it.

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Note: Authors are listed in alphabetical order. In the case of multiple authors, the first-named author determines the alphabetical order for the listing, and the other, single authors are given only cross-references. Editors of volumes are not regarded as authors, but in entries involving edited volumes the editors names are included when known. Under an author, the arrangement of the listing is chronological, with works in preparation or in press at the end. Cross-references also occur at the end of an author's listings. It was intended that each entry should be be given in as complete a form as reasonable and without abbreviations. Unfortunately, a few entries remain incomplete.

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